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To cite this article: Metin Gürses 2022 *J. Phys.: Conf. Ser.* **2191** 012013

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Integrability of Three Dimensional Gravity Field Equations

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Abstract. We show that the tree dimensional Einstein vacuum field equations with cosmological constant are integrable. Using the $sl(2, R)$ valued soliton connections we obtain the metric of the spacetime in terms of the dynamical variables of the integrable nonlinear partial differential equations.

1. Introduction

Gravity in three dimensions is highly simple. It has no degrees of freedom and the constraint equations can be explicitly solved. The main motivation to study three dimensional gravity was quantization as a toy model. For this purpose, the Einstein-Hilbert action was written as a Chern-Simons action with a suitable gauge group [1], [2]. Very recently [3], this property has been used to construct highly nontrivial exact solutions of the Einstein field equations with cosmological constant. The method presented here uses the zero curvature formalism of the integrable systems. The soliton connection is the Chern-Simons gauge potential with $SL(2, R)$ group as the gauge group. The soliton equations corresponding to this gauge connection are known as the AKNS equations containing the nonlinear Schrodinger and modified Korteweg de Vries equations.

In this work we shall show that the field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \Lambda g_{\mu\nu},$$

where Λ is the cosmological constant and $\mu, \nu = 1, 2, \dots, D$ are integrable when $D = 3$, by using the Newmann-Penrose null tetrad formalism. We find that $SL(2, R)$ valued tetrad one form is the difference of two zero curvature connections without referring to the Chern-Simons theory.

Before starting to write the gravitational field equations in null tetrad formalism we give a brief introduction to the $sl(2, R)$ valued soliton connections and AKNS system. Let $x^\mu = (t, x, y)$ be the local coordinates of (M, g) and the soliton connection \mathbf{a} depends on the coordinates t and x . Hence

$$\mathbf{a} = P dx + Q dt,$$



where $P(t, x)$ and $Q(t, x)$ are some $sl(2, \mathbb{R})$ matrices. They are given by

$$P = \begin{pmatrix} 2\xi & p(t, x) \\ q(t, x) & -2\xi \end{pmatrix}, Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

Here ξ is the eigenvalue parameter and p and q are dynamical variables which satisfy nonlinear integrable system of equations of evolutionary type. The functions A, B , and C depend on the dynamical variables p and q and their partial derivatives with respect to x .

Since the soliton connection \mathbf{a} has no curvature then matrices P and Q satisfy the zero curvature condition

$$P_{,t} - Q_{,x} + [P, Q] = 0,$$

which gives the well known AKNS equations [4]

$$\begin{aligned} A_x - pC + qB &= 0, \\ q_t + C_x - 2qA - 2\xi C &= 0, \\ p_t + B_x + 2pA + 2\xi B &= 0, \end{aligned}$$

We expand A , B and C as polynomials of the eigenvalue parameter ξ , i.e.,

$$A = \sum_{n=0}^N A_n \xi^{N-n}, \quad B = \sum_{n=0}^N B_n \xi^{N-n}, \quad C = \sum_{n=0}^N C_n \xi^{N-n},$$

where the functions A_n , B_n and C_n depend on the dynamical variables p and q and their partial derivatives with respect to x .

The AKNS System: Taking $N = 2, 3$ we obtain the Nonlinear Schrodinger and modified Korteweg de Vries systems respectively. This means that when we begin with the Lax pair in $sl(2, \mathbb{R})$ algebra and assume them as a polynomial of the spectral parameter of degree less or equal to three then we obtain the following system of evolution equations :

$$q_t = a_2 \left(-\frac{1}{2} q_{xx} + q^2 p \right) + a_3 \left(-\frac{1}{4} q_{xxx} + \frac{3}{2} qpq_x \right), \quad (1)$$

$$p_t = a_2 \left(\frac{1}{2} p_{xx} - qp^2 \right) + a_3 \left(-\frac{1}{4} p_{xxx} + \frac{3}{2} qpp_x \right). \quad (2)$$

Here a_2 and a_3 are arbitrary constants. For general N we have a recursion relation $u_t = \mathcal{R}^N u_x$ ($N = 2, 3 \dots$) where $u = (p, q)^T$ and \mathcal{R} is the recursion operator of the AKNS system. All the members of the AKNS system have infinitely many conserved quantities, infinitely many symmetries, bi-Hamiltonian structure, and admit soliton solution generating techniques etc.

