SIMPLE FUNCTORS OF ADMISSIBLE LINEAR CATEGORIES

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

SIMPLE FUNCTORS OF ADMISSIBLE LINEAR CATEGORIES

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We review the notion of an admissible R-linear category for a commutative unital ring R and we prove the classification theorem for simple functors of such a category by Barker-Boltje which states that there is a bijective correspondence between the seeds of linear category and simple linear functors. We also review the application of this theorem by Bouc to the biset category by showing that the biset category is admissible. Finally, we classify the simple functors for the category of finite abelian p-groups and show that, for a natural number n, the n-th simple functor is non-zero on precisely the groups which have exponent at least p^n.

Keywords: biset functor, simple functors, admissible category.

ÖZET

UYGUN DOĞRUSAL KATEGORİLERİN BASİT İZLEÇLERİ

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Değişmeli ve birim elemanlı bir halka üzerine tanımlı uygun kategorileri inceleyecek ve Barker-Boltje tarafından ispatlanmış olan basit doğrusal izleçlerin sınıflandırılması teoremini göstereceğiz. Bu teorem basit doğrusal izleçler ile doğrusal kategorilerin çekirdekleri arasında bire bir eşleme olduğunu söyler. Ardından bu teoremi Bouc tarafından tanımlanmış olan ikili etki izleçlerini sınıflandırmada kullanacağız. Son olarak, bu teorem ile sonlu değişmeli p-öbekleri kategorisinin n'inci izlecinin görüntüsünün, kuvveti p^n'den büyük olan öbeklerde sıfırdan farklı olduğunu ispatlayacağız.

Anahtar sözcükler: ikili etki izleçleri, basit izleçler, uygun kategori.

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Contents

Chapter 1

Introduction

In this thesis, we review the notion of an admissible R-linear category for a commutative unital ring R and the classification theorem for simple functors of such a category by Barker-Boltje. We also review the application of this theorem by Bouc to biset category. Finally, we apply the classification theorem for simple functors to a special case, namely the category of finite abelian p-groups.

Let $\mathcal{C}$ be a category, we write $\mathrm{Obj}(\mathcal{C})$ for the class of objects of $\mathcal{C}$ and we write $\mathcal{C}(G, H)$ for the class of morphism from H to G. The category $\mathcal{C}$ is said to be locally small if each morphism class is a set and $\mathcal{C}$ is said to be small if the class of objects is also a set.

Let R be a commutative unital ring. An R-linear category is a locally small category satisfying that each morphism set has an R-module structure and the composition of morphisms is R-bilinear. Functors between R-linear categories are called R-linear functors.

The quiver algebra of a small R-linear category $\mathcal{X}$ is the direct sum of the morphisms of the category $\mathcal{X}$ which has R-algebra structure. There are R-linear equivalences between the category of R-linear functors from $\mathcal{X}$ to the category of R-modules and the category of modules of the quiver algebra of $\mathcal{X}$.

A preseed of the category $\mathcal{X}$ is defined to be a pair (G, V) where G is an object and V is a simple module of the endomorphism algebra of G. There exists an equivalence relation defined on preseeds. For a given preseed (G, V), there exists a unique simple $\mathcal{X}$-functor up to isomorphism. Also, equivalent preseeds give rise to isomorphic simple functors. However, for a given simple functor the associated preseed is not unique and so we need to define some conditions on preseeds to find a unique one associated to each simple functor.

The R-linear category $\mathcal{X}$ is said to be admissible with respect to some partial ordering defined on the set of isomorphism classes of objects if every morphism from H to G is a linear combination of morphisms which factor through objects smaller than or equal to H and G. The notion of admissibility gives us a way to choose the minimal of the associated preseeds of a simple functor which will be called a seed. Thus, in a sense which will be made precise in Chapter 2, a seed of an admissible R-linear category is a preseed which cannot be factored through smaller objects.

Theorem. [1] Let $\mathcal{X}$ be an admissible R-linear category. There is a bijective correspondence between the isomorphism classes of simple $\mathcal{X}$-functors and the equivalence classes of seeds of $\mathcal{X}$.

The biset category is one of the main examples of the admissible R-linear category which is firstly introduced by Bouc. There exists a proof of classification of simple biset functors presented by Bouc, but proving by using the classification theorem for simple functors of Barker-Boltje is one of our intentions in this thesis. Before introducing the biset category, let us recall some notions concerning bisets.

Let G and H be groups. Then an (H, G)-biset U is both a left H-set and a right G-set satisfying that the H-action and the G-action commute. The (H, G)-biset can also be defined as an $H \times G$-set, so all the properties of G-sets also apply to a biset.

The biset Burnside group $B(H, G)$ is defined as the Grothendieck group of the category of finite (H, G)-bisets.

The biset category $\mathcal{C}$ is defined as the category whose objects are finite groups and morphisms are biset Burnside groups. Also, for a commutative unital ring R, the category $R\mathcal{C}$ is also defined with objects are finite groups and the morphisms are extensions of biset Burnside groups $RB(H,G) = R \otimes B(H,G)$. The category $R\mathcal{C}$ is an admissible R-linear category. The $R\mathcal{C}$-functor is the functor from the category $R\mathcal{C}$ to the category of R-modules. The following theorem which will be proved in Chapter 3, is the classification of biset functors theorem proved by Bouc and is the conclusion that we are looking for. An earlier variant of this theorem was given by Thévenaz-Webb [2].

Theorem. [3] Let R be a commutative unital ring and $\mathcal{D}$ be an admissible subcategory of the biset category $\mathcal{C}$. There is a one-to-one correspondence between the set of isomorphism classes of simple $R\mathcal{D}$ functors $S_{G,V}$ and the isomorphism classes of seeds (G,V), where G is a group and V is a simple module of $R\mathrm{Out}(G)$.

In the final chapter, we will focus on another example of an admissible R-linear category, namely the category of finite abelian p-groups and classify its simple functors by using the classification theorem for simple functors.

Theorem. Let p be a prime and $\mathcal{A}$ be the category of abelian p-groups. Then the isomorphism classes of simple $\mathcal{A}$-functors S are in a bijective correspondence with those positive integers n such that p^n is the exponent of a group in the set of objects of $\mathcal{A}$. The correspondence $S \leftrightarrow n$ is characterized by C_{p^n} is the minimal element satisfying $S(C_{p^n}) \neq 0$.

