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Ramanujan's identities and representation of integers by certain binary and quaternary quadratic forms

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Abstract We revisit old conjectures of Fermat and Euler regarding the representation of integers by binary quadratic form x^2+5y^2. Making use of Ramanujan's ${}_1\psi_1$ summation formula, we establish a new Lambert series identity for $\sum_{n,m=-\infty}^{\infty} q^{n^2+5m^2}$. Conjectures of Fermat and Euler are shown to follow easily from this new formula. But we do not stop there. Employing various formulas found in Ramanujan's notebooks and using a bit of ingenuity, we obtain a collection of new Lambert series for certain infinite products associated with quadratic forms such as x^2+6y^2, $2x^2+3y^2$, x^2+15y^2, $3x^2+5y^2$, x^2+27y^2, $x^2+5(y^2+z^2+w^2)$, $5x^2+y^2+z^2+w^2$. In the process, we find many new multiplicative *eta*-quotients and determine their coefficients.

Keywords Quadratic forms · q-series identities · *eta*-quotients · Multiplicative functions

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1 Introduction

A binary quadratic form (BQF) is a function

$$Q(x,y)=ax^2+bxy+cy^2$$

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with $a, b, c \in \mathbb{Z}$. It will be denoted by (a, b, c). We say that n is represented by (a, b, c) if there exist x and $y \in \mathbb{Z}$ such that $Q(x, y) = n$.

The representation theory of BQF has a long history that goes back to antiquity. Diophantus' Arithmeticae contains the following important example of composition of two forms:

$$\left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2.$$

Influenced by Diophantus, Fermat studied representations by $(1, 0, a)$. For $a = 1, 2, 3$, he proved a number of important results such as the following:

A prime p can be written as a sum of two squares iff $p \equiv 1 \pmod 4$.

We remark that representation by $(1, 0, 3)$ played an important role in Euler's proof of Fermat's Last Theorem in the case of $n = 3$.

Fermat realized that $(1, 0, 5)$ was very different from the previous cases $(1, 0, 1)$, $(1, 0, 2)$, and $(1, 0, 3)$ considered by him. He made the following conjecture:

If p and q are two primes that are congruent to 3 or $7 \pmod{20}$, then pq is representable by $(1, 0, 5)$.

Euler made two conjectures that were very similar to those of Fermat:

a. *Prime p is representable by $(1, 0, 5)$ iff $p \equiv 1$ or $9 \pmod{20}$.*
b. *If p is prime, then $2p$ is representable by $(1, 0, 5)$ iff $p \equiv 3$ or $7 \pmod{20}$.*

However, his next conjecture for $(1, 0, 27)$ contained an unexpected cubic residue condition:

Prime p is representable by $(1, 0, 27)$ iff $p \equiv 1 \pmod 3$ and 2 is a cubic residue modulo p.

Lagrange and Legendre initiated systematic study of quadratic forms. But it was Gauss who brought the theory of BQF to essentially its modern state. He introduced class form groups and genus theory for BQF. He proved Euler's conjecture for $(1, 0, 27)$ and in the process discovered a so-called cubic reciprocity law. Gauss' work makes it clear why $(1, 0, 27)$ is much harder to deal with than $(1, 0, 5)$. Indeed, a class form group with discriminant -20 consists of two inequivalent classes $(1, 0, 5)$ and $(2, 2, 3)$. These forms cannot represent the same integer. On the other hand, a class form group with discriminant -108 consists of three classes $(1, 0, 27)$, $(4, 2, 7)$, $(4, -2, 7)$. These forms belong to the same genus. That is, they may represent the same integer. An interested reader may want to consult [9] and [14] for the wealth of historical information and [19] for the latest development.

In his recent book, Number Theory in the Spirit of Ramanujan, Bruce Berndt discusses representation problem for $(1, 0, 1)$, $(1, 0, 2)$, $(1, 1, 1)$, $(1, 0, 3)$. Central to this approach is Ramanujan's ${}_1\psi_1$ summation formula which implies in particular that [7, p. 58, Eq. (3.2.90)]

$$\sum_{x, y \in \mathbb{Z}} q^{x^2 + y^2} = 1 + 4 \sum_{n \geq 1} \frac{q^n}{1 + q^{2n}}. \tag{1.1}$$

Using geometric series, it is straightforward to write the right-hand side of (1.1) as

$$\begin{aligned}1+4\sum_{n\geq1,m\geq0}(-1)^m q^n q^{2nm} &= 1+4\sum_{n\geq1,m\geq0}(-1)^m q^{n(2m+1)}\\ &= 1+4\sum_{n\geq1,m\geq1}\left(\frac{-4}{m}\right)q^{nm}\\ &= 1+4\sum_{n\geq1}\sum_{d\mid n}\left(\frac{-4}{d}\right)q^n,\end{aligned}$$

where we used the Kronecker symbol to be defined in the next section and the well-known formula

$$\left(\frac{-4}{n}\right)=\begin{cases}0 & \text{if } n \text{ is even},\\ (-1)^{(n-1)/2} & \text{if } n \text{ is odd}.\end{cases}$$

Let $r(n)$ be the number of representations of a positive integer n by quadratic form k^2+l^2. Suppose the prime factorization of n is given by

$$n=2^a\prod_{i=1}^{r}p_i^{v_i}\prod_{j=1}^{s}q_j^{w_j},$$

where $p_i\equiv 1 \pmod 4$ and $q_i\equiv -1 \pmod 4$. Using the fact that $\sum_{d\mid n}(\frac{-4}{d})$ is multiplicative, we find that

$$r(n)=4\prod_{i=1}^{r}(1+v_i)\prod_{j=1}^{s}\frac{1+(-1)^{w_j}}{2}. \tag{1.2}$$

The reader may wish to consult [2] for background on multiplicative functions, convolution of multiplicative functions, and Legendre's symbol. Clearly, Fermat's Theorem is an immediate corollary of (1.2).

The main object of this manuscript is to reveal new and exciting connections between the work of Ramanujan and the theory of quadratic forms. This paper is organized as follows.

We collect necessary definitions and formulas in Sect. 2.

In Sect. 3 we use the ${}_1\psi_1$ summation formula to prove new generalized Lambert series identities for

$$\sum_{n,m=-\infty}^{\infty}q^{n^2+5m^2} \quad\text{and}\quad \sum_{n,m=-\infty}^{\infty}q^{2n^2+2nm+3m^2}.$$

These results enable us to derive simple formulas for the number of representations of an integer n by $(1,0,5)$ and $(2,2,3)$. Conjectures of Fermat and Euler for $(1,0,5)$ are easy corollaries of these formulas. Our treatment of $(1,0,6)$ and $(2,0,3)$ in Sect. 4 is very similar. However, in addition to the ${}_1\psi_1$ summation formula, we need to use two cubic identities of Ramanujan. In Sect. 5 we treat $(1,0,15)$ and $(3,0,5)$. The surprise here is that we need to employ one of the forty identities of Ramanujan for the Rogers–Ramanujan functions. Section 6 deals with $(1,0,27)$ and $(4,2,7)$. We do not confine our discussion solely to BQF. In Sect. 7 we boldly treat quaternary forms $x^2+5(y^2+z^2+w^2)$ and $5x^2+y^2+z^2+w^2$. We conclude with a brief description of the prospects for future work.

2 Definitions and useful formulas

Throughout the manuscript we assume that q is a complex number with $|q|<1$. We adopt the standard notation

$$(a;q)_n := (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right),$$

$$(a;q)_\infty := \prod_{n=0}^{\infty}\left(1-aq^n\right),$$

$$E(q) := (q;q)_\infty.$$

Next, we recall Ramanujan's definition for a general theta function. Let

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1. \tag{2.1}$$

The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [6, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2}$$

Two important special cases of (2.1) are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left(-q;q^2\right)_\infty^2\left(q^2;q^2\right)_\infty = \frac{E^5(q^2)}{E^2(q^4)E^2(q)} \tag{2.3}$$

and

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=-\infty}^{\infty} q^{2n^2-n} = \left(-q;q^4\right)_\infty\left(-q^3;q^4\right)_\infty\left(q^4;q^4\right)_\infty = \frac{E^2(q^2)}{E(q)}. \tag{2.4}$$

The product representations in (2.3)–(2.4) are special cases of (2.2). We shall use the famous quintuple product identity, which, in Ramanujan's notation, takes the form [6, p. 80, Entry 28(iv)]

$$E(q)\frac{f(-a^2,-a^{-2}q)}{f(-a,-a^{-1}q)} = f\left(-a^3q,-a^{-3}q^2\right) + af\left(-a^{-3}q,-a^3q^2\right), \tag{2.5}$$

where a is any complex number.

Function $f(a,b)$ also satisfies a useful addition formula. For each nonnegative integer n, let

$$U_n := a^{n(n+1)/2} b^{n(n-1)/2} \quad \text{and} \quad V_n := a^{n(n-1)/2} b^{n(n+1)/2}.$$

Then [6, p. 48, Entry 31]

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{2.6}$$

From (2.6) with $n=2$ we obtain

$$f(a,b) = f\left(a^3 b, ab^3\right) + af\left(\frac{b}{a}, \frac{a}{b}(ab)^4\right). \tag{2.7}$$

A special case of (2.7) which we frequently use is

$$\varphi(q) = \varphi\left(q^4\right) + 2q\psi\left(q^8\right). \tag{2.8}$$

With $a=b=q$ and $n=3$, we also find that

$$\varphi(q) = \varphi\left(q^9\right) + 2qf\left(q^3, q^{15}\right). \tag{2.9}$$

Our proofs employ a well-known special case of Ramanujan's ${}_1\psi_1$ summation formula: If $|q| < |a| < 1$, then [6, p. 32, Entry 17]

$$E^3(q) \frac{f(-ab, -q/ab)}{f(-a, -q/a) f(-b, -q/b)} = \sum_{n=-\infty}^{\infty} \frac{a^n}{1 - bq^n}. \tag{2.10}$$

We frequently use the elementary result [6, p. 45, Entry 29]. If $ab = cd$, then

$$f(a,b) f(c,d) = f(ac, bd) f(ad, bc) + af\left(\frac{b}{c}, \frac{c}{b}abcd\right) f\left(\frac{b}{d}, \frac{d}{b}abcd\right). \tag{2.11}$$

Next, we recall that for an odd prime p, Legendre's Symbol $(\frac{n}{p})$ or $(n \mid p)$ is defined by

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } n \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

Kronecker's Symbol $(\frac{n}{m})$ is defined as follows:

$$\left(\frac{n}{k}\right) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k \text{ is a prime dividing } n, \\ \text{Legendre's symbol} & \text{if } k \text{ is an odd prime}; \end{cases}$$

$$\left(\frac{n}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}, \ n \equiv \pm 1 \pmod 8, \\ -1 & \text{if } n \text{ is odd}, \ n \equiv \pm 3 \pmod 8. \end{cases}$$

In general,

$$\left(\frac{n}{m}\right)=\prod_{i=1}^{s}\left(\frac{n}{p_i}\right) \quad \text{if } m=\prod_{i=1}^{s} p_i$$

is a prime factorization of m.

It is easy to show that $(\frac{a}{bc})=(\frac{a}{b})(\frac{a}{c})$ and $(\frac{ab}{c})=(\frac{a}{c})(\frac{b}{c})$. Hence, $(\frac{n}{m})$ is a completely multiplicative function of n and also of m.

