**CONSISTENCY IN HOUSE ALLOCATION
PROBLEMS**

A THESIS

**SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE**

By

Haluk İ. Ergin

June, 1999

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I certify that I have read this thesis and that in my opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

CONSISTENCY IN HOUSE ALLOCATION PROBLEMS

Haluk İ. Ergin

M.S. in Mathematics

Supervisor: Asst. Prof. Dr. Alexander Goncharov

June, 1999

In *house allocation problems*, we look for a systematic way of assigning a set of indivisible objects, e.g. houses, to a group of individuals having preferences over these objects. Typical real life examples are graduate housing, assignment of offices and tasks. Once an allocation is decided upon, the actual assignments of the agents are not likely to take place simultaneously. Therefore, rules whose predictions are independent of the sequence in which the actual assignments are realized turn out to be very appealing. We model this property via the *consistency* principle and identify various classes of *consistent* rules and correspondences.

Keywords: Consistency, house allocation problems, assignment of indivisible objects.

ÖZET

EV TAHSİSAT PROBLEMLERİNDE TUTARLILIK

Haluk İ. Ergin

Matematik Bölümü Yüksek Lisans

Tez Yöneticisi: Yard. Doç. Dr. Alexander Goncharov

Haziran, 1999

Ev tahsisat problemlerinde, bir grup bölünmez malı, bu mallar üzerinde tercihleri bulunan bir grup kişiye tahsis etmenin sistematik yöntemleri araştırılmaktadır. Lojman, ofis ve görev tahsisatı bu problemlerin tipik örneklerini oluşturmaktadırlar. Bir tahsisata karar verildikten sonra, bireylerin fiili tahsisatlarının eşzamanlı olarak gerçekleşme olasılığı düşüktür. Bu yüzden, fiili tahsisatın izleyeceği sıradan bağımsız olan tahsisat kuralları cazip bir sınıf oluşturmaktadırlar. Bu tezde, yukarıda sözü geçen özellik *tutarlılık* prensibi ile modellenmekte ve çeşitli tek ve çok değerli *tutarlı* kural sınıfları belirlenmektedir.

Anahtar kelimeler: Tutarlılık, ev tahsisat problemleri, bölünmez mal tahsisatı.

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Chapter 1

Introduction

A *house allocation problem* is a one-sided matching problem, where a set of agents collectively own a set of indivisible goods, e.g. houses, and every agent has strict preferences over these indivisible goods. The number of agents and the number of houses are assumed to be finite and equal. An allocation is an assignment of the houses to the agents, such that each agent receives exactly one house. Assignment of dormitory rooms or offices at the beginning of the academic year are examples of house allocation problems.

House allocation problems are closely related to the *housing market* model introduced by Shapley and Scarf (1974). The only difference between the two classes is that, in the latter, each agent owns one house, whereas in the former, houses are owned collectively. The housing market model has been thoroughly investigated and many strong results have been obtained concerning the core (competitive) correspondence. Roth and Postlewaite (1977) show that the core correspondence is *singlevalued* and Roth (1982) shows that it is *strategy-proof*. Ma (1994) shows that the core correspondence is the only correspondence that is *Pareto optimal*, *individually rational* and

strategy-proof. Abdulkadiroğlu and Sönmez (1998a) introduce the core from random endowments as a lottery mechanism for house allocation problems. They show that the core from random endowments is equivalent to random serial dictatorship, which formally establishes the close relationship between the housing market model and the house allocation problems.

In the context of house allocation problems, a correspondence is a map that chooses a set of allocations for each problem. A rule is a *singlevalued* correspondence. In this thesis, we identify various classes of *consistent* and *conversely consistent* correspondences. Informally, *consistency* requires that, if an allocation is chosen for a problem, then for any subgroup of agents, the restriction of that allocation should be chosen for the smaller problem consisting of that subgroup and their original assignments. *Consistent* rules are coherent in their suggestions for problems involving different groups of agents. For example, in n -person bargaining problems, a rule that selects the egalitarian outcome when n equals 2 and a dictatorial outcome when n is greater than 2, is quite implausible because it is not *consistent*. The *consistency* principle has been analyzed in many contexts, such as game theory, public finance, and fair allocation.¹ As we illustrate in the next paragraph, *consistent* rules also have a very practical appeal in classes of resource allocation problems where individuals are likely to receive their material allocations sequentially. Examples of such classes are two-sided matching, rationing and house allocation problems where many strong results have already been obtained from the applications of the *consistency* principle. In economies with indivisible goods and money, Tadenuma and Thomson (1991) identify the

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¹A comprehensive survey of *consistency* for resource allocation problems can be found in Thomson (1996).

correspondences that satisfy *no-envy* and variants of *consistency*, *neutrality*, and *converse consistency*. In a large class of two-sided matching problems, Sasaki and Toda (1992) show that the stable correspondence (the core) is the only correspondence that satisfies *Pareto optimality*, *anonymity*, *consistency*, and *converse consistency*. Moulin (1997) investigates *consistent* rules in the context of rationing problems.

In the context of house allocation problems, *consistency* requires that once an allocation is chosen and a group of agents take their assigned houses before the others, the allocation rule should not change the assignments of the remaining agents in the reduced problem involving the remaining agents and houses. For example, suppose that a rule assigning dormitory rooms to students is not *consistent*. Then, if some students occupy their rooms before the others, the rule may require a change in the assignments of the remaining students! Such a change would not only impose operational and transactional costs, but it would also lead the agents and the authorities to question the plausibility of the rule. *Consistent* rules are robust to non-simultaneous allocations of the houses. Therefore, we believe that *consistent* rules are more likely to emerge than ‘*inconsistent*’ rules.

In a problem where every agent has the same preferences over the houses, *every allocation discriminates between agents*. Indeed, there is a one-to-one correspondence between allocations and priority orderings over the set of individuals, illustrating the impossibility of equal treatment of equals in this class of problems. For this reason, “sequential solutions” and “serial dictatorships” constitute a powerful class of rules when randomization or monetary compensations are not allowed. Given an exogenous priority ordering which

may for example be based on seniority, a serial dictatorship rule sequentially assigns every agent his most preferred house while respecting earlier assignments. Sequential solutions are a more general class of rules where certain agents receive their least preferred house when their turn comes. Simple sequential solutions are *consistent*, *conversely consistent*, and *neutral*. In Theorem 1, we show that simple sequential solutions are the only rules that satisfy a weak form of *consistency* and a weak form of *neutrality*, namely *pairwise consistency* and *pairwise neutrality*. Simple serial dictatorships are *Pareto optimal*, *strategy-proof*, *consistent*, *conversely consistent*, and *neutral*. In Corollary 1, we show that simple serial dictatorships are the only rules that are *weakly Pareto optimal*, *pairwise consistent*, and *pairwise neutral*. Besides its descriptive nature, Theorem 1 can be interpreted as a negative finding, since dropping *efficiency* does not allow us to recover rules having other properties of normative interest. Then, we drop *singlevaluedness*. In Proposition 5, we show that *anonymous* correspondences are not very appealing even in the *multivalued* case. In Corollary 2, we characterize *Pareto optimal*, *consistent*, and *conversely consistent* correspondences via their behavior in two-person problems. In Theorem 2, we show that a correspondence is *non-empty valued*, *Pareto optimal*, *consistent*, *conversely consistent*, and *neutral* if and only if it can be written as a union of serial dictatorships in a particular manner.

In Chapter 4, we relax some of our axioms with the hope of overcoming the dictatorial feature of consistent correspondences mentioned above. Firstly, we weaken the *pairwise neutrality* axiom and introduce *lower neutrality* which requires a rule to be independent of the agents' bottom-ranked

houses in two-person problems. “Serial bidictatorships” constitute a class which generalizes serial dictatorships. Given an exogenous priority ordering with possibly ties of size two and tie-breaking functions indexed by the houses, a serial bidictatorship rule assigns agents their most preferred houses while respecting earlier assignments and breaking ties in the priority ordering by using the tie-breaking functions. If two agents are in the same indifference class w.r.t. the priority ordering, the tie-breaking process may be interpreted as partitioning the set of houses into two, so that each of the two agents has priority over the possession of the houses in one of these subsets. For example, when allocating offices, it is reasonable to give priority to a professor doing computational research, over the offices equipped with advanced computers and to give priority to a professor who is mainly involved in teaching, over large sized offices, so that he can conduct his office-hours comfortably. Simple serial bidictatorships need not be dictatorial. They are *Pareto optimal*, *strategy-proof*, *consistent*, *conversely consistent* and *lower neutral*. In Theorem 3, we show that simple serial bidictatorships are the only rules that are *weakly Pareto optimal*, *pairwise consistent* and *lower neutral*.

Then, we allow rules to associate with each problem, a lottery over the allocations for that problem. We extend the *consistency* principle to *lottery-valued* rules by requiring that a priori, a decision to draw the lottery in arbitrary sequence should not change the lotteries faced in the beginning. However, the answer to the question of existence of *anonymous* and *consistent lottery-valued* rules turns out to be negative, as we show in Theorem 4 that there does not exist an *ex-post Pareto optimal*, *consistent* and *ex-ante anonymous lottery-valued* rule. Precise definitions of the above concepts are

provided in Chapter 2. Chapters 3 and 4 contain the results and Chapter 5 presents the concluding remarks. The independence of axioms is deferred to the appendix.

Chapter 2

Environments

Let $\mathcal{N}$ be a *set of potential agents* and $\mathcal{H}$ a *set of potential houses* such that $|\mathcal{N}| \geq 3$ and $|\mathcal{H}| \geq 3$. A house allocation problem or simply a **problem** is a triplet $\mathcal{E} = (N, H, (R_i)_{i \in N})$ where $\emptyset \neq N \subset \mathcal{N}$, $\emptyset \neq H \subset \mathcal{H}$, $|N| = |H|$ is finite, and for each $i \in N$, R_i is a linear order on H representing agent i 's preference over the houses in H .¹ For each $i \in N$, P_i denotes the asymmetric part of R_i .²

Given a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, an **allocation** $\mu : N \rightarrow H$ is a bijection, where $\mu(i)$ denotes the house assigned to agent i .

Let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ be any problem, μ any allocation for $\mathcal{E}$ and $i, j \in N$ any two agents. We say that **i envies j under μ** if $\mu(j) P_i \mu(i)$.

An allocation correspondence, or simply a **correspondence**, is a map

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¹A binary relation R_i on H is a **linear order** if it is **reflexive** ($\forall a \in H : a R_i a$), **complete** ($\forall a, b \in H : a \neq b \implies a R_i b$ or $b R_i a$), **transitive** ($\forall a, b, c \in H : a R_i b$ and $b R_i c \implies a R_i c$) and **antisymmetric** ($\forall a, b \in H : a R_i b$ and $b R_i a \implies a = b$). Indifference between different houses is not allowed.

²For any $a, b \in H$, we say $a P_i b$ if and only if $a R_i b$ and not $b R_i a$. In general, a relation P_i on H is **asymmetric** if for any $a, b \in H$, $a P_i b$ implies not $b P_i a$.

φ which associates with each problem a possibly empty set of allocations for that problem. An allocation rule, or simply a **rule**, is a map φ which associates with each problem *exactly one* allocation for that problem. A rule is a *singlevaled* correspondence.

Given a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, an allocation μ' for $\mathcal{E}$ **weakly Pareto dominates** another allocation μ for $\mathcal{E}$ if every agent in N is weakly better off and at least one agent is strictly better off under μ' than under μ . The allocation μ' **strongly Pareto dominates** μ for $\mathcal{E}$ if every agent in N is strictly better off under μ' than under μ . The **Pareto correspondence** associates with each problem the set of allocations that are not weakly Pareto dominated. The **weak Pareto correspondence** associates with each problem the set of allocations that are not strongly Pareto dominated. A correspondence is **Pareto optimal** if it never chooses allocations that are weakly Pareto dominated. Similarly, a correspondence is **weakly Pareto optimal** if it never chooses allocations that are strongly Pareto dominated.

Abdulkadiroğlu and Sönmez (1998a) show that serial dictatorships lead to *Pareto optimal* allocations. Serial dictatorships can be considered as the *Pareto optimal* subclass of a more general class of rules that we call sequential solutions. Given a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, a linear order $\succeq$ on N and a subset $M \subset N$, the **sequential allocation induced by $\succeq$ and M for $\mathcal{E}$** is defined inductively as follows. Let i^k be the k^{th} person from the top in N w.r.t. $\succeq$. First, if $i^1 \in M$, then i^1 is allocated his top-ranked house in H , otherwise i^1 is allocated his bottom-ranked house in H . At the k^{th} step, if $i^k \in M$, then i^k is allocated his top-ranked house among those that are not already allocated in earlier steps, otherwise i^k is allocated his bottom-

ranked house among the remaining ones. The set M identifies the set of agents whose welfares are maximized by the sequential solution. Let i^n be the bottom-ranked person in N w.r.t. $\succeq$. Note that the sequential allocation induced by $\succeq$ and M will be the same, whether $i^n \in M$ or not. Moreover, if $M \supset N \setminus \{i^n\}$, then the above sequential solution corresponds with the serial dictatorship induced by $\succeq$. Formally, given a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$ and a linear order $\succeq$ on N , the **serial dictatorship allocation induced by $\succeq$ for $\mathcal{E}$** is the sequential allocation induced by $\succeq$ and N for $\mathcal{E}$. Conversely, a sequential allocation coincides with the serial dictatorship allocation where the preferences of the agents in $N \setminus M$ are turned upside-down.

We next introduce natural extensions of sequential solutions to the variable population case. For any linear order $\succeq$ on $\mathcal{N}$ and any $\emptyset \neq N \subset \mathcal{N}$, let $\succeq|_N$ be the restriction of $\succeq$ to N . A rule is a **simple sequential solution** if there exists a linear order $\succeq$ on $\mathcal{N}$ and a subset $\mathcal{M} \subset \mathcal{N}$ such that for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, the rule selects the sequential allocation induced by $\succeq|_N$ and $\mathcal{M} \cap N$. In this case, the rule is denoted by $\varphi^{\succeq, \mathcal{M}}$.³ A rule is a **simple serial dictatorship** if it coincides with $\varphi^{\succeq, \mathcal{N}}$ for some linear order $\succeq$ on $\mathcal{N}$. For simplicity, we will denote such a rule by $\varphi^{\succeq}$.

For any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, any $\emptyset \neq N' \subset N$ and any allocation μ for $\mathcal{E}$, the **reduced problem of $\mathcal{E}$ w.r.t. N' at μ** is:

$$r_{N'}^\mu(\mathcal{E}) = \left(N', \mu(N'), (R_i|_{\mu(N')})_{i \in N'} \right)$$

where $\mu(N')$ is the set of remaining houses after the agents in $N \setminus N'$ have

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³Since agents have strict preferences, given a linear order $\succeq$, a subset $\mathcal{M} \subset \mathcal{N}$ and a problem $\mathcal{E}$, there *exists a unique* sequential allocation induced by $\succeq|_N$ and $\mathcal{M} \cap N$ for $\mathcal{E}$. Therefore, $\varphi^{\succeq, \mathcal{M}}$ is well defined as a rule.

left with their assigned houses, and $R_i|_{\mu(N')}$ is the restriction of agent i 's preference to the remaining houses. The **reduced allocation w.r.t. N'** , $\mu_{N'}: N' \rightarrow \mu(N')$ is the bijection defined by $\mu_{N'}(i) = \mu(i)$, for each $i \in N'$.