Chapter 2

Simple Functors of Linear Categories

In this chapter, we will introduce the notion of linear categories and linear functors defined on linear categories. We will define preseeds and seeds to prove the theorem called classification theorem for simple functors of linear categories.

2.1 Linear Categories

Let $\mathcal{C}$ be a category, we write $\mathrm{Obj}(\mathcal{C})$ for the class of objects of $\mathcal{C}$ and, for each $G, H \in \mathrm{Obj}(\mathcal{C})$, we write $\mathcal{C}(G, H)$ for the *class of morphism* from H to G. The category $\mathcal{C}$ is said to be *locally small* if each class $\mathcal{C}(G, H)$ is a set. Furthermore $\mathcal{C}$ is said to be *small* if the class $\mathrm{Obj}(\mathcal{C})$ is a set.

Let us consider a commutative unital ring R. An *R-linear category* is defined to be a locally small category such that each morphism set $\mathcal{C}(G, H)$ is equipped with the structure of an R-module and, for all $G, H, K \in \mathrm{Obj}(\mathcal{C})$, the composition function $\mathcal{C}(G, H) \times \mathcal{C}(H, K) \to \mathcal{C}(G, K)$ is R-bilinear. Functors between R-linear categories are called *R-linear functors*.

Let L and M be R-linear functors. Then, M is said to be a *subfunctor* of L if

$M(G) \subseteq L(G)$ for every $G \in \mathrm{Obj}(\mathcal{C})$. Also, a simple R-linear functor is a functor whose only nontrivial subfunctor is itself.

Consider a small R-linear category $\mathcal{X}$. We define the quiver algebra ${}^{\oplus}\mathcal{X}$ of $\mathcal{X}$ to be the direct sum

$$ {}^{\oplus}\mathcal{X} = \bigoplus_{G,H \in \mathrm{Obj}(\mathcal{X})} \mathcal{X}(G,H) $$

as an algebra over R, where addition is defined by usual addition of morphisms and multiplication is defined by composition of morphisms such that products of incompatible morphisms are zero. Thus, any element ${}^{\oplus}x \in {}^{\oplus}\mathcal{X}$ can be written uniquely in the form

$$ {}^{\oplus}x = \sum_{G,H \in \mathrm{Obj}(\mathcal{X})} {}_{G}x_{H} $$

where each ${}_{G}x_{H} \in \mathcal{X}(G,H)$ and only finitely many of the terms ${}_{G}x_{H}$ are nonzero. For $x, y \in {}^{\oplus}\mathcal{X}$ and $G, H, K \in \mathrm{Obj}(\mathcal{X})$, we write ${}_{G}x_{H}y_{K} = {}_{G}x_{H} \cdot {}_{H}y_{K}$. The multiplication operation can be expressed as

$$ {}_{G}xy_{K} = \sum_{H \in \mathrm{Obj}(\mathcal{X})} {}_{G}x_{H}y_{K} $$

Let us observe that the smallness of $\mathcal{X}$ is necessary for the definition of the quiver algebra because the indices of a direct sum are required to run over the elements of a set.

We define the *extended quiver algebra* ${}^{\Pi}\mathcal{X}$ to be the set of formal sums $\sum_{G,H \in \mathrm{Obj}(\mathcal{X})} {}_{G}x_{H}$ where each ${}_{G}x_{H} \in \mathcal{X}(G,H)$ such that for each $G \in \mathrm{Obj}(\mathcal{X})$, there are only finitely many $H \in \mathrm{Obj}(\mathcal{X})$ satisfying ${}_{G}x_{H} \neq 0$ and for each $H \in \mathrm{Obj}(\mathcal{X})$, there are only finitely many $G \in \mathrm{Obj}(\mathcal{X})$ satisfying ${}_{G}x_{H} \neq 0$. The multiplication is defined as before. Thus, as R-modules,

$$ {}^{\oplus}\mathcal{X} \leq {}^{\Pi}\mathcal{X} \leq \prod_{G,H \in \mathrm{Obj}(\mathcal{X})} \mathcal{X}(G,H) $$

We point out that ${}^{\Pi}\mathcal{X}$ is a unital ring with unity element

$$ 1 = \sum_{G \in \mathrm{Obj}(\mathcal{X})} \mathrm{Id}_{G} $$

where Id_G denotes the identity morphism on G. The ring ${}^{\oplus}\mathcal{X}$ is unital if and only if the set of objects is finite.

Theorem 2.1.1. *[1] There are R-linear equivalences between*

- *the category of R-linear functors L from $\mathcal{X}$ to the category of R-modules,*
- *the category of ${}^{\oplus}\mathcal{X}$-modules ${}^{\oplus}L$ such that ${}^{\oplus}\mathcal{X}.{}^{\oplus}L = {}^{\oplus}L$,*
- *the category of ${}^{\Pi}\mathcal{X}$-modules ${}^{\Pi}L$ such that ${}^{\Pi}\mathcal{X}.{}^{\Pi}L = {}^{\Pi}L$.*

Proof. The correspondence of $L \leftrightarrow {}^{\oplus}L \leftrightarrow {}^{\Pi}L$ can be characterized by the conditions $\bigoplus_G L(G) = {}^{\oplus}L = {}^{\Pi}L$ and $L(G) = \mathrm{Id}_G.{}^{\oplus}L = \mathrm{Id}_G.{}^{\Pi}L$. The map from the category of $\mathcal{X}$-functors to the category of ${}^{\oplus}\mathcal{X}$-modules is well-defined since the condition ${}^{\oplus}\mathcal{X}.{}^{\oplus}L ={}^{\oplus} L$ ensures that ${}^{\oplus}L = \bigoplus_G \mathrm{Id}_G.{}^{\oplus}L$. Also, any natural transformation $\theta : L \to L^{'}$ can be mapped to a ${}^{\oplus}\mathcal{X}$-map as $\bigoplus_G L(G) \to \bigoplus_G L^{'}(G)$ and any ${}^{\oplus}\mathcal{X}$-map ${}^{\oplus}L \to {}^{\oplus}L^{'}$ gives rise to the natural transformation of R-linear functors $L \to L^{'}$ by the correspondence map. The correspondence of the category of ${}^{\Pi}\mathcal{X}$-modules can be showed in a similar way. □

2.2 Simple Functors and Preseeds

In this section, we explain what can be said about the simple functors of any R-linear category.

