

ON THE BOUNDARY CONTROL OF BEAM EQUATION

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Abstract: A flexible system described by Euler-Bernoulli beam equation is considered. The beam is clamped at one end, and is free at the other end. Boundary control force and torque inputs are applied at the free end of the beam. The transfer functions of the controllers are marginally stable and may contain some poles on the imaginary axis. Various stability results are shown and the application of the proposed control law to disturbance rejection problem is considered.

Keywords: Flexible beam, distributed parameter systems, stability.

1. INTRODUCTION

The progress in robotics and space technology has resulted in the use of lightweight materials in constructions of such systems, for various reasons including the convenience in transportation. Another reason is the need for fast rotating systems. Such mechanical systems contain parts which can be modeled as flexible structures. To achieve high precision demands for such systems, one has to take the effect of flexibility into account in designing the controllers.

Consider a mechanical system which has coupled flexible and rigid parts, such as a robot arm with a flexible link or a spacecraft with flexible appendages. The equations of motion for such systems are generally a set of coupled partial and ordinary differential equations with appropriate boundary conditions. Once the equations of motion for such systems are obtained, the commonly used approach is to express the solutions as an infinite sum in terms of the eigenfunctions corresponding to the relevant partial differential equation, and then to consider only finitely many terms in this sum, see e.g. Meirovitch (1967). This approach is called "modal" analysis and reduces the original set of equations, to a finite, although often very large, set of coupled ordinary differential equations. However, having established a control law for this reduced set of equations does not always guarantee that the same law

will work on the original set of equations, (e.g., one might encounter the so-called "spillover" problems, Balas (1978)). Also note that the actual number of modes of an elastic system, in theory, is infinite and the number of modes that should be retained is not known a priori.

In recent years, the boundary control of flexible systems, (i.e., controls applied to the *boundaries* of the flexible parts as opposed to the controls *distributed* over the flexible parts), has become an important research area. This idea was first applied to the systems described by wave equation, (e.g., strings), Chen (1979), and was extended to the Euler-Bernoulli beam equation, Chen et. al. (1987). In particular, in Chen et. al. (1987), it has been proven that, in a cantilever beam, a single actuator applied at the free end of the beam is sufficient to uniformly stabilize the beam deflections. Recently, the boundary control techniques has been applied to the stabilization of a flexible spacecraft performing planar motion, Morgül (1991), and three dimensional motion Morgül (1990). For more references and technical information on this subject, the reader is referred to Luo et. al. (1999).

In this note, we consider a linear time invariant system which is represented by one-dimensional Euler-Bernoulli beam equation in a bounded domain. We assume that the system is clamped at one end and

the boundary control inputs (force and torque) are applied at the other end. For this system, we propose finite dimensional *dynamic* boundary controllers, which generate these inputs. This introduces extra degrees of freedom in designing controllers which could be exploited in solving a variety of control problems, such as disturbance rejection, pole assignment, etc., while maintaining stability. The transfer functions of the controllers are proper rational functions of the complex variable s , and may contain some poles on the imaginary axis, provided that the residues corresponding to these poles are nonnegative; the rest of the transfer functions are required to be strictly positive real. This type of controllers have been proposed before for the stabilization of flexible structures, see Morgül (1994) for the wave equation, Morgül (1998) for disturbance rejection, and Morgül (1992) for the beam equation (except for the poles on the imaginary axis). We then show that if the poles on the imaginary axis do not belong to a countable set (e.g. the zeroes of a transcendence function), then the closed loop system is asymptotically stable. We also consider the case where the outputs of the controllers are corrupted by disturbance. We show that if the structure of the disturbance is known (i.e. the frequency spectrum), then it may be possible to choose the controller accordingly to attenuate the effect of the disturbance at the system output.

This paper is organized as follows. In the next section we introduce the system considered and propose a class of controllers for stabilization. In section 3 we give some stability results. In section 4 we consider disturbance rejection problem and finally we give some concluding remarks.

