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Entanglement, local measurements and symmetry

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Abstract
A definition of entanglement in terms of local measurements is discussed. Namely, the maximum entanglement corresponds to the states that cause the highest level of quantum fluctuations in all local measurements determined by the dynamic symmetry group of the system. A number of examples illustrating this definition is considered.

Keywords: Coherent states, dynamic symmetry, entanglement, quantum communication channel, quantum information

1. Introduction

Celebrating the centenary of Eugene Paul Wigner, one cannot but acclaim Wigner's approach to quantum mechanics that has been formulated in his famous papers (Wigner 1931, 1939). According to this approach, the general properties of a quantum mechanical system are specified by the dynamical symmetry of the corresponding Hilbert space. For years, the approach has been used in quantum mechanics and quantum field theory and has demonstrated an 'unexpected efficiency' (Wigner 1967). The main aim of this paper is to apply Wigner's approach to the investigation of the phenomenon of quantum entanglement.

It has been recognized that the notion of entanglement has a deep conceptual meaning, touching on the problems of locality and reality in quantum mechanics. At the same time, entanglement is considered to be a base of quantum computing, communications and cryptography (see Bowmeester *et al* 2000 and Tombesi and Hirota 2001 and references therein). In spite of great success in engineered entanglement (for a recent review, see Zeilinger 1999, Raimond *et al* 2001, Gisin *et al* 2002), there is still no agreement among the experts on the very definition of entanglement and its proper measure (e.g., see Peres 1998, Verdal and Plenio 1998, Brukner *et al* 2001).

In the usual treatment, entanglement is associated with nonseparability of corresponding states. It should be stressed that nonseparability is not a sufficient condition of maximum entanglement, and probably of entanglement at all (Horodecki *et al* 1998).

It has been shown recently (Can *et al* 2002a) that the entangled states of physical systems obey a certain condition, namely the local measurements have the maximum uncertainty in comparison with the other states allowed for a system under consideration. This condition can be used as an operational definition of maximum entanglement (definition in terms of what can be measured). At the same time, there is an adequate mathematics hidden behind this physical definition that has been unveiled recently (Klyachko 2002).

Let us note first that this novel definition has a deep physical meaning. One can choose to interpret the entangled state shared between Alice and Bob as a quantum communication channel, in which the information is carried mostly by the correlations between the sides of the channel (Brukner *et al* 2001). These correlations manifest themselves in terms of local measurements performed at the ends of the channel and the maximum correlation corresponds to the maximum uncertainty of local measurements (Klyachko and Shumovsky 2002).

We now note that the set of possible independent measurements for a given system is specified by the symmetry properties of the Hilbert space, corresponding to this physical system. Beginning with this fact, reflecting the key idea of Wigner's approach, it is possible to examine the notion of entanglement in terms of the geometric invariant theory (Klyachko 2002) (for references on geometric invariant theory see Mumford *et al* 1994).

In this paper we continue the discussion of the new definition of maximum entanglement and consider a number of physical examples.

The paper is arranged as follows. In section 2, we consider the definition of entangled states in terms of the maximum uncertainty of local measurements. This definition is illustrated by a number of examples involving the two- and three-qubit systems. In section 3, we show that the above definition of entanglement can also be expressed in terms of a certain property of the matrix of coefficients, specifying the entangled state. Namely, the parallel slices of this matrix should be orthogonal and should have the same measure. In section 4, we consider a realization of long-lived, easily monitored entanglement in a system of three-level Λ-type atoms. Finally, in section 5 we briefly discuss the obtained results and their implementation.

2. Definition of entanglement

In the usual treatment, the entanglement is associated with the states of the composite systems. Consider a composite system defined in the Hilbert space

$$\mathcal{H} = \bigotimes_{\ell=1}^{N} \mathcal{H}_\ell \tag{1}$$

where $N \geqslant 2$ is the number of components and each component has the dimension n_ℓ (the number of independent quantum degrees of freedom). Then, the dynamic symmetry group corresponding to a component is

$$G_\ell = SU(n_\ell). \tag{2}$$

An example of some considerable interest is provided by the N-qubit system consisting of spin-1/2 particles. In this case, for all ℓ, $n_\ell = 2$ and $G_\ell = SU(2)$.