Definition 2.2.1. A *preseed* of the category $\mathcal{X}$ is defined as the pair (G, V) where $G \in \mathrm{Obj}(\mathcal{X})$ and V is a simple module of the endomorphism algebra $\mathcal{X}(G, G) = \mathrm{End}_{\mathcal{X}} G$. Two preseeds (G, V) and $(G^{'}, V^{'})$ are said to be *equivalent* if there exists an isomorphism $\theta : G \to G^{'}$ in $\mathcal{X}$ sending the isomorphism class of V to the isomorphism class of $V^{'}$.

Our aim now is to define a relation between preseeds and the simple functors for R-linear categories by using the following lemma.

Lemma 2.2.2. *[4] Let A be a unital ring and i be an idempotent of A and let $B = iAi$. Then $T \cong iS$ gives a bijective correspondence between the isomorphism classes of simple B-modules T and the isomorphism classes of simple A-modules S such that $iS \neq 0$.*

We will skip the proof of this lemma but the details can be found in [4].

Proposition 2.2.3. *[1] Given a preseed (G, V) of an R-linear category $\mathcal{X}$, then there exists a simple $\mathcal{X}$-functor $S_{G,V}$ unique up to isomorphism such that $S_{G,V}(G) \cong V$ as $\mathcal{X}(G, G)$-modules. Any two equivalent preseeds (G, V) and (G', V') give rise to isomorphic simple functors $S_{G,V} \cong S_{G',V'}$*

Proof. Since Id_G is an idempotent for every $G \in \mathrm{Obj}(\mathcal{X})$,we have

$$\mathrm{Id}_G.{}^{\oplus}\mathcal{X}.\mathrm{Id}_G \cong \mathcal{X}(G, G)$$

Thus, by using the previous lemma, every simple module of $\mathcal{X}(G, G)$ corresponds to a simple ${}^{\oplus}\mathcal{X}$-module which gives us the simple functor $S_{G,V}(G) \cong V$. □

2.3 Simple Functors and Seeds

Let us recall that a binary relation P is *well founded* on a class X if and only if every nonempty subset of X has a minimal element with respect to P. The definition of well foundedness is also valid for posets which we will use in the definition of admissible R-linear category.

Definition 2.3.1. Let $\leq$ be a well founded partial ordering on the isomorphism classes of $\mathrm{Obj}(\mathcal{X})$. The R-linear category $\mathcal{X}$ is said to be *admissible* with respect to $\leq$ provided, for all $G, H \in \mathrm{Obj}(\mathcal{X})$, every morphism $x : H \to G$ can be written as a linear combination

$$x = \sum_I {}_G a_I b_H$$

where ${}_G a_I \in \mathcal{X}(G, I)$, ${}_I b_H \in \mathcal{X}(I, H)$ and I runs through the elements of $\mathrm{Obj}(\mathcal{X})$ such that $[I]_{\mathcal{X}} \leq [G]_{\mathcal{X}}$ and $[I]_{\mathcal{X}} \leq [H]_{\mathcal{X}}$.

Here, $[G]_{\mathcal{X}}$ denotes the *isomorphism class* of G in $\mathcal{X}$.

Now, we assume that $\mathcal{X}$ is admissible with respect to a given well founded partial ordering in the isomorphism classes. We denote the ideal of $\mathcal{X}(G,G)$ consisting of the endomorphisms x that can be written as a linear combination as in the *Definition 2.3.1* where I runs through the elements of $\mathrm{Obj}(\mathcal{X})$ such that $[I]_{\mathcal{X}} < [G]_{\mathcal{X}}$ by $\mathcal{X}_<(G,G)$. Also, we define the quotient algebra $\overline{\mathcal{X}}(G,G)$ over R by

$$\overline{\mathcal{X}}(G,G) = \mathcal{X}(G,G)/\mathcal{X}_<(G,G)$$

Definition 2.3.2. A *seed* of $\mathcal{X}$ is defined to be a preseed (G,V) of $\mathcal{X}$ such that the simple $\mathcal{X}(G,G)$-module V is annihilated by $\mathcal{X}_<(G,G)$, that is, V is a simple $\overline{\mathcal{X}}(G,G)$-module.

Proposition 2.3.3. *[1] Let $\mathcal{X}$ be an admissible category and (G,V) be a seed in $\mathcal{X}$. For any $H \in \mathrm{Obj}(\mathcal{X})$ satisfying $S_{G,V}(H) \neq 0$, we have $[G]_{\mathcal{X}} \leq [H]_{\mathcal{X}}$.*

Proof. Let us choose $s \in S(G) - \{0\}$ and $s^{'} \in S(H) - \{0\}$. Since S is simple, there exists some $x \in {}^{\oplus}\mathcal{X}$ such that $s^{'} = xs$. Now, $s = \mathrm{Id}_G s$ and $s^{'} = \mathrm{Id}_H s^{'} = \mathrm{Id}_H x \mathrm{Id}_G s$, so we have $x \in \mathcal{X}(G,H)$. By admissibility, we can represent $x = \sum {}_H u_I v_G$ where ${}_H u_I \in \mathcal{X}(H,I)$ and ${}_I v_G \in \mathcal{X}(I,G)$ and I runs over finitely many groups such that $[H] \geq [I] \leq [G]$. Since $0 \neq s^{-1} = \sum {}_I {}_H u_I v_G s$, there exists some I such that ${}_H u_I v_G s \neq 0$. Hence, the element ${}_I v_G s$ of $S(I)$ is nonzero. Thus, there exists $y \in {}^{\oplus}\mathcal{X}$ such that $s = yt$. Arguing as before, $y \in \mathcal{X}(G,I)$. Thus, $s = {}_G y_I v_G s$ and so $[I] = [G] \leq [H]$.

□

Theorem 2.3.4. *[1] Let $\mathcal{X}$ be an admissible category, then the condition $S \cong S_{G,V}$ characterizes a bijective correspondence between the isomorphism classes of simple $\mathcal{X}$-functors and the equivalence classes of seeds (G,V) of $\mathcal{X}$.*

Proof. We have already showed that every seed (G,V), which is also a preseed, gives rise to a simple functor $S_{G,V}$.