2. PROBLEM STATEMENT

We consider a flexible beam clamped at one end and is free at the other end. Without loss of generality, we assume that the beam length, mass density and the flexural rigidity are given as $L = 1, \rho = 1$ and $EI = 1$, respectively. We denote the displacement of the beam by $u(x, t)$ at $x \in (0, 1)$ and $t \geq 0$. The beam is clamped at one end and is controlled by a boundary control force at the other end. The equations are given as ($x \in (0, 1), t \geq 0$):

$$u_{tt} + u_{xxxx} = 0 \quad , \quad (1)$$

$$u(0, t) = 0 \quad , \quad u_x(0, t) = 0 \quad , \quad (2)$$

$$-u_{xx}(1, t) = \alpha_1 f_1(t), u_{xxx}(1, t) = \alpha_2 f_2(t) \quad (3)$$

where a subscript, as in u_t denotes a partial differential with respect to the corresponding variable, and $f_1(\cdot), f_2(\cdot) : \mathbf{R}_+ \rightarrow \mathbf{R}$ are the boundary control torque and force applied at the free end of the beam, respectively. Here $\alpha_i \in \{0, 1\}$, and $\alpha_i = 0$ means that the corresponding controller is not applied, whereas $\alpha_i = 1$ means that the corresponding controller is applied.

We assume that $f_i(t)$, $i = 1, 2$, is generated by the following controller :

$$\dot{z}_i = A_i z_i + b_i r_i(t) \quad , \quad (4)$$

$$\dot{x}_{i1} = \omega_i x_{i2} \quad , \quad \dot{x}_{i2} = -\omega_i x_{i1} + r_i(t) \quad , \quad (5)$$

$$f_i(t) = c_i^T z_i + d_i r_i + k_i x_{i2} \quad , \quad (6)$$

where $z_i \in \mathbf{R}^{n_i}$, for some natural number n_i , is the controller state, $A_i \in \mathbf{R}^{n_i \times n_i}$ is a constant matrix, $b_i, c_i \in \mathbf{R}^{n_i}$ are constant column vectors, d_i, k_i, ω_i are positive constants, and the superscript T denotes transpose. The controller inputs r_i are defined as :

$$r_1(t) = u_{xt}(1, t) \quad , \quad r_2(t) = u_t(1, t) \quad . \quad (7)$$

If we take the Laplace transform, then the controller transfer function $g_i(s)$ between its input r_i and output $f_i(t)$ may be found as

$$g_i(s) = h_i(s) + \frac{k_i s}{s^2 + \omega_i^2} \quad , \quad (8)$$

where $h_i(s) = c_i^T (sI - A_i)^{-1} b_i + d_i$.

We make the following assumptions concerning the actuator given by (4)-(6) throughout this work (for $i = 1, 2$).

Assumption 1 : All eigenvalues of $A_i \in \mathbf{R}^{n_i \times n_i}$ have negative real parts.

Assumption 2 : (A_i, b_i) is controllable and (c_i, A_i) is observable.

Assumption 3 : $d_i \geq 0, k_i \geq 0$; moreover there exists a constant $\gamma_i, d_i \geq \gamma_i \geq 0$, such that the following holds :

$$\Re\{h_i(j\omega)\} > \gamma_i \quad , \quad \omega \in \mathbf{R} \quad , \quad (9)$$

where $h_i(s)$ is given in (8). Moreover for $d_i > 0$, we assume $\gamma_i > 0$ as well.

The assumptions given above implies that $h_i(s)$ is a strictly positive real transfer function, and hence $g_i(s)$ given by (8) is only positive real. The dynamic controller given by (8) with $\alpha_1 = 0, \alpha_2 = 1, k_2 = 0$ was considered in Morgül (1992) and it was shown that when $g_2(s)$ satisfies the above assumptions with $\gamma_2 > 0$, then the resulting closed-loop system is exponentially stable. The case $\alpha_1 = 0$ was also considered in Morgül (2001).