The local measurements, providing the information about entanglement, are defined by the observables g_ℓ from the Lie algebra LieG_ℓ of the dynamic symmetry group G_ℓ (2).

Assume that $g_\ell \in \mathrm{Lie}G\ell$ is a local measurement that gives the spin projection on a given axis. Then, the result of the local measurements is specified by the expectation values

$$\langle g_\ell \rangle = \langle \psi | g_\ell | \psi \rangle \tag{3}$$

and by the variances

$$\langle (\Delta g_\ell)^2 \rangle = \langle \psi | (g_\ell)^2 | \psi \rangle - \langle \psi | g_\ell | \psi \rangle^2 \tag{4}$$

determining the quantum error of measurements. Here $|\psi\rangle$ denotes a state in (1).

Consider the variance (4). First of all, it is well known that

$$\langle (\Delta g_\ell)^2 \rangle \geqslant 0$$

for all $|\psi\rangle \in \mathcal{H}$. Then, the operators $(g_\ell)^2$ always have diagonal form and eigenvalues that can only be equal to unity and zero (Serre 1992). Therefore, the maximum uncertainty of a local measurement is achieved when the second term in (4), corresponding to the squared expectation value (3), is equal to zero. It is clear that this condition requires a special choice of the state $|\psi\rangle \in \mathcal{H}$.

Following our previous discussions (Can *et al* 2002a, Klyachko and Shumovsky 2002, Klyachko 2002), let us define the *maximum entangled state* in (1) by the condition

$$\forall \ell \qquad \langle g_\ell \rangle = 0 \qquad g_\ell \in \mathrm{Lie}G_\ell. \tag{5}$$

This means that the perfect entanglement of a composite system provides the maximum uncertainty of all local measurements performed over all components. In other words, the maximum entanglement corresponds to a state in which all projections of the spin are equal to zero.

Before we begin to discuss this definition in detail, let us note that the coherent states of photons are widely used for decades in quantum optics. It is interesting that these states can also be defined in terms of the dynamic symmetry approach (Perelomov 1986). According to Perelomov's analysis, the coherent states provide the *minimum* uncertainty of local measurements. That is why the coherent states are usually considered as *almost classical states*.

It is clear that the maximum entangled states defined in terms of condition (5) represent the very reverse case with respect to the coherent states. Thus, the perfect entangled states, corresponding to the *maximum* uncertainty of local measurements, should be considered as the *fundamentally quantum states*.

Let us return to the example of the N-qubit system. Then, each subspace in (1) is spanned by the two vectors

$$e_\ell^{(1)} = |+_\ell\rangle \qquad e_\ell^{(2)} = |-_\ell\rangle$$

where $|\pm_\ell\rangle$ denotes the spin-up and spin-down states of the ℓth spin, respectively. The physical realization of the 'spin' variable can be chosen differently. For example, it can be polarization of photons or state of a two-level atom. In this local basis, the infinitesimal generators of the $SU(2)$ group have the following form:

$$\begin{aligned} \sigma_\ell^x &= |+_\ell\rangle\langle -_\ell| + |-_\ell\rangle\langle +_\ell| \\ \sigma_\ell^y &= -\mathrm{i}|+_\ell\rangle\langle -_\ell| + \mathrm{i}|-_\ell\rangle\langle +_\ell| \\ \sigma_\ell^z &= |+_\ell\rangle\langle +_\ell| - |-_\ell\rangle\langle -_\ell| \end{aligned} \tag{6}$$

so that

$$\forall \ell \qquad j = x, y, z \qquad (\sigma_\ell^j)^2 = \mathbf{1}$$

where $\mathbf{1}$ is the unit operator.