2. Null tetrad approach

Newmann-Penrose formalism [5] in four dimensions is much elegant when written in $sl(2, \mathbb{C})$ algebra. It is known that the Einstein field equations admit prolonged Cartan frames [6] in this formalism. We start first with $SL(2, \mathbb{C})$ valued tetrad one form in four dimensions

$$\sigma = \begin{pmatrix} \mathbf{l} & \mathbf{m} \\ \bar{\mathbf{m}} & \mathbf{n} \end{pmatrix}, \quad (3)$$

where $\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}$ are null tetrad one forms. The metric takes the form

$$ds^2 = \mathbf{l} \oplus \mathbf{n} + \mathbf{n} \oplus \mathbf{l} + \mathbf{m} \oplus \bar{\mathbf{m}} + \bar{\mathbf{m}} \oplus \mathbf{m}. \quad (4)$$

Here a bar over a letter denotes complex conjugation. In our approach we will use the following $SL(2, \mathbb{C})$ tetrad one form

$$\tilde{\sigma} = \begin{pmatrix} \mathbf{n} & -\bar{\mathbf{m}} \\ -\mathbf{m} & \mathbf{l} \end{pmatrix} = \varepsilon \sigma \varepsilon^{-1}, \quad (5)$$

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (6)$$

Spin Connection: w

$$d\tilde{\sigma} = -w \tilde{\sigma} + \tilde{\sigma} w^\dagger, \quad (7)$$

where w is the $sl(2, \mathbb{C})$ valued connection one form given as

$$w = \begin{pmatrix} w_0 & w_2 \\ w_1 & -w_0 \end{pmatrix}, \quad (8)$$

where

$$w_0 = \gamma \mathbf{l} + \epsilon \mathbf{n} - \alpha \mathbf{m} - \beta \bar{\mathbf{m}}, \quad (9)$$

$$w_1 = -\tau \mathbf{l} + -\kappa \mathbf{n} + \rho \mathbf{m} + \sigma \bar{\mathbf{m}}, \quad (10)$$

$$w_2 = \nu \mathbf{l} + \pi \mathbf{n} - \lambda \mathbf{m} - \mu \bar{\mathbf{m}}, \quad (11)$$

where $\gamma, \epsilon, \alpha, \beta, \dots$ are the Newman-Penrose spin coefficients. In the construction of the metric tensor we don't need these spin coefficients. In the sequel of the text we may use these Greek letters for different purposes.

Curvature two form: $\mathbf{R} = dw + w w$

$$\mathbf{R} = \begin{pmatrix} R_0 & R_2 \\ R_1 & -R_0 \end{pmatrix}, \quad (12)$$

where

$$R_0 = (\Lambda - \Phi_{11} + \psi_2) \mathbf{l} \mathbf{n} + \psi_3 \mathbf{l} \mathbf{m} + \Phi_{12} \mathbf{n} \bar{\mathbf{m}} - \psi_1 \mathbf{n} \bar{\mathbf{m}} + (\psi_2 - \Phi_{11} - \Lambda) \mathbf{m} \bar{\mathbf{m}}, \quad (13)$$

$$R_1 = -(\psi_1 + \Phi_{01}) \mathbf{l} \mathbf{n} - (\psi_2 + 2\Lambda) \mathbf{l} \mathbf{m} - \Phi_{02} \mathbf{l} \bar{\mathbf{m}} + \Phi_{00} \mathbf{n} \mathbf{m} + \psi_0 \mathbf{n} \bar{\mathbf{m}} + (\Phi_{01} - \psi_1) \mathbf{m} \bar{\mathbf{m}} \quad (14)$$

$$R_2 = (\psi_3 + \Phi_{21}) \mathbf{l} \mathbf{n} - (\psi_2 + 2\Lambda) \mathbf{n} \bar{\mathbf{m}} - \Phi_{20} \mathbf{n} \mathbf{m} + \Phi_{22} \mathbf{l} \bar{\mathbf{m}} + \psi_4 \mathbf{l} \mathbf{m} - (\Phi_{21} - \psi_3) \mathbf{m} \bar{\mathbf{m}}, \quad (15)$$

