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# Statistical arbitrage in jump-diffusion models with compound Poisson processes

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**Abstract**

We prove the existence of statistical arbitrage opportunities for jump-diffusion models of stock prices when the jump-size distribution is assumed to have finite moments. We show that to obtain statistical arbitrage, the risky asset holding must go to zero in time. Existence of statistical arbitrage is demonstrated via 'buy-and-hold until barrier' and 'short until barrier' strategies with both single and double barrier. In order to exploit statistical arbitrage opportunities, the investor needs to have a good approximation of the physical probability measure and the drift of the stochastic process for a given asset.

**Keywords** Statistical arbitrage · Jump-diffusion model · Compound Poisson process · Monte Carlo simulation

**JEL Classification** C60 · G11 · G12

## 1 Introduction

Statistical arbitrage strategies are quantitative strategies that focus on investing in securities, both long and short positions, centered on a mathematical or statistical algorithm that identifies a co-moving relationship between the two securities (Naccarato et al. 2019). These strategies exploit systematic relationships among equity securities with similar characteristics. Typically, univariate statistical pairs trading strategies are structured as follows: In a formation period, two securities are identified whose prices have moved together historically.

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We say these securities have an equilibrium relationship which is identified by a statistical model. Next the spread of their price movements are monitored in a subsequent trading period. If the prices diverge and the spread widens between the pair, a trade is placed where the better performing stock is sold short and the worst performing stock is purchased. Because of this assumed equilibrium relationship between the two securities, the expectation is that the spread will revert to its historical mean, resulting in a profit.[1] More specifically, the profit from a statistical arbitrage strategy is a riskless profit opportunity in the limit as the number of trades, assets or time goes to infinity.

Since traders have at their disposal sophisticated statistical models to quickly identify and eliminate regular arbitrage opportunities, more research is needed to uncover ways to generate statistical arbitrage profits.[2] The fundamental issue of identifying high alpha stocks is still at the core of the problem. However, assuming that a trader is able to identify those high or low performing stocks, two problems remain. The first problem faced by the trader is establishing the conditions that guarantee the existence of statistical arbitrage opportunities for the two candidate stocks. The trader's second problem is to identify the type of trading strategies that allow or do not allow for statistical arbitrage profits. In this paper, we address these two problems in terms of the mathematical definition of statistical arbitrage proposed in the literature, assuming that the trader can identify high or low alpha generating stocks.

To offer a solution to the two problems, in this paper we apply the mathematical definition of statistical arbitrage provided by Hogan et al. (2004) and present a solution when such opportunities arise under a general class of stock price models (i.e. jump-diffusion models with compound Poisson processes). Göncü (2015) derives the statistical arbitrage condition for the Black–Scholes model framework. In this paper, we derive the condition for the existence of statistical arbitrage opportunities for a widely used class of jump-diffusion models with a compound Poisson process.

Our motivation for using jump-diffusion models is because of the well-known limitations of the Black–Scholes framework in modelling stock price dynamics. The assumption of normality is not supported by numerous empirical evidence. For example, the leptokurtic feature of the stock return distributions is one of the stylized facts identified by Cont (2001). Another empirical feature is the "volatility smile" observed for implied volatility obtained from option prices. This observed behavior is inconsistent with the assumption of constant volatility assumed in the Black–Scholes framework. Jump-diffusion models provide a better fit to these empirical stylized facts compared to the Black–Scholes model. Therefore, it is important to consider the more general case of jump-diffusion models and prove the existence of statistical arbitrage opportunities within this class of asset pricing models.

Jump-diffusion models fall within the class of Lévy processes with finite frequency of jumps. In his seminal paper, Merton (1976) provides one of the earliest attempts to extend the Black–Scholes model to include jumps in the diffusion process. His formula for pricing a European call option on a single stock includes a Poisson jump component in addition to the usual diffusion part. However, Merton's model has been the target of several criticisms because of the assumption that jump risk is not systematic. Later studies documented that jump risk is indeed systematic (e.g. see Bjork and Naslund (1998) for details). Various jump-diffusion models within the framework proposed by Merton have been suggested in the

[1] The straightforward concept of univariate pairs trading is often extended into more sophisticated strategies. We could implement a strategy in a quasi-multivariate framework where one security is traded against a weighted portfolio of co-moving securities. Another strategy involves trading groups of stocks against other groups of stocks that co-move together.