Consider the simplest case of only two components ($N = 2$). Then, the Hilbert space (1) is spanned by the four base vectors

$$|\psi_{ik}\rangle = e_1^{(i)} \otimes e_2^{(k)} \qquad i, k = 1, 2.$$

Any state in such a space can be represented as follows:

$$|\psi\rangle = \sum_{i,k=1}^{2} \psi_{ik} |\psi_{ik}\rangle \tag{7}$$

where the complex coefficients ψ_{ik} obey the standard normalization condition

$$|\psi_{11}|^2 + |\psi_{12}|^2 + |\psi_{21}|^2 + |\psi_{22}|^2 = 1. \tag{8}$$

Employing the definition (5) with the measurements defined by (6) then gives the following set of six equations:

$$\begin{aligned} \mathrm{Re}(\psi_{11}\psi_{21}^* + \psi_{12}\psi_{22}^*) &= 0 \\ \mathrm{Im}(\psi_{11}\psi_{21}^* + \psi_{12}\psi_{22}^*) &= 0 \\ \mathrm{Re}(\psi_{11}\psi_{12}^* + \psi_{22}\psi_{21}^*) &= 0 \\ \mathrm{Im}(\psi_{11}\psi_{12}^* + \psi_{21}\psi_{22}^*) &= 0 \\ |\psi_{11}|^2 + |\psi_{12}|^2 - |\psi_{21}|^2 - |\psi_{22}|^2 &= 0 \\ |\psi_{11}|^2 - |\psi_{12}|^2 + |\psi_{21}|^2 - |\psi_{22}|^2 &= 0. \end{aligned} \tag{9}$$

Thus, state (7) is characterized by eight real coefficients (absolute values and phases of ψ_{ij}), while the normalization condition (8) together with conditions (9) give only seven equations. Since one parameter remains free, there are infinitely many maximum entangled states in the two-qubit system.

In general, a state of an N-qubit system is specified by 2^{N+1} real parameters, while conditions (5) together with the normalization condition give only $(3N + 1)$ equations. Thus, there are infinitely many maximum entangled states in an arbitrary N-qubit composite system ($N \geqslant 2$).

It follows from the normalization condition (8) and equations (9) that

$$\begin{gathered} |\psi_{22}| = |\psi_{11}| \\ |\psi_{21}| = |\psi_{12}| \\ |\psi_{11}|^2 + |\psi_{12}|^2 = 1/2 \\ \cos\left(\frac{\phi_{11} + \phi_{22} - \phi_{12} - \phi_{21}}{2}\right) = 0 \end{gathered} \tag{10}$$

where $\phi_{ik} \equiv \arg\psi_{ik}$. Consider some realization of equations (10). It is easily seen that the choice of either $|\psi_{11}| = 0$ or $|\psi_{12}| = 0$ leads to the conventional Einstein–Podolsky–Rosen (EPR) and Bell states

$$\begin{aligned} |\psi_{EPR}\rangle &= \frac{1}{\sqrt{2}}(|+_1 -_2\rangle \pm |-_1 +_2\rangle) \\ |\psi_{Bell}\rangle &= \frac{1}{\sqrt{2}}(|+_1 +_2\rangle \pm |-_1 -_2\rangle) \end{aligned} \tag{11}$$

respectively. The states (11) form a basis in the four-dimensional Hilbert space. Let us stress that each state in (11) contains only two base vectors $|\psi_{ij}\rangle$ out of four. The conditions (10) permit us to construct the maximum entangled states containing all four base vectors. For example,

$$|\psi\rangle = \tfrac{1}{2}(|+_1 +_1\rangle + \mathrm{i}|+_1 -_2\rangle + \mathrm{i}|-_1 +_2\rangle + |-_1 -_2\rangle) \tag{12}$$

is the maximum entangled two-qubit state.

Let us stress that, from the mathematical point of view, there is only one maximum entangled state of the two-qubit system, namely the EPR state. All other maximum entangled states defined by the conditions (10) are equivalent to the EPR state to within the action of the dynamic symmetry group. At the same time, these states can be different from the physical point of view because they are realized under different conditions caused by the physical environment of the system.

We now note that equations (9) can be obtained in a different way. Let us note that the coefficients ψ_{ij} in (7) form a (2×2) matrix $[\psi]$. Then, it is easily seen that the above equations express the orthogonality conditions for the parallel rows and columns of this matrix $[\psi]$ and the condition that different rows and columns have the same norm. The generalization of this result is discussed in the next section.