3 Lambert series identities for $\sum_{n,m=-\infty}^{\infty} q^{n^2+5m^2}$

Theorem 3.1

$$\varphi(q)\varphi(q^5)=2\left\{\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{10n}}-\sum_{n=-\infty}^{\infty}\frac{q^{5n+2}}{1+q^{10n+4}}\right\} \tag{3.1}$$

$$=2\left\{\sum_{n=-\infty}^{\infty}\frac{q^{3n}}{1+q^{10n}}+\sum_{n=-\infty}^{\infty}\frac{q^{5n+1}}{1+q^{10n+2}}\right\} \tag{3.2}$$

$$=1+\sum_{n=1}^{\infty}\left(\frac{-20}{n}\right)\frac{q^n}{1-q^n}+\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{1+q^{2n}}. \tag{3.3}$$

Furthermore,

$$1+\sum_{n=1}^{\infty}\left(\frac{-20}{n}\right)\frac{q^n}{1-q^n}=\frac{E(q^2)E(q^4)E(q^5)E(q^{10})}{E(q)E(q^{20})} \tag{3.4}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{1+q^{2n}}=q\frac{E(q)E(q^2)E(q^{10})E(q^{20})}{E(q^4)E(q^5)}. \tag{3.5}$$

Proof Employing (2.10) with q, a, and b replaced by q^{10}, q, and -1, respectively, we find that

$$\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{10n}}=E^3(q^{10})\frac{f(q,q^9)}{f(-q,-q^9)f(1,q^{10})}. \tag{3.6}$$

From (2.10) we similarly find that

$$\sum_{n=-\infty}^{\infty}\frac{q^{5n+2}}{1+q^{10n+4}}=q^2E^3(q^{10})\frac{f(q,q^9)}{f(-q^5,-q^5)f(q^4,q^6)}. \tag{3.7}$$

From (2.11), with $a=c=-q^2$ and $b=d=q^3$, we obtain

$$f(-q^2,q^3)f(-q^2,q^3)=f(-q^5,-q^5)f(q^4,q^6)-q^2f(1,q^{10})f(-q,-q^9). \tag{3.8}$$

By (3.6), (3.7), and (3.8), we conclude that

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{q^n}{1+q^{10n}} - \sum_{n=-\infty}^{\infty} \frac{q^{5n+2}}{1+q^{10n+4}} \\
&\quad = E^3(q^{10}) \frac{f(q,q^9)}{f(-q,-q^9)f(1,q^{10})} - q^2 E^3(q^{10}) \frac{f(q,q^9)}{f(-q^5,-q^5)f(q^4,q^6)} \\
&\quad = \frac{E^3(q^{10})f(q,q^9)}{f(-q,-q^9)f(1,q^{10})f(-q^5,-q^5)f(q^4,q^6)} \\
&\qquad \times \{f(-q^5,-q^5)f(q^4,q^6) - q^2 f(1,q^{10})f(-q,-q^9)\} \\
&\quad = \frac{E^3(q^{10})f(q,q^9)}{f(-q,-q^9)f(1,q^{10})f(-q^5,-q^5)f(q^4,q^6)} f(-q^2,q^3)f(-q^2,q^3) \\
&\quad = \frac{1}{2}\varphi(q)\varphi(q^5),
\end{aligned}$$

after several applications of (2.2). Similarly, we find that

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1+q^{10n}} + \sum_{n=-\infty}^{\infty} \frac{q^{5n+1}}{1+q^{10n+2}} \\
&\quad = E^3(q^{10}) \frac{f(q^3,q^7)}{f(-q^3,-q^7)f(1,q^{10})} + q E^3(q^{10}) \frac{f(q^3,q^7)}{f(-q^5,-q^5)f(q^2,q^8)} \\
&\quad = \frac{E^3(q^{10})f(q^3,q^7)}{f(-q^3,-q^7)f(1,q^{10})f(-q^5,-q^5)f(q^2,q^8)} \\
&\qquad \times \{f(-q^5,-q^5)f(q^2,q^8) + q f(1,q^{10})f(-q^3,-q^7)\} \\
&\quad = \frac{E^3(q^{10})f(q^3,q^7)}{f(-q^3,-q^7)f(1,q^{10})f(-q^5,-q^5)f(q^2,q^8)} f(q,-q^4)f(q,-q^4) \\
&\quad = \frac{1}{2}\varphi(q)\varphi(q^5).
\end{aligned}$$

Before we move on, we would like to make the following:

Remark 3.2 The following generalized Lambert series identity for $\varphi(q)\varphi(q^5)$ is given in [8, Cor. 6.5]

$$\varphi(-q)\varphi(-q^5) = 2\sum_{k=-\infty}^{\infty} \frac{q^{k(5k+3)/2}}{1+q^{5k}} - 2q \sum_{k=-\infty}^{\infty} \frac{q^{k(5k+7)/2}}{1+q^{5k+2}}.$$

It would be interesting to find a direct proof that

$$\sum_{k=-\infty}^{\infty} (-1)^k \left(\frac{q^k}{1+q^{10k}} - \frac{q^{5k+2}}{1+q^{10k+4}} \right) = \sum_{k=-\infty}^{\infty} \frac{q^{k(5k+3)/2}}{1+q^{5k}} - q \sum_{k=-\infty}^{\infty} \frac{q^{k(5k+7)/2}}{1+q^{5k+2}}.$$

Next, we prove (3.3).

$$\sum_{n=-\infty}^{\infty}\left\{\frac{q^{5n+1}}{1+q^{10n+2}}-\frac{q^{5n+2}}{1+q^{10n+4}}\right\}$$
$$=\sum_{n=0}^{\infty}\left\{\frac{q^{5n+1}}{1+q^{10n+2}}-\frac{q^{5n+2}}{1+q^{10n+4}}-\frac{q^{5n+3}}{1+q^{10n+6}}+\frac{q^{5n+4}}{1+q^{10n+8}}\right\}$$
$$=\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{1+q^{2n}}. \tag{3.9}$$

Also,

$$\sum_{n=-\infty}^{\infty}\frac{q^n+q^{3n}}{1+q^{10n}}=1+\sum_{j\in\{1,3,7,9\}}\sum_{n=1}^{\infty}\frac{q^{jn}}{1+q^{10n}}$$
$$=1+\sum_{j\in\{1,3,7,9\}}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m q^{jn}q^{10nm}$$
$$=1+\sum_{j\in\{1,3,7,9\}}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m q^{n(10m+j)}$$
$$=1+\sum_{j\in\{1,3,7,9\}}\sum_{m=0}^{\infty}(-1)^m\frac{q^{10m+j}}{1-q^{10m+j}}$$
$$=1+\sum_{n=1}^{\infty}\left(\frac{-20}{n}\right)\frac{q^n}{1-q^n}. \tag{3.10}$$

Now using (3.9) and (3.10) together with (3.1) and (3.2), we see that (3.3) is proved.

Equations (3.4) and (3.5) are essentially given in [17, Eqs. 3.2 and 3.29]. Moreover, the *eta*-quotients that appear in these equations are included in a list of certain multiplicative functions determined by Martin [13]. We should emphasize that (3.1), (3.2), and (3.3) are new. We observe directly that the coefficients of the two Lambert series in (3.3) are multiplicative and that they differ at most by a sign. This enables us to compute the coefficients of $\varphi(q)\varphi(q^5)$. □

Corollary 3.3 *Let $a(n)$ be the number of representations of a positive integer n by quadratic form k^2+5l^2. If the prime factorization of n is given by*

$$n=2^a5^b\prod_{i=1}^{r}p_i^{v_i}\prod_{j=1}^{s}q_j^{w_j},$$

where $p_i \equiv 1, 3, 7,$ *or* $9 \pmod{20}$ *and* $q_i \equiv 11, 13, 17,$ *or* $19 \pmod{20}$, *then*

$$a(n) = \left(1 + (-1)^{a+t}\right) \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \frac{1 + (-1)^{w_j}}{2}, \tag{3.11}$$

where t *is the number of prime factors of* n, *counting multiplicity, that are congruent to* 3 *or* $7 \pmod{20}$.

Proof Observe that

$$\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{1+q^{2n}} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m \left(\frac{n}{5}\right) q^{n(2m+1)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{-4}{m}\right) \left(\frac{n}{5}\right) q^{nm}$$
$$= \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{-4}{d}\right) \left(\frac{n/d}{5}\right)\right) q^n. \tag{3.12}$$

Similarly,

$$\sum_{n=1}^{\infty} \left(\frac{-20}{n}\right) \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{-20}{d}\right)\right) q^n. \tag{3.13}$$

Define

$$b(n) = \sum_{d|n} \left(\frac{-20}{d}\right) \quad \text{and} \quad c(n) = \sum_{d|n} \left(\frac{-4}{d}\right) \left(\frac{n/d}{5}\right). \tag{3.14}$$

We have by (3.3) that $a(n) = b(n) + c(n)$. Clearly both $b(n)$ and $c(n)$ are multiplicative functions. Therefore one only needs to find their values at prime powers. It is easy to check that for a prime p,

$$b(p^\alpha) = \begin{cases} 1 & \text{if } p = 2 \text{ or } 5, \\ 1+\alpha & \text{if } p \equiv 1, 3, 7, 9 \pmod{20}, \\ \frac{1+(-1)^\alpha}{2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}, \end{cases} \tag{3.15}$$

and

$$c(p^\alpha) = \begin{cases} (-1)^\alpha & \text{if } p = 2, \\ 1 & \text{if } p = 5, \\ 1+\alpha & \text{if } p \equiv 1, 9 \pmod{20}, \\ (-1)^\alpha (1+\alpha) & \text{if } p \equiv 3, 7 \pmod{20}, \\ \frac{1+(-1)^\alpha}{2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases} \tag{3.16}$$

Equivalent reformulations of (3.11) can also be found in [10, p. 84, Ex. 1] and [12, Thr. 7]. We should remark that (3.11) implies conjectures of Fermat and Euler for $(1, 0, 5)$ stated in the introduction. The last two equations immediately imply (3.11). □

We now determine the representations of integers by the quadratic form $(2, 2, 3)$ and make some further observations.

Corollary 3.4 *Let $d(n)$ be a number of representations of a positive integer n by the quadratic form $2k^2+2kl+3l^2$. If the prime factorization of n is given by*

$$n = 2^a 5^b \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j},$$

where $p_i \equiv 1, 3, 7,$ or $9 \pmod{20}$ and $q_i \equiv 11, 13, 17,$ or $19 \pmod{20}$, then

$$d(n) = \left(1-(-1)^{a+t}\right) \prod_{i=1}^{r} (1+v_i) \prod_{j=1}^{s} \frac{1+(-1)^{w_j}}{2}, \tag{3.17}$$

where t is the number of prime factors of n, counting multiplicity, that are congruent to 3 or $7 \pmod{20}$.