A correspondence φ is **consistent** if for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, any $\emptyset \neq N' \subset N$ and any $\mu \in \varphi(\mathcal{E})$, one has $\mu_{N'} \in \varphi(r_{N'}^\mu(\mathcal{E}))$. Note that the union of *consistent* correspondences is *consistent*. A correspondence φ is **pairwise consistent** if for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, any $N' \subset N$ with $|N'| = 2$ and any $\mu \in \varphi(\mathcal{E})$, one has $\mu_{N'} \in \varphi(r_{N'}^\mu(\mathcal{E}))$. It is **conversely consistent** if for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$ with $|N| \geq 2$ and any allocation μ for $\mathcal{E}$ such that for any $N' \subset N$ with $|N'| = 2$ we have $\mu_{N'} \in \varphi(r_{N'}^\mu(\mathcal{E}))$, we have $\mu \in \varphi(\mathcal{E})$.⁴ By changing set memberships to equalities, one obtains the definitions of *consistency*, *pairwise consistency* and *converse consistency* for rules.

Anonymity requires that a correspondence should be independent of the names of the agents. Formally, a correspondence φ is **anonymous** if for any $\emptyset \neq H \subset \mathcal{H}$, any two problems $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $\mathcal{E}' = (N', H, (R'_i)_{i \in N'})$

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⁴An alternative definition of *converse consistency* would require that if for any proper subset N' of N with $|N'| \geq 2$, one has $\mu_{N'} \in \varphi(r_{N'}^\mu(\mathcal{E}))$ then $\mu \in \varphi(\mathcal{E})$. These two definitions turn out to be equivalent in the context of house allocation problems. It is straightforward to check the equivalence of these definitions via induction on the number of players, by using the following two **transitivity** properties of reduction: if $\emptyset \neq N'' \subset N' \subset N$, $\mathcal{E} = (N, H, (R_i)_{i \in N})$ is a problem and μ is an allocation for $\mathcal{E}$ then $r_{N''}^{\mu_{N'}}(r_{N'}^\mu(\mathcal{E})) = r_{N''}^\mu(\mathcal{E})$ and $(\mu_{N'})_{N''} = \mu_{N''}$. By using the same properties, one can also show that *pairwise consistency* and *converse consistency* imply *consistency*. The latter statement is a direct consequence of Lemma 2. Thomson (1996) points out that in any class of allocation problems where admissible problems involve finitely many agents and reduction is *transitive*, the two forms of *converse consistency* are equivalent.

and any $\mu \in \varphi(\mathcal{E})$, if $\pi: N \rightarrow N'$ is a bijection satisfying:

$$\forall i \in N, \forall a, b \in H : \quad aR_i b \iff aR'_{\pi(i)} b,$$

then $\mu \circ \pi^{-1} \in \varphi(\mathcal{E}')$.

Neutrality requires that a correspondence should be independent of the particular labeling of the houses. More precisely, a correspondence φ is **neutral** if for any $\emptyset \neq N \subset \mathcal{N}$, any two problems $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $\mathcal{E}' = (N, H', (R'_i)_{i \in N})$ and any $\mu \in \varphi(\mathcal{E})$, if $\pi: H \rightarrow H'$ is a bijection satisfying:

$$\forall i \in N, \forall a, b \in H : \quad aR_i b \iff \pi(a)R'_i \pi(b),$$

then $\pi \circ \mu \in \varphi(\mathcal{E}')$. By changing the quantifier “for any $\emptyset \neq N \subset \mathcal{N}$ ” to “for any $N \subset \mathcal{N}$ with $|N| = 2$ ”, we obtain the definition of **pairwise neutrality**.

Chapter 3

Main Results

3.1 Consistent Rules

For any $i \in \mathcal{N}$, any linear order $\succeq$ on $\mathcal{N}$ and any $\emptyset \neq N \subset \mathcal{N}$, let $L(i, \succeq, N) = \{j \in N \mid i \succeq j\}$. A property that *characterizes* serial dictatorships in the context of assignment problems is that an agent never envies those who are ranked below him in the serial dictatorship order. This idea is generalized to sequential solutions in the following lemma.

Lemma 1 *Let $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $M \subset N$ and let $\succeq$ be a linear order on N . An allocation μ for $\mathcal{E}$ is the **sequential allocation induced by $\succeq$ and M for $\mathcal{E}$** if and only if the following are true:*

1. $\mu(i) R_i \mu(j)$ for any $i \in N \cap M$ and any $j \in L(i, \succeq, N)$,
2. $\mu(j) R_i \mu(i)$ for any $i \in N \setminus M$ and any $j \in L(i, \succeq, N)$.

Proof Let $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $M \subset N$ and let $\succeq$ be a linear order on N .

First, assume that μ is the sequential allocation induced by $\succeq$ and M for $\mathcal{E}$. Let $i \in N$ and $j \in L(i, \succeq, N)$. Since $i \succeq j$, j does not come before i

in the sequential solution order. Therefore, $\mu(j)$ is not previously allocated at the step when i receives his house. If $i \in M$, then $\mu(i)$ is the top-ranked house among the remaining ones w.r.t. R_i , at the step when i 's assignment is made. In particular, $\mu(i) R_i \mu(j)$. Similarly, if $i \notin M$, then $\mu(i)$ is the bottom-ranked house among the remaining ones w.r.t. R_i , at the step when i 's assignment is made. In particular, $\mu(j) R_i \mu(i)$.

For the converse, assume that the allocation μ for $\mathcal{E}$ is such that Conditions 1 and 2 are satisfied. For each $k \in \{1, \dots, |N|\}$, let $i^k \in N$ be the k^{th} person from the top in N w.r.t. $\succeq$. Let $k \in \{1, \dots, |N|\}$ and $a \in \{\mu(i^k), \mu(i^{k+1}), \dots, \mu(i^{|N|})\}$. Then $a = \mu(j)$ for some $j \in L(i^k, \succeq, N)$. Therefore, if $i^k \in M$, we have $\mu(i^k) R_{i^k} a$ by Condition 1, and $\mu(i^k)$ is the top-ranked house in $\{\mu(i^k), \mu(i^{k+1}), \dots, \mu(i^{|N|})\}$ w.r.t. R_{i^k} . Similarly, if $i^k \notin M$, by Condition 2, we have that $a R_{i^k} \mu(i^k)$, and $\mu(i^k)$ is the bottom-ranked house in $\{\mu(i^k), \mu(i^{k+1}), \dots, \mu(i^{|N|})\}$ w.r.t. R_{i^k} . So, initially, if $i^1 \in M$, then i^1 receives his top-ranked house in H . Otherwise, he receives his bottom-ranked house in H . At the k^{th} step, if $i^k \in M$, then i^k receives his top-ranked house among those that are not already allocated in earlier steps, otherwise i^k receives his bottom-ranked house among the remaining ones. Therefore, μ is the sequential allocation induced by $\succeq$ and M for $\mathcal{E}$.

QED

Proposition 1 *Simple sequential solutions are consistent, conversely consistent, and neutral.*

Proof: Let $\succeq$ be a linear order on $\mathcal{N}$ and let $\mathcal{M} \subset \mathcal{N}$.

To see that $\varphi^{\succeq, \mathcal{M}}$ is consistent, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $\mu = \varphi^{\succeq, \mathcal{M}}(\mathcal{E})$ and $\emptyset \neq N' \subset N$. Let $M = \mathcal{M} \cap N$ and $M' = \mathcal{M} \cap N'$. Since μ is the sequential

allocation induced by $\succeq|_N$ and M for $\mathcal{E}$, by Lemma 1, we have:

1. $\mu(i) R_i \mu(j)$ for any $i \in N \cap M$ and any $j \in L(i, \succeq|_N, N)$,
2. $\mu(j) R_i \mu(i)$ for any $i \in N \setminus M$ and any $j \in L(i, \succeq|_N, N)$.

Since for any $i \in N$, it is true that $L(i, \succeq|_{N'}, N') \subset L(i, \succeq|_N, N)$, we in particular have:

1. $\mu_{N'}(i) R_i|_{\mu(N')} \mu_{N'}(j)$ for any $i \in N' \cap M'$ and any $j \in L(i, \succeq|_{N'}, N')$,
2. $\mu_{N'}(j) R_i|_{\mu(N')} \mu_{N'}(i)$ for any $i \in N' \setminus M'$ and any $j \in L(i, \succeq|_{N'}, N')$.

By Lemma 1 and Conditions 1 and 2 above, we have that $\mu_{N'}$ is the sequential allocation induced by $\succeq|_{N'}$ and $M' = \mathcal{M} \cap N'$ for $r_{N'}^\mu(\mathcal{E})$. Therefore, $\mu_{N'} = \varphi^{\succeq, \mathcal{M}}(r_{N'}^\mu(\mathcal{E}))$, showing that $\varphi^{\succeq, \mathcal{M}}$ is *consistent*.

To see that $\varphi^{\succeq, \mathcal{M}}$ is *conversely consistent*, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ be any problem with $|N| \geq 2$ and let μ be an allocation for $\mathcal{E}$ such that for any $\emptyset \neq N' \subset N$ with $|N'| = 2$, we have $\mu_{N'} \in \varphi^{\succeq, \mathcal{M}}(r_{N'}^\mu(\mathcal{E}))$. Let $i \in N$, $j \in L(i, \succeq, N)$, $M = \mathcal{M} \cap N$ and $M' = \mathcal{M} \cap \{i, j\}$. Since $\mu_{\{i, j\}} = \varphi^{\succeq, \mathcal{M}}(r_{\{i, j\}}^\mu(\mathcal{E}))$, the allocation $\mu_{\{i, j\}}$ is the sequential allocation induced by $\succeq|_{\{i, j\}}$ and M' for $r_{\{i, j\}}^\mu(\mathcal{E})$. By Lemma 1, we have:

1. $\mu_{\{i, j\}}(i) R_i|_{\mu(\{i, j\})} \mu_{\{i, j\}}(j)$ if $i \in M'$,
2. $\mu_{\{i, j\}}(j) R_i|_{\mu(\{i, j\})} \mu_{\{i, j\}}(i)$ if $i \notin M'$.

Equivalently,

1. $\mu(i) R_i \mu(j)$ for any $i \in N \cap M$ and any $j \in L(i, \succeq|_N, N)$,
2. $\mu(j) R_i \mu(i)$ for any $i \in N \setminus M$ and any $j \in L(i, \succeq|_N, N)$.

So, by Lemma 1, μ is the sequential allocation induced by $\succeq|_N$ and $M = \mathcal{M} \cap N$ for $\mathcal{E}$, i.e. $\mu = \varphi^{\succeq, \mathcal{M}}(\mathcal{E})$. Therefore, φ is *conversely consistent*.

To see that $\varphi^{\succeq, \mathcal{M}}$ is *neutral*, let $\emptyset \neq N \subset \mathcal{N}$, $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $\mathcal{E}' = (N, H', (R'_i)_{i \in N})$ and let $\pi: H \rightarrow H'$ be a bijection satisfying:

$$\forall i \in N, \forall a, b \in H : \quad a R_i b \iff \pi(a) R'_i \pi(b).$$

Let $M = \mathcal{M} \cap N$, $\mu = \varphi^{\succeq, \mathcal{M}}(\mathcal{E})$, $\mu' = \varphi^{\succeq, \mathcal{M}}(\mathcal{E}')$ and for each $k \in \{1, \dots, |N|\}$, let $i^k \in N$ be the k^{th} person from the top in N w.r.t. $\succeq$. Suppose that $\pi(\mu(i^1)) \neq \mu'(i^1)$. If $i^1 \in M$, then $\mu'(i^1)$ is the top-ranked house in H' w.r.t. R'_{i^1} and one has $\mu'(i^1) P'_{i^1} \pi(\mu(i^1))$. By assumption, $H \ni \pi^{-1}(\mu'(i^1)) P_{i^1} \pi^{-1}(\pi(\mu(i^1))) = \mu(i^1)$, a contradiction to $\mu(i^1)$ being the top-ranked house in H w.r.t. R_{i^1} . Otherwise, if $i^1 \notin M$, then $\mu'(i^1)$ is the bottom-ranked house in H' w.r.t. R'_{i^1} and one has $\pi(\mu(i^1)) P'_{i^1} \mu'(i^1)$. Then, $\mu(i^1) = \pi^{-1}(\pi(\mu(i^1))) P_{i^1} \pi^{-1}(\mu'(i^1)) \in H$, a contradiction to $\mu(i^1)$ being the bottom-ranked house in H w.r.t. R_{i^1} . Therefore, $\pi(\mu(i^1)) = \mu'(i^1)$. Now, assume that there exists $k \in \{2, \dots, |N|\}$, such that for all $l \in \{1, \dots, k-1\}$ one has $\pi(\mu(i^l)) = \mu'(i^l)$. Suppose that $\pi(\mu(i^k)) \neq \mu'(i^k)$. If $i^k \in M$, then $\mu'(i^k)$ is the top-ranked house in $\{\mu'(i^k), \mu'(i^{k+1}), \dots, \mu'(i^{|N|})\} \ni \pi(\mu(i^k))$ w.r.t. R'_{i^k} and one has $\mu'(i^k) P'_{i^k} \pi(\mu(i^k))$. But then $\{\mu(i^k), \mu(i^{k+1}), \dots, \mu(i^{|N|})\} \ni \pi^{-1}(\mu'(i^k)) P_{i^k} \pi^{-1}(\pi(\mu(i^k))) = \mu(i^k)$, a contradiction to $\mu(i^k)$ being the top-ranked house in $\{\mu(i^k), \mu(i^{k+1}), \dots, \mu(i^{|N|})\}$ w.r.t. R_{i^k} . Otherwise, if $i^k \notin M$, then $\mu'(i^k)$ is the bottom-ranked house in $\{\mu'(i^k), \mu'(i^{k+1}), \dots, \mu'(i^{|N|})\} \ni \pi(\mu(i^k))$ w.r.t. R'_{i^k} and one has $\pi(\mu(i^k)) P'_{i^k} \mu'(i^k)$. So, $\mu(i^k) = \pi^{-1}(\pi(\mu(i^k))) P_{i^k} \pi^{-1}(\mu'(i^k)) \in \{\mu(i^k), \mu(i^{k+1}), \dots, \mu(i^{|N|})\}$, a contradiction to $\mu(i^k)$ being the bottom-ranked house in $\{\mu(i^k), \mu(i^{k+1}), \dots, \mu(i^{|N|})\}$ w.r.t. R_{i^k} . Therefore, $\pi(\mu(i^k)) = \mu'(i^k)$. By induction, $\pi(\mu(i)) = \mu'(i)$ for any $i \in N$, showing

that $\varphi^{\succeq, \mathcal{M}}$ is neutral.

QED

Proposition 2 *Simple serial dictatorships are Pareto optimal, consistent, conversely consistent, and neutral.*

Proof By Abdulkadiroğlu and Sönmez (1998a), serial dictatorships lead to *Pareto optimal* allocations. Therefore, simple serial dictatorships are *Pareto optimal*. The other claims directly follow from Proposition 1.

QED

Theorem 1 *If a rule is pairwise consistent and pairwise neutral, then it is a simple sequential solution.*

Proof Let φ be any *pairwise consistent* and *pairwise neutral* rule. Let $a, b \in \mathcal{H}$ be two distinct houses and $i, j \in \mathcal{N}$ two distinct agents. Let the problems $\mathcal{E}^1$ and $\mathcal{E}^2$ be as follows:

$\mathcal{E}^1$		$\mathcal{E}^2$	
P_i^1	P_j^1	P_i^2	P_j^2
a	a	a	b
b	b	b	a

Depending on the values that φ takes (the underlined selections below) in the problems $\mathcal{E}^1$ and $\mathcal{E}^2$, exactly one of the following four cases prevails:

• Case 1: $\bar{i} \succeq j$

P_i^1	P_j^1	P_i^2	P_j^2
<u>a</u>	a	<u>a</u>	<u>b</u>
b	<u>b</u>	b	a

- Case 2: $i \succeq j$

P_i^1	P_j^1	P_i^2	P_j^2
a	$\underline{a}$	a	b
$\underline{b}$	b	$\underline{b}$	$\underline{a}$

- Case 3: $\bar{j} \succeq i$

P_i^1	P_j^1	P_i^2	P_j^2
a	$\underline{a}$	$\underline{a}$	$\underline{b}$
$\underline{b}$	b	b	a

- Case 4: $j \succeq i$

P_i^1	P_j^1	P_i^2	P_j^2
$\underline{a}$	a	a	b
b	$\underline{b}$	$\underline{b}$	$\underline{a}$

By the *pairwise neutrality* assumption, the four cases above are independent of the choice of houses a and b . Therefore, we have:

- Case 1: $\bar{i} \succeq j$ In any problem $\mathcal{E}$ involving i and j , i does not envy j under $\varphi(\mathcal{E})$.