Consider now another example of some considerable importance provided by the three-qubit states. The simplest case is represented by the Grinberger–Horn–Zeilinger (GHZ) states

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|+_1 +_2 +_3\rangle \pm |-_1 -_2 -_3\rangle). \tag{13}$$

The general three-qubit state has the following form:

$$|\psi\rangle = \sum_{i,k,m} \psi_{ikm} e_1^{(i)} \otimes e_2^{(k)} \otimes e_3^{(m)} \qquad i, k, m = 1, 2 \tag{14}$$

where $e_\ell^{(i)}$ are the same base vectors as above and the coefficients ψ_{ikm} obey the normalization condition

$$\sum_{i,k,m} |\psi_{ikm}|^2 = 1. \tag{15}$$

In this case, the (2×3) matrix $[\psi]$ is specified by eight complex or 16 real parameters. In turn, the conditions (5) together with (15) give only ten equations. Thus, there are infinitely many maximum entangled three-qubit states.

Through the use of definition (5) with Pauli operators (6), we can get a number of restrictions on the coefficients $\{\psi_{ikm}\}$, providing the entanglement in (14). Leaving aside the general case, we restrict our consideration to the two examples. Consider first the state

$$|\psi_1\rangle = \psi_{111}|+_1 +_2 +_3\rangle + \psi_{121}|+_1 -_2 +_3\rangle + \psi_{222}|-_1 -_2 -_3\rangle. \tag{16}$$

Clearly, this is a nonseparable space in (1) and thus it can be considered as a candidate for an entangled state. Then, the use of definition (5) gives

$$\begin{gathered} |\psi_{111}||\psi_{121}|\cos(\phi_{111} - \phi_{121}) = 0 \\ |\psi_{111}||\psi_{121}|\sin(\phi_{111} - \phi_{121}) = 0 \\ |\psi_{111}|^2 + |\psi_{121}|^2 - |\psi_{222}|^2 = 0 \\ |\psi_{111}|^2 - |\psi_{121}|^2 - |\psi_{222}|^2 = 0 \end{gathered}$$

where ϕ_{ikm} again denotes the phase of the complex coefficients. It is seen that the only solution of these equations is

$$|\psi_{111}| = |\psi_{222}| = \frac{1}{\sqrt{2}} \qquad |\psi_{121}| = 0.$$

This solution reduces state (16) to one of the GHZ states (13) that definitely obey the definition of entanglement in terms of the maximum uncertainty of local measurements (see Can *et al* 2002a). At any $|\psi_{121}| \neq 0$, the nonseparable state (16) does not manifest entanglement. It should be stressed in this connection that the nonseparability by itself is not a sufficient condition of entanglement (Horodecki *et al* 1998).

Consider now another, more symmetric realization of the three-qubit state (14)

$$\begin{aligned} |\psi_2\rangle = \psi_{111}|+_1 +_2 +_3\rangle + \psi_{121}|+_1 -_2 +_3\rangle + \psi_{212}|-_1 +_2 -_3\rangle \\ + \psi_{222}|-_1 -_2 -_3\rangle. \end{aligned} \tag{17}$$

Through the use of definition (5) and normalization (15) we can obtain the set of ten equations that can be reduced to the following conditions:

$$\begin{gathered} |\psi_{111}|^2 + |\psi_{121}|^2 = \tfrac{1}{2} \\ |\psi_{222}| = |\psi_{111}| \\ |\psi_{212}| = |\psi_{121}| \\ \phi_{111} - \phi_{121} - \phi_{212} + \phi_{222} = \pm\pi + 2n\pi. \end{gathered} \tag{18}$$

In contrast to the GHZ states (13), the coefficients here do not have fixed values but lie on a circle of radius 1/2. Again, state (18) is equivalent to the GHZ state (13) to within the action of the dynamic symmetry group $SU(2) \times SU(2) \times SU(2)$. The conditions (5) can also be used to construct the basis of eight three-qubit maximum entangled states.

3. Maximum entanglement and the matrix of coefficients in (7)

The maximum entanglement of a nonseparable state is usually defined in terms of the reduced density matrix. Namely, the reduced entropy should have the maximum value, the same for all components of the composite system (e.g., see Scully and Zubairy 1997). It is then a straightforward matter to show that this condition follows from the definition of entanglement in terms of local measurements (5).