Proof Recall that

$$\sum_{n=0}^{\infty} a(n) q^n := \sum_{k,l=-\infty}^{\infty} q^{k^2+5l^2}.$$

By comparing (3.11) and (3.17), it suffices to show that $d(n) = a(2n)$ for all $n \in \mathbb{N}$. To that end we observe

$$\begin{aligned}
\sum_{n=0}^{\infty} d(n) q^n &= \sum_{n,m=-\infty}^{\infty} q^{2n^2+2nm+3m^2} \\
&= \sum_{n,m=-\infty}^{\infty} q^{2n^2+2n(2m)+3(2m)^2} + \sum_{n,m=-\infty}^{\infty} q^{2n^2+2n(2m+1)+3(2m+1)^2} \\
&= \sum_{n,m=-\infty}^{\infty} q^{2((n+m)^2+5m^2)} + \sum_{n,m=-\infty}^{\infty} q^{\frac{(2n+2m+1)^2+5(2m+1)^2}{2}} \\
&= \sum_{n,m=-\infty}^{\infty} q^{\frac{(2n)^2+5(2m)^2}{2}} + \sum_{n,m=-\infty}^{\infty} q^{\frac{(2n+1)^2+5(2m+1)^2}{2}} \\
&= \sum_{n=0}^{\infty} a(2n) q^n.
\end{aligned}$$

□

By (3.3), (3.11), and (3.17) we find that

$$\sum_{n,m=-\infty}^{\infty} q^{2n^2+2nm+3m^2} = 1 + \sum_{n=1}^{\infty} \left(\frac{-20}{n}\right) \frac{q^n}{1-q^n} - \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{1+q^{2n}}. \tag{3.18}$$

Also by adding identities in (3.3) and (3.18), we conclude that

$$\sum_{n,m=-\infty}^{\infty} q^{n^2+5m^2} + \sum_{n,m=-\infty}^{\infty} q^{2n^2+2nm+3m^2} = 2 + 2\sum_{n=1}^{\infty} \left(\frac{-20}{n}\right) \frac{q^n}{1-q^n}.$$

This last equation is a special case of Dirichlet's formula [18, p. 123, Thr. 4]. Comparing (3.11) and (3.17), we see that $a(n)d(n) = 0$. This means that a positive integer cannot be represented by $(1, 0, 5)$ and $(2, 2, 3)$ at the same time.

We end this section by proving a Lambert series representation for $\psi(q)\psi(q^5)$.

Theorem 3.5

$$\psi(q)\psi\left(q^5\right) = \sum_{n=-\infty}^{\infty} \frac{q^{3n}+q^{7n+1}}{1-q^{20n+5}} = \sum_{n=-\infty}^{\infty} \frac{q^n+q^{9n+6}}{1-q^{20n+15}}. \tag{3.19}$$

Proof By two applications of (2.10) with q replaced by q^{20}, $a = q^3, q^7$, and $b = q^5$, we find that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{q^{3n}+q^{7n+1}}{1-q^{20n+5}} &= E^3\left(q^{20}\right) \frac{f(-q^8,-q^{12})}{f(-q^3,-q^{17})f(-q^5,-q^{15})} \\
&\quad + qE^3\left(q^{20}\right) \frac{f(-q^8,-q^{12})}{f(-q^7,-q^{13})f(-q^5,-q^{15})} \\
&= \frac{E^3(q^{20})f(-q^8,-q^{12})}{f(-q^5,-q^{15})f(-q^3,-q^{17})f(-q^7,-q^{13})} \\
&\quad \times \left\{f\left(-q^7,-q^{13}\right) + qf\left(-q^3,-q^{17}\right)\right\} \\
&= \frac{E^3(q^{20})f(-q^8,-q^{12})}{f(-q^5,-q^{15})f(-q^3,-q^{17})f(-q^7,-q^{13})} \\
&\quad \times f\left(q,-q^4\right),
\end{aligned} \tag{3.20}$$

where in the last step we use (2.7) with $a = q$ and $b = -q^4$. It is now easy to verify by several applications of (2.2) that (3.20) is equal to $\psi(q)\psi(q^5)$. The proof of the second representation given in (3.19) is very similar to that of the first one, and so we forego its proof. □

4 Lambert series identities for $\sum_{n,m=-\infty}^{\infty} q^{n^2+6m^2}$ and $\sum_{n,m=-\infty}^{\infty} q^{2n^2+3m^2}$

Theorem 4.1 *Let*

$$P(q) := \frac{E(q^2)E(q^3)E(q^8)E(q^{12})}{E(q)E(q^{24})} \quad \textit{and} \quad Q(q) := q\frac{E(q)E(q^4)E(q^6)E(q^{24})}{E(q^3)E(q^8)}. \tag{4.1}$$

Then

$$P(q)=\sum_{n=-\infty}^{\infty}\frac{q^n+q^{5n}}{1+q^{12n}}=1+\sum_{n=1}^{\infty}\left(\frac{-6}{n}\right)\frac{q^n}{1-q^n}, \tag{4.2}$$

$$Q(q)=\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}-q^{9n+3}}{1+q^{12n+4}}=\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n(1-q^{2n})}{1+q^{4n}}. \tag{4.3}$$

Moreover,

$$\begin{align}
\varphi(q)\varphi\bigl(q^6\bigr)&=P(q)+Q(q) \tag{4.4}\\
&=\sum_{n=-\infty}^{\infty}\frac{q^n+q^{5n}}{1+q^{12n}}+\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}-q^{9n+3}}{1+q^{12n+4}} \tag{4.5}\\
&=2\left\{\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{12n}}-\sum_{n=-\infty}^{\infty}\frac{q^{9n+3}}{1+q^{12n+4}}\right\} \tag{4.6}\\
&=2\left\{\sum_{n=-\infty}^{\infty}\frac{q^{5n}}{1+q^{12n}}+\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}}{1+q^{12n+4}}\right\} \tag{4.7}\\
&=1+\sum_{n=1}^{\infty}\left(\frac{-6}{n}\right)\frac{q^n}{1-q^n}+\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n(1-q^{2n})}{1+q^{4n}} \tag{4.8}
\end{align}$$

and

$$\begin{align}
\varphi\bigl(q^2\bigr)\varphi\bigl(q^3\bigr)&=P(q)-Q(q) \tag{4.9}\\
&=\sum_{n=-\infty}^{\infty}\frac{q^n+q^{5n}}{1+q^{12n}}-\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}-q^{9n+3}}{1+q^{12n+4}} \tag{4.10}\\
&=2\left\{\sum_{n=-\infty}^{\infty}\frac{q^n}{1+q^{12n}}-\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}}{1+q^{12n+4}}\right\} \tag{4.11}\\
&=2\left\{\sum_{n=-\infty}^{\infty}\frac{q^{5n}}{1+q^{12n}}+\sum_{n=-\infty}^{\infty}\frac{q^{9n+3}}{1+q^{12n+4}}\right\} \tag{4.12}\\
&=1+\sum_{n=1}^{\infty}\left(\frac{-6}{n}\right)\frac{q^n}{1-q^n}-\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n(1-q^{2n})}{1+q^{4n}}. \tag{4.13}
\end{align}$$

Proof By employing (2.11) with $a=q$, $b=q^{11}$, $c=-q^5$, and $d=-q^7$, we find that

$$f\bigl(q,q^{11}\bigr)f\bigl(-q^5,-q^7\bigr)=f\bigl(-q^6,-q^{18}\bigr)f\bigl(-q^8,-q^{16}\bigr)$$

$$+qf\big(-q^4,-q^{20}\big)f\big(-q^6,-q^{18}\big)$$
$$=\psi\big(-q^6\big)E\big(q^8\big)+q\psi\big(-q^6\big)f\big(-q^4,-q^{20}\big). \quad (4.14)$$

By two applications of (2.10) with q replaced by q^{12}, $a=q,q^5$, $b=-1$, and by (4.14), we find that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{q^n+q^{5n}}{1+q^{12n}} &= E^3\big(q^{12}\big)\frac{f(q,q^{11})}{f(1,q^{12})f(-q,-q^{11})}\\ &\quad+E^3\big(q^{12}\big)\frac{f(q^5,q^7)}{f(1,q^{12})f(-q^5,-q^7)}\\ &=\frac{E^3(q^{12})}{f(1,q^{12})f(-q,-q^{11})f(-q^5,-q^7)}\\ &\quad\times\big\{f\big(q,q^{11}\big)f\big(-q^5,-q^7\big)+f\big(-q,-q^{11}\big)f\big(q^5,q^7\big)\big\}\\ &=2\frac{E^3(q^{12})}{f(1,q^{12})f(-q,-q^{11})f(-q^5,-q^7)}\psi\big(-q^6\big)E\big(q^8\big)\\ &=\frac{E(q^2)E(q^3)E(q^8)E(q^{12})}{E(q)E(q^{24})},\end{aligned} \quad (4.15)$$

after several applications of (2.2).

By (2.7), we observe that

$$E(q)=f\big(-q,-q^2\big)=f\big(q^5,q^7\big)-qf\big(q,q^{11}\big). \quad (4.16)$$

Arguing as above and using (4.16), we conclude that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}-q^{9n+3}}{1+q^{12n+4}} &= qE^3\big(q^{12}\big)\frac{f(q^5,q^7)}{f(-q^3,-q^9)f(q^4,q^8)}\\ &\quad-q^3E^3\big(q^{12}\big)\frac{f(q^{-1},q^{13})}{f(-q^3,-q^9)f(q^4,q^8)}\\ &=qE^3\big(q^{12}\big)\frac{f(q^5,q^7)}{f(-q^3,-q^9)f(q^4,q^8)}\\ &\quad-q^2E^3\big(q^{12}\big)\frac{f(q,q^{11})}{f(-q^3,-q^9)f(q^4,q^8)}\\ &=q\frac{E^3(q^{12})}{\psi(-q^3)f(q^4,q^8)}\big\{f\big(q^5,q^7\big)-qf\big(q,q^{11}\big)\big\}\\ &=q\frac{E^3(q^{12})E(q)}{\psi(-q^3)f(q^4,q^8)}=q\frac{E(q)E(q^4)E(q^6)E(q^{24})}{E(q^3)E(q^8)},\end{aligned} \quad (4.17)$$

after several applications of (2.2).

The proofs of the second part of (4.2) and that of (4.3) are similar to those of (3.9) and (3.10), and so we omit their proofs.

We prove (4.4) and (4.9) simultaneously by proving

$$2P=\varphi(q)\varphi\bigl(q^6\bigr)+\varphi\bigl(q^2\bigr)\varphi\bigl(q^3\bigr) \tag{4.18}$$

and

$$2Q=\varphi(q)\varphi\bigl(q^6\bigr)-\varphi\bigl(q^2\bigr)\varphi\bigl(q^3\bigr). \tag{4.19}$$

First we prove (4.19). We will need two identities of Ramanujan [6, p. 232], namely

$$2\frac{\psi^3(q)}{\psi(q^3)}=\frac{\varphi^3(q)}{\varphi(q^3)}+\frac{\varphi^3(-q^2)}{\varphi(-q^6)} \tag{4.20}$$

and

$$4q\psi\bigl(q^2\bigr)\psi\bigl(q^6\bigr)=\varphi(q)\varphi\bigl(q^3\bigr)-\varphi(-q)\varphi\bigl(-q^3\bigr). \tag{4.21}$$

From (4.21) with (2.8) we find that

$$\begin{aligned}4q\psi\bigl(q^2\bigr)\psi\bigl(q^6\bigr)&=\varphi(q)\varphi\bigl(q^3\bigr)-\varphi(-q)\varphi\bigl(-q^3\bigr)\\&=\bigl(\varphi\bigl(q^4\bigr)+2q\psi\bigl(q^8\bigr)\bigr)\bigl(\varphi\bigl(q^{12}\bigr)+2q^3\psi\bigl(q^{24}\bigr)\bigr)\\&\quad-\bigl(\varphi\bigl(q^4\bigr)-2q\psi\bigl(q^8\bigr)\bigr)\bigl(\varphi\bigl(q^{12}\bigr)-2q^3\psi\bigl(q^{24}\bigr)\bigr)\\&=4q\bigl\{\psi\bigl(q^8\bigr)\varphi\bigl(q^{12}\bigr)+q^2\varphi\bigl(q^4\bigr)\psi\bigl(q^{24}\bigr)\bigr\}.\end{aligned} \tag{4.22}$$

Upon replacing q^2 by q in (4.22), we conclude that

$$\psi(q)\psi\bigl(q^3\bigr)=\psi\bigl(q^4\bigr)\varphi\bigl(q^6\bigr)+q\varphi\bigl(q^2\bigr)\psi\bigl(q^{12}\bigr). \tag{4.23}$$