[2] This not only applies to the arbitrage strategies, but to more general investment problems as well. For recent research, see Ben Saida and Prigent (2018) and Amedee-Manasme et al. (2019).

literature in the last few decades.[3] Among them, the model proposed by Kou (2002) with the double exponential jump-size distribution deserves special attention. This is because Kou's model is able to generate closed-form solutions not only for standard call and put options, but also for exotic options,[4] as well as interest rate options including caps and floors.[5] Moreover, Kou's model is flexible enough to model upward and downward jumps in asset prices with different parameters. From an econometric viewpoint, Ramezani and Zeng (2004) show that the double exponential jump-diffusion model is more compatible with empirical data than Merton's jump-diffusion model and fits the data better than the Black–Scholes model.[6]

In the studies by Bertram (2009, 2010), optimal statistical arbitrage trading is analyzed via the first passage time distribution of diffusion processes. With the exception of the recent study by Göncü (2015), the conditions that guarantee the existence of statistical arbitrage opportunities with respect to a mathematical definition of statistical arbitrage have been ignored in the literature. A common approach has been to consider statistical arbitrage within the pairs trading framework (e.g. see Kestner 2003; Vidyamurthy 2004; Elliott et al. 2005). However, in this study we provide the first analysis of the existence of statistical arbitrage opportunities in the framework of Lévy processes. We do this by building upon the framework of Göncü (2015).

The rest of the paper is organized as follows. In Sect. 2, we provide the definition of statistical arbitrage that we use in this paper. In Sect. 3, we present the jump-diffusion models in the case of buy-and-hold strategy. Our results for the first passage time of stock prices applied to the barrier level in jump-diffusion models are reported and discussed in Sect. 4. In Sect. 5, we prove the existence of statistical arbitrage opportunities in jump-diffusion models. Section 6 summarizes our findings.

## 2 Mathematical definition of statistical arbitrage

In this section, we state the definition of statistical arbitrage used in this paper following Hogan et al. (2004). Given the stochastic process for the discounted cumulative trading profits, which we denote by $\{v(t) : t \geq 0\}$ and define on a probability space $(\Omega, \mathcal{F}, P)$, statistical arbitrage is defined as follows.

**Definition 1** A statistical arbitrage is a zero initial cost, self-financing trading strategy $\{v(t) : t \geq 0\}$ with cumulative discounted value $v(t)$ such that

1. $v(0) = 0$
2. $\lim_{t\to\infty} E[v(t)] > 0$,
3. $\lim_{t\to\infty} P(v(t) < 0) = 0$, and
4. $\lim_{t\to\infty} \frac{var(v(t))}{t} = 0$ if $P(v(t) < 0) > 0, \quad \forall t < \infty$.

A standard arbitrage opportunity can be considered as a special case of statistical arbitrage. In a standard arbitrage opportunity, we have a positive probability of profit while the probability of loss is zero. Hence, any profit obtained from an arbitrage opportunity can be deposited in a riskless money market account for the rest of the infinite time horizon, which also satisfies the definition of statistical arbitrage above.

[3] For a recent application, see Perera et al. (2018).

[4] See Kou and Wang (2003), Kou and Wang (2004) and Kou et al. (2005).

[5] See Glasserman and Kou (2011).

[6] For a further discussion of jump-diffusion processes, see Tankov and Cont (2004) and Singleton (2009).

## 3 Jump-diffusion models for asset prices

We consider a general class of jump-diffusion models, which is a Lévy process with compound Poisson jumps. Our results apply to jump-diffusion models with compound Poisson processes for any jump-size distribution with finite moments. Within this class of models, we consider Merton (1976) and Kou (2002) jump-diffusion models as examples of special cases. Merton assumes that the asset price jumps are independently and identically normally distributed, whereas Kou allows possible asymmetries in positive and negative jumps and assumes a double exponential distribution for modelling jumps. For both models there exists a change of measure that allows us to switch between the physical and risk-neutral stock price dynamics, which reflects itself as a condition on the drift term. However, primarily we are interested in the population parameters of the stock price process and conditions on these parameters to analyze the existence of statistical arbitrage strategies.

We assume that the stock price dynamics is given by

$$S_t = S_0 \exp\left((\alpha - \sigma^2/2)dt + \sigma dW_t + \sum_{i=1}^{N(t)} Y_i\right) \tag{1}$$

where $W_t$ is a standard Brownian motion process, $\sigma$ is the stock's price volatility, $\alpha$ is the stock's expected growth rate, and $\sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process. To obtain the risk-free growth of the stock price process one would set $\alpha + \lambda k$ equal to the risk-free rate, where $k = E[\exp(Y)] - 1$. By doing so, we have $k = \exp(\mu_y + \sigma_y^2/2) - 1$ and $k = p\frac{\eta_1}{\eta_1 - 1} + q\frac{\eta_2}{\eta_2 + 1} - 1$ ($\eta_1 > 1, \eta_2 > 0$) in the Merton (1976) and Kou (2002) models, respectively.