Indeed, suppose that there exists a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$ involving i and j , such that i envies j under $\mu = \varphi(\mathcal{E})$. Then, $\mu(j) P_i \mu(i)$, i.e. $\mu_{\{i,j\}}(j) P_i|_{\mu(\{i,j\})} \mu_{\{i,j\}}(i)$. In conjunction with the *pairwise consistency* of φ , this implies that one of the following cases prevails in the reduced problem $r_{\{i,j\}}^\mu(\mathcal{E})$:

$P_i _{\mu(\{i,j\})}$	$P_j _{\mu(\{i,j\})}$	$P_i _{\mu(\{i,j\})}$	$P_j _{\mu(\{i,j\})}$
$\mu_{\{i,j\}}(j)$	$\underline{\mu_{\{i,j\}}(j)}$	$\mu_{\{i,j\}}(j)$	$\mu_{\{i,j\}}(i)$
$\underline{\mu_{\{i,j\}}(i)}$	$\mu_{\{i,j\}}(i)$	$\underline{\mu_{\{i,j\}}(i)}$	$\underline{\mu_{\{i,j\}}(j)}$

where the underlined allocations represent φ 's selection for the reduced problem. In either case, we obtain a contradiction to $\bar{i} \succeq j$ by setting $a = \mu_{\{i,j\}}(j)$ and $b = \mu_{\{i,j\}}(i)$.

• Case 2: $\underline{i} \succeq j$ In any problem $\mathcal{E}$ involving i and j , i envies j under $\varphi(\mathcal{E})$.

Suppose that there exists a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$ involving i and j , such that i does not envy j under $\mu = \varphi(\mathcal{E})$. Then, $\mu(i) P_i \mu(j)$, i.e. $\mu_{\{i,j\}}(i) P_i|_{\mu(\{i,j\})} \mu_{\{i,j\}}(j)$. By *pairwise consistency* of φ , one of the following cases prevails in the reduced problem $r_{\{i,j\}}^\mu(\mathcal{E})$:

$$\begin{array}{c|c} P_i|_{\mu(\{i,j\})} & P_j|_{\mu(\{i,j\})} \\ \hline \underline{\mu_{\{i,j\}}(i)} & \mu_{\{i,j\}}(i) \\ \mu_{\{i,j\}}(j) & \underline{\mu_{\{i,j\}}(j)} \end{array} \quad \begin{array}{c|c} P_i|_{\mu(\{i,j\})} & P_j|_{\mu(\{i,j\})} \\ \hline \underline{\mu_{\{i,j\}}(i)} & \underline{\mu_{\{i,j\}}(j)} \\ \mu_{\{i,j\}}(j) & \mu_{\{i,j\}}(i) \end{array},$$

where the underlined allocations represent φ 's selection for the reduced problem. In either case, we obtain a contradiction to $\underline{i} \succeq j$ by setting $a = \mu_{\{i,j\}}(i)$ and $b = \mu_{\{i,j\}}(j)$.

The two other cases are exactly symmetric. Since the four cases considered are independent of the choice of houses a and b , we can define a *reflexive* relation $\succeq$ on $\mathcal{N}$ by letting $i \succeq j$ if and only if $\bar{i} \succeq j$ or $\underline{i} \succeq j$, for any two distinct $i, j \in \mathcal{N}$. The relation $\succeq$ is *complete* since one of the four cases prevails and it is *antisymmetric* since the four cases are mutually exclusive. We will now show that for any three distinct agents $i, j, k \in \mathcal{N}$, the following implications hold:

- $\bar{i} \succeq j$ and $\bar{j} \succeq k \implies \bar{i} \succeq k$,
- $\bar{i} \succeq j$ and $\underline{j} \succeq k \implies \bar{i} \succeq k$,
- $\underline{i} \succeq j$ and $\bar{j} \succeq k \implies \underline{i} \succeq k$,
- $\underline{i} \succeq j$ and $\underline{j} \succeq k \implies \underline{i} \succeq k$.

Let $a, b, c \in \mathcal{H}$ be three distinct houses and let $i, j, k \in \mathcal{N}$ be any three distinct agents.

$\bar{i} \succeq j$ and $\bar{j} \succeq k \implies \bar{i} \succeq k$:

Suppose that $\bar{i} \succeq j$ and $\bar{j} \succeq k$. Then, in any problem involving i, j and k , we have that i does not envy j and j does not envy k . Consider the following problem $\mathcal{E}$ and the underlined allocation μ :

P_i	P_j	P_k
<u>a</u>	a	a
b	<u>b</u>	b
c	c	<u>c</u>

Note that μ is the unique allocation for $\mathcal{E}$ under which i does not envy j and j does not envy k . Therefore, $\mu = \varphi(\mathcal{E})$. Consider the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$:

P_i	P_k
<u>a</u>	a
c	<u>c</u>

By *pairwise consistency* of φ , the underlined selection $\mu_{\{i,k\}} \in \varphi(r_{\{i,k\}}^\mu(\mathcal{E}))$.

Therefore, either $\bar{i} \succeq k$ or $\underline{k} \succeq i$.

Consider the following problem $\mathcal{E}$ and the underlined allocation μ :

P_i	P_j	P_k
<u>a</u>	a	<u>c</u>
b	<u>b</u>	b
c	c	a

Note that μ is the unique allocation for $\mathcal{E}$ under which i does not envy j and j does not envy k . Therefore, $\mu = \varphi(\mathcal{E})$. Consider the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$:

P_i	P_k
<u>a</u>	<u>c</u>
c	a

By *pairwise consistency* of φ , the underlined selection $\mu_{\{i,k\}} \in \varphi(r_{\{i,k\}}^\mu(\mathcal{E}))$. Therefore, either $\bar{i} \succeq k$ or $\bar{k} \succeq i$. By the above paragraph, $\bar{i} \succeq k$.

$\bar{i} \succeq j$ and $\underline{j} \succeq k \implies \bar{i} \succeq k$:

Suppose that $\bar{i} \succeq j$ and $\underline{j} \succeq k$. Then, in any problem involving i , j and k , we have that i does not envy j and j envies k . Consider the following problem $\mathcal{E}$ and the underlined allocation μ :

P_i	P_j	P_k
<u>a</u>	c	<u>c</u>
b	<u>b</u>	b
c	a	a

Note that μ is the unique allocation for $\mathcal{E}$ under which i does not envy j and j envies k . Therefore, $\mu = \varphi(\mathcal{E})$. By considering the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$ and by *pairwise consistency* of φ , either $\bar{i} \succeq k$ or $\bar{k} \succeq i$.

Consider the following problem $\mathcal{E}$ and the underlined allocation μ :

P_i	P_j	P_k
<u>a</u>	c	a
b	<u>b</u>	b
c	a	<u>c</u>

Here, μ is the unique allocation for $\mathcal{E}$ under which i does not envy j and j envies k . Therefore, $\mu = \varphi(\mathcal{E})$. By considering the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$ and by *pairwise consistency* of φ , either $\bar{i} \succeq k$ or $\underline{k} \succeq i$. By the above paragraph, $\bar{i} \succeq k$.

$\underline{i} \succeq j$ and $\bar{j} \succeq k \implies \underline{i} \succeq k$:

Suppose that $\underline{i} \succeq j$ and $\bar{j} \succeq k$. Then, in any problem involving i , j and k , we have that i envies j and j does not envy k . Consider the following problem $\mathcal{E}$ and the underlined allocation μ :

P_i	P_j	P_k
a	c	c
b	<u>b</u>	b
<u>c</u>	a	<u>a</u>

It can be seen that μ is the unique allocation for $\mathcal{E}$ under which i envies j and j does not envy k . Therefore, $\mu = \varphi(\mathcal{E})$. By considering the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$ and by *pairwise consistency* of φ , either $\underline{i} \succeq k$ or $\underline{k} \succeq i$.

Consider the following problem $\mathcal{E}$ and the underlined allocation μ :

P_i	P_j	P_k
a	c	<u>a</u>
b	<u>b</u>	b
<u>c</u>	a	c

Here, μ is the unique allocation for $\mathcal{E}$ under which i envies j and j does not envy k . Therefore, $\mu = \varphi(\mathcal{E})$. By considering the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$ and by *pairwise consistency* of φ , either $\underline{i} \succeq k$ or $\bar{k} \succeq i$. By the above paragraph, $\underline{i} \succeq k$.

$\underline{i} \succeq j$ and $\underline{j} \succeq k \implies \underline{i} \succeq k$:

Suppose that $\underline{i} \succeq j$ and $\underline{j} \succeq k$. Then, in any problem involving i , j and k , we have that i envies j and j envies k . Consider the following problem $\mathcal{E}$ and the underlined allocation μ :

P_i	P_j	P_k
a	a	<u>a</u>
b	<u>b</u>	b
<u>c</u>	c	c

Note that μ is the unique allocation for $\mathcal{E}$ under which i envies j and j envies k . Therefore, $\mu = \varphi(\mathcal{E})$. By considering the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$ and by *pairwise consistency* of φ , either $\underline{i} \succeq k$ or $\bar{k} \succeq i$.

Consider the following problem $\mathcal{E}$ and the underlined allocation μ :

P_i	P_j	P_k
a	a	c
b	<u>b</u>	b
<u>c</u>	c	<u>a</u>

In this case, μ is the unique allocation for $\mathcal{E}$ under which i envies j and j envies k . Therefore, $\mu = \varphi(\mathcal{E})$. By considering the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$ and by *pairwise consistency* of φ , either $\underline{i} \succeq k$ or $\underline{k} \succeq i$. By the above paragraph, $\underline{i} \succeq k$.

The four implications shown above imply that for any three distinct agents $i, j, k \in \mathcal{N}$:

$$\left(\bar{i} \succeq j \text{ and } j \succeq k \implies \bar{i} \succeq k \right) \quad \text{and} \quad \left(\underline{i} \succeq j \text{ and } j \succeq k \implies \underline{i} \succeq k \right).$$

In particular, the relation $\succeq$ is *transitive*.

Let $i \in \mathcal{N}$ not be the minimal element¹ of $\mathcal{N}$ w.r.t. $\succeq$. Then, there exists $k \in L(i, \succeq, \mathcal{N})$ with $i \succ k$.² Let $j \in L(i, \succeq, \mathcal{N})$ such that $j \neq k$ and $i \succ j$, then:

- $\bar{i} \succeq k \implies \bar{i} \succeq j$: Suppose that $\bar{i} \succeq k$. If $k \succeq j$, we immediately have that $\bar{i} \succeq j$. Otherwise if $j \succeq k$, suppose that it is not true that $\bar{i} \succeq j$. But then since $i \succeq j$, we must have $\underline{i} \succeq j$. Along with $j \succeq k$, this implies that $\underline{i} \succeq k$, a contradiction. Therefore, $\bar{i} \succeq j$.
- $\underline{i} \succeq k \implies \underline{i} \succeq j$: Suppose that $\underline{i} \succeq k$. Similarly, if $k \succeq j$, we immediately have that $\underline{i} \succeq j$. Otherwise if $j \succeq k$, suppose that it is not true that $\underline{i} \succeq j$. But then since $i \succeq j$, we must have $\bar{i} \succeq j$. Along with $j \succeq k$, this implies that $\bar{i} \succeq k$, a contradiction. Therefore, $\underline{i} \succeq j$.

Therefore, we may validly define the set $\mathcal{M} \subset \mathcal{N}$ as follows. If there exists a minimal element of $\mathcal{N}$ w.r.t. $\succeq$, let it belong to $\mathcal{M}$. For any other $i \in \mathcal{N}$, let $i \in \mathcal{M}$ if and only if $\bar{i} \succeq k$ for some—or for any $k \in L(i, \succeq, \mathcal{N})$ with $i \succ k$.

Finally, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $i \in N$, $j \in L(i, \succeq, N)$ and $\mu = \varphi(\mathcal{E})$. If $i = j$ then $\mu(i) = \mu(j)$ and therefore $\mu(i) R_i \mu(j)$ and $\mu(j) R_i \mu(i)$. Otherwise $i \neq j$, so we have $i \succ j$. In this case, if $i \in N \cap \mathcal{M}$ then by construction $\bar{i} \succeq j$, i.e. i never envies j , i.e. $\mu(i) R_i \mu(j)$. Otherwise if $i \in N \setminus \mathcal{M}$ then by construction $\underline{i} \succeq j$, i.e. i always envies j , i.e. $\mu(j) R_i \mu(i)$. By Lemma 1, μ is the sequential allocation induced by $\succeq|_N$ and $\mathcal{M} \cap N$ for $\mathcal{E}$. Therefore, φ

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¹For any $i \in \mathcal{N}$, i is the *minimal element* of $\mathcal{N}$ w.r.t. $\succeq$, if for any $j \in \mathcal{N}$, we have $j \succeq i$. For the case when $\mathcal{N}$ is finite, this is equivalent to saying that i is the bottom-ranked agent in $\mathcal{N}$ w.r.t. $\succeq$. By the *antisymmetry* of $\succeq$, there exists at most one minimal element of $\mathcal{N}$ w.r.t. $\succeq$.

² $\succ$ is used to denote the asymmetric part of $\succeq$.

is the simple sequential solution induced by $\succeq$ and $\mathcal{M}$.

QED

Corollary 1 *If a rule is weakly Pareto optimal, pairwise consistent, and pairwise neutral, then it is a simple serial dictatorship.*

Proof Let φ be a *weakly Pareto optimal, pairwise consistent, and pairwise neutral* rule. By Theorem 1, φ is a simple sequential solution, i.e. there exists $\succeq$ and $\mathcal{M} \subset \mathcal{N}$ such that $\varphi = \varphi^{\succeq, \mathcal{M}}$. Suppose that $\varphi \neq \varphi^{\succeq}$. Then there exists $i \in \mathcal{N} \setminus \mathcal{M}$ such that i is not the minimal element in $\mathcal{N}$ w.r.t. $\succeq$. Let $j \in \mathcal{N}$ be such that $i \succ j$ and let the problem $\mathcal{E}$ be as follows:

P_i	P_j
a	b
b	a

Since $i \notin \mathcal{M}$, under the allocation $\varphi(\mathcal{E}) = \varphi^{\succeq, \mathcal{M}}(\mathcal{E})$, agents i and j receive b and a , respectively. However, both are made strictly better off by exchanging their assigned houses, a contradiction to φ being *weakly Pareto optimal*. Therefore, $\varphi = \varphi^{\succeq}$.

QED

3.2 Consistent Correspondences

Thomson (1998) points out that the Pareto correspondence is *consistent* in allocation problems where goods are privately appropriable. We start by proving this remark in our special context of house allocation problems.