Similarly,

$$\begin{aligned}&\varphi(q)\varphi\bigl(-q^3\bigr)-\varphi(-q)\varphi\bigl(q^3\bigr)\\&\quad=\bigl(\varphi\bigl(q^4\bigr)+2q\psi\bigl(q^8\bigr)\bigr)\bigl(\varphi\bigl(q^{12}\bigr)-2q^3\psi\bigl(q^{24}\bigr)\bigr)\\&\qquad-\bigl(\varphi\bigl(q^4\bigr)-2q\psi\bigl(q^8\bigr)\bigr)\bigl(\varphi\bigl(q^{12}\bigr)+2q^3\psi\bigl(q^{24}\bigr)\bigr)\\&\quad=4q\bigl\{\psi\bigl(q^8\bigr)\varphi\bigl(q^{12}\bigr)-q^2\varphi\bigl(q^4\bigr)\psi\bigl(q^{24}\bigr)\bigr\}\\&\quad=4q\psi\bigl(-q^2\bigr)\psi\bigl(-q^6\bigr),\end{aligned} \tag{4.24}$$

where in the last step we used (4.23) with q replaced by $-q^2$. We are now ready to prove (4.19). Recall that $Q(q)$ is defined by (4.1). By several applications of (2.2), we see that (4.19) is equivalent to

$$2q\frac{\psi(-q)\psi(-q^2)\psi(-q^3)\psi(-q^6)}{\psi(q^4)\varphi(-q^3)}=\varphi(q)\varphi\bigl(q^6\bigr)-\varphi\bigl(q^2\bigr)\varphi\bigl(q^3\bigr), \tag{4.25}$$

or

$$\begin{aligned}
&2q\psi(-q)\psi(-q^2)\psi(-q^3)\psi(-q^6)\\
&\quad=\varphi(q)\varphi(q^6)\psi(q^4)\varphi(-q^3)-\varphi(q^2)\varphi(q^3)\psi(q^4)\varphi(-q^3)\\
&\quad=\varphi(q)\varphi(-q^3)\psi(q^4)\varphi(q^6)-\psi^2(q^2)\varphi^2(-q^6),
\end{aligned}\tag{4.26}$$

where we used the trivial identities

$$\varphi(q)\varphi(-q)=\varphi^2(-q^2)\quad\text{and}\quad\psi^2(q)=\psi(q^2)\phi(q).\tag{4.27}$$

When we employ (4.24) on the far left-hand side of (4.26) and (4.23) on the right-hand side of (4.26), we find that

$$\begin{aligned}
&\psi(-q)\psi(-q^3)\big(\varphi(q)\varphi(-q^3)-\varphi(-q)\varphi(q^3)\big)\\
&\quad=\big(\psi(q)\psi(q^3)+\psi(-q)\psi(-q^3)\big)\varphi(q)\varphi(-q^3)-2\psi^2(q^2)\varphi^2(-q^6).
\end{aligned}\tag{4.28}$$

Upon cancellation, we see that

$$-\psi(-q)\psi(-q^3)\varphi(-q)\varphi(q^3)=\psi(q)\psi(q^3)\varphi(q)\varphi(-q^3)-2\psi^2(q^2)\varphi^2(-q^6).\tag{4.29}$$

Next we multiply both sides of (4.29) with $\frac{\varphi^2(q)}{\psi(q)\psi(q^3)\varphi^2(-q^6)}$ and obtain, after several applications of (2.2), that

$$-\frac{\varphi^3(-q^2)}{\varphi(-q^6)}=\frac{\varphi^3(q)}{\varphi(q^3)}-2\frac{\psi^3(q)}{\psi(q^3)},\tag{4.30}$$

which is (4.20). Hence the proof of (4.19) is complete.

The proof of (4.18) is very similar to that of (4.19). Recall that $Q(q)$ is defined by (4.1). By several applications of (2.2), we see that (4.18) is equivalent to

$$2\frac{\psi(-q)\psi(-q^2)\psi(-q^3)\psi(-q^6)}{\psi(q^{12})\varphi(-q)}=\varphi(q)\varphi(q^6)+\varphi(q^2)\varphi(q^3)\tag{4.31}$$

or

$$\begin{aligned}
&\psi(-q)\psi(-q^2)\psi(-q^3)\psi(-q^6)\\
&\quad=\varphi(q)\varphi(q^6)\psi(q^{12})\varphi(-q)+\varphi(q^2)\varphi(q^3)\psi(q^{12})\varphi(-q)\\
&\quad=\varphi^2(-q^2)\psi^2(q^6)+\varphi(-q)\varphi(q^3)\varphi(q^2)\psi(q^{12}).
\end{aligned}\tag{4.32}$$

If we employ (4.24) on the far left-hand side of (4.32), and (4.23) on the right-hand side of (4.32) and multiply both sides by $2q$, we find that

$$\begin{aligned}
&\psi(-q)\psi(-q^3)\big(\varphi(q)\varphi(-q^3)-\varphi(-q)\varphi(q^3)\big)\\
&\quad=2q\psi^2(q^6)\varphi^2(-q^2)+\varphi(-q)\varphi(q^3)\big(\psi(q)\psi(q^3)-\psi(-q)\psi(-q^3)\big).
\end{aligned}\tag{4.33}$$

Upon cancellation, we find that

$$\psi(-q)\psi(-q^3)\varphi(q)\varphi(-q^3)=\psi(q)\psi(q^3)\varphi(-q)\varphi(q^3)+2q\psi^2(q^6)\varphi^2(-q^2). \tag{4.34}$$

It is easy to see that (4.34) and (4.29) are "reciprocals" of each other. For related definitions and modular equations corresponding to (4.18) and (4.19), see [6, p. 230, Entry 5 (i)]. Hence, the proof of (4.18) is complete.

As an immediate corollary of (4.18) and (4.19), we note the following two interesting theta function identities:

$$\frac{\varphi(q)\varphi(q^6)-\varphi(q^2)\varphi(q^3)}{\varphi(q)\varphi(q^6)+\varphi(q^2)\varphi(q^3)}=q\frac{\varphi(-q)\psi(q^{12})}{\varphi(-q^3)\psi(q^4)} \tag{4.35}$$

and

$$\begin{aligned}\varphi^2(q)\varphi^2(q^6)-\varphi^2(q^2)\varphi^2(q^3)&=4qE(q^2)E(q^4)E(q^6)E(q^{12})\\&=4q\psi(q)\psi(-q)\psi(-q^3)\psi(-q^6).\end{aligned}$$

Identities (4.2) and (4.3), together with (4.4) and (4.9), clearly imply (4.5), (4.8), (4.10), and (4.13). To prove the remaining identities (4.6), (4.7), (4.11), and (4.12), one only needs to prove that

$$\sum_{n=-\infty}^{\infty}\frac{q^n-q^{5n}}{1+q^{12n}}=\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}+q^{9n+3}}{1+q^{12n+4}}. \tag{4.36}$$

Arguing as in (4.15) and (4.17), one can easily show that

$$\sum_{n=-\infty}^{\infty}\frac{q^n-q^{5n}}{1+q^{12n}}=\sum_{n=-\infty}^{\infty}\frac{q^{3n+1}+q^{9n+3}}{1+q^{12n+4}}=q\frac{E(q^2)E(q^3)E(q^4)E(q^{24})}{E(q)E(q^8)}. \tag{4.37}$$

Hence, the proof of the Theorem 4.1 is complete. □

Corollary 4.2 *Let $a(n)$ and $b(n)$ be the number of representations of a positive integer n by quadratic form k^2+6l^2 and $2k^2+3l^2$, respectively. If the prime factorization of n is given by*

$$n=2^a3^b\prod_{i=1}^{r}p_i^{v_i}\prod_{j=1}^{s}q_j^{w_j},$$

where $p_i\equiv 1,5,7,$ *or* $11 \pmod{24}$ *and* $q_i\equiv 13,17,19,$ *or* $23 \pmod{24}$, *then*

$$a(n)=\left(1+(-1)^{a+b+t}\right)\prod_{i=1}^{r}(1+v_i)\prod_{j=1}^{s}\frac{1+(-1)^{w_j}}{2} \tag{4.38}$$

and

$$b(n)=\left(1-(-1)^{a+b+t}\right)\prod_{i=1}^{r}(1+v_i)\prod_{j=1}^{s}\frac{1+(-1)^{w_j}}{2}, \tag{4.39}$$

where t is a number of prime factors of n, counting multiplicity, that are congruent to 5 *or* 11 (mod 24).

Proof Observe that

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^{n}(1-q^{2n})}{1+q^{4n}} &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^{m}\left(\frac{n}{3}\right)\left(q^{n(4m+1)}-q^{n(4m+3)}\right)\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m}\left(\frac{n}{3}\right)\left(q^{n(4m-1)}-q^{n(4m-3)}\right)\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{2}\right)\left(\frac{n}{3}\right)q^{nm}\\
&= \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{d}{2}\right)\left(\frac{n/d}{3}\right)\right)q^{n}.
\end{aligned}$$

Similarly,

$$\sum_{n=1}^{\infty}\left(\frac{-6}{n}\right)\frac{q^{n}}{1-q^{n}} = \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{-6}{d}\right)\right)q^{n}.$$

Define

$$c(n)=\sum_{d\mid n}\left(\frac{-24}{d}\right) \quad \text{and} \quad d(n)=\sum_{d\mid n}\left(\frac{d}{2}\right)\left(\frac{n/d}{3}\right).$$

Equations (4.8) and (4.13) imply that $a(n)=c(n)+d(n)$ and $b(n)=c(n)-d(n)$. Clearly both $c(n)$ and $d(n)$ are multiplicative functions. Therefore, one only needs to find their values at prime powers. It is easy to check that for a prime p,

$$c(p^{\alpha}) = \begin{cases} 1 & \text{if } p=2 \text{ or } 3,\\ 1+\alpha & \text{if } p\equiv 1,5,7,11 \pmod{24},\\ \frac{1+(-1)^{\alpha}}{2} & \text{if } p\equiv 13,17,19,23 \pmod{24} \end{cases} \tag{4.40}$$

and

$$d(p^{\alpha}) = \begin{cases} (-1)^{\alpha} & \text{if } p=2 \text{ or } 3,\\ 1+\alpha & \text{if } p\equiv 1,7 \pmod{24},\\ (-1)^{\alpha}(1+\alpha) & \text{if } p\equiv 5,11 \pmod{24},\\ \frac{1+(-1)^{\alpha}}{2} & \text{if } p\equiv 13,17,19,23 \pmod{24}. \end{cases} \tag{4.41}$$

From these two equations (4.38) and (4.39) are immediate. Equivalent reformulations of (4.38) and (4.39) can also be found in [10, p. 84, Ex. 2] and [12, Thr. 7]. □

5 Lambert series identities for $\sum_{n,m=-\infty}^{\infty} q^{n^2+15m^2}$ and $\sum_{n,m=-\infty}^{\infty} q^{3n^2+5m^2}$

Theorem 5.1 *Let*

$$P(q) := \frac{E(q)E(q^6)E(q^{10})E(q^{15})}{E(q^2)E(q^{30})} \quad \text{and} \quad Q(q) := q\frac{E(q^2)E(q^3)E(q^5)E(q^{30})}{E(q^6)E(q^{10})}. \tag{5.1}$$

Then

$$P(q) = 1 - \sum_{n=1}^{\infty} \left(\frac{-15}{n}\right) \frac{q^n}{1+q^n}, \tag{5.2}$$

$$Q(q) = \sum_{n=1}^{\infty} \left(\frac{5}{n}\right) \frac{q^n(1+q^n)}{1+q^{3n}}. \tag{5.3}$$