Let S be a simple module of ${}^{\oplus}\mathcal{X}$ and let us choose a minimal object G satisfying $S(G) \neq 0$ and $S(G) = V$ as a $\mathcal{X}(G,G)$-module. Since $\mathcal{X}(G,G) = \mathrm{Id}_G.{}^{\oplus}\mathcal{X}.\mathrm{Id}_G$, we can write $S(G) = \mathrm{Id}_G.S$ and conclude that V is a simple $\mathcal{X}(G,G)$-module. Thus, (G,V) is a preseed.

Indeed, the preseed (G,V) is a seed. If it is not annihilated by $\mathcal{X}_{<}(G,G)$, then there exists an object H such that $S_{G,V}(H) \neq 0$ such that $[H] < [G]$ which contradicts with the minimality of G.

Also, the associated seed (G,V) for the simple functor S is unique. Suppose S has two seeds (G,V) and $(G^{'},V^{'})$. First let us show that $G = G^{'}$ by choosing $s \in S(G) - \{0\}$ and $s^{'} \in S(G^{'}) - \{0\}$. Since S is a simple module and $s, s^{'} \in S$, then there exists an element $u \in^{\oplus} \mathcal{X}$ such that $s^{'} = us$. By admissibility of $\mathcal{X}$, we can decompose u as $u = \sum_{I\ G'} a_I b_H$ with $[I] \leq [G]$ and $[I] \leq [G^{'}]$. By minimality of G and $G^{'}$, we conclude that $[G] = [G^{'}]$. Then we can write $V = \mathrm{Id}_G.S$ and $V^{'} = \mathrm{Id}_G.S$ implies that $V = V^{'}$. $\square$

Chapter 3

Simple Biset Functors

In this chapter, we will first introduce the notion of bisets and then the biset category as an R-linear category. Our intention is to classify the simple biset functors by using the classification theorem for simple functors that we have proved in the previous chapter. Bouc shows how to find simple biset functors in [3], but by using the generalized version of the theorem, now we have an alternative way to find them directly.

3.1 Bisets

This section defines the notion of bisets and introduces some properties of bisets which will be used to show that it is an admissible R-linear category.

Definition 3.1.1. Let G and H be groups. Then an (H, G)-*biset* U is a both left H-set and a right G-set, such that the H-action and G-action commute, in other words, $(h.u).g = h.(u.g)$ for every $h \in H$, $u \in U$ and $g \in G$. This element will be denoted by hug.

We can regard every (H, G)-biset as a $(H \times G)$-set by defining the action as $(h, g).u := hug^{-1}$. Thus, all properties of G-sets are also valid for bisets.

Let U and V be (H, G)-bisets. A *biset homomorphism* $f : U \to V$ is a morphism such that $f(hug) = h.f(u).g$ for every $h \in H$, $u \in U$ and $g \in G$.

Let U be an (H, G)-biset, then $(H \times G)/U$ is called the set of (H, G)-orbits on U and it is denoted as double cosets $H \setminus U/G$. The biset U is called *transitive* if $H \setminus U/G$ has cardinality 1, or equivalently for every $u, v \in U$, there exist $h \in H$ and $g \in G$ such that $hug = v$.

Since, an (H, G)-biset is defined as an $(H \times G)$-set, we have the following lemma.

Lemma 3.1.2. *[3] Let G and H be groups. Then any transitive (H, G)-biset U is isomorphic to $(H \times G)/L$ for some subgroup L of $H \times G$.*

So, let U be an (H, G)-biset, then choose a set $H \setminus U/G$ of representatives of (H, G)-orbits on U. Then there is an isomorphism of (H, G)-bisets

$$U \cong \coprod_{u \in [H \setminus U/G]} (H \times G)/(H, G)_u$$

where $(H, G)_u = \{(h, g) \in H \times G \mid h \cdot u = u \cdot g\}$

Now, let us define some elementary bisets that we will use in the rest of this thesis.

Let G be a finite group. The group G is a (G, G)-biset for the left and right actions defined by group multiplications. This biset is called the *identity* (G, G)-*biset* and denoted by Id_G.

Let H be a subgroup of G, then the set G is an (H, G)-biset for the actions defined by left and right multiplications. This biset is called *restriction* and it is denoted by ${}_H\mathrm{Res}_G$. Similarly, the set G is an (G, H)-biset for the actions defined by the left and right multiplications. This biset is called *induction* and it is denoted by ${}_G\mathrm{Ind}_H$.

Let N be a normal subgroup of G, then the set G/N is a $(G, G/N)$-biset for the right action given by the multiplication and the left action given by the projection onto G/N and then multiplication by a representative element in G/N. This biset is called *inflation* and it is denoted by ${}_G\mathrm{Inf}_{G/N}$. Similarly, the set G/N is a $(G/N, G)$-biset for the left action defined by multiplication and the right action defined by projection onto G/N and then multiplication by a representative element in G/N. This biset is called *deflation* and it is denoted by ${}_{G/N}\mathrm{Def}_G$.

Let $f : G \to H$ be a group isomorphism, then the set H is an (H, G)-biset for the left action of H is defined by multiplication and right action of G is given by taking the image by f and then multiplication. This biset is called *isogation* and it is denoted by ${}_H\mathrm{Iso}_G(f)$.

3.1.1 Product of Bisets

Definition 3.1.3. Let G, H and K be groups. Let us define an H-action on $V \times U$ where U is an (H, G)-biset and V is a (K, H)-biset as $(v, u) \cdot h = (v \cdot h, h^{-1} \cdot u)$. Then the *product* of V and U is defined as the set of orbits of this action on the cartesian product $V \times U$. It is denoted by $V \times_H U$ and the H-orbit, which is the orbit of this H-action, is denoted by $(v,_H u)$. The set $V \times_H U$ is a (K, G)-biset with the actions $k \cdot (v,_H u) \cdot g = (kv,_H ug)$.

We give two main examples of the product of elementary bisets that we introduce in the previous section. We need to define the notion of section before giving the examples of products.

Let G be a group. The *section* (T, S) of G is a pair of subgroups of G such that S is a normal subgroup of T.