Proposition 3 *The Pareto correspondence is consistent.*

Proof Let φ be the Pareto correspondence. Also, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $\mu \in \varphi(\mathcal{E})$ and $\emptyset \neq N' \subset N$. Suppose that $\mu_{N'} \notin \varphi(r_{N'}^\mu(\mathcal{E}))$. Then, there is an allocation $\bar{\mu}$ for $r_{N'}^\mu(\mathcal{E})$ that weakly Pareto dominates $\mu_{N'}$. Define the allocation μ' for $\mathcal{E}$ by:

$$\mu'(i) = \begin{cases} \bar{\mu}(i) & \text{if } i \in N', \\ \mu(i) & \text{otherwise.} \end{cases}$$

Then, μ' weakly Pareto dominates μ for $\mathcal{E}$, a contradiction to μ being a *Pareto optimal* allocation. Therefore, $\mu_{N'} \in \varphi(r_{N'}^\mu(\mathcal{E}))$, showing that φ is *consistent*.

QED

The following proposition asserts that the Pareto correspondence is not *conversely consistent*. It is analogous to Tadenuma and Thomson's (1991) result about the lack of *converse consistency* of the Pareto correspondence in economies with indivisible goods and money.

Proposition 4 *The Pareto correspondence is not conversely consistent. The weak Pareto correspondence is neither pairwise consistent nor conversely consistent.*

Proof Let 1, 2, 3 be three distinct potential agents and a, b, c three distinct potential houses. To see that the Pareto correspondence is not *conversely consistent*, consider the following problem $\mathcal{E}$:

P_1	P_2	P_3
a	c	b
<u>b</u>	<u>a</u>	<u>c</u>
c	b	a

Let μ be the allocation corresponding to the underlined selection. Note that for any $\{i, j\} \subset \{1, 2, 3\}$ with $i \neq j$, the allocation $\mu_{\{i, j\}}$ is chosen by the Pareto correspondence in the reduced problem $r_{\{i, j\}}^\mu(\mathcal{E})$. Thus, if the Pareto correspondence were *conversely consistent*, then μ should be in the set of *Pareto optimal* allocations for $\mathcal{E}$. However, μ is strongly and therefore weakly Pareto dominated in $\mathcal{E}$, so it is not in the Pareto correspondence for $\mathcal{E}$, showing that the Pareto correspondence is not *conversely consistent*. The same example shows that the weak Pareto correspondence is not *conversely consistent*.

To see that the weak Pareto correspondence is not *pairwise consistent*, consider the following problem $\mathcal{E}$:

P_1	P_2	P_3
<u>a</u>	c	b
b	<u>b</u>	<u>c</u>
c	a	a

Let μ be the allocation corresponding to the underlined selection. The allocation μ is not strongly Pareto dominated in $\mathcal{E}$, therefore it is in the weak Pareto correspondence for $\mathcal{E}$. However, the reduced allocation $\mu_{\{2, 3\}}$ is

strongly Pareto dominated in the reduced problem $r_{\{2,3\}}^\mu(\mathcal{E})$, thus $\mu_{\{2,3\}}$ is not in the weak Pareto correspondence for $r_{\{2,3\}}^\mu(\mathcal{E})$, showing that the weak Pareto correspondence is not *pairwise consistent*.

QED

The Pareto correspondence is *anonymous* but not *conversely consistent*. The next proposition shows that there does not exist a *non-empty valued* correspondence that is *weakly Pareto optimal*, *anonymous*, and *conversely consistent*.

Proposition 5 *Any non-empty valued, anonymous, and conversely consistent correspondence is not weakly Pareto optimal.*

Proof Let φ be a *non-empty valued*, *anonymous*, and *conversely consistent* correspondence. Then, for any distinct $i, j \in \mathcal{N}$ and $a, b \in \mathcal{H}$, the correspondence φ will choose both allocations from the problem:

P_i	P_j
a	a
b	b

since it is *non-empty valued* and *anonymous*. By *converse consistency* of φ , the underlined allocation μ will be chosen by φ from the following problem $\mathcal{E}$:

P_1	P_2	P_3
a	c	b
<u>b</u>	<u>a</u>	<u>c</u>
c	b	a

Note that the allocation μ is strongly Pareto dominated in $\mathcal{E}$, showing that φ is not *weakly Pareto optimal*.

QED

For any correspondence or rule φ , let $\varphi|_2$ be its restriction to two-person problems. A correspondence φ is an **extension** of a correspondence $\bar{\varphi}$ defined for two-person problems if $\varphi|_2 = \bar{\varphi}$. The following lemma states that *consistent* and *conversely consistent* correspondences are characterized by their restrictions to two-person problems.

Lemma 2 *For any correspondence $\bar{\varphi}$ defined for two-person problems, there exists a consistent and conversely consistent extension φ that is unique up to one-person problems.³ The extension φ is defined as follows. For any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$ with $|N| \geq 2$ and for any allocation μ for $\mathcal{E}$:*

$$\mu \in \varphi(\mathcal{E}) \iff \forall N' \subset N \text{ with } |N'| = 2 : \quad \mu_{N'} \in \bar{\varphi}(r_{N'}^\mu(\mathcal{E})).$$

In particular, any consistent and conversely consistent correspondence φ is expressed as in above where $\bar{\varphi} = \varphi|_2$.

Proof Let $\bar{\varphi}$ be any correspondence defined for two-person problems. Let the correspondence φ be defined as in above for problems involving more than one person and W.L.O.G. let φ select the unique allocation in one-person problems. Since $\bar{\varphi} = \varphi|_2$, φ is *conversely consistent* by definition. To see that φ is *consistent*, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $\emptyset \neq N' \subset N$ and $\mu \in \varphi(\mathcal{E})$. Assume W.L.O.G. that $|N'| \geq 2$. Consider the reduced problem $\mathcal{E}' = r_{N'}^\mu(\mathcal{E})$ and the allocation $\mu_{N'}$ for $\mathcal{E}'$. For any $N'' \subset N'$ with $|N''| = 2$, we have $(\mu_{N'})_{N''} =$

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³In other words, if there exist two different *consistent* and *conversely consistent* extensions of $\bar{\varphi}$, they would only differ in their selections from one-person problems.

$\mu_{N''} \in \bar{\varphi}(r_{N''}^{\mu}(\mathcal{E})) = \varphi(r_{N''}^{\mu}(\mathcal{E})) = \varphi(r_{N''}^{\mu_{N'}}(r_{N'}^{\mu}(\mathcal{E}))) = \bar{\varphi}(r_{N''}^{\mu_{N'}}(\mathcal{E}'))$, since $N'' \subset N$ with $|N''| = 2$ and $\mu \in \varphi(\mathcal{E})$. Therefore, by definition of φ , we have $\mu_{N'} \in \varphi(\mathcal{E}') = \varphi(r_{N'}^{\mu}(\mathcal{E}))$, showing that φ is *consistent*.

To show uniqueness of φ up to one-person problems, let φ' be any other *consistent* and *conversely consistent* extension of $\bar{\varphi}$. *Consistency* of φ' requires that $\varphi' \subset \varphi$. Similarly, *converse consistency* of φ' requires that $\varphi \subset \varphi'$ in problems involving more than one person, showing the uniqueness of the extension up to one-person problems.⁴

Any *consistent* and *conversely consistent* correspondence φ is in particular a *consistent* and *conversely consistent* extension of $\varphi|_2$. By uniqueness of the *consistent* and *conversely consistent* extension of $\varphi|_2$ up to one-person problems, φ is expressed as in above where $\bar{\varphi} = \varphi|_2$.

QED

For any correspondence $\bar{\varphi}$ defined for two-person problems, let $Ext(\bar{\varphi})$ be the *consistent* and *conversely consistent* extension of $\bar{\varphi}$ selecting the unique allocation in one-person problems. The correspondence $Ext(\bar{\varphi})$ is uniquely defined by Lemma 2. For any two correspondences $\bar{\varphi}$ and $\bar{\varphi}'$ defined for two-person problems such that $\bar{\varphi} \subset \bar{\varphi}'$, we have $Ext(\bar{\varphi}) \subset Ext(\bar{\varphi}')$. In particular, if $\{\bar{\varphi}_{\alpha}\}_{\alpha \in I}$ is any collection of correspondences defined for two-person problems, we have:

$$\bigcup_{\alpha \in I} Ext(\bar{\varphi}_{\alpha}) \subset Ext\left(\bigcup_{\alpha \in I} \bar{\varphi}_{\alpha}\right).$$

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⁴The proof of the uniqueness part makes implicit use of the “Elevator Lemma” in Thomson (1998). The Elevator Lemma states that if $\varphi|_2 \subset \varphi'|_2$, φ is *consistent* and φ' is *conversely consistent*, then $\varphi \subset \varphi'$ up to one-person problems.

Moreover, for any linear order $\succeq$ on $\mathcal{N}$ and any $\mathcal{M} \subset \mathcal{N}$, we have:

$$Ext(\varphi^{\succeq, \mathcal{M}}|_2) = \varphi^{\succeq, \mathcal{M}},$$

by Proposition 1 and Lemma 2.

Let $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ be an indexed family of relations on $\mathcal{N}$. The family $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ is **3^+ -acyclic** if there do not exist distinct elements $i_1, i_2, \dots, i_n \in \mathcal{N}$ and $a_1, a_2, \dots, a_n \in \mathcal{H}$ with $n \geq 3$ such that $i_1 \succeq_{a_1, a_2} i_2 \succeq_{a_2, a_3} \dots \succeq_{a_{n-1}, a_n} i_n \succeq_{a_n, a_1} i_1$.

For any correspondence $\bar{\varphi}$ defined for two-person problems, we can naturally induce a family of relations $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ on $\mathcal{N}$ as follows. For any $a, b \in \mathcal{H}$, if $a = b$, let $\succeq_{a,b} = \emptyset$, otherwise if $a \neq b$, let $\succeq_{a,b}$ be the *reflexive* relation such that for any distinct $i, j \in \mathcal{N}$, we have $i \succeq_{a,b} j$ if and only if $\bar{\varphi}$ chooses the underlined allocation from the following problem:

P_i	P_j
<u>a</u>	a
b	<u>b</u>

Lemma 3 *If a Pareto optimal and conversely consistent correspondence φ is such that $\varphi|_2$ is non-empty valued, then $\varphi|_2$ induces a 3^+ -acyclic family of relations on $\mathcal{N}$. Conversely, for any consistent correspondence φ such that $\varphi|_2$ induces a 3^+ -acyclic family of relations on $\mathcal{N}$ and is Pareto optimal, the correspondence φ is Pareto optimal.*

Proof: Let φ be any Pareto optimal and conversely consistent correspondence such that $\varphi|_2$ is non-empty valued. Let $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ be the family of relations induced by $\varphi|_2$ on $\mathcal{N}$. Suppose that there exist distinct elements $i_1, i_2, \dots, i_n \in \mathcal{N}$ and $a_1, a_2, \dots, a_n \in \mathcal{H}$ with $n \geq 3$ such that

$i_1 \succeq_{a_1, a_2} i_2 \succeq_{a_2, a_3} \dots \succeq_{a_{n-1}, a_n} i_n \succeq_{a_n, a_1} i_1$. Consider the following problem $\mathcal{E}$ and the underlined allocation μ :

P_{i_1}	P_{i_2}	P_{i_3}	$\dots$	P_{i_n}
a_n	a_1	a_2		a_{n-1}
<u>a_1</u>	<u>a_2</u>	<u>a_3</u>		<u>a_n</u>
$\vdots$	$\vdots$	$\vdots$		$\vdots$

Note that for any two distinct integers $l, k \in \{1, 2, \dots, n\}$, we have that $\mu_{\{i_l, i_k\}} \in \varphi|_2 \left(r_{\{i_l, i_k\}}^\mu(\mathcal{E}) \right) = \varphi \left(r_{\{i_l, i_k\}}^\mu(\mathcal{E}) \right)$. But then, by *converse consistency* of φ , we have $\mu \in \varphi(\mathcal{E})$, a contradiction to φ being *Pareto optimal* and μ being Pareto dominated in $\mathcal{E}$. Therefore, the family $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ is 3^+ -acyclic.

For the converse, let φ be any *consistent* correspondence such that $\varphi|_2$ induces a 3^+ -acyclic family of relations $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ on $\mathcal{N}$ and is *Pareto optimal*. Suppose that φ is not *Pareto optimal*. Then there exists $\mathcal{E} = (N, H, (R_i)_{i \in N})$ and $\mu \in \varphi(\mathcal{E})$ such that μ is weakly Pareto dominated by another allocation μ' for $\mathcal{E}$. We will show that there exists a cycle of agents $i_0, i_1, \dots, i_n = i_0 \in N$ such that each one envies the next, under μ . Let $\emptyset \neq N' \subset N$ be the set of agents who are strictly better off under μ' . Since preferences are *antisymmetric* and the agents in $N \setminus N'$ are indifferent between μ and μ' , their assignments are the same in either case, i.e. $\mu'|_{N \setminus N'} = \mu|_{N \setminus N'}$. Therefore, we have that $\mu'(N') = \mu(N')$. Now, consider the bijection $\pi = \mu^{-1} \circ \mu' : N \rightarrow N$. Since $\mu'|_{N \setminus N'} = \mu|_{N \setminus N'}$ and $\mu'(N') = \mu(N')$, we have that $\pi|_{N \setminus N'}$ is the identity map on $N \setminus N'$ and $\pi|_{N'}$ is a permutation of N' . Choose $i \in N'$ and let $N_i = \left\{ \pi^k(i) \mid k \in \{0, 1, 2, \dots\} \right\} \subset N'$.⁵ Let

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⁵Here, π^k denotes the map π composed k times with itself and $\pi^{-k} = (\pi^{-1})^k$. The map π^0 denotes the identity.

$j \in N_i$. Since $\mu(\pi(j)) = \mu \circ (\mu^{-1} \circ \mu')(j) = \mu'(j)$ and $j \in N'$, we have that $\mu(\pi(j)) = \mu'(j) P_j \mu(j)$, i.e. j envies $\pi(j)$ under μ . In particular, $j \neq \pi(j)$ and $j, \pi(j) \in N_i$, therefore, $n = |N_i| \geq 2$. Note that for any $j = \pi^l(i) \in N_i$ and any positive integer k such that $\pi^k(j) = j$, we have $\pi^k(i) = \pi^{-l} \circ \pi^k \circ \pi^l(i) = \pi^{-l} \circ \pi^k(j) = \pi^{-l}(j) = \pi^{-l} \circ \pi^l(i) = i$. Then, $N_i = \{i, \pi(i), \pi^2(i), \dots, \pi^{k-1}(i)\}$, so $k \geq |N_i| = n$. Therefore, the agents $i, \pi(i), \pi^2(i), \dots, \pi^{n-1}(i)$ are all distinct, for otherwise there exists $j \in N_i$ and a positive integer $k < n$ such that $\pi^k(j) = j$, a contradiction. Then, we have $N_i = \{i, \pi(i), \pi^2(i), \dots, \pi^{n-1}(i)\}$. Moreover, since $\pi^n(i) \in N_i$, by a similar argument, we can only have that $\pi^n(i) = i$. Letting $i_k = \pi^k(i)$ and $a_k = \mu(i_k)$, we have that $a_0 = a_n P_{i_{n-1}} a_{n-1} \dots a_2 P_{i_1} a_1 P_{i_0} a_0$. Moreover, for any $k \in \{0, 1, \dots, n-1\}$, we have $a_{k+1} P_{i_{k+1}} a_k$, for otherwise there exists $k \in \{0, 1, \dots, n-1\}$ such that $a_k P_{i_{k+1}} a_{k+1} P_{i_k} a_k$. But then, by *consistency* of φ , the following underlined allocation is chosen by φ and hence by $\varphi|_2$, from the reduced problem $r_{\{i_k, i_{k+1}\}}^\mu(\mathcal{E})$:

$P_{i_k} _{\{a_k, a_{k+1}\}}$	$P_{i_{k+1}} _{\{a_k, a_{k+1}\}}$
a_{k+1}	a_k
<u>a_k</u>	<u>a_{k+1}</u>

a contradiction to $\varphi|_2$ being *Pareto optimal*. In particular, we have $n \geq 3$, for otherwise, if $n = 2$, then $a_0 = a_2 P_{i_1} a_1$, a contradiction to $a_{k+1} P_{i_{k+1}} a_k$ for $k = 0$. Moreover, for every $k \in \{0, 1, \dots, n-1\}$, the reduced problem $r_{\{i_k, i_{k+1}\}}^\mu(\mathcal{E})$ is of the form:

$P_{i_k} _{\{a_k, a_{k+1}\}}$	$P_{i_{k+1}} _{\{a_k, a_{k+1}\}}$
a_{k+1}	<u>a_{k+1}</u>
<u>a_k</u>	a_k

where the underlined allocation is chosen by $\varphi|_2$, by *consistency* of φ . But then, $i_0 \succeq_{a_0, a_{n-1}} i_{n-1} \succeq_{a_{n-1}, a_{n-2}} \dots \succeq_{a_2, a_1} i_1 \succeq_{a_1, a_0} i_0$ and $n \geq 3$, a contradiction to $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ being 3^+ -acyclic. Therefore, φ is *Pareto optimal*, completing the proof of the lemma.