Moreover,

$$\varphi(-q)\varphi(-q^{15}) = P(q) - Q(q) \tag{5.4}$$

$$= 1 - \sum_{n=1}^{\infty} \left(\frac{-15}{n}\right) \frac{q^n}{1+q^n} - \sum_{n=1}^{\infty} \left(\frac{5}{n}\right) \frac{q^n(1+q^n)}{1+q^{3n}} \tag{5.5}$$

and

$$\varphi(-q^3)\varphi(-q^5) = P(q) + Q(q) \tag{5.6}$$

$$= 1 - \sum_{n=1}^{\infty} \left(\frac{-15}{n}\right) \frac{q^n}{1+q^n} + \sum_{n=1}^{\infty} \left(\frac{5}{n}\right) \frac{q^n(1+q^n)}{1+q^{3n}}. \tag{5.7}$$

Proof Identities (5.2), (5.4), and (5.6) were observed by Ramanujan [6, p. 379, Entry 10 (vi)], [22, Eq. 50] and [6, p. 377, Entry 9 (v), (vi)]. We prove (5.3).
It is easy to observe that

$$\sum_{n=1}^{\infty} \left(\frac{5}{n}\right) \frac{q^n(1+q^n)}{1+q^{3n}} = \sum_{n=-\infty}^{\infty} \frac{q^{5n+1}+q^{10n+2}}{1+q^{15n+3}} - \sum_{n=-\infty}^{\infty} \frac{q^{5n+2}+q^{10n+4}}{1+q^{15n+6}}. \tag{5.8}$$

Next by four applications of (2.10) on the right-hand side of (5.8), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(\frac{5}{n}\right) \frac{q^n(1+q^n)}{1+q^{3n}} \\
&\quad = E^3(q^{15})\left\{q\frac{f(q^7,q^8)}{E(q^5)f(q^3,q^{12})} + q^2\frac{f(q^2,q^{13})}{E(q^5)f(q^3,q^{12})}\right\} \\
&\qquad - E^3(q^{15})\left\{q^2\frac{f(q^4,q^{11})}{E(q^5)f(q^6,q^9)} + q^3\frac{f(q,q^{14})}{E(q^5)f(q^6,q^9)}\right\}
\end{aligned}$$

$$= q\frac{E^3(q^{15})}{E(q^5)f(q^3,q^{12})f(q^6,q^9)}\{(f(q^7,q^8)+qf(q^2,q^{13}))f(q^6,q^9) \\ - q(f(q^4,q^{11})+qf(q,q^{14}))f(q^3,q^{12})\}. \tag{5.9}$$

Now we employ (2.11) for each term of (5.9) inside the parenthesis; the identity in (5.9) now becomes

$$q\frac{E^3(q^{15})}{E(q^5)f(q^3,q^{12})f(q^6,q^9)} \\ \times \{(f(q^{14},q^{16})-q^2f(q^4,q^{26}))(f(q^{13},q^{17})-qf(q^7,q^{23})) \\ + q(f(q^8,q^{22})-q^2f(q^2,q^{28}))(f(q^{11},q^{19})-q^3f(q,q^{29}))\}. \tag{5.10}$$

Recall that the Rogers–Ramanujan functions are defined by

$$G(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}. \tag{5.11}$$

These functions satisfy the famous Rogers–Ramanujan identities [16, pp. 214–215]

$$G(q) = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \quad \text{and} \quad H(q) = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{5.12}$$

Our proof makes use of one of Ramanujan's forty identities for the Rogers–Ramanujan functions, namely [21]

$$G(q)G(q^4) - qH(q)H(q^4) = \frac{\varphi(q^5)}{E(q^2)}. \tag{5.13}$$

Next, we employ the quintuple product identity (2.5), with q replaced by q^{10} and $a=-q$, to find that

$$f(-q^{13},-q^{17}) + qf(-q^7,-q^{23}) = E(q^{10})\frac{f(-q^2,-q^8)}{f(-q,-q^9)} = E(q^2)G(q). \tag{5.14}$$

Similarly, from (2.5) we find

$$E(q^2)H(q) = f(-q^{11},-q^{19}) + q^3f(-q,-q^{29}), \tag{5.15}$$

$$E(q)G(q^2) = f(q^7,q^8) - qf(q^2,q^{13}), \tag{5.16}$$

$$E(q)H(q^2) = f(q^4,q^{11}) - qf(q,q^{14}). \tag{5.17}$$

Next, making use of (5.8), (5.9), (5.10), (5.14), (5.15), (5.16), (5.17), (5.13), and (5.1), we conclude that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{5}{n}\right)\frac{q^n(1+q^n)}{1+q^{3n}} \\
&\quad = q\frac{E^3(q^{15})}{E(q^5)f(q^3,q^{12})f(q^6,q^9)}E^2(q^2)\{G(q^4)G(-q)+qH(q^4)H(-q)\} \\
&\quad = q\frac{E^3(q^{15})}{E(q^5)f(q^3,q^{12})f(q^6,q^9)}E^2(q^2)\frac{\varphi(-q^5)}{E(q^2)} \\
&\quad = q\frac{E(q^2)E(q^3)E(q^5)E(q^{30})}{E(q^6)E(q^{10})} \\
&\quad = Q(q). \qquad \square
\end{aligned}
$$

Adding together (5.4) and (5.6) and replacing q by $-q$, we find that

$$\varphi(q)\varphi(q^{15})+\varphi(q^3)\varphi(q^5)=2-2\sum_{n=1}^{\infty}\left(\frac{-15}{n}\right)\frac{(-q)^n}{1+(-q)^n}.$$

It is instructive to compare this formula with an equivalent formula (50) in [22], which states that

$$\varphi(q)\varphi(q^{15})+\varphi(q^3)\varphi(q^5)=2+\sum_{n=1}^{\infty}\tilde{a}(n)\frac{q^n}{1-q^n},$$

where

$$\tilde{a}(n)=2\left(\frac{-60}{n}\right)-2\delta(2|n)\left(\frac{-60}{n/2}\right)+2\delta(4|n)\left(\frac{-15}{n/4}\right),$$

with

$$\delta(a|b)=\begin{cases}1 & \text{if } a|b,\\ 0 & \text{otherwise.}\end{cases}$$

Corollary 5.2 *Let $a(n)$ and $b(n)$ be the number of representations of a positive integer n by quadratic form k^2+15l^2 and $3k^2+5l^2$, respectively. If the prime factorization of n is given by*

$$n=2^a3^b5^c\prod_{i=1}^{r}p_i^{v_i}\prod_{j=1}^{s}q_j^{w_j},$$

where $p_i\equiv 1,2,4,$ *or* $8 \pmod{15}$, $p_i\neq 2$ *and* $q_i\equiv 7,11,13,$ *or* $14 \pmod{15}$, *then*

$$a(n)=|a-1|\left(1+(-1)^{a+b+c+t}\right)\prod_{i=1}^{r}(1+v_i)\prod_{j=1}^{s}\frac{1+(-1)^{w_j}}{2} \tag{5.18}$$

and

$$b(n)=|a-1|\bigl(1-(-1)^{a+b+c+t}\bigr)\prod_{i=1}^{r}(1+v_i)\prod_{j=1}^{s}\frac{1+(-1)^{w_j}}{2}, \tag{5.19}$$

where t is a number of odd prime factors of n, counting multiplicity, that are congruent to 2 *or* 8 (mod 15).

Proof Observe that

$$\begin{aligned}
Q(q)=\sum_{n=1}^{\infty}\left(\frac{5}{n}\right)\frac{q^n(1+q^n)}{1+q^{3n}} &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^m\left(\frac{5}{n}\right)\bigl(q^{n(3m+1)}+q^{n(3m+2)}\bigr)\\
&= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\left(\frac{5}{n}\right)\bigl(q^{n(3m-1)}+q^{n(3m-2)}\bigr)\\
&= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\left(\frac{-3}{m}\right)\left(\frac{5}{n}\right)q^{nm}\\
&= -\sum_{n=1}^{\infty}\Biggl(\sum_{d|n}(-1)^d\left(\frac{-3}{d}\right)\left(\frac{5}{n/d}\right)\Biggr)q^n.
\end{aligned}$$

Therefore,

$$Q(-q)=-\sum_{n=1}^{\infty}\Biggl(\sum_{d|n}(-1)^{n+d}\left(\frac{-3}{d}\right)\left(\frac{5}{n/d}\right)\Biggr)q^n. \tag{5.20}$$

Similarly,

$$P(-q)=1+\sum_{n=1}^{\infty}\Biggl(\sum_{d|n}(-1)^{n+d}\left(\frac{-15}{n/d}\right)\Biggr)q^n. \tag{5.21}$$

Now we define

$$c(n)=\sum_{d|n}(-1)^{n+d}\left(\frac{-15}{n/d}\right)\quad\text{and}\quad d(n)=\sum_{d|n}(-1)^{n+d}\left(\frac{-3}{d}\right)\left(\frac{5}{n/d}\right).$$

By (5.4) and (5.6) we have $a(n)=c(n)+d(n)$ and $b(n)=c(n)-d(n)$, for $n>0$. Using the fact that $(-1)^{n+1}$ is a multiplicative function of n, we conclude that $c(n)$ and $d(n)$ are multiplicative functions. From (5.1), (5.20), and (5.21) we also observe that the following *eta*-quotients are multiplicative:

$$\begin{aligned}
P(-q)&=\frac{E(-q)E(q^6)E(q^{10})E(-q^{15})}{E(q^2)E(q^{30})}\\
&=\frac{E^2(q^2)E(q^6)E(q^{10})E^2(q^{30})}{E(q)E(q^4)E(q^{15})E(q^{60})},
\end{aligned} \tag{5.22}$$

$$-Q(-q) = q\frac{E(q^2)E(-q^3)E(-q^5)E(q^{30})}{E(q^6)E(q^{10})}$$
$$= q\frac{E(q^2)E^2(q^6)E^2(q^{10})E(q^{30})}{E(q^3)E(q^5)E(q^{12})E(q^{20})}, \tag{5.23}$$
$$Q(q) = q\frac{E(q^2)E(q^3)E(q^5)E(q^{30})}{E(q^6)E(q^{10})}. \tag{5.24}$$

It is easy to check that for a prime p,

$$c(p^\alpha) = \begin{cases} |\alpha - 1| & \text{if } p = 2, \\ 1 & \text{if } p = 3 \text{ or } 5, \\ 1+\alpha & \text{if } p \equiv 1, 2, 4, 8 \pmod{15},\ p \neq 2, \\ \frac{1+(-1)^\alpha}{2} & \text{if } p \equiv 7, 11, 13, 14 \pmod{15}, \end{cases} \tag{5.25}$$

and

$$d(p^\alpha) = \begin{cases} (-1)^\alpha|\alpha - 1| & \text{if } p = 2, \\ (-1)^\alpha & \text{if } p = 3 \text{ or } 5, \\ 1+\alpha & \text{if } p \equiv 1, 4 \pmod{15}, \\ (-1)^\alpha(1+\alpha) & \text{if } p \equiv 2, 8 \pmod{15},\ p \neq 2, \\ \frac{1+(-1)^\alpha}{2} & \text{if } p \equiv 7, 11, 13, 14 \pmod{15}. \end{cases} \tag{5.26}$$

From these two equations (5.18) and (5.19) are immediate. Equivalent reformulations of (5.18) and (5.19) can also be found in [12]. □

6 Representations by the the quadratic form $k^2 + 27l^2$

In this section, we give a formula for the number of representations of a positive integer by the quadratic form $k^2 + 27l^2$.