3. STABILITY RESULTS

Let the assumptions 1-3 stated above hold. Then, since the transfer function $h_i(s)$, $i = 1, 2$, is strictly positive real it follows from the Meyer-Kalman-Yakubovich Lemma that there exist symmetric positive definite matrices $Q_i \in \mathbf{R}^{n_i \times n_i}$ and $P_i \in \mathbf{R}^{n_i \times n_i}$, a vector $q_i \in \mathbf{R}^{n_i}$ satisfying (see Slotine and Li (1991, p. 133)) :

$$A_i^T P_i + P_i A_i = -q_i q_i^T - Q_i \quad , \quad (10)$$

$$P_i b_i - c_i = \sqrt{2(d_i - \gamma_i)} q_i \quad (11)$$

To analyze the system given by (1)-(3), (4)-(6), we first define the function space \mathcal{H} as follows

$$\mathcal{H} = \{(u \ v \ z_1 \ z_2 \ x_{11} \ x_{12} \ x_{21} \ x_{22})^T \mid \begin{array}{l} u \in \mathbf{H}_0^2, \quad v \in \mathbf{L}^2, z_i \in \mathbf{R}^{n_i}, \\ x_{i1}, x_{i2} \in \mathbf{R}, i = 1, 2 \end{array} \} \quad (12)$$

where the spaces \mathbf{L}^2 , and \mathbf{H}_0^k are defined as follows

$$\mathbf{L}^2 = \{f : [0, 1] \rightarrow \mathbf{R} \mid \int_0^L f^2 dx < \infty\} \quad (13)$$

$$\mathbf{H}^k = \{f \in \mathbf{L}^2 \mid f', \dots, f^{(k)} \in \mathbf{L}^2\}, \quad (14)$$

$$\mathbf{H}_0^k = \{f \in \mathbf{H}^k \mid f(0) = f'(0) = 0\} \quad (15)$$

The equations (1)-(3), (4)-(6) can be written in the following abstract form :

$$\dot{z} = Lz \quad , \quad z(0) \in \mathcal{H} \quad , \quad (16)$$

where $z = (u \ u_t \ z_1 \ z_2 \ x_{11} \ x_{12} \ x_{21} \ x_{22})^T \in \mathcal{H}$, the operator $L : \mathcal{H} \rightarrow \mathcal{H}$ is a linear unbounded operator defined as

$$L \begin{pmatrix} u \\ v \\ z_1 \\ z_2 \\ x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} v \\ -u_{xxx} \\ A_1 z_1 + b_1 v_x(1) \\ A_2 z_2 + b_2 v(1) \\ \omega_1 x_{12} \\ -\omega_1 x_{11} + v_x(1) \\ \omega_2 x_{22} \\ -\omega_2 x_{21} + v(1) \end{pmatrix} \quad (17)$$

The domain $D(L)$ of the operator L is defined as

$$D(L) = \{z \in \mathcal{H} \mid u \in \mathbf{H}_0^4, v \in \mathbf{H}_0^2, z_i \in \mathbf{R}^{n_i}, x_{i1}, x_{i2} \in \mathbf{R}; u_{xx}(1) + \alpha_1 [c_1^T z_1 + d_1 v_x(1) + k_1 x_{12}] = 0, -u_{xxx}(1) + \alpha_2 [c_2^T z_2 + d_2 v(1) + k_2 x_{22}] = 0\} \quad (18)$$

where $z = (u \ v \ z_1 \ z_2 \ x_{11} \ x_{12} \ x_{21} \ x_{22})^T$.

Note that when $\alpha_i = 0$ (i.e. the corresponding controller is not applied), the lines and terms corresponding to z_i, x_{i1}, x_{i2} should be omitted.