QED

A restatement of Lemma 3 gives us the following characterization of *Pareto optimal*, *consistent*, and *conversely consistent* correspondences.

Corollary 2 *A correspondence φ is non-empty valued in one and two-person problems, Pareto optimal, consistent, and conversely consistent if and only if $\varphi = \text{Ext}(\bar{\varphi})$ for some non-empty valued and Pareto optimal correspondence $\bar{\varphi}$ defined for two-person problems that induces a 3^+ -acyclic family of relations on $\mathcal{N}$.*

A relation $\succeq$ on $\mathcal{N}$ is 3^+ -**acyclic** if there do not exist distinct elements $i_1, i_2, \dots, i_n \in \mathcal{N}$ with $n \geq 3$ such that $i_1 \succeq i_2 \succeq \dots \succeq i_n \succeq i_1$. Any *complete* and 3^+ -acyclic relation is *transitive*. A *transitive* relation is 3^+ -acyclic if and only if its indifference classes are of size smaller than 3. For any *complete* and 3^+ -acyclic relation $\succeq$ on $\mathcal{N}$, let $\bar{\varphi}^\succeq$ be the *non-empty valued*, *Pareto optimal* and *neutral* correspondence defined for two-person problems, such that for any two distinct agents $i, j \in \mathcal{N}$ and any problem $\mathcal{E}$ of type:

P_i	P_j
$\underline{a}$	a
b	$\underline{b}$

the set $\bar{\varphi}^\succeq(\mathcal{E})$ contains the underlined allocation if and only if $i \succeq j$. In this case, the family of relations $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ induced by $\bar{\varphi}^\succeq$ are such that for any distinct $a, b \in \mathcal{H}$, we have $\succeq_{a,b} = \succeq$.

By the axiom of choice, for any *reflexive*, *complete*, and 3^+ -*acyclic* relation $\succeq$ on $\mathcal{N}$, there exists a linear order $\succeq' \subset \succeq$. If $\succeq$ has n indifference classes of size 2, then there exist exactly 2^n such linear orders. For example, when $\mathcal{N} = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ with $|\mathcal{N}| = 6$, a typical *reflexive*, *complete* and 3^+ -*acyclic* relation $\succeq$ on $\mathcal{N}$ is depicted as follows:

$$\begin{array}{c} \hline \succeq \\ \hline i_3 \ i_6 \\ \\ i_2 \\ \\ i_1 \ i_5 \\ \\ i_4 \end{array}$$

Since $\succeq$ has two indifference classes of size 2, there are exactly 4 linear orders $\succeq_1, \succeq_2, \succeq_3, \succeq_4 \subset \succeq$, each obtained by arbitrarily breaking indifferences:

$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$
i_3	i_3	i_6	i_6
i_6	i_6	i_3	i_3
i_2	i_2	i_2	i_2
i_1	i_5	i_1	i_5
i_5	i_1	i_5	i_1
i_4	i_4	i_4	i_4

We also have that $\overline{\varphi}^\succeq = \varphi^{\succeq_1}|_2 \cup \varphi^{\succeq_2}|_2 \cup \varphi^{\succeq_3}|_2 \cup \varphi^{\succeq_4}|_2$.

Lemma 4 *A correspondence φ is non-empty valued, Pareto optimal, consistent, conversely consistent, and neutral if and only if there exists a reflexive, complete, and 3^+ -acyclic relation $\succeq$ on $\mathcal{N}$ such that $\varphi = \text{Ext}(\overline{\varphi}^\succeq)$.*

Proof: Let $\succeq$ be a *reflexive*, *complete*, and 3^+ -*acyclic* relation on $\mathcal{N}$. Let $\varphi = \text{Ext}(\overline{\varphi}^\succeq)$. By the axiom of choice, there exists a linear order $\succeq' \subset \succeq$.

Then, $\varphi^{\succeq'}|_2 \subset \bar{\varphi}^{\succeq}$, i.e. $\varphi^{\succeq'} = \text{Ext}(\varphi^{\succeq'}|_2) \subset \text{Ext}(\bar{\varphi}^{\succeq}) = \varphi$. Since φ contains the simple serial dictatorship $\varphi^{\succeq'}$, it is *non-empty valued*. The correspondence φ is *consistent* and *conversely consistent*. By Lemma 3, it is also *Pareto optimal*. To see that φ is *neutral*, let $\emptyset \neq N \subset \mathcal{N}$, $\mathcal{E} = (N, H, (R_i)_{i \in N})$ and $\mathcal{E}' = (N, H', (R'_i)_{i \in N})$. W.L.O.G., assume that $|N| \geq 2$. Suppose that there exists a bijection $\pi: H \rightarrow H'$ satisfying:

$$\forall i \in N, \forall a, b \in H : \quad a R_i b \iff \pi(a) R'_i \pi(b),$$

Let $\mu \in \varphi(\mathcal{E})$. By definition of φ , we have:

$$\forall N' \subset N \text{ with } |N'| = 2 : \quad \mu_{N'} \in \bar{\varphi}^{\succeq}(r_{N'}^{\mu}(\mathcal{E})).$$

Take any $N' \subset N$ with $|N'| = 2$. From above, $\mu_{N'} \in \bar{\varphi}^{\succeq}(r_{N'}^{\mu}(\mathcal{E}))$. Note that $\pi|_{\mu(N')}: \mu(N') \rightarrow \pi \circ \mu(N')$ is a bijection between the house sets of the reduced problems $r_{N'}^{\mu}(\mathcal{E})$ and $r_{N'}^{\pi \circ \mu}(\mathcal{E}')$ satisfying:

$$\forall i \in N', \forall a, b \in \mu(N') : \quad a R_i|_{\mu(N')} b \iff \pi|_{\mu(N')}(a) R'_i|_{\pi \circ \mu(N')} \pi|_{\mu(N')}(b),$$

So, by *neutrality* of $\bar{\varphi}^{\succeq}$ for two-person problems, $(\pi \circ \mu)|_{N'} = \pi|_{\mu(N')} \circ \mu_{N'} \in \bar{\varphi}^{\succeq}(r_{N'}^{\pi \circ \mu}(\mathcal{E}'))$. Since this is true for any such N' , by definition of φ , we have that $\pi \circ \mu \in \varphi(\mathcal{E}')$. Therefore, φ is *neutral*.

For the converse, let φ be a *non-empty valued*, *Pareto optimal*, *consistent*, *conversely consistent*, and *neutral* correspondence. By Lemma 2, $\varphi = \text{Ext}(\varphi|_2)$ up to one player problems, and by Lemma 3, $\varphi|_2$ induces a 3^+ -acyclic family of relations $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ on $\mathcal{N}$. Note that in this case, since φ is *non-empty valued*, the equality $\varphi = \text{Ext}(\varphi|_2)$ also holds for one-person problems and for any pair of distinct $a, b \in \mathcal{H}$, the relation $\succeq_{a,b}$ is *complete*. Since φ is *neutral*, in particular, $\varphi|_2$ is *neutral*. *Neutrality* of $\varphi|_2$

requires in turn that the *reflexive* and *complete* relation $\succeq_{a,b}$ is identical for any pair of distinct $a, b \in \mathcal{H}$. Let $\succeq = \succeq_{a,b}$ for some—or for any pair of distinct $a, b \in \mathcal{H}$. Then, $\varphi|_2 = \overline{\varphi}^\succeq$. Moreover, since the family $\{\succeq_{a,b}\}_{(a,b) \in \mathcal{H} \times \mathcal{H}}$ is 3^+ -*acyclic*, the relation $\succeq$ is 3^+ -*acyclic*. Therefore, $\varphi = \text{Ext}(\overline{\varphi}^\succeq)$, where $\succeq$ is a *reflexive*, *complete*, and 3^+ -*acyclic* relation on $\mathcal{N}$, completing the proof of the lemma.

QED

The following theorem states that a correspondence is *non-empty valued*, *Pareto optimal*, *consistent*, *conversely consistent*, and *neutral* if and only if it can be expressed as a union of simple serial dictatorships in a particular manner.

Theorem 2 *A correspondence φ is non-empty valued, Pareto optimal, consistent, conversely consistent, and neutral if and only if there exists a reflexive, complete, and 3^+ -acyclic relation $\succeq$ on $\mathcal{N}$ such that:*

$$\varphi = \bigcup_{\alpha \in I} \varphi^{\succeq_\alpha},$$

where $\{\succeq_\alpha\}_{\alpha \in I}$ is the set of linear orders contained in $\succeq$.

Proof Let $\succeq$ be any *reflexive*, *complete*, and 3^+ -*acyclic* relation on $\mathcal{N}$ and let $\{\succeq_\alpha\}_{\alpha \in I}$ be the set of linear orders contained in $\succeq$. We will show that

$$\text{Ext}(\overline{\varphi}^\succeq) = \bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}.$$

This will prove the theorem, by Lemma 4. We already know that

$$\overline{\varphi}^\succeq = \bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}|_2 = \left(\bigcup_{\alpha \in I} \varphi^{\succeq_\alpha} \right)|_2.$$

By showing that $\bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}$ is *consistent* and *conversely consistent*, we will have that $\bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}$ is a *consistent* and *conversely consistent* extension of

$\bar{\varphi}^{\succeq}$, which will imply the desired equality, by Lemma 2. By Proposition 2, simple serial dictatorships are *consistent*. Therefore, $\bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}$ is *consistent* as a union of *consistent* correspondences. To see that $\bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}$ is *conversely consistent*, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ with $|N| \geq 2$ and let μ be an allocation for $\mathcal{E}$, such that for any $N' \subset N$ with $|N'| = 2$, we have $\mu_{N'} \in \bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}(r_{N'}^\mu(\mathcal{E}))$. So, for any $\{i, j\} \subset N$ with $i \neq j$, we can choose $\alpha_{\{i, j\}} \in I$ such that $\mu_{\{i, j\}} \in \varphi^{\succeq_{\alpha_{\{i, j\}}}}(r_{\{i, j\}}^\mu(\mathcal{E}))$. Hence, we can define a *reflexive* and *complete* relation $\succeq'$ on N , such that for any distinct $i, j \in N$, we have $i \succeq' j$ if and only if $i \succeq_{\alpha_{\{i, j\}}} j$. Then, $\succeq'$ is 3^+ -*acyclic*, since $\succeq' \subset \bigcup_{\alpha \in I} (\succeq_\alpha|_N) = \succeq|_N$. Also note that $\succeq'$ is *transitive*, since it is *complete* and 3^+ -*acyclic*. Moreover, since each $\succeq_{\alpha_{\{i, j\}}}$ is *antisymmetric*, we have that $\succeq'$ is *antisymmetric*, showing that $\succeq'$ is a linear order on N . Since $\succeq' \subset \succeq|_N$, there exists $\beta \in I$ such that $\succeq' = \succeq_\beta|_N$. Moreover, since μ is the serial dictatorship allocation induced by $\succeq'$ for $\mathcal{E}$, for any $N' \subset N$ with $|N'| = 2$, we have $\mu_{N'} = \varphi^{\succeq_\beta}(r_{N'}^\mu(\mathcal{E}))$. But then, since $\varphi^{\succeq_\beta}$ is a simple serial dictatorship, it is *conversely consistent*, therefore we have $\{\mu\} = \varphi^{\succeq_\beta}(\mathcal{E}) \subset \bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}(\mathcal{E})$, showing that $\bigcup_{\alpha \in I} \varphi^{\succeq_\alpha}$ is *conversely consistent*.

QED

Chapter 4

Generalizations

In this chapter, we will relax some of the so far standing axioms, with the hope of identifying more ‘egalitarian’ classes of *consistent* rules. We will first present a weaker form of the *pairwise neutrality* axiom, namely *lower neutrality* and identify the class of *weakly Pareto optimal, pairwise consistent* and *lower neutral* rules in Theorem 3. Then, we will allow rules to associate with each problem a lottery over allocations, instead of restricting ourselves to sure allocations and we will extend the *consistency* principle to *lottery-valued* rules. Unexpectedly, the latter attempt will lead to a negative result, as we will show in Theorem 4 that there does not exist an *ex-post Pareto optimal, consistent* and *ex-ante anonymous lottery-valued* rule.

4.1 Weakening Neutrality

We will now introduce a weaker form of *pairwise neutrality* for rules, namely *lower neutrality* and characterize the class of *weakly Pareto Optimal*, *pairwise consistent* and *lower neutral* rules in Theorem 3. We will name these rules which generalize *simple serial dictatorships* as *simple serial bidictatorships*, as the reason will be clear shortly.

Lower neutrality requires that whenever in a two-person problem both agents rank some house a as their top choices, the agent who is assigned a should continue to receive it in any other two-person problem involving the same agents where they rank a as their top choices, irrespective of the other house involved in the problem. Formally, a rule φ is **lower neutral** if for any two distinct agents $i, j \in \mathcal{N}$ and any three distinct houses $a, b, c \in \mathcal{H}$, one has that:

$$\mu(i) = a \iff \mu'(i) = a,$$

where $\mu = \varphi(\mathcal{E})$, $\mu' = \varphi(\mathcal{E}')$ and the problems $\mathcal{E}$ and $\mathcal{E}'$ are as follows:

$\mathcal{E}$		$\mathcal{E}'$	
P_i	P_j	P'_i	P'_j
a	a	a	a
b	b	c	c

In other words, *lower neutrality* requires that whenever two agents compete for the possession of a certain house, the decision to favor one agent over the other may only depend on the labeling of their most preferred house.

Now, let a problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$ and a *complete* preorder $\succeq$ on N with the size of indifference classes being no larger than 2 be given. Denote

the partition of N into the indifference classes induced by $\succeq$ by $Ind(N, \succeq)$. Also, let a vector of tie-breaking functions $t = (t_I)_{I \in Ind(N, \succeq)}$ with $t_I: H \rightarrow I$, for every $I \in Ind(N, \succeq)$ be given. Let $m = |Ind(N, \succeq)| \leq |N|$ and for each positive integer $r \leq m$, let I^r be the r^{th} indifference class from the top in N w.r.t. $\succeq$. The **serial bidictatorship allocation induced by $\succeq$ and t for $\mathcal{E}$** is defined inductively as follows. First, if $|I^1| = 1$, let $I^1 = \{i\}$ and give agent i his most preferred house in H . Otherwise, if $|I^1| = 2$, let $I^1 = \{i, j\}$. If the top-ranked houses of i and j in H are the same, say a , then give a to $t_{I^1}(a)$ and give the other agent his second best house. Otherwise, if the top-ranked houses of i and j are different, then give i and j their top-ranked houses in H . At the r^{th} step, if $|I^r| = 1$, let $I^r = \{i\}$ and give agent i his most preferred house among those that are not already allocated in earlier steps. Otherwise, if $|I^r| = 2$, let $I^r = \{i, j\}$. If the top-ranked houses of i and j among those that are not already allocated in earlier steps are the same, say a , then give a to $t_{I^r}(a)$ and give the other agent his second best house among those that are not already allocated in earlier steps. Otherwise, if the top-ranked houses of i and j among those that are not already allocated in earlier steps are different, then give i and j their top-ranked houses among those that are not already allocated in earlier steps¹.