As a technical condition for the finite first passage time property of the jump-diffusion model, we assume that the jumps of the compound Poisson process satisfy the finiteness condition $E[e^{tY}] < \infty$ for any $t > 0$.

If we let $y_i = \exp(Y_i)$, we can re-write Eq. (1) as

$$S_t = S_0[\exp((\alpha - \sigma^2/2 - r_f)t + \sigma W_t)(\prod_{i=1}^{N(t)} y_i) - 1] \tag{2}$$

We consider two special cases for the distribution of independent and identically distributed (i.i.d.) jumps sizes, i.e. $Y_1, ..., Y_{N(t)}$. In the Merton jump-diffusion model, the jumps are assumed to be normally distributed with the probability density function $\phi(.)$ given by

$$\phi_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-(y-\mu_y)^2/2\sigma_y^2}, \quad -\infty < y < \infty, \tag{3}$$

where $\mu_y$ is the mean and $\sigma_y$ is the standard deviation for the jump-sizes.

Kou's jump-diffusion model is another special case of the class of jump-diffusion models that we consider. Kou gives the jump-size distribution $\phi(.)$ as

$$\phi_Y(y) = p\eta_1 e^{-\eta_1 y}\mathbf{1}_{y \geq 0} + q\eta_2 e^{-\eta_2 y}\mathbf{1}_{y<0}, \quad -\infty < y < \infty, \tag{4}$$

where $p, q \geq 0$, $p + q = 1$, $\eta_1 > 1$, and $\eta_2 > 0$.

### 3.1 Buy-and-hold strategy

Considering the class of jump-diffusion models given by Eq. (1), we analyze the expected trading profits from the buy-and-hold strategy in the following proposition.

**Proposition 2** *Assuming that a given risky asset follows the jump-diffusion model as in Eq. (1), expected discounted trading profits from the buy-and-hold strategy goes to infinity, i.e.* $\lim_{t\to\infty} E[v(t)] = \infty$, *for* $\alpha + \lambda k > r_f$.

***Proof*** We borrow $S_0$ at the risk-free rate $r_f$ at time 0 and invest in one unit of the risky asset. The value of the discounted cumulative trading profits at time $t$ is given by

$$v(t) = S_0[\exp((\alpha - \sigma^2/2 - r_f)t + \sigma W_t + \sum_{i=1}^{N(t)} Y_i) - 1]. \tag{5}$$

Since $E[\exp(u \sum_{i=1}^{N(t)} Y_i)] = \exp\left(\lambda t(E[e^{uY_i}] - 1)\right)$ where $E[e^{uY_i}] = \exp(u\mu_y + \frac{u^2\sigma_y^2}{2})$ and $W_t \sim N(0, t)$ for each $t$, we conclude that

$$E[v(t)] = S_0\left[exp((\alpha - r_f)t)exp(\lambda t(E[e^Y] - 1)) - 1\right] \tag{6}$$

$$= S_0\left[exp((\alpha - r_f + \lambda(E[e^Y] - 1))t) - 1\right] \tag{7}$$

$$= S_0\left[exp((\alpha - r_f + \lambda k)t) - 1\right] \tag{8}$$

Hence, we obtain $\lim_{t\to\infty} E[v(t)] = \infty$ iff $\alpha + \lambda k - r_f > 0$. □

***Remark 3*** Merton (1976) assumes that $Y$ is normally distributed with mean $\mu_y$ and variance $\sigma_y^2$. The value of $k$ is given by $E[e^Y] - 1 = \exp(\mu_y + \sigma_y^2/2) - 1$, and the result follows as $\lim_{t\to\infty} E[v(t)] = \infty$ iff $\alpha + \lambda[e^{\mu_y+\sigma_y^2/2} - 1] - r_f > 0$. In the jump-diffusion model proposed by Kou (2002), we have $k = E[e^Y] - 1 = \left(p\frac{\eta_1}{\eta_1-1} + q\frac{\eta_2}{\eta_2+1}\right) - 1$ for $\eta_1 > 1$ and $\eta_2 > 0$.

In the next proposition, we show that the condition of positive expected profits does not yield statistical arbitrage since the variance goes to infinity under the same condition.