Let the assumptions 1-3 hold, let $Q_i \in \mathbf{R}^{n_i \times n_i}$, $P_i \in \mathbf{R}^{n_i \times n_i}$, and $q_i \in \mathbf{R}^{n_i}$ be the solutions of (10) and (11) where P_i is also a symmetric and positive definite matrix. In \mathcal{H} , we define the following "energy" inner-product:

$$\langle y, \tilde{y} \rangle_E = \frac{1}{2} \int_0^1 v \tilde{v} dx + \frac{1}{2} \int_0^1 u_{xx} \tilde{u}_{xx} dx \quad (19)$$

$$+ \frac{1}{2} \sum_{i=1}^2 \tilde{z}_i^T P_i z_i$$

$$+ \frac{1}{2} \sum_{i=1}^2 k_i (x_{i1} \tilde{x}_{i1} + x_{i2} \tilde{x}_{i2})$$

where $y = (u \ v \ z_1 \ z_2 \ x_{11} \ x_{12} \ x_{21} \ x_{22})^T$, $\tilde{y} = (\tilde{u} \ \tilde{v} \ \tilde{z}_1 \ \tilde{z}_2 \ \tilde{x}_{11} \ \tilde{x}_{12} \ \tilde{x}_{21} \ \tilde{x}_{22})^T \in \mathcal{H}$. It can be shown that \mathcal{H} , together with the energy inner-product given by (19) becomes a Hilbert space. The "energy" norm induced by (19) (for the solution $z(t)$ of (16)) is given by :

$$E(t) = \frac{1}{2} \int_0^1 u_t^2 dx + \frac{1}{2} \int_0^1 u_{xx}^2 dx \quad (20)$$

$$+ \frac{1}{2} \sum_{i=1}^2 \tilde{z}_i^T P_i z_i$$

$$+ \frac{1}{2} \sum_{i=1}^2 k_i (x_{i1}^2 + x_{i2}^2) \quad .$$

Theorem 1 : Consider the system given by (16) with $d_i \geq 0$, and $k_i \geq 0$. The operator L generates a C_0 -semigroup of contractions $T(t)$ in \mathcal{H} , (for the terminology of semigroup theory, the reader is referred to e.g. Luo et. al. (1999)).

Proof : We use Lumer-Phillips theorem, to prove the assertion, see Luo et. al. (1999). To prove that L is dissipative, we differentiate (20) with respect to time. Then by using (1)-(3), (4)-(6), integrating by parts and using (10), (11), we obtain :

$$\dot{E} = \int_0^1 u_t u_{tt} dx + \int_0^1 u_{xx} u_{xxt} dx \quad (21)$$

$$+ \frac{1}{2} \sum_{i=1}^2 \tilde{z}_i^T (A_i^T P_i + P_i A_i) z_i$$

$$+ \sum_{i=1}^2 \tilde{z}_i^T P_i b_i r_i + \sum_{i=1}^2 k_i r_i x_{i2}$$

$$= -u_{xxx}(1, t) u_t(1, t) + u_{xx}(1, t) u_{xt}(1, t)$$

$$+ \frac{1}{2} \sum_{i=1}^2 \tilde{z}_i^T (A_i^T P_i + P_i A_i) z_i$$

$$+ \sum_{i=1}^2 \tilde{z}_i^T P_i b_i r_i + \sum_{i=1}^2 k_i r_i x_{i2}$$

$$= -\sum_{i=1}^2 \alpha_i f_i r_i + \sum_{i=1}^2 \tilde{z}_i^T P_i b_i r_i$$

$$- \frac{1}{2} \sum_{i=1}^2 \tilde{z}_i^T q_i q_i^T z_i - \frac{1}{2} \sum_{i=1}^2 \tilde{z}_i^T Q_i z_i$$

$$= -\sum_{i=1}^2 \alpha_i \gamma_i r_i^2 - \frac{1}{2} \sum_{i=1}^2 \tilde{z}_i^T Q_i z_i$$

$$- \frac{1}{2} \sum_{i=1}^2 [\alpha_i \sqrt{2(d_i - \gamma_i)} r_i - \sum_{i=1}^2 \tilde{z}_i^T q_i]^2$$

where r_i is given by (7). Since $\dot{E} \leq 0$, it follows that L is dissipative, (see (19), (20), (21)).