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¹Gibbard (1973) and Satterthwaite (1975) show that for social choice functions with unrestricted preference domain and cardinality of range greater than two, *strategy-proofness* is equivalent to *dictatorship*. However, note that serial bidictatorships are *strategy-proof* but not necessarily *dictatorial*, showing that *strategy-proofness* does not imply *dictatorship* in the context of house allocation problems. Serial bidictatorships constitute a subclass of “hierarchical exchange functions”, introduced in Papai (1997) and “top trading cycles mechanisms”, introduced in Abdulkadiroğlu and Sönmez (1998b).

Example: Consider the problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$ and the relation $\succeq$ on N depicted below:

$\mathcal{E}$						
P_1	P_2	P_3	P_4	P_5	P_6	$\succeq$
a	c	b	a	d	b	
f	d	c	b	c	d	3 6
b	f	f	c	f	c	2
e	e	a	e	a	f	1 5
d	a	d	f	e	a	4
c	b	e	d	b	e	

Note that $m = |Ind(N, \succeq)| = 4$ with $I^1 = \{3, 6\}$, $I^2 = \{2\}$, $I^3 = \{1, 5\}$ and $I^4 = \{4\}$. Let the vector of tie-breaking functions $t = (t_I)_{I \in Ind(N, \succeq)}$ be such that:

$$\begin{aligned}
t_{\{3,6\}}(a) &= t_{\{3,6\}}(c) = t_{\{3,6\}}(e) = t_{\{3,6\}}(f) = 3, \\
t_{\{3,6\}}(b) &= t_{\{3,6\}}(d) = 6, \\
t_{\{1,5\}}(b) &= t_{\{1,5\}}(d) = t_{\{1,5\}}(e) = 1, \\
t_{\{1,5\}}(a) &= t_{\{1,5\}}(c) = t_{\{1,5\}}(f) = 5.
\end{aligned}$$

We will now go through the algorithm described in the above paragraph in order to find the serial bidictatorship allocation induced by $\succeq$ and t for $\mathcal{E}$.

Step 1: Since $|I^1| = |\{3, 6\}| = 2$ and the top-ranked houses of agents 3 and 6 are both b , we give b to $6 = t_{\{3,6\}}(b)$ and agent 3 receives his second best house in H , namely c .

Step 2: Since $|I^2| = |\{2\}| = 1$, we give agent 2 his top-ranked house among those that are not already allocated in earlier steps, namely d .

Step 3: Since $|I^3| = |\{1, 5\}| = 2$ and the top-ranked houses of agents 1 and 5 among those that are not already allocated in earlier steps are different, we

give both agents 1 and 5 their best houses among those that are not already allocated in earlier steps. Namely, 1 receives a and 5 receives f .

Step 4: Since $|I^4| = |\{4\}| = 1$, we give agent 4 his top-ranked house among those that are not already allocated in earlier steps, namely e .

Then, the serial bidictatorship allocation induced by $\succeq$ and t for $\mathcal{E}$ is the underlined allocation depicted below:

P_1	P_2	P_3	P_4	P_5	P_6
<u>a</u>	c	b	a	d	<u>b</u>
f	<u>d</u>	<u>c</u>	b	c	d
b	f	f	c	<u>f</u>	c
e	e	a	<u>e</u>	a	f
d	a	d	f	e	a
c	b	e	d	b	e

We will now introduce natural extensions of serial bidictatorships to the variable population case. Let $\succeq$ be a *complete* preorder on $\mathcal{N}$ with the size of indifference classes being no larger than 2. Denote the partition of $\mathcal{N}$ into the indifference classes induced by $\succeq$ by $Ind(\mathcal{N}, \succeq)$. Let $t = (t_I)_{I \in Ind(\mathcal{N}, \succeq)}$ be a vector of tie-breaking functions with $t_I: \mathcal{H} \rightarrow I$ for every $I \in Ind(\mathcal{N}, \succeq)$. For any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, let $t(\mathcal{E}) = (t(\mathcal{E})_I)_{I \in Ind(N, \succeq|_N)}$ be a vector of tie-breaking functions with $t(\mathcal{E})_I: H \rightarrow I$ for every $I \in Ind(N, \succeq|_N)$, such that whenever $|I| = 2$, we have that $t(\mathcal{E})_I = t_I|_H$. A rule is ***the simple serial bidictatorship induced by $\succeq$ and t*** if for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, it chooses the serial bidictatorship allocation induced by $\succeq|_N$ and $t(\mathcal{E})$ for $\mathcal{E}$. In this case, the rule is denoted by $\varphi^{\succeq, t}$.

Lemma 5 Let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ be a problem, $\succeq$ a complete preorder on N with the size of indifference classes being no larger than 2 and let $t = (t_I)_{I \in \text{Ind}(N, \succeq)}$ be a vector of tie-breaking functions with $t_I: H \rightarrow I$ for every $I \in \text{Ind}(N, \succeq)$. An allocation μ for $\mathcal{E}$ is the serial bidictatorship allocation induced by $\succeq$ and t for $\mathcal{E}$ if and only if the following are true:

1. For any $i, j \in N$ with $i \succ j$: $\mu(i) R_i \mu(j)$,
2. For any $\{i, j\} \in \text{Ind}(N, \succeq)$ with $i \neq j$:

$$\mu(j) P_i \mu(i) \implies \mu(j) R_j \mu(i) \text{ and } j = t_{\{i, j\}}(\mu(j)).$$

Proof: Let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ be a problem, $\succeq$ a complete preorder on N with the size of indifference classes being no larger than 2 and let $t = (t_I)_{I \in \text{Ind}(N, \succeq)}$ be a vector of tie-breaking functions with $t_I: H \rightarrow I$ for every $I \in \text{Ind}(N, \succeq)$. Let $m = |\text{Ind}(N, \succeq)| \leq |N|$ and for each positive integer $r \leq m$, let I^r be the r^{th} indifference class from the top in N w.r.t. $\succeq$.

First, assume that μ is the serial bidictatorship allocation induced by $\succeq$ and t for $\mathcal{E}$. Let $i, j \in N$ with $i \succ j$. Then there exists positive integers r_i, r_j such that $r_i < r_j \leq m$, $i \in I^{r_i}$ and $j \in I^{r_j}$. Then, at step r_i , i will receive his house from the set $H \setminus \mu\left(\bigcup_{l=1}^{r_i-1} I^l\right) = \mu\left(\bigcup_{l=r_i}^m I^l\right)$ ². Let $I^{r_i} = \{i, k\}$. If the top-ranked houses of i and k in $\mu\left(\bigcup_{l=r_i}^m I^l\right)$ are distinct or if the top-ranked house of i in $\mu\left(\bigcup_{l=r_i}^m I^l\right)$ is in $t_{I^{r_i}}^{-1}(i)$, then $\mu(i)$ is the top-ranked house of i in $\mu\left(\bigcup_{l=r_i}^m I^l\right)$ and $\mu(j) \in \mu\left(\bigcup_{l=r_j}^m I^l\right) \subset \mu\left(\bigcup_{l=r_i}^m I^l\right)$, so $\mu(i) R_i \mu(j)$. Otherwise, the top-ranked houses of i and k in $\mu\left(\bigcup_{l=r_i}^m I^l\right)$ are the same, say a and $a \in t_{I^{r_i}}^{-1}(k)$. Then, $a = \mu(k)$, so $\mu(i)$ is the top-ranked

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²Here and after, we adhere to the convention that union over an empty collection of sets is empty.

house in $\mu \left(\bigcup_{l=r_i}^m I^l \right) \setminus \{a\}$ w.r.t. R_i and $\mu(j) \in \mu \left(\bigcup_{l=r_j}^m I^l \right) \subset \mu \left(\bigcup_{l=r_i}^m I^l \right) \setminus \{a\}$, implying that $\mu(i) R_i \mu(j)$. Therefore, Condition 1. is satisfied.

Now, let $\{i, j\} \in \text{Ind}(N, \succeq)$ with $i \neq j$. Then, there exists a positive integer r such that $I^r = \{i, j\}$. Then, at step r , agents i and j will receive their houses from the set $\mu \left(\bigcup_{l=r}^m I^l \right)$. Assume that $\mu(j) P_i \mu(i)$. Then, the top-ranked houses of i and j in $\mu \left(\bigcup_{l=r}^m I^l \right)$ are the same, say a and $a \in t_{j^r}^{-1}(j)$, for otherwise $\mu(i)$ is the top-ranked house in $\mu \left(\bigcup_{l=r}^m I^l \right)$ w.r.t. R_i and since $\mu(j) \in \mu \left(\bigcup_{l=r}^m I^l \right)$, we would have $\mu(i) R_i \mu(j)$, a contradiction to $\mu(j) P_i \mu(i)$. Thus, $a = \mu(j)$, so $j = t_{I^r}(a) = t_{\{i, j\}}(\mu(j))$. Moreover, since $\mu(j) = a$ is the top-ranked house in $\mu \left(\bigcup_{l=r}^m I^l \right)$ w.r.t. R_j and $\mu(i) \in \mu \left(\bigcup_{l=r}^m I^l \right)$, we have that $\mu(j) R_j \mu(i)$. Therefore, Condition 2. is satisfied.

For the necessity part, assume that the allocation μ for $\mathcal{E}$ is such that Conditions 1. and 2. are satisfied. Note that by Condition 1., for any positive integer $r \leq m$, for any $i \in I^r$ and for any $h \in \mu \left(\bigcup_{l=r+1}^m I^l \right)$, we have that $\mu(i) P_i h$, so that the top-ranked house of i in $\mu \left(\bigcup_{l=r}^m I^l \right)$ is in $\mu(I^r)$. Let r be a positive integer with $r \leq m$. If $|I^r| = 1$, let $I^r = \{i\}$. Then $\{\mu(i)\} = \mu(I^r)$ is the top-ranked house of i in $\mu \left(\bigcup_{l=r}^m I^l \right)$. If $|I^r| = 2$, let $I^r = \{i, j\}$ with $i \neq j$. Note that by Condition 2., it can not be the case that both $\mu(j) P_i \mu(i)$ and $\mu(i) P_j \mu(j)$, so there exists three possibilities:

- *Case 1:* $\mu(i) P_i \mu(j)$ and $\mu(j) P_j \mu(i)$.
- *Case 2:* $\mu(j) P_i \mu(i)$ and $\mu(j) P_j \mu(i)$.
- *Case 3:* $\mu(i) P_i \mu(j)$ and $\mu(i) P_j \mu(j)$.

Since the top-ranked houses of i and j in $\mu \left(\bigcup_{l=r}^m I^l \right)$ are in $\mu(I^r) = \{\mu(i), \mu(j)\}$, we have that:

- *Case 1:*
  - The top-ranked-house of i in $\mu\left(\bigcup_{l=r}^m I^l\right)$ is $\mu(i)$,
  - The top-ranked-house of j in $\mu\left(\bigcup_{l=r}^m I^l\right)$ is $\mu(j)$.
- *Case 2:*
  - The top-ranked-house of i in $\mu\left(\bigcup_{l=r}^m I^l\right)$ is $\mu(j)$,
  - The top-ranked-house of j in $\mu\left(\bigcup_{l=r}^m I^l\right)$ is $\mu(j)$.
- *Case 3:*
  - The top-ranked-house of i in $\mu\left(\bigcup_{l=r}^m I^l\right)$ is $\mu(i)$,
  - The top-ranked-house of j in $\mu\left(\bigcup_{l=r}^m I^l\right)$ is $\mu(i)$.

If the top-ranked houses of i and j in $\mu\left(\bigcup_{l=r}^m I^l\right)$ are different, then Case 1. above prevails, so, both i and j receive their top-ranked houses in $\mu\left(\bigcup_{l=r}^m I^l\right)$. If the top-ranked houses of i and j in $\mu\left(\bigcup_{l=r}^m I^l\right)$ are the same, say a , then either Case 2. or Case 3. will prevail. If $i = t_{I^r}(a)$, then Case 2. will not prevail, for otherwise $a = \mu(j)$, $\mu(j)P_i\mu(i)$ and $j \neq i = t_{I^r}(a) = t_{\{i,j\}}(\mu(j))$, a contradiction to Condition 2.. Therefore, if $i = t_{I^r}(a)$, then Case 3. prevails, i.e., $a = \mu(i)$, i.e., i receives his best house in $\mu\left(\bigcup_{l=r}^m I^l\right)$ and j receives his best house among the remaining houses $\{\mu(j)\} \cup \left(\mu\left(\bigcup_{l=r+1}^m I^l\right)\right)$, i.e., his second best house in $\mu\left(\bigcup_{l=r}^m I^l\right)$. Symmetrically, if $j = t_{I^r}(a)$ then j receives his best house in $\mu\left(\bigcup_{l=r}^m I^l\right)$ and i receives his second best house in $\mu\left(\bigcup_{l=r}^m I^l\right)$. So, initially, if $I^1 = \{i\}$, then agent i receives his most preferred house in H . Otherwise, if $I^1 = \{i, j\}$ with $i \neq j$, moreover if the top-ranked houses of i and j are different, then i and j receive their top-ranked houses in H . Otherwise, if the top-ranked houses of i and j in H are the same, say a , the

agent $t_{I^r}(a)$ receives a and the other agent receives his second best house in H . At the r^{th} step, if $I^r = \{i\}$, then agent i receives his most preferred house among those that are not already allocated in earlier steps. Otherwise, if $I^r = \{i, j\}$ with $i \neq j$, moreover if the top-ranked houses of i and j among those that are not already allocated in earlier steps are different, then i and j receive their top-ranked houses among those that are not already allocated in earlier steps. If the top-ranked houses of i and j among those that are not already allocated in earlier steps are the same, say a , the agent $t_{I^r}(a)$ receives a and the other agent receives his second best house among those that are not already allocated in earlier steps. Therefore, μ is the serial bidictatorship allocation induced by $\succeq$ and t for $\mathcal{E}$.

QED

Proposition 6 *Simple serial bidictatorships are Pareto optimal, consistent, conversely consistent and lower neutral.*

Proof: Let $\succeq$ be a *complete* preorder on $\mathcal{N}$ with the size of indifference classes being no larger than 2 and $t = (t_I)_{I \in \text{Ind}(\mathcal{N}, \succeq)}$ a vector of tie-breaking functions with $t_I: \mathcal{H} \rightarrow I$ for every $I \in \text{Ind}(\mathcal{N}, \succeq)$.

It is obvious that $\varphi^{\succeq, t}$ is *lower neutral*. To see that $\varphi^{\succeq, t}$ is *Pareto optimal*, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ be a problem and $\mu = \varphi^{\succeq, t}(\mathcal{E})$. Note that for each $I \in \text{Ind}(N, \succeq|_N)$, there exists $l_I \in I$ with $\mu(l_I) R_i \mu(j)$, for any $j \in I$. Define the *reflexive* relation $\succeq' \subset \succeq|_N$ as follows. For any distinct $i, j \in N$, let $i \succeq' j$ if and only if:

$$(i \succ j) \text{ or } (i \sim j \text{ and } i = l_{\{i, j\}}).$$

Then, $\succeq'$ is a linear order on N . Moreover, note that for any $i, j \in N$ with $i \succ' j$, we have that $\mu(i) R_i \mu(j)$. Therefore, by Lemma 1, μ is the serial

dictatorship allocation induced by $\succeq'$ for $\mathcal{E}$, in particular μ is *Pareto optimal*. Thus, $\varphi^{\succeq, t}$ is *Pareto optimal*.