Theorem 6.1

$$\varphi(q)\varphi(q^{27}) = \frac{\varphi(q)\varphi(q^3) - \varphi(q^3)\varphi(q^9)}{3} + \varphi(q^9)\varphi(q^{27}) + \frac{4}{3}qE(q^6)E(q^{18}). \tag{6.1}$$

Let $a(n)$ *and* $b(n)$ *be the number of representations of a positive integer* n *by quadratic forms* $(1, 0, 27)$ *and* $(4, 2, 7)$, *respectively.*
If $n \not\equiv 1 \pmod 6$, *then*

$$a(n) = b(n)$$
$$= \begin{cases} (3 - 2\delta_{\alpha,0})(1+,(-1)^\alpha)\prod_{i=1}^r (1+v_i)\prod_{j=1}^s \frac{1+(-1)^{w_j}}{2} & \textit{if } \beta \geq 2, \\ (1+(-1)^\alpha)\prod_{i=1}^r (1+v_i)\prod_{j=1}^s \frac{1+(-1)^{w_j}}{2} & \textit{if } \beta = 0 \textit{ and } \alpha > 0, \\ 0 & \textit{otherwise}, \end{cases} \tag{6.2}$$

where n has the prime factorization

$$n = 2^{\alpha} 3^{\beta} \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j}$$

with $p_i \equiv 1 \pmod 3$ *and* $2 \neq q_i \equiv 2 \pmod 3$, *and*

$$\delta_{j,0} := \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise}. \end{cases} \tag{6.3}$$

If $n \equiv 1 \pmod 6$, *then*

$$a(n) = \frac{2}{3} \prod_{i=1}^{r} (1 + v_i) \left(\prod_{i=1}^{s} (1 + u_i) + 2 \prod_{i=1}^{s} \big((1 + u_i) \mid 3 \big) \right) \prod_{i=1}^{t} \frac{1 + (-1)^{w_i}}{2}, \tag{6.4}$$

$$b(n) = \frac{2}{3} \prod_{i=1}^{r} (1 + v_i) \left(\prod_{i=1}^{s} (1 + u_i) - \prod_{i=1}^{s} \big((1 + u_i) \mid 3 \big) \right) \prod_{i=1}^{t} \frac{1 + (-1)^{w_i}}{2}, \tag{6.5}$$

where n has the prime factorization

$$\prod_{i=1}^{r} p_i^{v_i} \prod_{i=1}^{s} q_i^{u_i} \prod_{i=1}^{t} Q_j^{w_i} \tag{6.6}$$

with $p_i \equiv 1 \pmod 3$, $2^{\frac{p_i - 1}{3}} \equiv 1 \pmod{p_i}$, $q_i \equiv 1 \pmod 3$, $2^{\frac{q_i - 1}{3}} \not\equiv 1 \pmod{q_i}$, *and* $2 \neq Q_i \equiv 2 \pmod 3$.

Proof Observe that

$$\begin{aligned} \sum_{u,v=-\infty}^{\infty} q^{7u^2 + 2uv + 4v^2} &= \sum_{k=0}^{3} \sum_{s,v=-\infty}^{\infty} q^{7(4s+k)^2 + 2(4s+k)v + 4v^2} \\ &= \sum_{k=0}^{3} \sum_{s,r=-\infty}^{\infty} q^{7(4s+k)^2 + 2(4s+k)(r-s) + 4(r-s)^2} \\ &= \sum_{k=0}^{3} q^{7k^2} \sum_{r=-\infty}^{\infty} q^{2(2r^2 + kr)} \sum_{s=-\infty}^{\infty} q^{54(2s^2 + ks)} \\ &= f\big(q^4, q^4\big) f\big(q^{108}, q^{108}\big) + 2q^7 f\big(q^2, q^6\big) f\big(q^{54}, q^{162}\big) \\ &\quad + q^{28} f\big(1, q^8\big) f\big(1, q^{216}\big) \\ &= \big(\varphi(q)\varphi\big(q^{27}\big) + \varphi(-q)\varphi\big(-q^{27}\big)\big)/2 + 2q^7 \psi\big(q^2\big)\psi\big(q^{54}\big), \end{aligned} \tag{6.7}$$

where in the last step we used (2.8).

Similarly, we find that

$$
\begin{aligned}
&\sum_{u,v=-\infty}^{\infty} q^{7u^2+2uv+4v^2}\\
&\quad=\sum_{u,v=-\infty}^{\infty} q^{7(u-v)^2+2(u-v)v+4v^2}=\sum_{u,v=-\infty}^{\infty} q^{7u^2-12uv+9v^2}\\
&\quad=\sum_{k=0}^{2}\sum_{s,v=-\infty}^{\infty} q^{7(3s+k)^2-12(3s+k)v+9v^2}\\
&\quad=\sum_{k=0}^{2}\sum_{s,r=-\infty}^{\infty} q^{7(3s+k)^2-12(3s+k)(r+2s)+9(r+2s)^2}\\
&\quad=\sum_{k=0}^{2} q^{7k^2}\sum_{r=-\infty}^{\infty} q^{3(3r^2-4kr)}\sum_{s=-\infty}^{\infty} q^{9(3s^2+2ks)}\\
&\quad=f\left(q^9,q^9\right)f\left(q^{27},q^{27}\right)+2q^4 f\left(q^3,q^{15}\right)f\left(q^9,q^{45}\right)\\
&\quad=\varphi\left(q^9\right)\varphi\left(q^{27}\right)+\left(\varphi(q)-\varphi\left(q^9\right)\right)\left(\varphi\left(q^3\right)-\varphi\left(q^{27}\right)\right)/2\\
&\quad=\left(3\varphi\left(q^9\right)\varphi\left(q^{27}\right)+\varphi(q)\varphi\left(q^3\right)-\varphi(q)\varphi\left(q^{27}\right)-\varphi\left(q^3\right)\varphi\left(q^9\right)\right)/2, \qquad (6.8)
\end{aligned}
$$

where we used (2.9).

Lastly, we need the following identity of Ramanujan [6, p. 359, Entry. 4, (iv)]

$$
\left(\varphi(q)\varphi\left(q^{27}\right)-\varphi(-q)\varphi\left(-q^{27}\right)\right)/2-2q^7\psi\left(q^2\right)\psi\left(q^{54}\right)=2qE\left(q^6\right)E\left(q^{18}\right). \qquad (6.9)
$$

From (6.7), (6.8), and (6.9) we find that

$$
\begin{aligned}
\varphi(q)\varphi\left(q^{27}\right)=&\left(3\varphi\left(q^9\right)\varphi\left(q^{27}\right)+\varphi(q)\varphi\left(q^3\right)-\varphi(q)\varphi\left(q^{27}\right)-\varphi\left(q^3\right)\varphi\left(q^9\right)\right)/2\\
&+2qE\left(q^6\right)E\left(q^{18}\right), \qquad (6.10)
\end{aligned}
$$

which is (6.1). Formulas (6.7), (6.8), and (6.9) are special cases of a more general formula [23, Thr. (3.1), Cor. (3.3)]. In fact, Ramanujan's identity, (6.9), can be stated as follows:

$$
\begin{aligned}
&\sum_{u,v=\infty}^{\infty}(-1)^{u+v} q^{(7(2u+1)^2-12(2u+1)(2v+1)+9(2v+1)^2)/4}\\
&\quad=\left(\varphi(q)\varphi\left(q^{27}\right)-\varphi(-q)\varphi\left(-q^{27}\right)\right)/2-2q^7\psi\left(q^2\right)\psi\left(q^{54}\right)\\
&\quad=2qE\left(q^6\right)E\left(q^{18}\right).
\end{aligned}
$$

Recall that

$$1+\sum_{n=1}^{\infty} a(n)q^n := \varphi(q)\varphi\big(q^{27}\big) \quad \text{and} \quad 1+\sum_{n=1}^{\infty} b(n)q^n := \sum_{n,m=-\infty}^{\infty} q^{4n^2+2nm+7m^2}. \tag{6.11}$$

We also define $c(n)$ and $d(n)$ by

$$1+\sum_{n=1}^{\infty} c(n)q^n = \varphi(q)\varphi\big(q^3\big) \quad \text{and} \quad \sum_{n=1}^{\infty} d(n)q^n = qE\big(q^6\big)E\big(q^{18}\big). \tag{6.12}$$

Using the following Lambert series expansion for $\varphi(q)\varphi(q^3)$ (see, for example, [7, p. 75, Eq. (3.7.8)])

$$\varphi(q)\varphi\big(q^3\big) = 1+2\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^n} + 4\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^{4n}}{1-q^{4n}}, \tag{6.13}$$

it is easy to show that

$$c(n) = (3-2\delta_{\alpha,0})\big(1+(-1)^{\alpha}\big)\prod_{i=1}^{r}(1+v_i)\prod_{j=1}^{s}\frac{1+(-1)^{w_j}}{2}, \tag{6.14}$$

where n has the prime factorization

$$n = 2^{\alpha}3^{\beta}\prod_{i=1}^{r}p_i^{v_i}\prod_{j=1}^{s}q_j^{w_j},$$

where $p_i \equiv 1 \pmod 3$ and $2 \neq q_i \equiv 2 \pmod 3$.

From (6.1) we have

$$a(n) = \frac{c(n)-c(n/3)}{3} + c(n/9) + \frac{4}{3}d(n), \tag{6.15}$$

where we assume $c(n/l)=0$ if $l \nmid n$. If $n \not\equiv 1 \pmod 6$, then $d(n)\equiv 0$ and $a(n) = c(n/9)$ if $3|n$, while $a(n)=c(n)/3$ if $3 \nmid n$. This proves the claim in (6.2) for $a(n)$.

Now assume $n \equiv 1 \pmod 6$. If p is a prime and $p\equiv 1 \pmod 3$, then by (6.14) we see that $c(p)=4$. If p is represented by the form $(1,0,27)$, then $a(p)=4$. This is because $4 \le a(p) \le c(p) = 4$. Using (6.15), we see that $d(p)=2$. If p is not represented by $(1,0,27)$ then, by (6.15), $d(p)=-1$. Gauss proved that if p is a prime and $p\equiv 1 \pmod 3$, then p is represented by $(1,0,27)$ iff 2 is a cubic residue modulo p or, equivalently, iff $2^{(p-1)/3}\equiv 1 \pmod p$. Therefore,

$$d(p) = \begin{cases} 2 & \text{if } p\equiv 1 \pmod 3 \text{ and } 2^{(p-1)/3}\equiv 1 \pmod p, \\ -1 & \text{if } p\equiv 1 \pmod 3 \text{ and } 2^{(p-1)/3}\not\equiv 1 \pmod p, \\ 0 & \text{if } p \not\equiv 1 \pmod 3. \end{cases} \tag{6.16}$$

In [13], Y. Martin proved that $qE(q^6)E(q^{18})$ is a multiplicative cusp form in $S_1(\Gamma_0(108), (\frac{-108}{*}))$. That is, $d(n)$ is multiplicative, and for any prime p and $s \geq 0$,

$$d\big(p^{s+2}\big) = d(p)d\big(p^{s+1}\big) - \left(\frac{-108}{p}\right)d\big(p^s\big), \tag{6.17}$$

where $d(1) = 1$. Using this recursion formula together with (6.16), we find that

$$d\big(p^\alpha\big) = \begin{cases} \alpha + 1 & \text{if } p \equiv 1 \pmod 3 \text{ and } 2^{(p-1)/3} \equiv 1 \pmod p, \\ ((\alpha+1)|3) & \text{if } p \equiv 1 \pmod 3 \text{ and } 2^{(p-1)/3} \not\equiv 1 \pmod p, \\ (1 + (-1)^\alpha)/2 & \text{if } 2 \neq p \equiv 2 \pmod 3, \\ 0 & \text{if } p = 2 \text{ or } 3. \end{cases} \tag{6.18}$$

Therefore,

$$d(n) = \delta_{\alpha,0}\delta_{\beta,0}\prod_{i=1}^{r}(1+v_i)\prod_{i=1}^{s}\big((1+u_i) \mid 3)\big)\prod_{i=1}^{t}\frac{1+(-1)^{w_i}}{2}, \tag{6.19}$$

where n has the prime factorization

$$2^\alpha 3^\beta \prod_{i=1}^{r} p_i^{v_i} \prod_{i=1}^{s} q_i^{u_i} \prod_{i=1}^{t} Q_j^{w_i}, \tag{6.20}$$

where $p_i \equiv 1 \pmod 3$, $2^{\frac{p_i-1}{3}} \equiv 1 \pmod{p_i}$, $q_i \equiv 1 \pmod 3$, $2^{\frac{q_i-1}{3}} \not\equiv 1 \pmod{q_i}$, and $2 \neq Q_i \equiv 2 \pmod 3$. By (6.15), if $n \equiv 1 \pmod 6$, then $a(n) = \frac{c(n)+4d(n)}{3}$. Using (6.14) and (6.19), we arrive at the statement for $a(n)$ given in (6.4).

