HAMILTONIAN STRUCTURE OF THE LOTKA—VOLterra EQUATIONS

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The Lotka—Volterra equations governing predator—prey relations are shown to admit Hamiltonian structure with respect to a generalized Poisson bracket. These equations provide an example of a system for which the naive criterion for the existence of Hamiltonian structure fails. We show further that there is a three-component generalization of the Lotka—Volterra equations which is a bi-Hamiltonian system.

The conditions for a dynamical system

\[
\dot{x}^k = \dot{X}^k, \quad k = 1, 2, \ldots, 2n,
\]

to admit Hamiltonian structure are naively given by [1]

\[
\dot{X}^k = 0. \tag{2}
\]

Gonzalez-Gascon [2] has noted that this criterion is valid only when the variables \(x^i\) are chosen such that the symplectic two-form \(\omega\) is cast into the canonical form

\[
\omega = dx^1 \wedge dx^{n+1} + dx^2 \wedge dx^{n+2} + \ldots + dx^n \wedge dx^{2n} \tag{3}
\]

according to Darboux's theorem. When the original variables defining the dynamical system are not of this form, the criterion (2) is too restrictive. In fact Gonzalez-Gascon has given an example of a Hamiltonian system where this condition is violated. Gonzalez-Gascon's counter-example does not represent a familiar dynamical system. We shall show that the predator—prey equations of Lotka and Volterra provide another example of a Hamiltonian system for which the criterion (2) fails. It is surprising that the Hamiltonian structure of such a well-known system as the Lotka—Volterra equations has not been noted earlier.

The Lotka—Volterra equations are given by

\[
\dot{x} = (A-By)x, \quad \dot{y} = (Cx-D)y, \tag{4}
\]

where \(A, B, C\) and \(D\) are constants. Since the vector field

\[
X = (A-By)x \frac{\partial}{\partial x} + (Cx-D)y \frac{\partial}{\partial y} \tag{5}
\]

is not divergence free, the naive criterion for the existence of Hamiltonian structure fails. On the other hand we may consider the following ansatz for the symplectic two-form,

\[
\omega = f(x, y) \, dx \wedge dy, \tag{6}
\]

which is always closed in two dimensions. Hamilton's equations require that

\[
\omega | X = dH, \tag{7}
\]

where the Hamiltonian function \(H\) is a zero-form. The integrability conditions of eqs. (7) are obtained by applying the exterior derivative. Thus we find that \(\omega\) given by eq. (6) will be symplectic provided \(f\) satisfies

\[
[(A-By)x f]_x + [(Cx-D)y f]_y = 0. \tag{8}
\]

This first-order equation has the solution

\[
f = \frac{1}{xy} \tag{9}
\]

plus an arbitrary function of its characteristic,

\[
H = A \ln y + D \ln x - Cx - By, \tag{10}
\]

which also plays the role of the Hamiltonian func-
tion. Eq. (10) is well-known as the Liapunov function for the Lotka–Volterra equations.

In the dual representation [3] eqs. (4) can be written as
\[
\dot{x} = J^k \nabla_k H,
\]
with \(x^1 = x, x^2 = y\), where
\[
J = \begin{pmatrix}
0 & xy \\
-xy & 0
\end{pmatrix}
\]
are the structure functions. The Jacobi identities
\[
J^{k[m} \nabla_k J^{np]} = 0
\]
are satisfied automatically because we are in two dimensions.

The Lotka–Volterra equations are a Hamiltonian system with respect to the generalized Poisson bracket defined in terms of eq. (12). They do not admit a second Hamiltonian structure as an examination of the above general solution of eq. (8) reveals. There exist several generalizations of the Lotka–Volterra equations [4,5] which are going to admit a similar Hamiltonian structure and we shall now consider a three-component generalization which is a bi-Hamiltonian system.

Grammaticos et al. [5] have discussed the system
\[
\dot{x} = x(cy + z + \lambda), \quad \dot{y} = y(x + az + \mu),
\]
\[
\dot{z} = z(bx + y + \nu),
\]
where some of the constants appearing in these equations can be related to those in eqs. (4) by scaling the dynamical variables and time. It was pointed out in ref. [5] that subject to the conditions
\[
abc = -1, \quad \nu = \mu b - \lambda ab
\]
eqs. (14) admit two conserved quantities,
\[
H_1 = ab \ln x - b \ln y + \ln z,
\]
\[
H_2 = abx + y - az + \nu \ln y - \mu \ln z.
\]
This particular case is a bi-Hamiltonian system.

It can be readily verified that eqs. (14) can be written as Hamilton’s equations in two distinct ways,
\[
\dot{x}^i = J^k \nabla_k H_2 = J^k \nabla_k H_1,
\]
where the components of \(x^i\) are given by \(x, y, z\) respectively and
\[
J_1 = \begin{pmatrix}
0 & cxy & bcxz \\
-cxy & 0 & -yz \\
-bcxz & yz & 0
\end{pmatrix},
\]
\[
J_2 = \begin{pmatrix}
0 & cxy(az + \mu) & cxz(\nu + \nu) \\
-cxy(az + \mu) & 0 & xyz \\
-cxz(\nu + \nu) & -xyz & 0
\end{pmatrix}.
\]
In three dimensions the Jacobi identities (13) reduce to a single equation which is satisfied by any linear combination of \(J_1\) and \(J_2\) with constant coefficients. Thus they are compatible. No new conserved Hamiltonians are generated from the recursion relation (17) because
\[
J_1 \nabla H_1 = 0, \quad J_2 \nabla H_2 = 0,
\]
that is, \(H_1\) and \(H_2\) are Casimirs of \(J_1\) and \(J_2\) respectively.

The multi-Hamiltonian structure of Lotka–Volterra equations is evidently a rich subject as the above examples indicate.

References