To see that $\varphi^{\succeq, t}$ is *consistent*, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ be a problem, $\mu = \varphi^{\succeq, t}(\mathcal{E})$, $\emptyset \neq N' \subset N$ and $\mathcal{E}' = r_{N'}^\mu(\mathcal{E})$. Note that since μ is the serial bidictatorship allocation induced by $\succeq|_N$ and $t(\mathcal{E})$ for $\mathcal{E}$, we have that:

1. For any $i, j \in N$ with $i \succ|_N j$: $\mu(i) R_i \mu(j)$,
2. For any $\{i, j\} \in \text{Ind}(N, \succeq|_N)$ with $i \neq j$:

$$\mu(j) P_i \mu(i) \implies \mu(j) R_j \mu(i) \text{ and } j = t(\mathcal{E})_{\{i, j\}}(\mu(j)),$$

by Lemma 5. Let $i, j \in N'$ with $i \succ|_{N'} j$. Then, we have that $i, j \in N$ with $i \succ|_N j$, so by Condition 1. above, $\mu(i) R_i \mu(j)$, i.e., $\mu_{N'}(i) R_i|_{\mu(N')} \mu_{N'}(j)$. Therefore, Condition 1. is satisfied for $\succeq|_{N'}$, $t(\mathcal{E}')$, $\mu_{N'}$ and $\mathcal{E}'$. Now, let $\{i, j\} \in \text{Ind}(N', \succeq|_{N'})$ with $i \neq j$ and $\mu_{N'}(j) P_i|_{\mu(N')} \mu_{N'}(i)$. Then, we have that $\{i, j\} \in \text{Ind}(N, \succeq|_N)$, $i \neq j$ and $\mu(j) = \mu_{N'}(j) P_i \mu_{N'}(i) = \mu(i)$, so by Condition 2. above, we have that $\mu(j) R_j \mu(i)$ and $j = t(\mathcal{E})_{\{i, j\}}(\mu(j))$, i.e., $\mu_{N'}(j) R_j|_{\mu(N')} \mu_{N'}(i)$ and $j = t_{\{i, j\}}(\mu(j)) = t(\mathcal{E}')_{\{i, j\}}(\mu_{N'}(j))$. Thus, Condition 2. is satisfied for $\succeq|_{N'}$, $t(\mathcal{E}')$, $\mu_{N'}$ and $\mathcal{E}'$. Therefore, by Lemma 5, $\mu_{N'}$ is the serial bidictatorship allocation induced by $\succeq|_{N'}$ and $t(\mathcal{E}')$ for $\mathcal{E}'$, i.e., $\mu_{N'} = \varphi^{\succeq, t}(\mathcal{E}') = \varphi^{\succeq, t}(r_{N'}^\mu(\mathcal{E}))$. So, $\varphi^{\succeq, t}$ is *consistent*.

To see that $\varphi^{\succeq, t}$ is *conversely consistent*, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ be a problem and μ an allocation for $\mathcal{E}$ such that for any $\emptyset \neq N' \subset N$ with $|N'| = 2$, we have that $\mu_{N'} = \varphi^{\succeq, t}(r_{N'}^\mu(\mathcal{E}))$. Let $i, j \in N$ with $i \neq j$ and $\mathcal{E}' = r_{\{i, j\}}^\mu(\mathcal{E})$. First, let $i \succ|_N j$, i.e., $i \succ|_{\{i, j\}} j$. Then, since $\mu_{\{i, j\}}$ is the serial bidictatorship allocation induced by $\succeq|_{\{i, j\}}$ and $t(\mathcal{E}')$ for $\mathcal{E}'$, we have that $\mu_{\{i, j\}}(i) R_i|_{\mu(\{i, j\})} \mu_{\{i, j\}}(j)$, by Lemma 5. So, $\mu(i) R_i \mu(j)$, hence, Condition 1. in Lemma 5 is satisfied for $\succeq|_N$, $t(\mathcal{E})$, μ and $\mathcal{E}$. Now, let $i \sim_N j$

and $\mu(j)P_i\mu(i)$, i.e., $\{i, j\} \in \text{Ind}(\{i, j\}, \succeq|_{\{i, j\}})$ and $\mu_{\{i, j\}}(j)P_i|_{\mu(\{i, j\})}\mu_{\{i, j\}}(i)$. Then, since $\mu_{\{i, j\}}$ is the serial bidictatorship allocation induced by $\succeq|_{\{i, j\}}$ and $t(\mathcal{E}')$ for $\mathcal{E}'$, we have that $\mu_{\{i, j\}}(j)R_j|_{\mu(\{i, j\})}\mu_{\{i, j\}}(i)$ and $j = t(\mathcal{E}')_{\{i, j\}}(\mu_{\{i, j\}}(j))$, by Lemma 5. So, $\mu(j)R_j\mu(i)$ and $j = t(\mathcal{E})_{\{i, j\}}(\mu(j))$, hence, Condition 2. in Lemma 5 is satisfied for $\succeq|_N$, $t(\mathcal{E})$, μ and $\mathcal{E}$. Therefore, $\varphi^{\succeq, t}$ is *conversely consistent*, completing the proof of the proposition.

QED

Theorem 3 *If a rule is weakly Pareto optimal, pairwise consistent, and lower neutral, then it is a simple serial bidictatorship.*

Proof: Let φ be a *weakly Pareto optimal*, *pairwise consistent*, and *lower neutral* rule. Let $a \in \mathcal{H}$. Define the *reflexive* and *complete* relation $\succeq_a$ on $\mathcal{N}$ as follows. For any two distinct agents $i, j \in \mathcal{N}$, let $i \succeq_a j$ if there exists a house $b \in \mathcal{H} \setminus \{a\}$ such that φ chooses the underlined allocation from the following problem:

P_i	P_j
<u>a</u>	a
b	<u>b</u>

Note that by *lower neutrality* of φ , we have $i \succeq_a j$ if and only if for any $c \in \mathcal{H} \setminus \{a\}$, the rule φ chooses the underlined allocation from the following problem:

P_i	P_j
<u>a</u>	a
c	<u>c</u>

Therefore, the relation $\succeq_a$ is *antisymmetric*. Let $i, j, k \in \mathcal{N}$ be such that $i \succeq_a j$ and $j \succeq_a k$. If the agents i, j and k are not distinct, then we immediately have that $i \succeq_a k$. Otherwise, if the agents i, j and k are distinct, then let $b, c \in \mathcal{H} \setminus \{a\}$ with $b \neq c$ and consider the following problem $\mathcal{E}$:

P_i	P_j	P_k
a	a	a
b	b	b
c	c	c

Let $\mu = \varphi(\mathcal{E})$. Note that $\mu(k) \neq a$, for otherwise we obtain a contradiction to $j \succeq_a k$ and *pairwise consistency* of φ , by considering the reduced problem $r_{\{j,k\}}^\mu(\mathcal{E})$. Similarly, $\mu(j) \neq a$, for otherwise we obtain a contradiction to $i \succeq_a j$ and *pairwise consistency* of φ , by considering the reduced problem $r_{\{i,j\}}^\mu(\mathcal{E})$. Therefore, $\mu(i) = a$. But then, we have that the underlined allocation $\mu_{\{i,k\}}$ is selected by φ from the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$:

$P_i _{\mu(\{i,k\})}$	$P_k _{\mu(\{i,k\})}$	
<u>a</u>	a	,
$\mu(k)$	<u>$\mu(k)$</u>	

since φ is *pairwise consistent*. So, $i \succeq_a k$, i.e., $\succeq_a$ is *transitive*. Therefore, $\succeq_a$ is a linear order, for any $a \in \mathcal{H}$.

Let $a, b \in \mathcal{H}$ and $i, j, k \in \mathcal{N}$ be distinct. Suppose that k is bottom-ranked in $\{i, j, k\}$ w.r.t. $\succeq_a$ and top-ranked in $\{i, j, k\}$ w.r.t. $\succeq_b$. Let $c \in \mathcal{H} \setminus \{a, b\}$. W.L.O.G., let $i \succeq_a j$ and consider the following problem $\mathcal{E}$:

P_i	P_j	P_k
c	a	a
a	c	c
b	b	b

Let $\mu = \varphi(\mathcal{E})$. Note that $a \neq \mu(k)$, for otherwise we obtain a contradiction to $j \succeq_a k$ and *pairwise consistency* of φ , by considering the reduced problem $r_{\{j,k\}}^\mu(\mathcal{E})$. Suppose that $a = \mu(j)$, then $\mu(i) = c$, for otherwise $\mu(i) = b$ and we obtain a contradiction to $i \succeq_a j$ and *pairwise consistency* of φ , by considering the reduced problem $r_{\{i,j\}}^\mu(\mathcal{E})$. But then $\mu(i) = c$ and $\mu(j) = a$, so $\mu(k) = b$ contradicting $k \succeq_c i$ and *pairwise consistency* of φ , in the reduced problem $r_{\{i,k\}}^\mu(\mathcal{E})$. The only remaining possibility is that $a = \mu(i)$, but this contradicts *pairwise consistency* and *weak Pareto optimality* of φ , since the agents i and $\mu^{-1}(c)$ can become strictly better off by exchanging their assigned houses in $r_{\{i,\mu^{-1}(c)\}}^\mu(\mathcal{E})$. Therefore, for any distinct $a, b \in \mathcal{H}$ and $i, j, k \in \mathcal{N}$, k can not be both bottom-ranked in $\{i, j, k\}$ w.r.t. $\succeq_a$ and top-ranked in $\{i, j, k\}$ w.r.t. $\succeq_b$.

Suppose that there exists houses $a, b, c \in \mathcal{H}$ and distinct agents $i, j, k \in \mathcal{N}$ with $i \succeq_a j \succeq_b k \succeq_c i$. Note that it can not be the case that $a = b = c$, for otherwise we obtain a contradiction to the *antisymmetry* and *transitivity* of $\succeq_a = \succeq_b = \succeq_c$. Then, either $|\{a, b, c\}| = 2$ or $|\{a, b, c\}| = 3$. If $|\{a, b, c\}| = 2$, W.L.O.G. let $a = b \neq c$. Then, $i \succeq_a j \succeq_a k$, i.e., k is bottom-ranked in $\{i, j, k\}$ w.r.t. $\succeq_a$, therefore k can not be top-ranked in $\{i, j, k\}$ w.r.t. $\succeq_c$, by the previous paragraph, so we must have that $j \succeq_c k$. But then $j \succeq_c k \succeq_c i$ and $i \succeq_a j \succeq_a k$, i.e., i is bottom-ranked in $\{i, j, k\}$ w.r.t. $\succeq_c$ and top-ranked in $\{i, j, k\}$ w.r.t. $\succeq_a$, a contradiction to the previous paragraph. Thus,

$|\{a, b, c\}| \neq 2$, so the only remaining possibility is that $|\{a, b, c\}| = 3$, i.e., the houses a, b, c are distinct. But then, since for any agent $l \in \{i, j, k\}$:

$$\left| \left\{ h \in \{a, b, c\} \mid \begin{array}{l} l \text{ is top-ranked in} \\ \{i, j, k\} \text{ w.r.t. } \succeq_h \end{array} \right\} \right| \leq 2 \quad \text{and} \\ \left| \left\{ h \in \{a, b, c\} \mid \begin{array}{l} l \text{ is bottom-ranked} \\ \text{in } \{i, j, k\} \text{ w.r.t. } \succeq_h \end{array} \right\} \right| \leq 2 \quad ,$$

we have that:

$$\left| \left\{ l \in \{i, j, k\} \mid \exists h \in \{a, b, c\} : \begin{array}{l} l \text{ is top-ranked} \\ \text{in } \{i, j, k\} \text{ w.r.t. } \succeq_h \end{array} \right\} \right| \geq 2 \quad \text{and} \\ \left| \left\{ l \in \{i, j, k\} \mid \exists h \in \{a, b, c\} : \begin{array}{l} l \text{ is bottom-ranked} \\ \text{in } \{i, j, k\} \text{ w.r.t. } \succeq_h \end{array} \right\} \right| \geq 2$$

Therefore, by the pigeonhole principle, there exists an agent $l \in \{i, j, k\}$ such that l is top-ranked in $\{i, j, k\}$ w.r.t. $\succeq_h$ for some $h \in \{a, b, c\}$ and bottom-ranked in $\{i, j, k\}$ w.r.t. $\succeq_{h'}$ for some $h' \in \{a, b, c\}$, contradicting the above paragraph. Thus, there do not exist houses $a, b, c \in \mathcal{H}$ and distinct agents $i, j, k \in \mathcal{N}$ with $i \succeq_a j \succeq_b k \succeq_c i$.

Let $\succeq = \bigcup_{a \in \mathcal{H}} \succeq_a$. Note that the relation $\succeq$ on $\mathcal{N}$ is *reflexive* and *complete*. Let $i, j, k \in \mathcal{N}$ be such that $i \succeq j$ and $j \succeq k$. Then, there exists $a, b \in \mathcal{H}$ with $i \succeq_a j$ and $j \succeq_b k$. If agents i, j and k are not distinct, we immediately have that $i \succeq k$. Otherwise, if agents i, j and k are distinct, then let $c \in \mathcal{H}$. Note that it can not be the case that $k \succeq_c i$, for otherwise we have that $i \succeq_a j \succeq_b k \succeq_c i$ and $i, j, k \in \mathcal{N}$ are distinct, a contradiction. But then, by *completeness* of $\succeq_c$, we must have that $i \succeq_c k$, i.e., $i \succeq k$. Thus, $\succeq$ is *transitive*. Therefore, $\succeq$ is *complete* preorder on $\mathcal{N}$. Moreover, since there do not exist houses $a, b, c \in \mathcal{H}$ and distinct agents $i, j, k \in \mathcal{N}$ with $i \succeq_a j \succeq_b k \succeq_c i$,

$k \succeq_c i$, we have that the size of indifference classes of $\succeq$ are no larger than 2. Define the vector of tie-breaking functions $t = (t_I)_{I \in \text{Ind}(\mathcal{N}, \succeq)}$ with $t_I: \mathcal{H} \rightarrow I$ for every $I \in \text{Ind}(\mathcal{N}, \succeq)$, as follows. For any $I = \{i, j\} \in \text{Ind}(\mathcal{N}, \succeq)$ and $a \in \mathcal{H}$, set $t_I(a) = i$ if and only if $i \succeq_a j$.