It can be shown that $\lambda I - L : \mathcal{H} \rightarrow \mathcal{H}$ is onto for $\lambda > 0$, (see Morgül (1994) and Morgül (2001) for similar calculations). Then, it follows from the Lumer-Phillips theorem that L generates a C_0 -semigroup of contractions $T(t)$ on \mathcal{H} . \square

Note that the result stated above shows that the system under consideration is stable. Next we prove some asymptotic stability results.

Theorem 2 : Consider the system given by (16) with $d_i \geq 0$, and $k_i \geq 0$. Assume that $\alpha_1 = \alpha_2 = 1$, (i.e. both controllers are applied).

i : If $k_1 > 0, k_2 > 0$, and either $\tau = \sqrt{\omega_1}$ or $\tau = \sqrt{\omega_2}$ are not one of the roots of the following transcendental equation :

$$1 - \cosh \tau \cos \tau = 0 \quad , \quad (22)$$

then the semigroup $T(t)$ generated by L is asymptotically stable, that is all solutions of (16) asymptotically converge to zero.

ii : If $k_1 = 0, k_2 > 0$ then the semigroup $T(t)$ generated by L is asymptotically stable.

iii : If $k_1 > 0, k_2 = 0$ then the semigroup $T(t)$ generated by L is asymptotically stable.

Proof : Note that by Theorem 1, the operator L generates a C_0 -semigroup of contractions. To prove the assertions **i-iii**, we use LaSalle's invariance principle, extended to infinite dimensional systems, see Luo et. al. (1999). According to this principle, all solutions of (16) asymptotically tend to the maximal invariant subset of the following set :

$$\mathcal{S} = \{z \in \mathcal{H} | \dot{E} = 0\} \quad (23)$$

provided that the solution trajectories for $t \geq 0$ are *precompact* in \mathcal{H} . Since the operator $L : \mathcal{H} \rightarrow \mathcal{H}$ generates a C_0 -semigroup of *contractions* on \mathcal{H} (hence the solution trajectories are *bounded* on \mathcal{H} for $t \geq 0$), the precompactness of the solution trajectories are guaranteed if the operator $(\lambda I - L)^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ is compact for some $\lambda > 0$, see Luo et. al. (1999) To prove the last property, we first show that L^{-1} exists and is a compact operator on \mathcal{H} . To see this, let $q = (f \ h \ r_1 \ r_2 \ r_{11} \ r_{12} \ r_{21} \ r_{22})^T \in \mathcal{H}$ be given. We want to solve the equation $Lz = q$ for z , where $z = (u \ v \ z_1 \ z_2 \ x_{11} \ x_{12} \ x_{21} \ x_{22})^T \in D(L)$. After straightforward calculations, the required $z \in D(L)$ can be found uniquely. It follows that L^{-1} exists and maps \mathcal{H} into $\mathbf{H}^4 \times \mathbf{H}^2 \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}$. Since $q \in \mathcal{H}$ it follows that $f \in \mathbf{H}_0^2$, see (12). Hence, if $\|q\|$ is bounded in \mathcal{H} , it follows easily that that $f(1)$ is bounded as well. Therefore L^{-1} maps the bounded sets of \mathcal{H} into the bounded sets of $\mathbf{H}^4 \times \mathbf{H}^2 \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}$. Since the embedding of the latter into \mathcal{H} is compact, see Tanabe (1979, p. 14), it follows that L^{-1} is a compact operator. This also proves that the spectrum of L consists entirely of isolated eigenvalues, and that for any λ in the

resolvent set of L , the operator $(\lambda I - L)^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ is a compact operator, Kato (1980, p. 187). Furthermore, our argument above shows that $\lambda = 0$ is not an eigenvalue of L . Since the operator L generates a C_0 -semigroup of contractions on \mathcal{H} , by the argument given above it follows that the solutions trajectories of (16) are precompact in \mathcal{H} for $t \geq 0$, hence by LaSalle's invariance principle, the solutions asymptotically tend to the maximal invariant subset of \mathcal{S} (see (23)). Hence, to prove that all solutions of (16) asymptotically tend to the zero solution, it suffices to show that \mathcal{S} contains only the zero solution, which is a typical procedure in the application of LaSalle's invariance principle.