Example 3.1.4. Let G be a group and (T, S) be a section of G. There is an isomorphism of $(G, T/S)$-bisets

$${}_G\mathrm{Ind} \times_T \mathrm{Inf}_{T/S} \rightleftharpoons G/S$$

given by $\phi(g,_T tS) = gtS$ and $\phi^{-1}(gS) = (g,_T S)$. Thus, the $(G, T/S)$-biset G/S will be denoted by ${}_G\mathrm{Indinf}_{T/S}$.

Example 3.1.5. Let us assume the same assumptions as the previous example, there is an isomorphism of $(T/S, G)$-bisets

$$ {}_{T/S}\mathrm{Def} \times_T \mathrm{Res}_G \rightleftharpoons S \setminus G $$

given by $\psi(tS,_T g) = Stg$ and $\psi^{-1}(Sg) = (S,_T g)$. Thus, the $(T/S, G)$-biset $S \setminus G$ will be denoted by ${}_{T/S}\mathrm{Defres}_G$.

Notation 3.1.6. Let G, H and K be finite groups with L is a subgroup of $H \times G$ and M a subgroup of $K \times H$. Then the set

$$ M * L = \{(k, g) \in K \times G \mid (k, h) \in M \text{ and } (h, g) \in L \text{ for some } h \in H\} $$

is a subgroup of $K \times G$.

Lemma 3.1.7. *[3] Let G, H and K be finite groups. If L is a subgroup of $H \times G$ and M is a subgroup of $K \times H$, then we have*

$$ (K \times H)/M \times_H (H \times G)/L \cong \coprod_{h \in [p_2(M) \setminus H/p_1(L)]} K \times G/M * {}^{(h,1)}L $$

where $[p_2(M) \setminus H/p_1(L)]$ is a set of representatives of double cosets with

$$ p_1(L) = \{h \in H | (h, g) \in L \text{ for some } g \in G\} $$

$$ p_2(M) = \{h \in H | (k, h) \in M \text{ for some } k \in K\} $$

and ${}^{(h,1)}L$ is the left conjugation of L by $(h, 1)$.

Proof. Let us set $V = (K \times H)/M$ and $U = (H \times G)/L$. The map $f : K \setminus (V \times_H U/G) \to p_2(M) \setminus H/p_1(L)$ given by

$$ f(K((k, h)M,_H (h', g)L)G) = p_2(M)h^{-1}h' p_1(L) $$

is an isomorphism of bisets with the inverse map

$$ f^{-1}(p_2(M)hp_1(L)) = K((1, 1)M,_H (h, 1)L)G. $$

The stabilizer of the element $((1, 1)M,_H (h, 1)L)$ in $K \times G$ is equal to $M * {}^{(h,1)}L$. $\square$

This lemma is known as *Mackey formula* for bisets and it proves the following lemma which is called *Goursat theorem*.

Lemma 3.1.8. *[3] Let G and H be finite groups. There is a bijective correspondence between*

- *subgroups L of $H \times G$,*
- *quintuples (D, C, f, A, B) where (D, C) is a section of H, (B, A) is a section of G and $f : B/A \to D/C$ is an isomorphism.*

Proof. Let L be a subgroup of $H \times G$. The sections (D, C) and (B, A) are uniquely determined by the following identifications:

$$D = p_1(L) = \{h \in H | (h, g) \in L \text{ for some } g \in G\}$$

$$B = p_2(L) = \{g \in G | (h, g) \in L \text{ for some } h \in H\}$$

$$C = k_1(L) = \{h \in H | (h, 1) \in L\}$$

$$A = k_2(L) = \{g \in G | (1, g) \in L\}$$

and the group isomorphism $f : B/A \to D/C$ is determined by $f(bA) = dC$ where $(d, b) \in L$.

Conversely, let (D, C, f, A, B) be a quintaple such that (D, C) is a section of H, (B, A) is a section of G and $f : B/A \to D/C$ is an isomorphism, then

$$L = \{(h, g) \in H \times G | h \in D, g \in B, hC = f(gA)\}$$

is a subgroup of $H \times G$. □

We conclude the following factorization theorem for transitive bisets into elementary bisets by using the bijective correspondence that we proved in *Lemma 3.1.8.*

Theorem 3.1.9. *[3] Let G and H be finite groups. If L is a subgroup of $H \times G$ with the associated quintuple (D, C, f, A, B), then there is an isomorphism of (H, G)-bisets*

$$(H \times G)/L \cong_H \mathrm{Ind} \times_D \mathrm{Inf} \times_{D/C} \mathrm{Iso} \times_{B/A} \mathrm{Def} \times_B \mathrm{Res}_G.$$

Proof. The isomorphism of (H, G)-bisets is given by the maps

$$\theta((h,g)L) = (h,_D C,_{D/C} C,_{B/A} A,_B g^{-1}),$$

$$\psi((h,_D dC,_{D/C} d^{'} C,_{B/A} bA,_B g)) = (hdd^{'}, g^{-1}b^{-1})L$$

for $h \in H$, $d, d^{'} \in D$, $b \in B$ and $g \in G$. These maps are well-defined biset homomorphisms satisfying $\theta \circ \psi = \mathrm{Id}$ and $\psi \circ \theta = \mathrm{Id}$. Thus, they are mutual inverse isomorphisms of bisets. □

3.2 Biset Burnside Group

This section defines the notion of Burnside group and biset Burnside group. These concepts are required to define biset category which is an example of an R-linear category that we will review in the next section.

Definition 3.2.1. Let G be a finite group. The *Burnside group* $B(G)$ of G is defined as the Grothendieck group of the isomorphism classes of G-sets with addition operation defined by the disjoint union as

$$[X] + [Y] = [X \sqcup Y]$$

where $[X]$ denotes the isomorphism class of X.

Equivalently, Burnside group $B(G)$ is the quotient of the free abelian group of isomorphism classes of finite G-sets by the subgroup generated by the elements of the form $[X \sqcup Y] - [X] - [Y]$.

Let us observe that every element $[X]$ of the Burnside group $B(G)$ can be written as a linear combination of isomorphism classes of transitive G-sets which can be represented in the form G/H for some subgroup $H \leq G$. Thus, the quotients $[G/H]$ form a basis for $B(G)$ where H is some subgroup of G up to conjugacy.