**Proposition 4** *Assuming that a given risky asset follows the jump-diffusion model as given by Eq. (1), variance* $var(v(t))$ *and the time-averaged variance* $var(v(t))/t$ *of the discounted trading profits from the buy-and-hold strategy goes to infinity for* $\alpha - r_f + \lambda k > 0$. *Therefore, this strategy does not yield statistical arbitrage opportunities in the jump-diffusion models.*

***Proof***

$$E[v^2(t)] = S_0^2 - 2S_0 exp((\alpha - r_f)t)exp(\lambda t(E[e^Y] - 1)) \tag{9}$$

$$+S_0^2\left[exp\left((2\alpha + \sigma^2 - 2r_f)t + \lambda t(E[e^{2Y}] - 1)\right)\right]$$

$$= S_0^2 - 2S_0 exp((\alpha - r_f + \lambda k)t) \tag{10}$$

$$+S_0^2\left[exp\left((2\alpha + \sigma^2 - 2r_f + \lambda(E[e^{2Y}] - 1))t\right)\right] \tag{11}$$

$$E[v(t)]^2 = S_0^2 exp(2t(\alpha - r_f + \lambda k)) - 2S_0 exp(t(\alpha - r_f + \lambda k)) + S_0^2 \tag{12}$$

$$var(v(t)) = S_0^2\left[\exp((2\alpha - 2r_f + \sigma^2 + \lambda(E[e^{2Y}] - 1))t) - \exp(2(\alpha - r_f + \lambda k)t)\right], \tag{13}$$

$$= S_0^2 \exp(2(\alpha - r_f + \lambda k))\left[\exp((\sigma^2 + \lambda(E[e^{2Y}] - E[2e^Y] + 1))t) - 1\right], \tag{14}$$

$$= S_0^2 \exp(2(\alpha - r_f + \lambda k))\left[\exp((\sigma^2 + \lambda(E[(e^Y - 1)^2]))t) - 1\right] \geq 0, \tag{15}$$

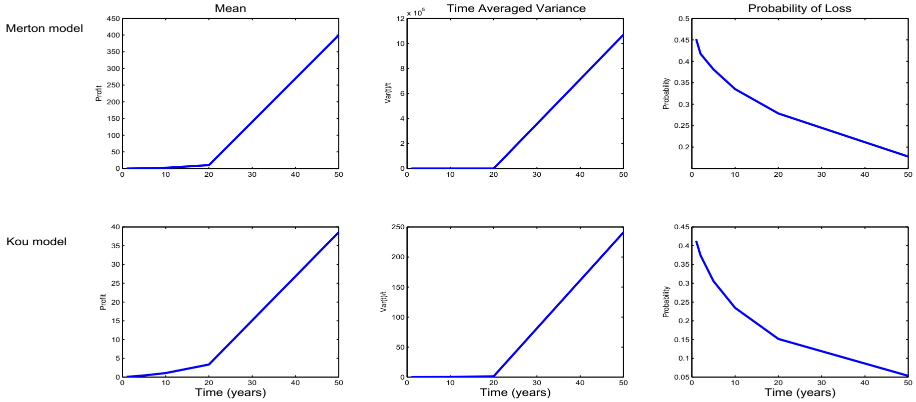


**Fig. 1** Evolution of mean, time averaged variance and probability of loss for the buy-and-hold strategy for $\alpha - r_f + \lambda k > 0$. Investment horizons considered: 1, 2, 5, 10, 20, 50 years

where the variance diverges to infinity for $\alpha + \lambda k - r_f > 0$ and we have $\lim_{t\to\infty} var(v(t))/t = \infty$. Since the condition of positive expected trading profits in the limit is attained for $\alpha + \lambda k - r_f > 0$, we cannot have positive trading profits and decaying time averaged variance at the same time. This strategy fails to satisfy the definition of statistical arbitrage. □

***Remark 5*** In Merton's jump-diffusion model, we have $E[e^{2Y}] = \exp(2\mu_y + 2\sigma_y^2)$ whereas in Kou's jump-diffusion model $E[e^{2Y}] = \left(p\frac{\eta_1}{\eta_1 - 2} + q\frac{\eta_2}{\eta_2 + 2}\right)$. The variance of the buy-and-hold strategy can be computed.

Buy-and-hold strategies that yield positive expected trading profits fail to satisfy the decay condition of the time averaged variance. Therefore, clearly the risky asset holding should go to zero in time; in other words, the risky asset should be sold in finite time. As we have shown in Proposition 4, there are no statistical arbitrage opportunities for a buy-and-hold strategy.