To prove that \mathcal{S} contains only the zero solution, we set $\dot{E} = 0$ in (21), which results in $z_i = 0, i = 1, 2$. This implies that $\dot{z}_i = 0$, hence by using (4) and (6) we obtain $r_i(t) = 0, f_i(t) = k_i x_{i2}, i = 1, 2$. Hence, all solutions of (16) in \mathcal{S} satisfy the following equation ($i = 1, 2$)

$$u_{tt} + u_{xxx} = 0 \quad , \quad (24)$$

$$\dot{x}_{i1} = \omega_i x_{i2} \quad , \quad \dot{x}_{i2} = -\omega_i x_{i1} \quad , \quad (25)$$

$$u(0, t) = 0 \quad , \quad u_x(0, t) = 0 \quad , \quad (26)$$

$$u_{xt}(1, t) = 0 \quad , \quad u_t(1, t) = 0 \quad , \quad (27)$$

$$-u_{xx}(1, t) = k_1 x_{12}, u_{xxx}(1, t) = k_2 x_{22} \quad (28)$$

The solution x_{i2} of (25) can be written as :

$$x_{i2} = a_i \cos(\omega_i t + \theta_i) \quad , \quad (29)$$

where a_i and θ_i are arbitrary constants.

Since the boundary conditions in (26), (27) are separable, the solution u of (24) can be found by using separation of variables, see Meirovitch (1967). That is, the solution of (24), (26), and (27) assumes the following form : $u(x, t) = A(t)B(x)$ where the functions $A : \mathbf{R}_+ \rightarrow \mathbf{R}$ and $B : [0, 1] \rightarrow \mathbf{R}$ are differentiable functions to be determined from the boundary conditions. We distinguish the following cases :

a : $\dot{A} \equiv 0$. In this case, the solutions of (24) is $u(x, t) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. From (26) and (27), it follows that $c_i = 0, i = 0, \dots, 4$, and $a_i = 0$. Hence, the only possible solution is $u(x, t) \equiv 0$.

b : $\dot{A} \neq 0$. In this case, the solution of (24) and (26) is in the following form :

$$A(t) = c \cos(\omega t + \theta) \quad , \quad (30)$$

$$B(x) = c_3 (\cosh \tau x - \cos \tau x) + c_4 (\sinh \tau x - \sin \tau x) \quad , \quad (31)$$

where $\tau = \sqrt{\omega}$, and c, θ, c_3, c_4 are arbitrary constants. By using (31) in (27), it can be shown that in order to have a nontrivial solution, τ should satisfy (22). By using (31) and (29) in (28), it can be shown that

to have a nontrivial solution, both $\tau = \sqrt{\omega_1}$ and $\tau = \sqrt{\omega_2}$ must be a root of (22). It can be easily shown that if this condition fails, then the only possible solution is the trivial solution. Hence, by LaSalle's invariance principle, we conclude that the solutions of (16) asymptotically tend to the zero solution.

ii, iii can be proven similarly. Note that in the case **ii**, in addition to the boundary conditions (26) and (27), we will have $u_{xx}(1, t) = 0$, whereas for the case **iii**, we will have $u_{xxx}(1, t) = 0$ as well. By using separation of variables, it can easily be shown that the only solution of (24) together with these boundary conditions is the zero solution. Hence, again by LaSalle's invariance theorem, we obtain the stated asymptotic stability result. \square

The case $\alpha_1 = 0, \alpha_2 = 1$ (i.e. only the boundary control force is applied) has been considered in Morgül (2001). Next we give a result for the remaining case $\alpha_1 = 1, \alpha_2 = 0$.

Theorem 3 : Consider the system given by (16) with $d_i \geq 0$, and $k_i \geq 0$. Assume that $\alpha_1 = 1, \alpha_2 = 0$, (i.e. only the boundary control torque is applied). If $k_1 > 0$ and $\tau = \sqrt{\omega_1}$ is not one of the roots of the following transcendental equation

$$\sinh \tau \cos \tau + \cosh \tau \sin \tau = 0 \quad (32)$$

then the semigroup $T(t)$ generated by L is asymptotically stable, that is all solutions of (16) asymptotically converge to zero.