Definition 3.2.2. Let G and H be finite groups. The *biset Burnside group* $B(H, G)$ is the Burnside group $B(H \times G)$, in other words, it is defined as the Grothendieck group of the category of finite (H, G)-bisets.

Let us observe that the product of bisets gives a unique bilinear map

$$\times_H : B(K,H) \times B(H,G) \to B(K,G)$$

such that $[V] \times_H [U] = [V \times_H U]$ where U is a finite (H,G)-biset and V is a finite (K,H)-biset.

Let G be a finite group. Then the biset Burnside group $B(G,G)$ has a natural ring structure with the product defined by $[V] \times_G [U] = [V \times_G U]$ where U and V are finite (G,G)-bisets. The identity element of this ring is the isomorphism class $[\mathrm{Id}_G]$ of the identity (G,G)-biset.

3.3 Classification of Simple Biset Functors

In this section, we will define the biset category and biset functors. Our aim is to classify the simple biset functors by showing that biset category is an admissible R-linear category and then by applying the classification theorem for simple functors on the biset category.

Definition 3.3.1. The *biset category* $\mathcal{C}$ of finite groups is the category defined as follows:

- The objects are finite groups,
- if G and H are finite groups, then $\mathrm{Hom}_{\mathcal{C}}(G,H) = B(H,G)$,
- if G, H and K are finite groups, then the composition of morphisms is such that $[V] \circ [U] = [V \times_H U]$ for a (G,H)-biset U and an (H,K)-biset V,
- for any finite group G, the identity morphism of G in $\mathcal{C}$ is equal to $[\mathrm{Id}_G]$.

Let us observe that this biset category is a linear category since the set of morphisms are abelian groups with respect to addition and the composition of morphisms is bilinear.

We have already defined the biset category with integer coefficients but it is also natural to consider other coefficient rings. Thus, we will define the following category.

Definition 3.3.2. Let R be a commutative unital ring. The category $R\mathcal{C}$ is defined as follows:

- The objects are finite groups,
- if G and H are finite groups, then $\mathrm{Hom}_{R\mathcal{C}}(G,H) = R \otimes_{\mathbb{Z}} B(H,G)$,
- the composition of morphisms in $R\mathcal{C}$ is the R-linear extension of the composition in $\mathcal{C}$,
- for any finite group G, the identity morphism of G in $R\mathcal{C}$ is equal to $[R \otimes_{\mathbb{Z}} \mathrm{Id}_G]$.

The category $R\mathcal{C}$ is an R-linear category since the sets of morphisms in $R\mathcal{C}$ are R-modules and the composition is R-bilinear.

We define a *biset functor* over R to be an R-linear functor from $R\mathcal{C}$ to the category of R-modules. Via Theorem 2.1.1 we can regard the biset functors over R as modules of the quiver algebra ${}^{\oplus}R\mathcal{C}$ or the extended quiver algebra ${}^{\Pi}R\mathcal{C}$.

Three examples are the *ordinary character functor* $A_{\mathbb{C}}$, the *Burnside functor* B and the *Burnside unit functor* $B^{\times}$. Here $A_{\mathbb{C}}(G)$ is the ordinary character ring, $B(G)$ is the Burnside ring of G, and $B^{\times}(G)$ is the unit group of $B(G)$. See, for instance, Yoshida [5] and Bouc [3],[6].

The only property of the biset category that we need to show is admissibility. Firstly, let us observe the basis elements of the set of morphisms in biset category.

Let G and H be finite groups, then any morphism from G to H in $\mathcal{C}$ is a linear combination of morphisms of the form $[(H \times G)/L]$ with integer coefficients, where L is some subgroup of $H \times G$. As we proved in the *Theorem 3.1.9*, any such morphism factors in the set of morphism in $\mathcal{C}$ as the composition

$$ {}_H\mathrm{Ind}_D \mathrm{Inf}_{D/C} \mathrm{Iso}_{B/A} \mathrm{Def}_B \mathrm{Res}_G $$

for suitable sections (B,A) and (D,C) of G and H respectively. Note that, we are abusing notation by writing ${}_G\mathrm{Ind}_H$ in place of $[{}_G\mathrm{Ind}_H]$. Thus, the sets of morphisms of the biset category $\mathcal{C}$ is generated by five types of morphisms above associated to elementary bisets.

We define the relation $\leq$ on the isomorphism classes of $\mathrm{Obj}(\mathcal{C})$ as $[H] \leq [G]$ if and only if H is isomorphic to a subgroup of G. This gives us a partial ordering on the isomorphism classes of $\mathrm{Obj}(\mathcal{C})$ and so we can conclude that the following category is admissible. The hypothesis in the next lemma is a little more general than the definition of admissibility by Bouc in [3].

Lemma 3.3.3. *Let $\mathcal{D}$ be a $\mathbb{Z}$-linear subcategory of the biset category $\mathcal{C}$ such that*

- $\mathrm{Obj}(\mathcal{D})$ *is closed under taking subquotients, in other words, for every $G \in \mathrm{Obj}(\mathcal{D})$ all the subgroups H and all the quotient groups G/N where N is a normal subgroup is also in the set of objects,*
- *for all morphisms $[(G \times H)/L] = {}_H\mathrm{Indinf}_{D/C_L}\mathrm{Iso}(f_L)_{B/A_L}\mathrm{Defres}_G$ in $\mathcal{D}$, the morphisms ${}_G\mathrm{Indinf}_{(D/C)_L}$ and ${}_{(D/C)_L}\mathrm{Iso}(f_L)_{(B/A)_L}$ and ${}_{(B/A)_L}\mathrm{Defres}_G$ are in $\mathcal{D}$.*

Then $\mathcal{D}$ is admissible.

Proof. Let U be an (H,G)-biset, then

$$U = \sum_{L \in H \times G} {}_G\mathrm{Indinf}_{(D/C)_L}\mathrm{Iso}(f_L)_{(B/A)_L}\mathrm{Defres}_G$$

where $[G] \geq [B/A] = [D/C] \leq [H]$. Hence, we can write every morphism U as a linear combination of transitive bisets that can be factored through smaller or equal objects. $\square$

The R-linear subcategory $R\mathcal{D}$ is also admissible in a similar way. We define an *$R\mathcal{D}$-functor* to be an R-linear functor from $R\mathcal{D}$ to the category of R-modules. Our next objective is to find the seeds of the admissible category $R\mathcal{D}$ which will give us a complete description of the simple biset functors from $R\mathcal{D}$ to the category of

R-modules which are also known as simple biset functors. The following lemma proved by Bouc characterizes the seeds of the biset category.