Note that for the case of $\alpha + \lambda k - r_f < 0$, the present value of the expected profits converges to $-S_0$ as $t \to \infty$. Furthermore, we require $2(\alpha - r_f) + \lambda(E(e^{2Y}) - 1) \leq -\sigma^2$ for the time averaged variance to decay to zero. Hence, if both of these conditions are satisfied, continuous short selling of the stock allows for positive expected profits with a decaying time averaged variance. For this to happen, the trader should be able to identify the underperforming stocks better than the market in general.

*Numerical examples for the buy-and-hold strategy* We simulate 10,000 stock price paths at investment horizons of 1, 2, 5, 10, 20, and 50 years and assume the following set of parameters in Merton's and Kou's jump-diffusion models: $r_f = 0.05, \alpha = 0.12, \sigma = 0.2, \lambda = 3, S_0 = 1$. For the jump-size distribution in the Merton model, we assume that $\mu_y = 0$, $\sigma_y = 0.1$, whereas for the Kou model, we assume $p = q = 0.5$ (symmetric case), $1/\eta_1 = 1/\eta_2 = 2\%$. In this case, the condition $\alpha - r_f + \lambda k > 0$ is satisfied for both models which is consistent with the average trading profits obtained in Fig. 1.

With our choice of parameters, the condition $2(\alpha - r_f) + \sigma^2 + \lambda(E[e^{2Y}] - 1) > 0$ is satisfied for both models. Therefore, as expected the time averaged variance of the trading profits diverges to infinity as shown in Fig. 1. Furthermore, for $\alpha - r_f + \lambda k > 0$, the probability of loss decays to zero as shown in Fig. 1.

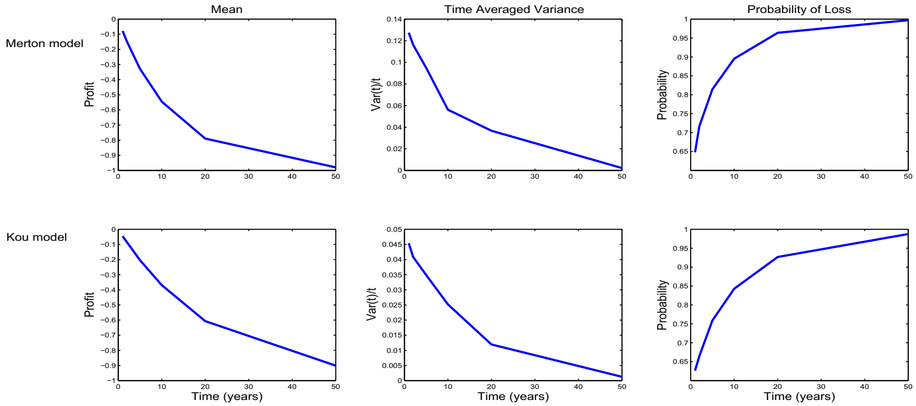


**Fig. 2** Evolution of mean, time averaged variance and probability of loss for the buy-and-hold strategy for $\alpha - r_f + \lambda k < 0$. Investment horizons considered: 1, 2, 5, 10, 20, 50 years

As a second example for the buy-and-hold strategy, we change the parameters of the jump-size distribution in both models. We assume that $\mu_y = -0.02$, $\sigma_y = 0.1$ in the Merton model and $p = 0.2$, $q = 0.8$, and $\eta_1 = \eta_2 = 1/0.02$ in the Kou model. In this case, observe that we have $\alpha - r_f + \lambda k < 0$ and $2(\alpha - r_f) + \sigma^2 + \lambda E[e^{2Y}] - 1) < 0$. This implies that the present value of the trading profits converges to $-S_0$ while the variance decays to zero. In Fig. 2, we observe the average trading profits, time averaged variance and the probability of loss. As can be seen, only the continuous short selling of the risky asset would yield statistical arbitrage.

## 4 First passage time

In this section, we analyze the first passage time property of the jump-diffusion models which is critical to prove the existence of statistical arbitrage opportunities. For this purpose, we are interested in the first passage time of the stock price process $S$ to the deterministic barrier level $S_0(1+\gamma)e^{r_f t}$.[7]

Let $H_t = \ln(S_t e^{-r_f t}/S_0)/\sigma$, then we can rewrite Eq. (1) as

$$H_t = \mu t + W_t + \sum_{i=1}^{N(t)} Y_i^* \tag{16}$$

where $\mu = (\alpha - r_f - \sigma^2/2)/\sigma$ and $Y_i^* = Y_i/\sigma$. The first passage time of the jump-diffusion process in Eq. (16) is given by

$$\tau_x = \min\{t > 0 : H_t \geq x\} \tag{17}$$

where $x = \ln(1+\gamma)/\sigma$. The next two corollaries are direct results of Theorem 2.1 in Coutin and Dorobantu (2011).