Let us now note that the scheme that has been discussed in the previous section can be reformulated through the use of the properties of the matrix of coefficients $[\psi]$. It was shown in a previous section that, in the case of the two-qubit system, this matrix obeys a certain condition. Consider now the generalization of this result.

Let the factor spaces in the Hilbert space (1) be spanned by the orthonormal bases $\{e_\ell^{(i)}\}$. Then, a state of a composite system defined in (1) can be described by the normalized state vector of the form

$$|\psi\rangle = \sum \psi_{i_1 i_2 \cdots i_N} e_1^{(i_1)} \otimes \cdots \otimes e_N^{(i_N)}. \qquad (19)$$

The results of the previous section show us that the entanglement of state (19) is specified by a certain choice of the many-dimensional matrix $[\psi]$ of the coefficients in (19).

It has been proven (Klyachko 2002) that the state $|\psi\rangle \in \mathcal{H}$ manifests the maximum entanglement if and only if parallel slices of its matrix $[\psi]$ are orthogonal and have the same norm. (Concerning parallel slices of multi-dimensional matrices see Gelfand *et al* (1994). In the simplest case of two-qubit system considered in the previous section, the parallel slices are represented by rows and columns of the (2×2) matrix $[\psi]$. In the case of the three-qubit system, this is a (2×3) matrix.)

This general statement can be illustrated in the simplest case by state (12) whose matrix of coefficients has the form

$$[\psi] = \begin{bmatrix} 1/2 & \mathrm{i}/2 \\ \mathrm{i}/2 & 1/2 \end{bmatrix}.$$

It is clear that

$$[\,1/2 \quad \mathrm{i}/2\,] \begin{bmatrix} -\mathrm{i}/2 \\ 1/2 \end{bmatrix} = 0$$

so that the parallel slices are orthogonal. In turn

$$\|\,[\,1/2 \quad \mathrm{i}/2\,]\,\| = \|\,[\,\mathrm{i}/2 \quad 1/2\,]\,\| = 1/\sqrt{2}.$$

The fact that (12) represents the maximum entangled state can also be verified through the calculation of reduced entropies. It is straightforward to show that, in the case of state (12)

$$S_1 = S_2 = \ln 2$$

as all one can expect for the maximum entangled two-qubit state (Scully and Zubairy 1997). Here

$$S_k = -\,\mathrm{Tr}_{i\neq k}(\rho \ln \rho) \qquad \rho = |\psi\rangle\langle\psi|$$

is the reduced entropy. The above condition of maximum entanglement together with equations (10) permits us to construct other maximum entangled two-qubit states that involve all four base vectors $\{e_{ell}^{(1)}, e_\ell^{(2)}\}$ at $\ell = 1, 2$. For example, the states

$$|\psi'\rangle = \tfrac{1}{2}(|+_1 +_2\rangle - \mathrm{i}|+_1 -_2\rangle + \mathrm{i}|-_1 +_2\rangle - |-_1 -_2\rangle)$$

$$|\psi''\rangle = \tfrac{1}{2}(\mathrm{i}|+_1 +_2\rangle + |+_1 -_2\rangle + |-_1 +_2\rangle + \mathrm{i}|-_1 -_2\rangle)$$

$$|\psi'''\rangle = \tfrac{1}{2}(-\mathrm{i}|+_1 +_2\rangle + |+_1 -_2\rangle - |-_1 +_2\rangle + \mathrm{i}|-_1 -_2\rangle)$$

are the maximum entangled two-qubit states, forming together with (12) an orthonormal basis in the Hilbert space (1) (Can *et al* 2002a). This basis is equivalent to (11) to within the action of the dynamical symmetry group $SU(2) \times SU(2)$.

In a more general case of a two-component entangled state

$$|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

the matrix of coefficients $[\psi]$ has the dimensionality $n_1 \times n_2$ where $n_\ell \equiv \dim \mathcal{H}_\ell$. The orthogonal rows and columns have the norms $1/\sqrt{n_1}$ and $1/\sqrt{n_2}$, respectively. Thus, the maximum entanglement is allowed only if $n_1 = n_2$. In this case, $[\psi]$ is similar to the unitary matrix. This implies the uniqueness of the maximum entangled state to within the action of the dynamic symmetry group $SU(n) \times SU(n)$.