where Φ_{ij} ($i, j = 0, 1, 2$) are the trace free Ricci spin coefficients, ψ_a ($a = 0, 1, 2, 3, 4$) are the Weyl spinors and $\Lambda = -R/6$, where R is the scalar curvature. Let $\Phi_{ij} = 0$ and $\psi_a = 0$ for all ($i, j = 0, 1, 2$) and for all ($a = 0, 1, 2, 3, 4$), then

$$\mathbf{R} = \Lambda \begin{pmatrix} \mathbf{l} \mathbf{n} - \mathbf{m} \bar{\mathbf{m}} & -2\mathbf{n} \bar{\mathbf{m}} \\ -2\mathbf{l} \mathbf{m} & -\mathbf{l} \mathbf{n} + \mathbf{m} \bar{\mathbf{m}} \end{pmatrix}. \quad (16)$$

Now let

$$\Gamma_{\pm} = w \pm \ell_0 \tilde{\sigma} \varepsilon, \quad (17)$$

where ℓ_0 is any constant. Curvatures Ω_{\pm} of the connection Γ_{\pm} are given by

$$\Omega_{\pm} = \mathbf{R} + \ell_0^2 \tilde{\sigma} \varepsilon \tilde{\sigma} \varepsilon \pm \ell_0 \tilde{\sigma} (w^\dagger \varepsilon + \varepsilon w). \quad (18)$$

To have $\Omega_{\pm} = 0$ we must have $\bar{w} = w$ and $\bar{\mathbf{m}} = \mathbf{m}$. In this case the group $SL(2, \mathbb{C})$ reduces to $SL(2, \mathbb{R})$ and dimension reduces to three and

$$\ell_0^2 = -\Lambda. \quad (19)$$

As a result we end up with a theorem saying that, the integrability of 4 dimensional gravity implies that null tetrad must be real and the spacetime dimension falls down to three and hence $SL(2, \mathbb{C})$ group is contracted to $SL(2, \mathbb{R})$. This is an Inönü-Wigner contraction [7].

3. Newman-Penrose Formalism in Three Dimensions

Following [8] $SL(2, \mathbb{R})$ valued tetrad one form is given by

$$\tilde{\sigma} = \begin{pmatrix} \mathbf{n} & -\frac{1}{\sqrt{2}} \mathbf{m} \\ -\frac{1}{\sqrt{2}} \mathbf{m} & \mathbf{1} \end{pmatrix} = \varepsilon \sigma \varepsilon^{-1} \quad (20)$$

Connection one form: w

$$d\tilde{\sigma} = -w \tilde{\sigma} + \tilde{\sigma} w^t, \quad (21)$$

where w is the $sl(2, \mathbb{R})$ valued connection one form given as

$$w = \begin{pmatrix} w_0 & w_2 \\ w_1 & -w_0 \end{pmatrix}, \quad (22)$$

where

$$w_0 = \frac{1}{2} (-\epsilon' \mathbf{1} + \epsilon \mathbf{n} - \alpha \mathbf{m}), \quad (23)$$

$$w_1 = \frac{1}{\sqrt{2}} (-\tau \mathbf{1} - \kappa \mathbf{n} + \sigma \mathbf{m}), \quad (24)$$

$$w_2 = \frac{1}{\sqrt{2}} (-\kappa' \mathbf{1} - \tau' \mathbf{n} + \sigma' \mathbf{m}), \quad (25)$$

where $\gamma, \epsilon, \alpha, \beta, \dots$ are the Newman-Penrose spin coefficients. Then

$$d\ell = -\epsilon \mathbf{l} \mathbf{n} + (\alpha - \tau) \mathbf{l} \mathbf{m} + \kappa' \mathbf{n} \mathbf{m}, \quad (26)$$

$$dn = \epsilon' \mathbf{l} \mathbf{n} - \kappa' \mathbf{l} \mathbf{m} - (\alpha + \tau') \mathbf{n} \mathbf{m}, \quad (27)$$

$$dm = (\tau' - \tau) \mathbf{l} \mathbf{n} - \sigma' \mathbf{l} \mathbf{m} - \sigma \mathbf{n} \mathbf{m}. \quad (28)$$