For $\mathcal{D}$ as above, an automorphism of $f \in \mathrm{Aut}(G)$ is called a *$\mathcal{D}$-automorphism* of G provided ${}_G\mathrm{Iso}(f)_G \in \mathcal{D}(G,G)$. Let $\mathrm{Aut}_{\mathcal{D}}(G)$ be the subgroup of $\mathrm{Aut}(G)$ consisting of the $\mathcal{D}$-automorphisms. Let $\mathrm{Inn}_{\mathcal{D}}$ be the set of inner automorphism which are the morphisms comes from conjugation. If $f \in \mathrm{Inn}_{\mathcal{D}}(G)$, then $\mathrm{Iso}(f) \cong \mathrm{Id}_G$. Indeed, $\mathrm{Inn}_{\mathcal{D}}(G) \trianglelefteq \mathrm{Aut}_{\mathcal{D}}(G)$. More generally, given $f, f' \in \mathrm{Aut}_{\mathcal{D}}(G)$ then two bisets ${}_G\mathrm{Iso}(f)_G$ and ${}_G\mathrm{Iso}(f')_G$ are isomorphic if and only if $f^{-1}f'$ is an inner automorphism of G. Defining $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Aut}_{\mathcal{D}}(G)/\mathrm{Inn}_{\mathcal{D}}(G)$, we conclude that the $\mathbb{Z}$-subalgebra of the representatives of the isomorphism classes of $\mathrm{Aut}_{\mathcal{D}}(G)$ is equal to $\mathrm{Out}_{\mathcal{D}}(G)$ with multiplication $[\mathrm{Iso}(f)] \times_G [\mathrm{Iso}(f')] = [\mathrm{Iso}(ff')]$.

Lemma 3.3.4. *[3] Let R be a commutative unital ring and $\mathcal{D}$ be a $\mathbb{Z}$-linear subcategory of the biset category $\mathcal{C}$ such that*

- $\mathrm{Obj}(\mathcal{D})$ *is closed under taking subquotients,*
- *for all morphisms $[(G \times H)/L] = {}_H\mathrm{Indinf}_{(D/C)_L}\mathrm{Iso}(f_L)_{(B/A)_L}\mathrm{Defres}_G$ in $\mathcal{D}$, the morphisms ${}_G\mathrm{Indinf}_{D/C_L}$, ${}_{D/C_L}\mathrm{Iso}(f_L)_{B/A_L}$ and ${}_{B/A_L}\mathrm{Defres}_G$ are in $\mathcal{D}$.*

Then the endomorphism algebra $\mathrm{End}_{R\mathcal{D}}(G)$ can be represented as

$$\mathrm{End}_{R\mathcal{D}}(G) = R\mathcal{D}_{<}(G,G) \oplus \overline{R\mathcal{D}}(G,G)$$

where $R\mathcal{D}_{<}(G,G)$ is the ideal of $R\mathcal{D}(G,G)$ consisting all the endomorphisms that can be factored through smaller objects $[H] < [G]$ and $\overline{R\mathcal{D}}(G,G)$ is the R-subalgebra isomorphic to the group algebra $R\mathrm{Out}_{\mathcal{D}}(G)$.

Proof. First, let us construct a free $\mathbb{Z}$-submodule I_G generated by the transitive bisets $[(G \times G)/L]$ such that $[q(L)] = [p_1(L)/k_1(L)] < [G]$. We will prove that $R\mathcal{D}_{<}(G,G) = I_G$.

Let us assume that $[q(L)] < [G]$. Since the transitive biset $[(G \times G)/L]$ can be written as

$$[(G \times G)/L] = {}_G\mathrm{Indinf}_{q(L)}\mathrm{Iso}(f_L)_{q(L)}\mathrm{Defres}_G.$$

It can be factored through smaller objects, so $I_G \subseteq R\mathcal{D}_<(G,G)$.

An element $\alpha \in R\mathcal{D}_<(G,G)$ can be written as

$$\alpha = \sum_{i=1}^{n} \psi_i \circ \phi_i,$$

where $\phi_i : G \to H_i$ and $\psi_i : H_i \to G$ such that $[H_i] \leq [G]$ for $i \in \{1, ..., n\}$. Since each ϕ_i is a linear combination of morphisms $[(H_i \times G)/L_{i,j}]$ for $j \in \{1, ..., m\}$, and each ψ_i is a linear combination of morphisms $[(G \times H_i)/M_{i,k}]$ for $k \in \{1, ..., l\}$, the MAckey formula implies that α is a linear combination of morphisms of the form $[(G \times G)/M_{i,k} * L'_{i,j}]$ for some integers j and k and some conjugate $L'_{i,j}$ of $L_{i,j}$ in $H_i \times G$. By the assumptions, ${}_{q(M_{i,k}*L'_{i,j})}\mathrm{Defres}_G$ and ${}_G\mathrm{Indinf}_{q(M_{i,k}*L'_{i,j})}$ and ${}_{q(M_{i,k}*L'_{i,j})}\mathrm{Iso}_{q(M_{i,k}*L'_{i,j})}$ are also in $\mathcal{D}$. Thus, $R\mathcal{D}_<(G,G) \subseteq I_G$.