[7] Actually, any barrier level greater than the initial stock price can be chosen but since we assume that the risk-free rate is greater than zero, the barrier level should also grow with the risk-free rate. The particular choice of the barrier level as $S_0(1+\gamma)e^{r_f t}$ is to simplify the mathematical notation since we are working with the discounted stock price process.

**Corollary 6** *The first passage time of the stock price process $S_t$ given by Eq. (1) to the barrier level $B_t = S_0(1+\gamma)e^{r_f t}$ is almost surely finite, i.e. $P(\tau_x < \infty)$, for $\mu + \lambda E[Y^*] > 0$.*

***Proof*** From Eq. (16), we show that the first passage time of the stock price process $S_t$ to the barrier level $B_t = S_0(1+\gamma)e^{r_f t}$ is equivalent to the first passage time of the process $H_t$ to level $x$, i.e. $\tau_B \equiv \tau_x$ and applying Theorem 2.1 in Coutin and Dorobantu (2011), the result follows. □

**Corollary 7** *The first passage time of the stock price process $S_t$ given by Eq. (1) to the barrier level $B_t = S_0(1+\gamma)^{-1}e^{r_f t}$ is almost surely finite, i.e. $P(\tau_{x^*} < \infty)$, for $x^* = -\ln(1+\gamma)/\sigma$ and $-\mu + \lambda E[-Y^*] \geq 0$.*

***Proof*** Consider $-W_t \equiv W_t$ and $-Y_i^*$ to re-define the process $H_t = \mu^* t + W_t + \sum_{i=1}^{N(t)} \bar{Y}_i^*$ where $\mu^* = \mu$ and $\bar{Y}_i^* = -Y_i^*$. Then let $\tau_{x^*} = \min\{t > 0 : H_t \leq x^*\}$ with $x^* = -\ln(1+\gamma)/\sigma$. The corollary follows for the finite first passage time after applying Theorem 2.1 in Coutin and Dorobantu (2011) for $-\mu + \lambda E[-Y^*] \leq 0$, i.e. for $\mu + \lambda E[Y^*] \geq 0$. □

***Remark 8*** Corollary 6 can be applied to a wide range of jump-diffusion models. For example, under Merton's jump-diffusion model, this result implies the finite first passage time of the stock price process for $\mu + \lambda\mu_y/\sigma \geq 0$, i.e. $\alpha - r_f + \lambda\mu_y > \sigma^2/2$. If $\mu_y = 0$, we obtain the result of Theorem 4 in Göncü (2015).

Alternatively, under Kou's double exponential jump-diffusion model, this condition is equivalent to $\alpha - r_f + \lambda(\frac{p}{\eta_1} - \frac{q}{\eta_2}) > \sigma^2/2$, and if the jumps are symmetric, this again yields the condition in Theorem 4 of Göncü (2015).

In Coutin and Dorobantu (2011) (see Theorem 2.1 p. 1128), the density function of $\tau_x$ is derived for Lévy processes with compound Poisson process in Eq. (16) and it is given as follows:

$$f_\tau(t,x) = \lambda E[\mathbf{1}_{\{\tau_x > t\}}(1 - F_Y)(x - H_t)] + E[\mathbf{1}_{\{\tau_x > T_{N_t}\}} g(t - T_{N_t}, x - H_{T_{N_t}})] \tag{18}$$

where $F_Y$ is the cumulative distribution function (cdf) of the i.i.d. jump-sizes $Y_i$, $T_{N_t} = \sum_{i=1}^{N(t)} Y_i^*$, and

$$g(u,z) = \frac{|z|}{\sqrt{2\pi u^3}} \exp\left[-\frac{(z-\mu u)^2}{2u}\right] \mathbf{1}_{\{u \in (0,\infty)\}} \tag{19}$$

For example, in the Merton jump-diffusion model, $F_Y$ is the normal cdf, whereas for the double exponential jump-diffusion model of Kou, $F_Y$ is the cdf of the double exponential distribution.

Next, based on the finite first passage time results above, we construct a trading strategy that satisfies the statistical arbitrage definition (i.e. Definition 1) for the jump-diffusion models with compound Poisson processes.