Finally, let $\mathcal{E} = (N, H, (R_i)_{i \in N})$ be a problem and set $\mu = \varphi(\mathcal{E})$. Let $i, j \in N$ with $i \succ_N j$, in particular $i \neq j$ and it is not the case that $j \succeq i$. So, for any $a \in \mathcal{H}$, we have that $i \succeq_a j$. Suppose that $\mu(j)P_i\mu(i)$, then we must have that $\mu(j)P_j\mu(i)$, for otherwise we obtain a contradiction to *pairwise consistency* and *weak Pareto optimality* of φ by considering the reduced problem $r_{\{i,j\}}^\mu(\mathcal{E})$. By *pairwise consistency* of φ , the following underlined allocation is chosen by φ from the reduced problem $r_{\{i,j\}}^\mu(\mathcal{E})$:

$$\begin{array}{c|c} P_i|_{\mu(\{i,j\})} & P_j|_{\mu(\{i,j\})} \\ \hline \mu(j) & \underline{\mu(j)} \\ \mu(i) & \mu(i) \end{array},$$

a contradiction to $i \succeq_{\mu(j)} j$. Therefore, $\mu(i)R_i\mu(j)$, hence Condition 1. in Lemma 5 is satisfied. Now, let $\{i, j\} \in \text{Ind}(N, \succeq_N)$ with $i \neq j$, then $\{i, j\} \in \text{Ind}(\mathcal{N}, \succeq)$. Assume that $\mu(j)P_i\mu(i)$, then we must have that $\mu(j)P_j\mu(i)$, for otherwise we obtain a contradiction to *pairwise consistency* and *weak Pareto optimality* of φ by considering the reduced problem $r_{\{i,j\}}^\mu(\mathcal{E})$. Moreover, by *pairwise consistency* of φ , the following underlined allocation is chosen by φ from the reduced problem $r_{\{i,j\}}^\mu(\mathcal{E})$:

$$\begin{array}{c|c} P_i|_{\mu(\{i,j\})} & P_j|_{\mu(\{i,j\})} \\ \hline \mu(j) & \underline{\mu(j)} \\ \underline{\mu(i)} & \mu(i) \end{array},$$

i.e., $j \succeq_{\mu(j)} i$, i.e., $j = t_{\{i,j\}}(\mu(j))$. So, Condition 2. in Lemma 5 is also satisfied. Therefore, μ is the serial bidictatorship allocation induced by $\succeq|_N$ and $t(\mathcal{E})$ for $\mathcal{E}$. Therefore, φ is the simple serial bidictatorship induced by $\succeq$ and t .

QED

4.2 Lottery-valued Rules

With the hope of overcoming the dictatorial feature of *Pareto optimal* and *consistent* correspondences that we have confronted in our previous results, we will now allow rules to associate with each problem, a lottery over the allocations for that problem. In the following subsection, we will present the natural extension of the *consistency* principle to *lottery-valued* rules. In the second subsection, we will present our main impossibility result stating that there does not exist an *ex-post Pareto optimal*, *consistent* and *ex-ante anonymous lottery-valued* rule.

4.2.1 Consistency for Lottery-valued Rules

Given a problem $\mathcal{E}$, let $\mathcal{A}(\mathcal{E})$ denote the set of allocations for $\mathcal{E}$. Then, a **lottery over $\mathcal{A}(\mathcal{E})$** is a probability distribution $p: \mathcal{A} \rightarrow [0, 1]$ over $\mathcal{A}(\mathcal{E})$ ($\sum_{\mu \in \mathcal{A}(\mathcal{E})} p(\mu) = 1$). Given a problem $\mathcal{E}$ and a lottery p over $\mathcal{A}(\mathcal{E})$, let the **support of p** be $\text{Supp}(p) = \{\mu \in \mathcal{A}(\mathcal{E}) \mid p(\mu) > 0\}$.

A **lottery-valued rule** φ is a map that associates with each problem a lottery over the set of allocations for that problem.

The problem of extending the *consistency* principle to *lottery-valued* rules is not straightforward. In the context of *lottery-valued* rules, a non-degenerate solution associates positive probability to at least two different allocations for some problem. In this case, it is not clear what is meant by a subset of agents receiving their allocations before the others, since a priori, the agents do not face particular allocations but lotteries over them. It is therefore not immediate how to obtain a reduced problem. Even when the subset of agents w.r.t. whom the problem will be reduced is determined, there may be more

than one reduced problem, each associated with a different allocation in the support of the lottery. Even if we somehow reduce the problem w.r.t. one of the allocations in the support of the lottery, it is no more possible to guarantee the remaining agents the lotteries that they were facing in the beginning, since some of the houses are no more available. Therefore, we will generalize *consistency* as follows: A priori, a decision to draw the lottery in arbitrary sequence should not change the lotteries faced in the beginning. Formally, a *lottery-valued* rule φ is **consistent** if for any problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, any $\mu \in \text{Supp}(\varphi(\mathcal{E}))$ and any $\emptyset \neq N' \subset N$, we have that:

$$\forall \bar{\nu} \in \mathcal{A}(r_{N'}^\mu(\mathcal{E})) : \quad p'(\bar{\nu}) = \frac{p(\mu_{N \setminus N'}, \bar{\nu})}{\sum_{\nu \in \mathcal{A}(r_{N'}^\mu(\mathcal{E}))} p(\mu_{N \setminus N'}, \nu)}$$

where $p = \varphi(\mathcal{E})$ and $p' = \varphi(r_{N'}^\mu(\mathcal{E}))$.

A *lottery-valued* rule φ is **ex-post Pareto optimal** if for every problem $\mathcal{E}$, any allocation $\mu \in \text{Supp}(\varphi(\mathcal{E}))$ is *Pareto optimal*.

In the context of *lottery-valued* rules, it is clearly not possible to ‘equally treat the equals’ after a particular allocation has been realized. Therefore, we can only have *anonymity* before the lottery is drawn. *Ex-ante anonymity* requires that a *lottery-valued* rule should be independent of the names of the agents. Formally, a *lottery-valued* rule φ is **ex-ante anonymous** if for any $\emptyset \neq H \subset \mathcal{H}$, any two problems $\mathcal{E} = (N, H, (R_i)_{i \in N})$, $\mathcal{E}' = (N', H, (R'_i)_{i \in N'})$ and any allocation μ for $\mathcal{E}$, if $\pi: N \rightarrow N'$ is a bijection satisfying:

$$\forall i \in N, \forall a, b \in H : \quad a R_i b \iff a R'_{\pi(i)} b,$$

then $p(\mu) = p'(\mu \circ \pi^{-1})$, where $p = \varphi(\mathcal{E})$ and $p' = \varphi(\mathcal{E}')$.

4.2.2 The Main Impossibility

Theorem 4 *There does not exist an ex-post Pareto optimal, consistent and ex-ante anonymous lottery-valued rule.*

Proof: Suppose that φ is an *ex-post Pareto optimal, consistent and ex-ante anonymous lottery-valued rule*. Then, for any distinct $i, j \in \mathcal{N}$ and $a, b \in \mathcal{H}$, the rule φ will associate equal probability of $\frac{1}{2}$ to both allocations for the following problem:

P_i	P_j	
a	a	,
b	b	

since φ is *ex-ante anonymous*.

Let 1, 2, 3 be three distinct potential agents and a, b, c three distinct potential houses. Consider the following problem $\mathcal{E}$:

P_1	P_2	P_3
a	a	a
b	c	c
c	b	b

and let $p = \varphi(\mathcal{E})$. Note that there are exactly four *Pareto optimal* allocations μ^1, μ^2, μ^3 and μ^4 for $\mathcal{E}$ which are depicted below:

μ^1			μ^2			μ^3			μ^4		
P_1	P_2	P_3	P_1	P_2	P_3	P_1	P_2	P_3	P_1	P_2	P_3
<u>a</u>	a	a	<u>a</u>	a	a	a	a	<u>a</u>	a	<u>a</u>	a
b	<u>c</u>	c	b	c	<u>c</u>	<u>b</u>	<u>c</u>	c	<u>b</u>	c	<u>c</u>
c	b	<u>b</u>	c	<u>b</u>	b	c	b	b	c	b	b

Suppose that $\mu^1 \in \text{Supp}(p)$, i.e., $p(\mu^1) > 0$. Consider the reduced problem $r_{\{1,2\}}^{\mu^1}(\mathcal{E})$ and let $p' = \varphi(r_{\{1,2\}}^{\mu^1}(\mathcal{E}))$. Let the allocation $\bar{\nu}$ for $r_{\{1,2\}}^{\mu^1}(\mathcal{E})$ be as follows:

$\bar{\nu}$	
P_1	P_2
a	$\underline{a}$
$\underline{c}$	c

By the *ex-ante anonymity* of φ , we have that $p'(\bar{\nu}) = p'(\mu_{\{1,2\}}^1) = \frac{1}{2}$. Moreover, by *consistency* of φ , we have that:

$$p'(\bar{\nu}) = \frac{p(\mu_{\{3\}}^1, \bar{\nu})}{\sum_{\nu \in \mathcal{A}(r_{\{1,2\}}^{\mu^1}(\mathcal{E}))} p(\mu_{\{3\}}^1, \nu)} = \frac{p(\mu_{\{3\}}^1, \bar{\nu})}{p(\mu^1) + p(\mu_{\{3\}}^1, \bar{\nu})} ,$$

which implies that $p(\mu_{\{3\}}^1, \bar{\nu}) = p(\mu^1) > 0$, a contradiction to *ex-post Pareto optimality* of φ since the allocation $(\mu_{\{3\}}^1, \bar{\nu})$ for $\mathcal{E}$ is not *Pareto optimal*. Therefore, $\mu^1 \notin \text{Supp}(p)$. By exchanging the roles of agents 2 and 3 and by using exactly symmetric arguments, we also have that $\mu^2 \notin \text{Supp}(p)$.

Now, suppose that $\mu^3 \in \text{Supp}(p)$, i.e., $p(\mu^3) > 0$. Consider the reduced problem $r_{\{1,3\}}^{\mu^3}(\mathcal{E})$ and let $p' = \varphi(r_{\{1,3\}}^{\mu^3}(\mathcal{E}))$. Consider the allocation $\bar{\nu} = \mu_{\{1,3\}}^1$ for $r_{\{1,3\}}^{\mu^3}(\mathcal{E})$:

$\bar{\nu}$	
P_1	P_3
$\underline{a}$	a
b	$\underline{b}$

By the *ex-ante anonymity* of φ , we have that $p'(\bar{\nu}) = p'(\mu_{\{1,3\}}^3) = \frac{1}{2}$. Moreover, by *consistency* of φ , we have that:

$$p'(\bar{\nu}) = \frac{p(\mu_{\{2\}}^3, \bar{\nu})}{\sum_{\nu \in \mathcal{A}(r_{\{1,3\}}^{\mu^3}(\mathcal{E}))} p(\mu_{\{2\}}^3, \nu)} = \frac{p(\mu_{\{2\}}^3, \bar{\nu})}{p(\mu^3) + p(\mu_{\{2\}}^3, \bar{\nu})} = \frac{p(\mu^1)}{p(\mu^3) + p(\mu^1)} ,$$

which implies that $p(\mu^1) = p(\mu^3) > 0$, a contradiction to $\mu^1 \notin \text{Supp}(p)$ which was established in the previous paragraph. Therefore, $\mu^3 \notin \text{Supp}(p)$. By exchanging the roles of agents 2 and 3 and by using exactly symmetric arguments, we also have that $\mu^4 \notin \text{Supp}(p)$.

But then, by *ex-post Pareto optimality* of φ , we have that:

$$\sum_{\mu \in \mathcal{A}(\mathcal{E})} p(\mu) = p(\mu^1) + p(\mu^2) + p(\mu^3) + p(\mu^4) = 0,$$

a contradiction to p being a lottery over $\mathcal{A}(\mathcal{E})$. Therefore, there does not exist an *ex-post Pareto optimal, consistent and ex-ante anonymous lottery-valued* rule.

QED

Chapter 5

Concluding remarks

This thesis investigates the role of the *consistency* principle in *house allocation problems*. Classes of allocation rules and correspondences satisfying *consistency* and its *converse* are identified. The class of rules satisfying weak forms of *efficiency*, *consistency*, and *neutrality* are characterized by serial dictatorships where each agent is assigned his best house, following a sequence determined by an exogenous priority ordering. The more general class of *consistent* and *neutral* rules turn out to be characterized by sequential solutions generalizing serial dictatorships, where certain agents receive their least preferred house when their turn comes. The latter result is negative in the sense that one can not recover other properties of interest by dropping the *efficiency* axiom. Another way to generalize simple serial dictatorships is by weakening *neutrality* to *lower neutrality*. The class of *Pareto optimal*, *consistent* and *lower neutral* rules are characterized by simple serial dictatorships where each agent is assigned his best house, following a sequence determined by an exogenous priority ordering allowing ties of size two, which are in turn resolved by giving each agent a priority over the possession of cer-

tain houses. The impossibilities concerning *anonymity* and equal treatment of equals remain present even in the case of *multivalued* correspondences and *lottery-valued* rules. A promising direction for further research and a way to recover some *anonymity* seems to be the identification of the general class of *Pareto optimal* and *consistent* rules which do not necessarily satisfy a form of *neutrality*.

Appendix A

Independence of Axioms

Let $\mathcal{N} = \{1, 2, 3\}$ and $\mathcal{H} = \{a, b, c\}$ with $|\mathcal{N}| = |\mathcal{H}| = 3$ in the following examples which establish the independence of axioms in Theorem 1, Corollary 1 and Theorem 3.

(i) Let φ select the serial dictatorship allocation induced by the order $\succ$: $1 \succ 2 \succ 3$ in three-person problems, and the serial dictatorship allocation induced by the order $\succ'$: $2 \succ' 1 \succ' 3$ in all other problems. The rule φ is *Pareto optimal* and *neutral*. To see that φ is not *pairwise consistent*, consider the problem $\mathcal{E}$ depicted below:

P_1	P_2	P_3
<u>a</u>	a	a
b	<u>b</u>	b
c	c	<u>c</u>

Note that φ chooses the underlined allocation μ for $\mathcal{E}$. Let $\mu' = \varphi(r_{\{1,2\}}^\mu(\mathcal{E}))$, then $\mu'(1) = b \neq a = \mu_{\{1,2\}}(1)$, i.e., $\varphi(r_{\{1,2\}}^\mu(\mathcal{E})) = \mu' \neq \mu_{\{1,2\}}$. Therefore, φ

is not *pairwise consistent*.

(ii) Let $\succeq: 1 \succ 2 \succ 3$ and consider the simple sequential solution $\varphi^{\succeq, \emptyset}$. By Proposition 1, $\varphi^{\succeq, \emptyset}$ is *neutral* and *consistent*. To see that $\varphi^{\succeq, \emptyset}$ is not *weakly Pareto optimal* consider the following problem $\mathcal{E}$:

P_1	P_2
a	b
<u>b</u>	<u>a</u>

where the underlined allocation $\varphi^{\succeq, \emptyset}(\mathcal{E})$ is strongly Pareto dominated in $\mathcal{E}$.

(iii) In any three-person problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, let $\varphi(\mathcal{E})$ be the core from the endowment $\nu: N \rightarrow H$, where $\nu(1) = a$, $\nu(2) = b$ and $\nu(3) = c$. In any two-person problem $\mathcal{E} = (N, H, (R_i)_{i \in N})$, if $H = \nu(N)$, then let $\varphi(\mathcal{E})$ be the core from the endowment ν_N , otherwise if $H \neq \nu(N)$, i.e., $N = \{i, j\}$, $H = \{h, h'\}$ with $\nu(i) = h$ and $\nu(j) \neq h'$, then let $\varphi(\mathcal{E})$ be the core from the endowment $\bar{\nu}: N \rightarrow H$, where $\bar{\nu}(i) = h$ and $\bar{\nu}(j) = h'$. The rule φ is *Pareto optimal* and *consistent*.¹ Consider the underlined selections of φ from the following problems:

$\mathcal{E}$		$\mathcal{E}'$	
P_1	P_2	P'_1	P'_2
<u>a</u>	a	b	<u>b</u>
b	<u>b</u>	<u>a</u>	a

---

¹Note that φ is *Pareto optimal* and *non-empty valued* and $\varphi|_2$ does not induce a 3^+ -acyclic family of relations on $\mathcal{N}$. Therefore, by Lemma 3, φ is not *conversely consistent*.

Let $\mu = \varphi(\mathcal{E})$ and $\mu' = \varphi(\mathcal{E}')$. Agent 1 is assigned a under μ . Therefore, if φ is *pairwise neutral*, agent 1 should be assigned b under μ' , but this is not the case. Thus, φ is not *pairwise neutral*.

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