Proof : The proof of this fact is similar to that of Theorem 2, and hence is omitted here. \square

4. DISTURBANCE REJECTION

In this section we show the effect of the proposed control law given by (4)-(6) on the solutions of the system given by (1)-(3), when the output of the controller is corrupted by a disturbance $n_i(t)$, that is (6) has the following form :

$$f_i(t) = c_i^T z_i + d_i r_i + k_i x_{i2} + n_i(t) \quad (33)$$

or equivalently we have the following :

$$\hat{f}_i(s) = g_i(s)\hat{r}_i(s) + \hat{n}_i(s) \quad (34)$$

for $i = 1, 2$, where a hat denotes the Laplace transform of the corresponding variable $g_i(s)$ is given by (8).

To find the transfer function from n_i to r_i , first we need to find the transfer function from f_i to r_i . By taking the Laplace transform of (1)-(3) and using zero initial conditions, after some straightforward calculations we obtain the following : ($i = 1, 2$)

$$\hat{r}_i(s) = -h_{i1}(s)\hat{f}_1(s) - h_{i2}(s)\hat{f}_2(s) , \quad (35)$$

where $h_{ij}(s)$ are appropriate functions, which are not given here due to space limitations. After straightforward calculations, from (34), (35) we obtain ($i = 1, 2$):

$$\hat{r}_i(s) = \tilde{h}_{i1}(s)\hat{n}_1(s) + \tilde{h}_{i2}(s)\hat{n}_2(s) , \quad (36)$$

where $\tilde{h}_{ij}(s)$ are the closed-loop transfer functions.

From (36) we can also derive a procedure to design $g_i(s)$ if we know the structure of $n_i(t)$. For example if $n_i(t)$ has a band-limited frequency spectrum, (i.e. has frequency components in an interval of frequencies $[\Omega_1, \Omega_2]$), then we can choose $g_i(s)$ to minimize $|\tilde{h}_{ij}(j\omega)|$ for $\omega \in [\Omega_1, \Omega_2]$, for $i, j = 1, 2$. Note that to ensure the stability of the closed-loop system, $g_i(s)$ should be a positive real function as well, (see (8)). As a simple example, assume that $n_1(t) = 0, n_2(t) = a \cos \omega_0(t)$. Then we may choose $\alpha_1 = 0, \alpha_2 = 1$ (i.e. only boundary control force is applied), and $g_2(s)$ in the form (8) with $\omega_2 = \omega_0$. Provided that the assumptions 1-3 are satisfied and that $j\omega_0$ is not a zero of $h_{22}(s)$, the closed-loop system is asymptotically stable. Moreover, if $k_2 > 0$, then we have $\tilde{h}_{22}(j\omega_0) = 0$. From the above discussions we may conclude that this eliminates the effect of the disturbance at the output $u_i(1, t)$, see also Morgül (1998), Morgül (2001).

5. CONCLUSION

In this paper, we considered a linear time invariant system which is represented by one-dimensional Euler-Bernoulli beam equation in a bounded domain. We assumed that the system is clamped at one end and the boundary control force and torque inputs are applied at the other end. For this system, we proposed finite dimensional *dynamic* boundary controllers to generate the input force and torque. This introduces extra degrees of freedom in designing controllers which could be exploited in solving a variety of control problems, such as disturbance rejection, pole assignment, etc., while maintaining stability. The transfer function of the controllers are proper rational functions of the complex variable s , and may contain poles on the imaginary axis, provided that the residues corresponding to these poles are nonnegative; the rest of the transfer function is required to be strictly positive real function. We then proved that the closed-loop system is stable in general and asymptotically stable under certain conditions. These conditions depend on the location of the imaginary axis poles. We also discussed the case where the output of the controller is corrupted by a disturbance. We showed that, if the frequency spectrum of the controller is known, then by choosing the controller appropriately we may obtain better disturbance rejection.

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