## 5 Existence of statistical arbitrage

We show that even if an investor is able to identify high alpha stocks, buy-and-hold type of investment strategies do not yield statistical arbitrage opportunities in the sense of Definition 1. Therefore, the risky asset holding should decay to zero in time in order to bound the variance and realize statistical arbitrage profits. Following Göncü (2015), we introduce a stopping boundary for the risky asset and whenever the stock price hits the deterministic barrier level (or exceeds the barrier as a result of a jump), we sell the stock and invest the proceeds in the money market account.

Let $B_t = S_0(1+\gamma)e^{r_f t}$ denote the deterministic barrier level. Our buy-and-hold until barrier strategy is as follows: we borrow $S_0$ at time 0 and invest in one unit of the stock. We hold the stock until it hits the deterministic barrier level $B_t$. Hence, effectively we are locking in a profit of $S_0\gamma$ until it is realized. The discounted cumulative trading profits of this strategy can be written as

$$v(t) = \begin{cases} S_0(\gamma+\epsilon) & \text{if } \tau_B \in [0,t] \\ S_t e^{-r_f t} - S_0 & \text{otherwise,} \end{cases} \tag{20}$$

where $\tau_B = \min(t \geq 0 : S_t \geq B_t)$ is the first passage time to the barrier level $B_t$ and $\epsilon \geq 0$ is the amount exceeding the barrier level as a result of a possible jump. If the stock price process hits the barrier without a jump, then we have $\epsilon = 0$. However, if there is a jump at the first passage time, then $\epsilon > 0$ and we obtain a profit level of $\gamma + \epsilon$, where we overshoot the barrier level.

The limit of the discounted cumulative trading profits is obtained as

$$\lim_{t\to\infty} E[v(t)] = (\gamma+\epsilon)S_0 \tag{21}$$

since the stock price paths hits the barrier almost surely in finite time for $\mu + \lambda E[Y] \geq 0$. Hence, the following condition must be satisfied for finite hitting time to the barrier level:

$$\frac{\alpha - r_f - \sigma^2/2}{\sigma} + \lambda E[Y] > 0 \iff \alpha - r_f + \sigma\lambda E[Y] > \sigma^2/2 \tag{22}$$

Again, due to the almost surely finite first passage time property of the stock price paths, the probability of loss decays to zero

$$\lim_{t\to\infty} P(v(t) < 0) = 0 \tag{23}$$

almost surely if the condition in Eq. (22) is satisfied. Furthermore, for sufficiently large time $t$, the risky asset holding becomes zero and thus the time averaged variance of the trading strategy decays to zero, satisfying

$$\lim_{t\to\infty} var(v(t))/t = 0 \tag{24}$$

Therefore, for $\alpha + \sigma\lambda\mu_y - r_f \geq \sigma^2/2$, statistical arbitrage exists in the sense of Definition 1.

### 5.1 Probability of loss

Probability of loss $P(v(t) < 0)$ can be written as

$$P(v(t) < 0) = P(S_t < S_0 e^{r_f t}, \tau_x > t) = P(S_t < S_0 e^{r_f t} | \tau_x > t) P(\tau_x > t) \tag{25}$$

where the probability of loss converges to zero as $t \to \infty$ for $\alpha - r_f + \lambda E[Y] \geq \sigma^2/2$, since $P(\tau_x = \infty) = 0$. For $\alpha - r_f + \lambda E[Y] < \sigma^2/2$, we have $\lim_{t\to\infty} P(S_t < S_0 e^{r_f t} | \tau_x > t) P(\tau_x > t) > 0$ as $t \to \infty$ and thus the probability of loss does not decay to zero for the buy-and-hold until barrier strategy.

We have $P(\tau_x > t) = 1 - P(\tau_x \leq t)$ for a given time $t$ and barrier level $x$ and the cdf of the first passage time $\tau_x$ is given by

$$P(\tau_x \leq t) = \lambda E[\int_0^t \mathbf{1}_{\{\tau_x > s\}}(1 - F_Y)(x - H_s)ds] + E[\int_0^t \mathbf{1}_{\{\tau_x > T_{N_s}\}} g(s - T_{N_s}, x - H_{T_{N_s}})ds] \tag{26}$$

where $F_Y$ is the cdf of the i.i.d. jump-sizes $Y_i$, $T_{N_t} = \sum_{i=1}^{N(t)} Y_i^*$, and

$$g(u, z) = \frac{|z|}{\sqrt{2\pi u^3}} \exp\left[-\frac{(z - \mu u)^2}{2u}\right] \mathbf{1}_{\{u \in (0,\infty)\}} \tag{27}$$

Next, we show that there exists statistical arbitrage opportunities in the jump-diffusion models given in Eq. (1).

