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800 conics on a smooth quartic surface

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ABSTRACT

We construct an example of a smooth spatial quartic surface that contains 800 irreducible conics.

1. Introduction

This short note was motivated by Barth, Bauer [1], Bauer [2], and my recent paper [6]. Generalizing [2], define $N_{2n}(d)$ as the maximal number of smooth rational curves of degree d that can lie on a smooth degree $2n$ $K3$-surface $X \subset \mathbb{P}^{n+1}$. (All algebraic varieties considered in this note are over $\mathbb{C}$.) The bounds $N_{2n}(1)$ have a long history and currently are well known, whereas for $d = 2$ the only known value is $N_6(2) = 285$ (see [6]). In the most classical case $2n = 4$ (spatial quartics), the best known examples have 352 or 432 conics (see [1,2]), whereas the best known upper bound is 5016 (see [2], with a reference to S.A. Strømme).

For $d = 1$, the extremal configurations (for various values of n) tend to exhibit similar behaviour. Hence, contemplating the findings of [6], one may speculate that

- it is easier to count *all* conics, both irreducible and reducible, but
- nevertheless, in extremal configurations all conics are irreducible.

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On the other hand, famous *Schur's quartic* (the one on which the maximum $N_4(1)$ is attained) has 720 conics (mostly reducible), suggesting that 432 should be far from the maximum $N_4(2)$. Therefore, in this note I suggest a very simple (although also implicit) construction of a smooth quartic with 800 irreducible conics.

Theorem 1.1 *(see §3.3). There exists a smooth quartic surface $X_4 \subset \mathbb{P}^3$ containing 800 irreducible conics.*

The quartic X_4 is Kummer in the sense of [1,2]: it contains 16 disjoint conics. I conjecture that $N_4(2) = 800$ and, moreover, 800 is the sharp upper bound on the total number of conics (irreducible or reducible) on a smooth spatial quartic. This conjecture is substantiated by the following two facts:

- the configuration of conics on X_4 is the maximum that can be embedded, *via* the construction of [6] (see §3.1 *vs.* §3.2), to the Leech lattice;
- 800 is the maximal number of conics in the Barth–Bauer family (see [1]) of Kummer quartics (see [5] and Remark 1.2).

Remark 1.2. There has been a considerable development after this note appeared in the `arXiv`. X. Roulleau observed that, computing the projective automorphism group and using [3], X_4 in Theorem 1.1 must be given by the Mukai polynomial

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 + 12z_0z_1z_2z_3 = 0,$$

even though only 320 conics were found in [3]. Then, B. Naskręcki [8] found explicit equations of all 800 conics. Finally, upon observing that X_4 has 16 pairwise disjoint conics, the same quartic was constructed in [5] within the Barth–Bauer family.

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2. The Leech lattice (see [4])

2.1. The Golay code

The (*extended binary*) *Golay code* is the only binary code of length 24, dimension 12, and minimal Hamming distance 8. We regard codewords as subsets of $\Omega := \{1, \dots, 24\}$ and denote this collection of subsets by $\mathcal{C}$; clearly, $|\mathcal{C}| = 2^{12}$. The code $\mathcal{C}$ is invariant under the complement $o \mapsto \Omega \smallsetminus o$. Apart from $\varnothing$ and Ω itself, it consists of 759 *octads* (codewords of length 8), 759 complements thereof, and 2576 *dodecads* (codewords of length 12).

The setwise stabilizer of $\mathcal{C}$ in the full symmetric group $\mathbb{S}(\Omega)$ is the Mathieu group M_{24} of order 244823040; the actions of this group on Ω and $\mathcal{C}$ are described in detail in § 2 of [4, Chapter 10].

2.2. The square 4 vectors

The *Leech lattice* is the only root-free unimodular even positive definite lattice of rank 24. For the construction, consider the standard Euclidean lattice $E := \bigoplus_i \mathbb{Z}e_i$, $i \in \Omega$, and divide the form by 8, so that $e_i^2 = 1/8$. (Thus, we avoid the factor $8^{-1/2}$ appearing throughout in [4].) Then, the Leech lattice is the sublattice $\Lambda \subset E$ spanned over $\mathbb{Z}$ by the square 4 vectors of the form

$$(\mp 3, \pm 1^{23}) \qquad \text{(the upper signs are taken on a codeword } o \in \mathcal{C}\text{).} \tag{2.1}$$