Curvature two form: $\mathbf{R} = dw + w w$

$$\mathbf{R} = \begin{pmatrix} R_0 & R_2 \\ R_1 & -R_0 \end{pmatrix}, \quad (29)$$

where

$$R_0 = (2\Phi_{11} + \Lambda) \mathbf{l} \mathbf{n} - \Phi_{12} \mathbf{l} \mathbf{m} + \Phi_{10} \mathbf{n} \mathbf{m},$$

$$R_1 = -\sqrt{2}\Phi_{01} \mathbf{l} \mathbf{n} + \frac{1}{\sqrt{2}}(\Phi_{02} - 2\Lambda) \mathbf{l} \mathbf{m} - \sqrt{2}\Phi_{00} \mathbf{n} \mathbf{m},$$

$$R_2 = \sqrt{2}\Phi_{12} \mathbf{l} \mathbf{n} - \sqrt{2}\Phi_{22} \mathbf{l} \mathbf{m} + \frac{1}{\sqrt{2}}(\Phi_{02} - 2\Lambda) \mathbf{n} \mathbf{m},$$

where Φ_{ij} ($i, j = 0, 1, 2$) are the trace free Ricci spin coefficients and $\Lambda = -R/6$. Let $\Phi_{ij} = 0$ for all ($i, j = 0, 1, 2$), then

$$\mathbf{R} = \Lambda \begin{pmatrix} \mathbf{ln} & -\sqrt{2} \mathbf{nm} \\ -\sqrt{2} \mathbf{lm} & -\mathbf{ln} \end{pmatrix}, \quad (30)$$

and

$$\tilde{\sigma} \varepsilon = \begin{pmatrix} \frac{1}{\sqrt{2}} \mathbf{m} & \mathbf{n} \\ -\mathbf{l} & -\frac{1}{\sqrt{2}} \mathbf{m} \end{pmatrix}. \quad (31)$$

Hence

$$\tilde{\sigma} \varepsilon \tilde{\sigma} \varepsilon = \begin{pmatrix} \mathbf{ln} & -\sqrt{2} \mathbf{nm} \\ -\sqrt{2} \mathbf{lm} & -\mathbf{ln} \end{pmatrix} \quad (32)$$

Now let

$$\Gamma_{\pm} = w \pm \lambda \tilde{\sigma} \varepsilon, \quad (33)$$

where λ is any constant. Curvatures $\Omega_{\pm} = d\Gamma_{\pm} + \Gamma_{\pm} \Gamma_{\pm}$ of the connections Γ_{\pm} are found as

$$\Omega_{\pm} = \mathbf{R} + \lambda^2 \tilde{\sigma} \varepsilon \tilde{\sigma} \varepsilon \pm \lambda \tilde{\sigma} (w^t \varepsilon + \varepsilon w) \quad (34)$$

Since w is an $\mathfrak{sl}(2, \mathbb{R})$ connection then $w^t \varepsilon + \varepsilon w = 0$. Taking $\lambda^2 = -\Lambda$. then

$$\mathbf{R} + \lambda^2 \tilde{\sigma} \varepsilon \tilde{\sigma} \varepsilon = 0$$

Hence both Γ_+ and Γ_- are zero curvature $\mathfrak{sl}(2, \mathbb{R})$ connections. Then three dimensional gravity is integrable and as a result we have the soliton $\mathfrak{sl}(2, \mathbb{R})$ connections

$$\Gamma_{\pm} = w \pm \lambda \tilde{\sigma} \varepsilon$$

are the zero curvature connections for three dimensional AdS spacetimes. Writing Γ_{\pm} in a suitable form for the integrable systems we have the tetrad one form

$$\tilde{\sigma} \varepsilon = \frac{1}{2\lambda} (\Gamma_+ - \Gamma_-) \quad (35)$$