By performing a similar analysis, it is a straightforward matter to show that the three-qubit state (17), (18) also manifests the maximum entanglement.

Our considerations so far have dealt with composite systems of spin-1/2 particles (qubit systems). The scheme can be generalized to the case of an arbitrary spin $s \geqslant 1/2$ as well. Examples of spin-1 entangled states were discussed by Burlakov *et al* (1999) in the context of photon pairs in symmetric Fock states and by Can *et al* (2002a) in connection with polarization of multipole waves of photons. The application of such a state to quantum cryptography was considered by Bechman-Pasquinucci and Peres (2000).

According to the definition of maximum entanglement (5) discussed in section 2, the entangled state should give the average spin projection onto every direction equal to zero. Let us denote the spin-1 states by $|+\rangle$, $|0\rangle$ and $|-\rangle$. Consider the cascade decay of a two-level atom with the excited state specified by the angular momentum $j = 2$ and projection of the angular momentum on the quantization axis $m = 0$, and the ground state $j' = 0$, $m' = 0$. This transition gives rise to photon twins (Mandel and Wolf 1995) that can be observed in the states

$$|+_1 -_2\rangle \qquad |0_1 0_2\rangle \qquad |-_1 +_2\rangle \qquad (20)$$

because of the conservation of the total angular momentum in the process of radiation. It is then easily seen that the so-called $SU(2)$ phase states of photons (Shumovsky 2001)

$$|\phi_k\rangle = \frac{1}{\sqrt{3}}(|+_1 -_2\rangle + \mathrm{e}^{\mathrm{i}\phi_k}|0_1 0_2\rangle + \mathrm{e}^{2\mathrm{i}\phi_k}|-_1 +_2\rangle) \qquad (21)$$

where

$$\phi_k = \frac{2k\pi}{3} \qquad k = 0, 1, 2$$

obey the condition (5) and form a basis of entangled states dual to (20).

The principal difference between the systems with spin 1/2 and spin 1 is that the maximum entangled state (in the sense of definition (5)) can be realized in the composite system, consisting of at least two particles in the former case and in a single-particle system in the latter case. Consider, for example, the superposition state

$$|\psi\rangle = \lambda_+|+\rangle + \lambda_0|0\rangle + \lambda_-|-\rangle \qquad \sum_i |\lambda_i|^2 = 1. \tag{22}$$

Then, the measurement of projections gives in view of the definition (5) the following equations:

$$\begin{aligned} &|\lambda_+|^2 - |\lambda_-|^2 = 0 \\ &|\lambda_+||\lambda_0|\cos(\varphi_0 - \varphi_+) + |\lambda_0||\lambda_-|\cos(\varphi_- - \varphi_0) = 0 \\ &|\lambda_+||\lambda_0|\sin(\varphi_0 - \varphi_+) + |\lambda_0||\lambda_-|\sin(\varphi_- - \varphi_0) = 0 \end{aligned} \tag{23}$$

where $\varphi_i \equiv \arg \lambda_i$. One of the possible solutions, manifesting the single spin-1 particle entanglement, is $|\lambda_0| = 0$ and

$$|\psi_I\rangle = \frac{1}{\sqrt{2}}(|+\rangle + \mathrm{e}^{\mathrm{i}\phi}|-\rangle) \tag{24}$$

where ϕ is an arbitrary complex number. Another solution has the form $|\lambda_+| = |\lambda_-| = 0$, $|\lambda_0| = 1$, so that

$$|\psi_{II}\rangle = |0\rangle. \tag{25}$$

One more solution of (23) is specified by the conditions

$$\begin{aligned} &|\lambda_+| = |\lambda_-| \qquad 2|\lambda_+|^2 + |\lambda_0|^2 = 1 \\ &\cos\left(\frac{\varphi_+ + \varphi_- - 2\varphi_0}{2}\right) = 0. \end{aligned} \tag{26}$$

It is seen that the conditions (26) permit us to construct infinitely many entangled single-particle states. For example,

$$|\psi_{III}\rangle = |\lambda_+|^2\left(|+\rangle + \frac{1+\mathrm{i}}{\sqrt{2}|\lambda_+|^2}\sqrt{1-2|\lambda_+|^2}|0\rangle + |-\rangle\right) \tag{27}$$

is the single-particle entangled state. From the physical point of view, these states can be constructed for massive particles like ρ and K mesons that have reasonably long lifetime and for the alkaline atoms used in the experiments on Bose–Einstein condensation. The problem of interpretation, preparing entangled single-particle states and performing the necessary measurements deserves special consideration.