Now, let A_G denote the subset of endomorphisms generated by the transitive bisets $[(G \times G)/L]$ satisfying that $[q(L)] = [G]$. Since we have

$$[(G \times G)/L] = {}_G\mathrm{Indinf}_G\mathrm{Iso}(f_L)_G\mathrm{Defres}_G = {}_G\mathrm{Iso}(f)_G,$$

where f is an automorphism of G,A_G is generated by the transitive bisets of the form $\mathrm{Iso}(f)$ where f is an automorphism of G and $A_G \cong \mathrm{Out}_{\mathcal{D}}(G)$. Since transitive bisets gives a basis for the morphisms, we can decompose endomorphism algebra as

$$\mathcal{D}(G,G) = \mathcal{D}_<(G,G) \oplus \bigoplus_{f \in \mathrm{Out}_{\mathcal{D}}(G)} \mathbb{Z}_G\mathrm{Iso}(f)_G,$$

and so

$$R\mathcal{D}(G,G) = R\mathcal{D}_<(G,G) \oplus R\mathrm{Out}_{\mathcal{D}}(G).$$

□

Thus, a seed of an admissible subcategory $R\mathcal{D}$ of the category $R\mathcal{C}$ is defined to be a pair (G,V), where G is an object of $\mathcal{D}$ and V is a simple $R\mathrm{Out}(G)$-module. This together with *Theorem 2.3.4* gives us the following theorem which is a specific variant of Bouc [3].

Theorem 3.3.5. *[3] Let R be a commutative unital ring and $\mathcal{D}$ be a subcategory of $\mathcal{C}$ satisfying conditions in Lemma 3.3.3. There is a bijective correspondence between the set of isomorphism classes of simple $R\mathcal{D}$-functors S and the equivalence* *classes of seeds* (G, V) *where* $G \in \mathrm{Obj}(\mathcal{D})$ *and* V *is a simple* $R\mathrm{Out}_{\mathcal{D}}(G)$*-module. The correspondence is characterized by the map* $S \to (G, S(G))$ *where* G *is the minimal group for* S*.*

The above version of the theorem is more general than the original version in Bouc [3]. Our version covers the case of a Mackey functor associated with a fixed group G as in Thévenaz-Webb [2] and it also covers the Mackey functors associated with the fusion systems as in Boltje-Danz [7].

Chapter 4

Simple Functors for the Category of Finite Abelian p-groups

In this chapter, we will give an example of the admissible R-linear category and we will classify the simple functors by using the classification theorem for simple functors of admissible R-linear categories.

4.1 The Category of Finite Abelian p-groups

Let p be a prime, $\mathbb{F}$ be a field of characteristic p and let A be a set of finite abelian p-groups that is closed under taking subqroups up to isomorphism. Let $\mathcal{A}$ be the category of abelian p-groups belonging to A. Let $\mathcal{X}$ be the quotient category of $\mathcal{A}$ with the same objects and where morphisms are defined as

$$\mathcal{X}(G, H) = \mathbb{F} \otimes_{\mathbb{Z}} \mathrm{Hom}(G \leftarrow H)$$

where $\mathrm{Hom}(G \leftarrow H)$ denotes the $\mathbb{Z}$-module of group homomorphisms. The category $\mathcal{X}$ is an $\mathbb{F}$-linear category with compositions coming from the compositions of group homomorphism.

Note that the simple $\mathcal{A}$-functors can be identified with the simple $\mathcal{X}$-functors because any simple $\mathcal{A}$-functor is annihilated by multiplication by p.

The category of finite abelian p-groups $\mathcal{X}$ is an admissible category with partial ordering defined on the isomorphism classes of $\mathrm{Obj}(\mathcal{X})$ as $[H] \leq [G]$ if and only if H is isomorphic to a subquotient of G and every morphism $\theta : H \to G$ can be decomposed as $\theta = \theta_1 \circ \theta_2$ where $\theta_1 : H \to H/\mathrm{Ker}\theta$ is the canonical projection map and $\theta_2 : \mathrm{Im}\theta \hookrightarrow G$ is the inclusion map. Thus, the category $\mathcal{X}$ is admissible and we can use the classification theorem for simple functors of admissible categories to find the simple functors. The following lemmas will lead us to the simple functors.

Lemma 4.1.1. *Let S be a simple ${}^{\oplus}\mathcal{X}$-module and $G \in \mathrm{Obj}(\mathcal{X})$ such that $S(G) \neq 0$. Then $S(G_i) \neq 0$ for some cyclic group G_i.*

Proof. Let us write $G = \prod_{i=1}^{r} G_i$ as a direct product of cyclic groups of prime power order by using the fundamental theorem of finite abelian groups. Also, define $\phi : G \to G_i$ be the canonical projection and $\nu_i : G_i \hookrightarrow G$ be the inclusion then we can define the identity map on G as

$$\mathrm{Id}_G = \sum_{i=1}^{r} \nu_i \phi_i.$$

Thus, for a given nonzero $s \in S$, $\mathrm{Id}_G(s) = s$, so $\sum_{i=1}^{r} \nu_i \phi_i(s) = s$ provided that $\phi_i \neq 0$ for some i. □

Lemma 4.1.2. *There exists a unique simple module associated to C,which is the minimal subgroup of a simple $\mathcal{X}$-functor.*

Proof. Let C be a cyclic group of order p^k. Then

$$\mathbb{F} \otimes \mathrm{Hom}(C, C) = \{0, \mathrm{Id}_C, 2\mathrm{Id}_C, ..., (p-1)\mathrm{Id}_C\} \cong \mathbb{F}$$

and $\mathcal{X}_<(C, C) = 0$ since all morphisms are isomorphisms. Thus,

$$\overline{\mathcal{X}}(C, C) = \mathcal{X}(C, C)/\mathcal{X}_<(C, C) \cong \mathbb{F}$$

has only one simple module associated to the cyclic group C. □

Now, the classification of the simple functors for the category of abelian p-groups can be stated and proved.

Theorem 4.1.3. *Let $\mathcal{A}$ be the category of abelian p-groups. Then the isomorphism classes of simple $\mathcal{A}$-functors S are in a bijective correspondence with those positive integers n such that p^n is the exponent of a group in the set of objects of $\mathcal{A}$. The correspondence $S \leftrightarrow n$ is characterized by C_{p^n}, the minimal element satisfying $S(C_{p^n}) \neq 0$.*

Proof. Assume that the maximal cyclic group is C_{p^n} in the set of objects. Then the subsets $C_{p^1}, C_{p^2}, ..., C_{p^n}$ are also in the set of objects, implying there exists n cyclic groups associated to the simple functors by the first lemma that we have proved in this section. Also, by using the second lemma, we conclude that there exists a unique simple module associated to each minimal object. Thus, there exists n simple $\mathcal{X}$-functors and so n simple functors for the category of abelian p-groups. □

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