From (6.7) and (6.9) we have that

$$\begin{aligned} &\varphi(q)\varphi\big(q^{27}\big) - \sum_{n,m=-\infty}^{\infty} q^{4n^2+2nm+7m^2} \\ &\quad = \big(\varphi(q)\varphi\big(q^{27}\big) - \varphi(-q)\varphi\big(-q^{27}\big)\big)/2 - 2q^7\psi\big(q^2\big)\psi\big(q^{54}\big) \\ &\quad = 2qE\big(q^6\big)E\big(q^{18}\big). \end{aligned}$$

Therefore, $b(n) = a(n) - 2d(n)$. The formulas for $b(n)$ in (6.2) and (6.5) now follow from those for $a(n)$ and $d(n)$. Observe also from $b(n) = a(n) - 2d(n)$ that if p is a prime and $p \equiv 1 \pmod 3$, then $b(p) = 0$ if p is represented by $(1, 0, 27)$ and $b(p) = 2$, otherwise. Hence, these primes cannot be represented by $(1, 0, 27)$ and $(4, 2, 7)$ at the same time. □

While they are not explicitly stated there, the formulas for $a(n)$ and $b(n)$ given by (6.2), (6.3), (6.4), and (6.5) can be deduced from Theorems 4.1, 10.1, and 10.2 of [19] and Gauss' cubic reciprocity law.

7 Representations by the forms $n^2+5m^2+5k^2+5l^2$ and $5n^2+m^2+k^2+l^2$

In this section, we give formulas for the number of representations of positive integers by the quaternary forms $n^2+5m^2+5k^2+5l^2$ and $5n^2+m^2+k^2+l^2$ and also by the restricted forms $n+5m+5k+5l$ and $5n+m+k+l$ with $n, m, k,$ and l being triangular numbers.

Theorem 7.1

$$\varphi(-q)\varphi^3(-q^5)=\left(\frac{E^5(q)}{E(q^5)}+4\frac{E^5(q^2)}{E(q^{10})}\right)\Big/5-\left(q\frac{E^5(q^5)}{E(q)}-4q^2\frac{E^5(q^{10})}{E(q^2)}\right), \tag{7.1}$$

$$\varphi^3(-q)\varphi(-q^5)=\left(\frac{E^5(q)}{E(q^5)}+4\frac{E^5(q^2)}{E(q^{10})}\right)\Big/5-5\left(q\frac{E^5(q^5)}{E(q)}-4q^2\frac{E^5(q^{10})}{E(q^2)}\right), \tag{7.2}$$

$$4q\psi^3(q)\psi(q^5)=\left(\frac{E^5(q)}{E(q^5)}-\frac{E^5(q^2)}{E(q^{10})}\right)\Big/5+5\left(q\frac{E^5(q^5)}{E(q)}+q^2\frac{E^5(q^{10})}{E(q^2)}\right), \tag{7.3}$$

$$4q^2\psi(q)\psi^3(q^5)=\left(\frac{E^5(q)}{E(q^5)}-\frac{E^5(q^2)}{E(q^{10})}\right)\Big/5+\left(q\frac{E^5(q^5)}{E(q)}+q^2\frac{E^5(q^{10})}{E(q^2)}\right). \tag{7.4}$$

Furthermore, if $a(n)$, $b(n)$, $c(n)$, and $d(n)$ are defined by

$$\varphi(q)\varphi^3(q^5)=:1+\sum_{n=1}^{\infty}a(n)q^n, \qquad \varphi^3(q)\varphi(q^5)=:1+\sum_{n=1}^{\infty}b(n)q^n,$$

$$4q\psi^3(q)\psi(q^5)=:\sum_{n=1}^{\infty}c(n)q^n, \qquad 4q^2\psi(q)\psi^3(q^5)=:\sum_{n=1}^{\infty}d(n)q^n,$$

then

$$a(n)=(-1)^{n-1}\left(1+5^d(-1)^{g+t}\right)\frac{(5+(-2)^{g+1})}{3}\times\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}\frac{1-(-q_j)^{w_j+1}}{1+q_j}, \tag{7.5}$$

$$b(n)=(-1)^{n-1}\left(1+5^{d+1}(-1)^{g+t}\right)\frac{(5+(-2)^{g+1})}{3}\times\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}\frac{1-(-q_j)^{w_j+1}}{1+q_j}, \tag{7.6}$$

$$c(n)=(-2)^g\left(-1+5^{d+1}(-1)^{g+t}\right)\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}\frac{1-(-q_j)^{w_j+1}}{1+q_j}, \tag{7.7}$$

$$d(n)=(-2)^g\left(-1+5^{d}(-1)^{g+t}\right)\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}\frac{1-(-q_j)^{w_j+1}}{1+q_j}, \tag{7.8}$$

where n has the prime factorization

$$n = 2^g 5^d \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j}$$

with $p_i \equiv \pm 1 \pmod 5$ *and* $q_i \equiv \pm 2 \pmod 5$, q_i *is odd, and t is the number of odd prime divisors of n, counting multiplicities, that are congruent to* $\pm 2 \pmod 5$.

Proof Our proof employs the following well-known Lambert series identities of Ramanujan:

$$\frac{E^5(q)}{E(q^5)} = 1 - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}, \tag{7.9}$$

$$q\frac{E^5(q^5)}{E(q)} = \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}. \tag{7.10}$$

For the history of these and many related identities, see [1], [6, pp. 249–263]. We also use the theta function identities [6, p. 262, Entry 10]

$$\varphi^2(q) - \varphi^2\left(q^5\right) = 4qf\left(q,q^9\right)f\left(q^3,q^7\right) \tag{7.11}$$

and

$$\psi^2(q) - q\psi^2\left(q^5\right) = f\left(q^2,q^3\right)f\left(q,q^4\right). \tag{7.12}$$

By multiplying both sides of (7.11) with $\varphi^3(q^5)/\varphi(q)$, we find that

$$\varphi(q)\varphi^3\left(q^5\right) - \frac{\varphi^5(q^5)}{\varphi(q)} = 4q\frac{E^5(-q^5)}{E(-q)}. \tag{7.13}$$

From (7.13) we deduce that

$$\begin{aligned}
16q^2\frac{E^5(q^{10})}{E(q^2)} &= 16q^2\frac{\varphi(q)}{\varphi^5(q^5)}\frac{E^{10}(-q^5)}{E^2(-q)} \\
&= \frac{\varphi(q)}{\varphi^5(q^5)}\left\{\varphi(q)\varphi^3\left(q^5\right) - \frac{\varphi^5(q^5)}{\varphi(q)}\right\}^2 \\
&= \varphi^3(q)\varphi\left(q^5\right) - 2\varphi(q)\varphi^3\left(q^5\right) + \frac{\varphi^5(q^5)}{\varphi(q)}.
\end{aligned} \tag{7.14}$$

Using the imaginary transformation on (7.13) and (7.14), we obtain, respectively, that

$$5\varphi^3(q)\varphi\left(q^5\right) - \frac{\varphi^5(q)}{\varphi\left(q^5\right)} = 4\frac{E^5(-q)}{E(-q^5)} \tag{7.15}$$

and

$$25\varphi(q)\varphi^3\left(q^5\right) - 10\varphi^3(q)\varphi\left(q^5\right) + \frac{\varphi^5(q)}{\varphi(q^5)} = 16\frac{E^5(q^2)}{E(q^{10})}. \tag{7.16}$$

By multiplying both sides of (7.12) with $\psi^3(q^5)/\psi(q)$, we find that

$$\psi(q)\psi^3\big(q^5\big) - q\frac{\psi^5(q^5)}{\psi(q)} = \frac{E^5(q^{10})}{E(q^2)}. \tag{7.17}$$

From (7.17) we also find that

$$\begin{aligned}\frac{E^5(q^5)}{E(q)} &= \frac{\psi(q)}{\psi^5(q^5)}\frac{E^{10}(q^{10})}{E^2(q^2)} \\ &= \frac{\psi(q)}{\psi^5(q^5)}\left\{\psi(q)\psi^3\big(q^5\big) - q\frac{\psi^5(q^5)}{\psi(q)}\right\}^2 \\ &= \psi^3(q)\psi\big(q^5\big) - 2q\psi(q)\psi^3\big(q^5\big) + q^2\frac{\psi^5(q^5)}{\psi(q)}. \end{aligned} \tag{7.18}$$

Using the imaginary transformation on (7.17) and (7.18), we obtain, respectively, that

$$-5q\psi^3(q)\psi\big(q^5\big) + \frac{\psi^5(q)}{\psi(q^5)} = \frac{E^5(q^2)}{E(q^{10})} \tag{7.19}$$

and

$$25q^2\psi(q)\psi^3\big(q^5\big) - 10q\psi^3(q)\psi\big(q^5\big) + \frac{\psi^5(q)}{\psi(q^5)} = \frac{E^5(q)}{E(q^5)}. \tag{7.20}$$

Using (7.13), (7.14), (7.15), (7.16), (7.17), (7.18), (7.19), and (7.20), we easily derive (7.1), (7.2), (7.3), and (7.4).

Next, we sketch a proof of (7.5). We omit the proofs of (7.6), (7.7), and (7.8) since their proofs are similar to that of (7.5). For convenience, $[q^n]V(q)$ will denote the coefficient of q^n in the Taylor series expansion of $V(q)$.

From (7.9) we have

$$\frac{E^5(q)}{E(q^5)} = 1 - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n} = 1 - 5\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{d}{5}\right)d\right)q^n. \tag{7.21}$$

Using the fact that the coefficients are given by the multiplicative function $\sum_{d|n}(\frac{d}{5})d$, we conclude that for $n > 1$,

$$[q^n]\frac{E^5(q)}{E(q^5)} = -5\prod_{i=1}^{r}\frac{1-p_i^{v_i+1}}{1-p_i}\prod_{j=1}^{s}\frac{1-(-q_j)^{w_j+1}}{1+q_j}, \tag{7.22}$$

where n has the prime factorization

$$n = 5^d\prod_{i=1}^{r}p_i^{v_i}\prod_{j=1}^{s}q_j^{w_j} \tag{7.23}$$

with $p_i \equiv \pm 1 \pmod 5$ and $q_i \equiv \pm 2 \pmod 5$.