4. Entangled states in atomic systems

As a possible physical realization of the above-discussed formalism, consider now entangled states in atomic systems. It should be stressed that engineered entanglement in systems of trapped atoms and atomic beams has recently attracted a great deal of interest (e.g., see Bederson and Walther 2000, Myatt *et al* 2000, Rempe 2000, Raimond *et al* 2001, Julsgaard *et al* 2001 and references therein). In particular, single-photon exchange between two two-level atoms in a cavity can lead to a maximum entangled atomic state (Plenio *et al* 1999, Beige *et al* 2000).

It was then shown (Can *et al* 2002a) that the atomic entangled states in a cavity belong to a special class of the so-called $SU(2)$ phase states that has been introduced by Vourdas (1990) and generalized by one of the present authors (see Shumovsky 2001 and references therein). In particular, it was shown that these states obey the definition of maximum entanglement (5). The $SU(2)$ phase states can also be used in quantum coding (Vourdas 2002).

One of the important requirements dictated by the practical applications of entanglement in the field of quantum information technologies is that the lifetime of an entangled state should be long enough. Under this condition, it seems to be much more convenient to use three-level atoms with the Λ-type transitions instead of two-level atoms (Can *et al* 2002b). Consider this idea in more details.

The system of three-level Λ-type atoms interacting with the two cavity modes can be described by the following Hamiltonian:

$$\begin{aligned} H &= H_0 + H_{int} \\ H_0 &= \omega_P a_P^+ a_P + \omega_{Sk} a_S^+ a_S + \omega_{12}\sum_f R_{22}(f) + \omega_{13}\sum_f R_{33}(f) \\ H_{int} &= \sum_k \sum_f \{g_P R_{21}(f) a_P + g_S R_{32}(f) a_S + \text{H.c.}\}. \end{aligned} \tag{28}$$

Here a_P and a_S are the photon operators of the 'pumping' and Stokes modes, respectively. It is supposed that the cavity is an ideal one with respect to the pumping, while it strongly absorbs the Stokes photons. The operator $R_{21}(f)$ describes the transition in the fth atom from the ground to the highest excited level. In turn, $R_{32}(f)$ gives the transition from the highest excited level to an intermediate level 3 separated from the ground level by ω_{13}. The dipole transition between atomic levels 3 and 1 is forbidden because of parity conservation.

Assume first that the system consists of only two atoms and is initially prepared in the state

$$|\Psi_0\rangle = |1,1\rangle \otimes |1_P\rangle \tag{29}$$

so that both atoms are in the ground state while the cavity contains a single photon of the pumping mode. The evolution of the system in the cavity damped with respect to the Stokes photons is then described by the master equation

$$\dot{\rho} = -\mathrm{i}[H,\rho] + \kappa\{2a_S\rho a_S^+ - a_S^+ a_S\rho - \rho a_S^+ a_S\} \tag{30}$$

where $1/\kappa$ is the lifetime of a Stokes photon in the cavity defining the quality factor. The so-called Liouville term in the right-hand side of (30) takes into account the absorption of the Stokes photon. The density matrix ρ involves all eigenstates of the Hamiltonian (28) including the state

$$|\psi_{fin}\rangle = \frac{1}{\sqrt{2}}(|3,1\rangle + |1,3\rangle) \otimes |0_P\rangle \otimes |0_S\rangle. \tag{31}$$

The lifetime of this state is determined by the nonradiative processes and is therefore quite long.

As a consequence of evolution generated by the Hamiltonian (28), the pumping photon can be absorbed by either atom with equal probability, so that the system passes into the entangled state

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|2, 1\rangle + |1, 2\rangle) \otimes |0_P\rangle \otimes |0_S\rangle.$$

The lifetime of this state is completely defined by the dipole radiative processes $2 \to 1$ and $2 \to 3$ and therefore is very short.