Here in our approach the connections Γ_{\pm} are any zero curvature soliton connections, not coming from a Chern-Simons theory. Hence they do not depend on any boundary conditions as assumed by Cardenas et al [3]. Now we shall relate the flat gauge potential one forms Γ_{\pm} to the well known AKNS soliton connection. Let $\Gamma_+ = b_+^{-1} db_+ + b_+^{-1} \mathbf{a}^+ b_+$ where b_+ is nonsingular matrix and \mathbf{a}^+ is a soliton connection one form. Then $\Omega_{\pm} = d\Gamma_{\pm} + \Gamma_{\pm} \Gamma_{\pm} = 0$ implies that \mathbf{a}^+ is also a flat connection

$$d\mathbf{a}^+ + \mathbf{a}^+ \mathbf{a}^+ = 0$$

Similarly let $\Gamma_- = b_-^{-1} db_- + b_-^{-1} \mathbf{a}^- b_-$ where b_- is another nonsingular matrix and

$$d\mathbf{a}^- + \mathbf{a}^- \mathbf{a}^- = 0.$$

Choosing

$$\Gamma_{\pm} = \begin{pmatrix} X_{\pm} & Y_{\pm} \\ Z_{\pm} & -X_{\pm} \end{pmatrix} \quad (36)$$

where X_{\pm} , Y_{\pm} and Z_{\pm} are one forms. Then from (35) we find that

$$\mathbf{m} = \frac{1}{\sqrt{2\lambda}} (X_+ - X_-), \quad (37)$$

$$\mathbf{n} = \frac{1}{2\lambda} (Y_+ - Y_-), \quad (38)$$

$$\mathbf{l} = -\frac{1}{2\lambda} (Z_+ - Z_-), \quad (39)$$

4. Soliton Connection and the Metric

Let $x^{\mu} = (t, x, y)$ be the local coordinates of (M, g) and the matrices b_{\pm} depend on all coordinates and the soliton connections a^{\pm} depend on the coordinates t and x . Hence

$$\mathbf{a}^{\pm} = P^{\pm} dx + Q^{\pm} dt$$

where $P(t, x)$ and $Q(t, x)$ are some $sL(2, R)$ matrices. They are given by

$$P^{\pm} = \begin{pmatrix} 2\xi^{\pm} & p^{\pm}(t, x) \\ q^{\pm}(t, x) & -2\xi^{\pm} \end{pmatrix}, Q^{\pm} = \begin{pmatrix} A^{\pm} & B^{\pm} \\ C^{\pm} & -A^{\pm} \end{pmatrix}.$$

Here ξ^{\pm} are the eigenvalue parameters and p^{\pm} and q^{\pm} are dynamical variables which satisfy nonlinear integrable system of equations of evolutionary type. The functions A^{\pm} , B^{\pm} and C^{\pm} depend on the dynamical variables p^{\pm} and q^{\pm} and their partial derivatives with respect to x . Choosing the b matrices as

$$b = \begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}, \quad (40)$$

with unit determinant $\alpha\sigma - \beta\gamma = 1$. Functions α , β , γ and σ depend on all coordinates t, x, y . For each zero curvature connection Γ_+ and Γ_- we use different gauge matrices as b_{\pm} . Then we find the one forms X , Y and Z (ignoring \pm subscripts) as

$$X = [2(\alpha\sigma + \beta\gamma)\xi + \sigma\gamma p - \alpha\beta q] dx + [(\alpha\sigma + \beta\gamma)A + \sigma\gamma B - \alpha\beta C] dt + \sigma d\alpha - \beta d\gamma,$$

$$Y = [4\sigma\beta\xi + \sigma^2 p - \beta^2 q] dx + [2\sigma\beta A + \sigma^2 B - \beta^2 C] dt + \sigma d\beta - \beta d\sigma,$$

$$Z = [-4\xi\gamma\alpha - \gamma^2 p + \alpha^2 q] dx + [-2\alpha\gamma A - \gamma^2 B + \alpha^2 C] dt - \gamma d\alpha + \alpha d\gamma.$$

Finally by using the above expressions we obtain the tetrad one forms \mathbf{l} , \mathbf{n} , \mathbf{m} from (37)-(39) and the metric tensor is given by

$$g_{\mu\nu} = l_{\mu} n_{\nu} + l_{\nu} n_{\mu} - m_{\mu} m_{\nu}. \quad (41)$$

The only gravitational field equations are the following six AKNS equations

$$A_x^{\pm} - p^{\pm} C^{\pm} + q^{\pm} B^{\pm} = 0, \quad (42)$$

$$q_t^{\pm} + C_x^{\pm} - 2q^{\pm} A^{\pm} - 2\xi^{\pm} C^{\pm} = 0, \quad (43)$$

$$p_t^{\pm} + B_x^{\pm} + 2p^{\pm} A^{\pm} + 2\xi^{\pm} B^{\pm} = 0. \quad (44)$$