(We use the notation of [4]: $a^m, b^n, \ldots$ means that there are m coordinates equal to a, n coordinates equal to b, *etc.*) Apart from (2.1), the square 4 vectors in Λ are

$$(\pm 2^8, 0^{16}) \qquad (\pm 2 \text{ are taken on an octad, the number of } + \text{ is even}), \text{ or} \tag{2.2}$$

$$(\pm 4^2, 0^{22}) \qquad \text{(no further restrictions).} \tag{2.3}$$

Altogether, there are 196560 square 4 vectors: $24 \cdot |\mathcal{C}| = 98304$ vectors as in (2.1), $2^7 \cdot 759 = 97152$ vectors as in (2.2), and $2^2 \cdot C(24, 2) = 1104$ vectors as in (2.3).

3. The construction

In this section, we prove Theorem 1.1.

3.1. The lattice S

Consider the lattice $V := \mathbb{Z}\hbar + \mathbb{Z}a + \mathbb{Z}u_1 + \mathbb{Z}u_2 + \mathbb{Z}u_3$ with the Gram matrix

$$\begin{bmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 \\ 0 & 2 & 4 & 2 & -1 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 1 & -1 & 0 & 4 \end{bmatrix}.$$

It can be shown that, up to $O(\Lambda)$, there is a unique primitive isometric embedding $V \to \Lambda$; however, for our example, we merely choose a particular model. Fix an ordered quintuple $Q := (1, 2, 3, 4, 5) \subset \Omega$ and choose one of the four octads O such that $O \cap Q = \{1, 2, 4, 5\}$ (*cf. sextets* in §2.5 of [4, Chapter 10]); upon reordering Ω, we can assume that $O = \{1, 2, 4, 5, 6, 7, 8, 9\}$ (the underlined positions in the top row of Table 1). Then, the generators of V can be chosen as shown in the upper part of Table 1. (For better readability, we represent zeros by dots; all components beyond $\bar{O} := Q \cup O$ are zeros.)

The choice of Q and O is unique up to M_{24}; furthermore, the subgroup $G \subset M_{24}$ stabilising Q pointwise and O as a set can be identified with the alternating group $\mathbb{A}(O \smallsetminus Q)$; in particular, it acts simply transitively on the set of ordered pairs

Table 1
The lattice V and the conics.

#	$\underline{1}$	$\underline{2}$	3	$\underline{4}$	$\underline{5}$	$\underline{6}$	$\underline{7}$	$\underline{8}$	$\underline{9}$	
$\hbar$	4	4	·	·	·	·	·	·	·	
a	·	4	4	·	·	·	·	·	·	
u_1	·	·	4	4	·	·	·	·	·	
u_2	·	·	·	4	4	·	·	·	·	
u_3	−2	2	·	−2	2	2	2	2	2	
1	1	3	−1	1	−1	1	1	−1*	−1*	$\pm 1^{15}$
2	3	1	1	−1	1	1	1	−1*	−1*	$\pm 1^{15}$
3	2	2	·	·	·	·	·	·	·	$\pm 2^6, 0^9$
4	2	2	·	·	·	·	·	2*	−2*	$\pm 2^4, 0^{11}$
	fixed $= Q$					movable in $O \smallsetminus Q$				

Table 2
The number of conics in S.

1:	$C(4,2)\cdot\underline{16} = 96$	(codewords $o\in\mathcal{C}$ such that $o\cap\bar{O}=\{2,3,5,p,q\}$),
2:	$C(4,2)\cdot\underline{16} = 96$	(codewords $o\in\mathcal{C}$ such that $o\cap\bar{O}=\{1,4,p,q\}$),
3:	$2^5\cdot\underline{10} = 320$	(octads $o\in\mathcal{C}$ such that $o\cap\bar{O}=\{1,2\}$),
4:	$2^3\cdot P(4,2)\cdot\underline{3} = 288$	(octads $o\in\mathcal{C}$ such that $o\cap\bar{O}=\{1,2,p,q\}$).

$$(p,q)\colon \quad p,q\in O\smallsetminus Q=\{6,7,8,9\}, \quad p\neq q. \tag{3.1}$$