As the next step, the first term in the right-hand side of (30) generates evolution to another entangled state

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|3, 1\rangle + |1, 3\rangle) \otimes |0_P\rangle \otimes |1_S\rangle.$$

The absorption of the Stokes photon described by the Liouville term in (30) then leads to the final, long-lived state (31).

The scheme can be easily realized with a modern experimental technique. First of all, the single-photon excitation of the cavity field can be prepared (see Walther 1997, 2001 and references therein). One of the atoms can be trapped inside the cavity, while the other atom should pass through the cavity in the same way as in the experiments on excitation of Fock states of photons (Walther 2001). Another way is to send a beam of three-level atoms through the cavity with a single pumping photon so that every time there would be just two atoms inside the cavity.

Concerning the experimental realization, let us note that the Raman-type process with emission of a Stokes photon in a single atom has been observed recently (Henrich *et al* 2000).

In principle, the process can be realized in a system of more than two three-level atoms, interacting with a single cavity photon. In fact, Fock states with more than one photon have been successfully generated (Walther 2001). The use of the definition of entanglement (5) then shows that if the number of pumping photons in the cavity is n, then the number of atoms interacting at once with these photons should be $2n$ (Can *et al* 2002a). In this case, the entangled atomic states can be constructed as the $SU(2)$ phase states have been discussed by Can *et al* (2002a).

Another realization of long-lived entanglement in the system of three-level atoms that has been considered by Can *et al* (2002b) assumes that the Stokes photons can leave the cavity freely. In this case, detection of Stokes photons outside the cavity can be considered as a signal that the long-lived atomic entangled state has been created.

5. Conclusion

Let us briefly discuss the obtained results. The general scheme discussed in sections 2 and 3 has the following structure. To specify the entangled states of a composite system defined in the Hilbert space (1) it is necessary

(1) to specify the dynamic symmetry group G of the factor spaces in (1),
(2) to specify the local measurements defined by the Lie algebra, corresponding to the dynamic symmetry group G, and
(3) to apply the condition (5) that determines the matrix of coefficients of a general state in (1), corresponding to the entanglement (maximum entanglement).

The maximum entangled state can also be specified by the condition that the parallel slices of matrix $[\psi]$ are mutually orthogonal and have the same norm. It can be proven that this definition entails the conventional condition expressed in terms of reduced entropy (Klyachko 2002).

As follows from definition (5), the entangled states show the maximum level of quantum fluctuations in all local measurements. Therefore, they should be considered as fundamentally quantum states in contrast to the almost classical coherent states, showing the minimum of quantum fluctuations.

The scheme discussed in section 2 superposes the elements of the operational approach (zero result for all local measurements allowed for an entangled state) with the deep mathematics lying behind the definition of entanglement (5). In particular, it is possible to show that an arbitrary entangled state (not necessarily the maximum entangled state) can be defined to be the semistable vector in the Hilbert space $\mathcal{H}$ (1) and that the rate of entanglement can be specified by the length of the minimal vector in a complex orbit of the entangled state (Klyachko 2002).

It is interesting that the definition of entanglement represented by condition (5) permits us to consider the single-particle entangled states in the case of spin $s \geqslant 1$ in addition to the conventional composite-system states.

The practical realization of long-lived easily monitored entanglement discussed in section 4 seems to be accessible with the present experimental technique. Let us note that, instead of the three levels interacting with the cavity mode, another environment can be used. An interesting example is provided by the system of atoms in the presence of dispersive and absorbing objects (Dung *et al* 1998, 2002, Welsch *et al* 2002).

The definition of entanglement in terms of condition (5) is a general one and may exceed the limits of quantum optics and quantum information. For example, the combinations of quarks corresponding to π mesons can be treated in terms of the possible states with the dynamic symmetry provided by the hadron group $SU(3)$. Then, definition (5) shows that π^0 corresponds to the entangled combination of quarks, while $\pi^{\pm}$ are specified by the coherent combinations (Klyachko 2002). It is tempting to associate the short lifetime of π^0 with respect to $\pi^{\pm}$ with the strong quantum fluctuations in the entangled state and very weak ones in the coherent state.

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