5. Connections with Non-vanishing Curvatures

Let Γ_+ and Γ_- be two $\mathfrak{sl}(2, \mathbb{R})$ valued connection one forms with curvatures Ω_+ and Ω_- respectively. The three dimensional spacetime geometry has an $\mathrm{SL}(2, \mathbb{R})$ valued null tetrad one form $\tilde{\sigma}$ and $\mathfrak{sl}(2, \mathbb{R})$ valued connection one form w given in terms of the connections Γ_+ and Γ_- as follows

$$\tilde{\sigma} = \frac{1}{2\lambda}(\Gamma_+ - \Gamma_-)\varepsilon, \quad w = \frac{1}{2}(\Gamma_+ + \Gamma_-), \quad (45)$$

where the matrix ε is defined in (6) and λ is a constant. Then we find that

$$d\tilde{\sigma} + w\tilde{\sigma} - \tilde{\sigma}w^t = T, \quad (46)$$

where T is the $\mathfrak{sl}(2, \mathbb{R})$ valued torsion two form and found as $T = \Omega_+ - \Omega_-$. The curvature two form \mathbf{R} of the spacetime geometry is found as

$$\mathbf{R} + \lambda^2 \tilde{\sigma}\varepsilon\tilde{\sigma}\varepsilon = \frac{1}{2}(\Omega_+ + \Omega_-). \quad (47)$$

The last two equations (46) and (47) imply a 3 dimensional gravity with torsion and nonzero matter. When we let torsion to vanish then the connections Γ_+ and Γ_- have the same curvature, i.e., $\Omega_+ = \Omega_-$. Let us call this curvature as Ω . Hence the associated three dimensional gravity field equations are

$$d\tilde{\sigma} + w\tilde{\sigma} - \tilde{\sigma}w^t = 0, \quad (48)$$

$$\mathbf{R} + \lambda^2 \tilde{\sigma}\varepsilon\tilde{\sigma}\varepsilon = \Omega. \quad (49)$$

By choosing connections Γ_+ and Γ_- properly the above field equations may correspond to a well defined non-vacuum case. We may consider the equations (45) as Backlund transformations from vacuum to non-vacuum solutions of the Einstein field equations.

6. Conclusion

Using the null tetrad formalism in three dimensions we showed the the vacuum equations with cosmological constant are integrable. This means that we determine the tetrad one forms in terms of the curvature free connection one forms. Choosing the zero curvature connection one forms (soliton connections) properly we find the metric of the spacetime in terms of the variables of the well known AKNS system including the nonlinear Schrodinger, KdV and Modified KdV equations. In our formalism the gauge matrices b_{\pm} are taken independent and depend on all coordinates t, x, y . In [3] these matrices are assumed to be the inverses of each other and depend only on one coordinate. Some of these functions can be eliminated by using the $\mathrm{SL}(2, \mathbb{R})$ gauge transformation but we keep them for having the metric with maximum number of free functions. Since we have derived the tetrad one form $\tilde{\sigma}$ in terms of the zero curvature connections Γ_{\pm} without referring to the Chern-Simons theory we do not necessarily need any boundary conditions to be satisfied by the tetrad functions.

We have some number of comments and questions to be answered. One of them is the infinite number of symmetries and conservation laws of the AKNS system. How these be perceived in the three dimensional gravity is unclear. Another point is the recursion operator of the AKNS system. By using the recursion operator of AKNS system one can go from one system to another one. How the corresponding three dimensional metrics are related is another puzzle. Finally, how we should interpret solitons and soliton collisions in three dimensional gravity needs further efforts.

In the last part of this paper we have considered the connections with non-vanishing curvatures and discussed possibility of obtaining non-vacuum solutions.

7. Acknowledgment

I would like to thank Bayram Tekin and Tahsin Çağrı Şişman for their kind interest in this work.

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