Define a *conic* as a square 4 vector $l\in\Lambda$ such that

$$l\cdot\hbar=2, \quad l\cdot a=1, \quad l\cdot u_1=l\cdot u_2=l\cdot u_3=0.$$

This strange condition can be recast as follows: $l\cdot\hbar=2$ and l (as well as $\hbar$) lies in the rank 20 lattice

$$S:=\bar{V}^\perp\subset\Lambda, \quad \text{where } \bar{V}:=\hbar_V^\perp \text{ (the orthogonal complement in } V).$$

Using §2.2, we conclude that each conic fits one of the four patterns shown at the bottom of Table 1: there are two for (2.1) and two for (2.2). (If l is as in (2.3), we have $l\cdot a=0 \bmod 2$.) The number of conics within each pattern is computed as shown in Table 2, where

- the ordered or unordered pair (p,q) as in (3.1) designates the two variable special positions marked with a * in Table 1,
- the underlined factor counts certain codewords $o\in\mathcal{C}$; the restrictions given by (2.1) or (2.2) are described in the parentheses, and
- the other factors account for the choice of (p,q) and/or signs in ± 2.

These counts sum up to 800.

3.2. The Néron–Severi lattice

Observe that $\hbar\in 2S^\vee$: indeed, $\hbar-2a\in\bar{V}$ and we have $x\cdot\hbar=2x\cdot a=0 \bmod 2$ for any $x\in S$. Thus, we can apply to $S\ni\hbar$ the construction of [6], *i.e.*, consider the orthogonal complement $\hbar_S^\perp=V^\perp\subset\Lambda$, reverse the sign of the form, and pass to the index 2 extension

$$N:=\bigl(-(\hbar_S^\perp)\oplus\mathbb{Z}h\bigr)_2^\sim, \quad h^2=4,$$

containing the vector $c:=c(l):=l-\frac{1}{2}\hbar+\frac{1}{2}h$ for some (equivalently, any) conic $l\in S$. These 800 new vectors $c\in N$ are also called *conics*; one obviously has

$$c^2=-2 \quad\text{and}\quad c\cdot h=2. \tag{3.2}$$

Starting from

$$\operatorname{discr} V\cong\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}\oplus\begin{bmatrix}\frac{1}{8}\end{bmatrix}\oplus\begin{bmatrix}\frac{2}{5}\end{bmatrix}$$

(see Nikulin [9] for the concept of *discriminant form* $\operatorname{discr} V := V^\vee/V$ and related techniques), we easily compute

$$\mathcal{N} := \operatorname{discr} N \cong \left[\tfrac{5}{4}\right] \oplus \left[\tfrac{1}{8}\right] \oplus \left[\tfrac{2}{5}\right] \cong \left[-\tfrac{1}{4}\right] \oplus \left[-\tfrac{5}{8}\right] \oplus \left[\tfrac{2}{5}\right].$$

Therefore, $-\mathcal{N} \cong \operatorname{discr} T$, where $T := \mathbb{Z}b \oplus \mathbb{Z}c$, $b^2 = 4$, $c^2 = 40$. Then, it follows from [9, Thm 1.12.2] that there is a primitive isometric embedding of the hyperbolic lattice N to the intersection lattice H_2 of a $K3$-surface, so that $T \cong N^\perp$ plays the rôle of the transcendental lattice. Finally, by the surjectivity of the period map [7], we conclude that there exists a $K3$-surface X with $NS(X) \cong N$.

3.3. Proof of Theorem 1.1

The Néron–Severi lattice $NS(X) \cong N$ constructed in the previous section is equipped with a distinguished polarisation $h \in N$, $h^2 = 4$. Since the original lattice $S \subset \Lambda$ is root free, N does *not* contain any of the following "bad" vectors:

- $e \in N$ such that $e^2 = -2$ and $e \cdot h = 0$ (*exceptional divisors*) or
- $e \in N$ such that $e^2 = 0$ and $e \cdot h = 2$ (2-*isotropic vectors*)

(see [6] for details). Hence, by Nikulin [10] and Saint-Donat [11], the linear system $|h|$ is fixed point free and maps X onto a smooth quartic surface $X_4 \subset \mathbb{P}^3$.

The lattice N contains 800 conics c as in (3.2). By the Riemann–Roch theorem, each class c is effective, *i.e.*, represented by a curve $C \subset X_4$ of projective degree 2. Since X is smooth and contains no lines (or other curves of odd degree, as we have $h \in 2N^\vee$ by the construction), each of these curves C is irreducible. This concludes the proof of Theorem 1.1. □

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