It is easy to show [11, Thr. 4]

$$[q^n]\frac{qE^5(q^5)}{E(q)} = 5^d \prod_{i=1}^{r} \frac{1-p_i^{v_i+1}}{1-p_i} \prod_{j=1}^{s} (-1)^{w_j} \frac{1-(-q_j)^{w_j+1}}{1+q_j}, \tag{7.24}$$

where $n > 0$ has the prime factorization

$$n = 5^d \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j}$$

with $p_i \equiv \pm 1 \pmod 5$ and $q_i \equiv \pm 2 \pmod 5$. Using (7.22) and (7.24) together with (7.1), we arrive at (7.5). $\square$

Theorem 7.1 has the following consequence:

Corollary 7.2

(a) $[q^n]\big(\varphi^3(q)\varphi(q^5)\big) > 0, \qquad [q^n]\big(\psi^3(q)\psi(q^5)\big) > 0$ *for any* $n \geq 0$,

(b) $[q^n]\big(\varphi(q)\varphi^3(q^5)\big) = 0, \qquad [q^n]\big(\psi(q)\psi^3(q^5)\big) > 0$ *iff* $n \equiv 2$ *or* $3 \pmod 5$.

Note that our corollary is in agreement with Ramanujan's observation in [15], where the quadratic form $x^2 + y^2 + z^2 + 5w^2$ is listed as universal. It means that this form represents all positive integers. Interested reader may want to check [3] for new results about universal quadratic forms.

Next, we use (7.13), (7.14), and (7.15) to derive

$$\begin{aligned}
&\left(\frac{E^5(-q)}{E(-q^5)} + 4\frac{E^5(q^2)}{E(q^{10})}\right)\Big/5 \\
&\quad = \big(5\varphi(q)\varphi^3(q^5) - \varphi^3(q)\varphi(q^5)\big)/4 && (7.25)\\
&\quad = \frac{1}{4}\frac{\varphi^2(q^5)}{\varphi^2(q)}\left\{5\varphi^3(q)\varphi(q^5) - \frac{\varphi^5(q)}{\varphi(q^5)}\right\} && (7.26)\\
&\quad = \frac{\varphi^2(q^5)E^5(-q)}{\varphi^2(q)E(-q^5)} && (7.27)\\
&\quad = \frac{E^5(q^2)E^7(q^{10})}{E(q)E(q^4)E^3(q^5)E^3(q^{20})}. && (7.28)
\end{aligned}$$

Moreover, using (7.9) and (7.10), we obtain

$$\begin{aligned}
&\left(\frac{E^5(-q)}{E(-q^5)} + 4\frac{E^5(q^2)}{E(q^{10})}\right)\Big/5 \\
&\quad = 1 + \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-(-q)^n} && (7.29)\\
&\quad = 1 + \sum_{n=1}^{\infty}\left(\sum_{d\mid n}(-1)^{n+d} d\left(\frac{d}{5}\right)\right)q^n. && (7.30)
\end{aligned}$$

Arguing as above, one finds

$$q\frac{E^5(-q^5)}{E(-q)}+4q^2\frac{E^5(q^{10})}{E(q^2)}=q\frac{\varphi^2(q)E^5(-q^5)}{\varphi^2(q^5)E(-q)} \tag{7.31}$$

$$=q\frac{E^7(q^2)E^5(q^{10})}{E^3(q)E^3(q^4)E(q^5)E(q^{20})} \tag{7.32}$$

$$=-\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{(-q)^n}{(1+(-q)^n)^2} \tag{7.33}$$

$$=\sum_{n=1}^{\infty}\left(\sum_{d|n}(-1)^{n+d}d\left(\frac{n/d}{5}\right)\right)q^n, \tag{7.34}$$

and

$$-\left(\frac{E^5(q)}{E(q^5)}-\frac{E^5(q^2)}{E(q^{10})}\right)\Big/5=q\frac{\psi^2(q^5)E^5(q^2)}{\psi^2(q)E(q^{10})} \tag{7.35}$$

$$=q\frac{E^2(q)E(q^2)E^3(q^{10})}{E^2(q^5)} \tag{7.36}$$

$$=\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^{2n}} \tag{7.37}$$

$$=\sum_{n=1}^{\infty}\left(\sum_{d|n}d\left(\frac{d}{5}\right)\gamma(n/d)\right)q^n, \tag{7.38}$$

where

$$\gamma(n):=\begin{cases}1 & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases}$$

Lastly,

$$q\frac{E^5(q^5)}{E(q)}+q^2\frac{E^5(q^{10})}{E(q^2)}=q\frac{\psi^2(q)E^5(q^{10})}{\psi^2(q^5)E(q^2)} \tag{7.39}$$

$$=q\frac{E^3(q^2)E^2(q^5)E(q^{10})}{E^2(q)} \tag{7.40}$$

$$=\sum_{\substack{n=1\\ n\text{ is odd}}}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2} \tag{7.41}$$

$$=\sum_{n=1}^{\infty}\left(\sum_{d|n}\gamma(d)\left(\frac{d}{5}\right)n/d\right)q^n. \tag{7.42}$$

The *eta*-quotients given by (7.36) and (7.40) are clearly multiplicative. Using the fact that $(-1)^{n+1}$ is a multiplicative function of n, we also see that the first two *eta*-quotients given in (7.28) and (7.32) are multiplicative. These four multiplicative *eta*-quotients are not included in Martin's list. Lambert series representations of (7.29) and (7.37) are given by Ramanujan [6, p. 249, Entry 8 (i), (ii)]. Identity (7.39)–(7.40) is identity (6.12) of [4]. Using (7.1), (7.2), (7.3), (7.4), (7.29), (7.33), (7.37), and (7.41), we have

Corollary 7.3

$$\varphi(q)\varphi^3(q^5) = 1 + \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-(-q)^n} - \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{(-q)^n}{(1+(-q)^n)^2}, \tag{7.43}$$

$$\varphi^3(q)\varphi(q^5) = 1 + \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-(-q)^n} - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{(-q)^n}{(1+(-q)^n)^2}, \tag{7.44}$$

$$4q\psi^3(q)\psi(q^5) = -\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^{2n}} + 5\sum_{\substack{n=1\\ n \text{ is odd}}}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}, \tag{7.45}$$

$$4q^2\psi(q)\psi^3(q^5) = -\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^{2n}} + \sum_{\substack{n=1\\ n \text{ is odd}}}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}. \tag{7.46}$$

8 Outlook

Clearly, this manuscript does not exhaust all potential connections between the Ramanujan identities and quadratic forms. We believe that Ramanujan identities can be employed to find coefficients of many sextenary forms. For example, in [5] we will show how to use our new identity

$$\begin{aligned}
7\varphi^3(-q)\varphi^3(-q^7) = {} & -49\left(q^2\frac{E^7(q^7)}{E(q)} + qE^3(q)E^3(q^7)\right) \\
& + 56\left(7q^4\frac{E^7(q^{14})}{E(q^2)} + q^2E^3(q^2)E^3(q^{14})\right) \\
& - \frac{E^7(q)}{E(q^7)} + 8\frac{E^7(q^2)}{E(q^{14})},
\end{aligned}$$

together with two identities of Ramanujan, to determine the coefficients of $\varphi^3(q)\varphi^3(q^7)$. There we shall also prove the following intriguing inequalities:

$$[q^n]\left(\psi^3(q)\psi^3(q^7) - q\frac{E^7(q^{14})}{E(q^2)}\right) \ge 0,$$

$$[q^n]\left(\varphi^3(q)\varphi^3(q^7) + q^7 - q^2\frac{E^7(q^7)}{E(q)}\right) \ge 0.$$

We will also determine the coefficients of $\varphi^5(q)\varphi(q^3)$ and $\varphi(q)\varphi^5(q^3)$ by proving that

$$\varphi^5(q)\varphi\big(q^3\big) = 1+\sum_{n=1}^{\infty}\Bigg(\sum_{d|n}(-1)^{n+d}d^2\bigg(\frac{d}{3}\bigg)\Bigg)q^n+9\sum_{n=1}^{\infty}\Bigg(\sum_{d|n}(-1)^{n+d}d^2\bigg(\frac{n/d}{3}\bigg)\Bigg)q^n$$

and

$$\varphi(q)\varphi^5\big(q^3\big) = 1+\sum_{n=1}^{\infty}\Bigg(\sum_{d|n}(-1)^{n+d}d^2\bigg(\frac{d}{3}\bigg)\Bigg)q^n+\sum_{n=1}^{\infty}\Bigg(\sum_{d|n}(-1)^{n+d}d^2\bigg(\frac{n/d}{3}\bigg)\Bigg)q^n.$$

In the course of this investigation we have determined the coefficients of many multiplicative *eta*-quotients. Those *eta*-quotients given by (3.4), (3.5), (4.1), and (6.9) are on Martin's list [13], while those that are given by (5.22), (5.23), (5.24), (7.28), (7.32), (7.36), and (7.40) are not in his list. This is because Martin only considered multiplicative *eta*-quotients which are eigenforms for all Hecke operators. As an example, it seems that the multiplicative *eta*-quotient $-Q(-q)$, defined by (5.23), is an eigenform for all Hecke operators T_p with odd prime p, but it is not hard to show that

$$T_2\big(-Q(-q)\big) = \frac{E^2(q^6)E^2(q^{10})}{E(q^2)E(q^{30})}.$$

The *eta*-quotients in (7.9) and (7.10) also appear in Martin's list [13]. In our future publications we plan to discuss the coefficients of all multiplicative *eta*-products that appear on this list. To this end we proved the following:

Theorem 8.1 *Suppose the prime factorization of n is*

$$2^{\alpha}\prod_{i=1}^{r}p_i^{u_i}\prod_{i=1}^{s}q_i^{v_j}\prod_{i=1}^{l}P_i^{w_i}\prod_{i=1}^{t}Q_j^{d_i},$$

where

$$\begin{aligned}
&Q_i \equiv 3 \pmod 4,\\
&p_i \equiv 5 \pmod 8,\\
&q_i \equiv 1 \pmod 8 \quad \textit{and} \quad 2^{(q_i-1)/4} \equiv 1 \pmod{q_i},\\
&P_i \equiv 1 \pmod 8 \quad \textit{and} \quad 2^{(P_i-1)/4} \equiv -1 \pmod{P_i}.
\end{aligned}$$

Then

$$\begin{aligned}
\big[q^n\big]\bigg(q\frac{E^4(q^{16})}{E(q^{32})E(q^8)}\bigg) &= \delta_{\alpha,0}\prod_{i=1}^{r}(-1)^{u_i}\big(1+(-1)^{u_i}\big)/2\prod_{i=1}^{s}(1+v_i)\\
&\quad\times\prod_{i=1}^{l}(-1)^{w_j}(1+w_j)\prod_{i=1}^{t}\big(1+(-1)^{d_i}\big)/2.
\end{aligned}$$

We would like to conclude with the following remarkable identity:

$$q\,\frac{\widetilde{Q}(2,0,7)+\widetilde{Q}(3,2,5)}{\widetilde{Q}(1,0,14)-\widetilde{Q}(3,2,5)}=\frac{E^2(q^4)E^2(q^{14})}{E^2(q^2)E^2(q^{28})},$$

where

$$\widetilde{Q}(a,b,c):=\sum_{n,m=-\infty}^{\infty} q^{an^2+bnm+cm^2}.$$

It is important to observe that neither $\widetilde{Q}(2,0,7)+\widetilde{Q}(3,2,5)$ nor $\widetilde{Q}(1,0,14)-\widetilde{Q}(3,2,5)$ is an *eta*-quotient. This result along with other similar type identities will be discussed elsewhere.

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