**Proposition 9** *Assume that there exists a trader who can identify over/under performing stocks and the stock prices follow the class of jump-diffusion models given by Eq. (1). Then, there always exists statistical arbitrage opportunities in the sense of Definition 1.*

***Proof*** We have to consider two cases: In the first case, we have a risky asset that is drifted upward that satisfies the condition $\mu + \lambda E[Y] \geq 0$. Then, we consider the buy-and-hold until barrier type of strategy, borrowing $S_0$ at the risk-free rate and immediately taking a long position in the risky asset at time zero. We hold the risky asset until it hits the barrier level $B_t = S_0(1 + \gamma)e^{r_f t}$ locking in the $\gamma$ percent profit for any $\gamma > 0$. By Corollary 6, we know that the first passage time to the barrier level $B_t$ is almost surely finite, i.e. $P(\tau_B < \infty) = 1$. This implies that we realize the expected profit of $\gamma$ percent almost surely, and probability of loss converges to zero. Furthermore, we have $\lim_{t\to\infty} var(v(t))/t = 0$, and thus Definition 1 is satisfied.

In the second case, we consider $\mu + \lambda E[Y] < 0$. Then, we short sell the risky asset that satisfies this condition at time zero and invest the proceeds immediately into the money market account at the risk-free rate $r_f$. We keep the short position until the barrier is reached. Once the risky asset hits the barrier level or exceeds $B_t = S_0(1 + \gamma)^{-1}e^{r_f t}$, we close the short position in the risky asset. □

*Numerical examples for the single barrier long and short strategies* To verify our theoretical results in Proposition 9, we consider Monte Carlo simulation of stock price processes under the Merton and Kou jump-diffusion models. Similar to the numerical examples for the buy-and-hold strategy in the previous section, we simulate 10,000 stock price paths at investment horizons of 1, 2, 5, 10, 20, and 50 years. We assume that $r_f = 0.05$, $\gamma = 0.05$, $\alpha = 0.1$, $\sigma = 0.2$, $\lambda = 3$, and $S_0 = 1$. For the Merton jump-distribution model, we assume that $\mu_y = 0$, $\sigma_y = 0.1$, whereas for the Kou jump-diffusion model, we assume $p = q = 0.5$ (symmetric case), $1/\eta_1 = 1/\eta_2 = 2\%$. In this case, the condition $\alpha - r_f + \lambda k > 0$ is satisfied for both models which is consistent with the results shown in Fig. 3.

As a second example, we verify the existence of statistical arbitrage under the short until barrier type of strategy. We consider an under-performing stock with $\alpha < r_f$. In this case, we change the value of $\alpha$ to 0.01, the value of $\mu_y$ to -0.01, and the value of $p$ to 0.45, while keeping the same values for the other parameters. In Fig. 4, we observe the average trading profits, time averaged variance and the probability of loss, which is consistent with the result obtained in Proposition 9.

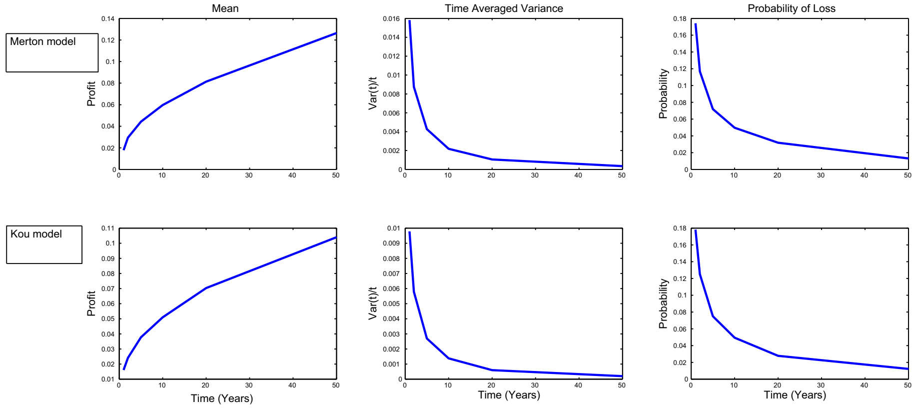


**Fig. 3** Evolution of mean, time averaged variance and probability of loss for the buy-and-hold until barrier strategy. Investment horizons considered: 1, 2, 5, 10, 20, 50 years

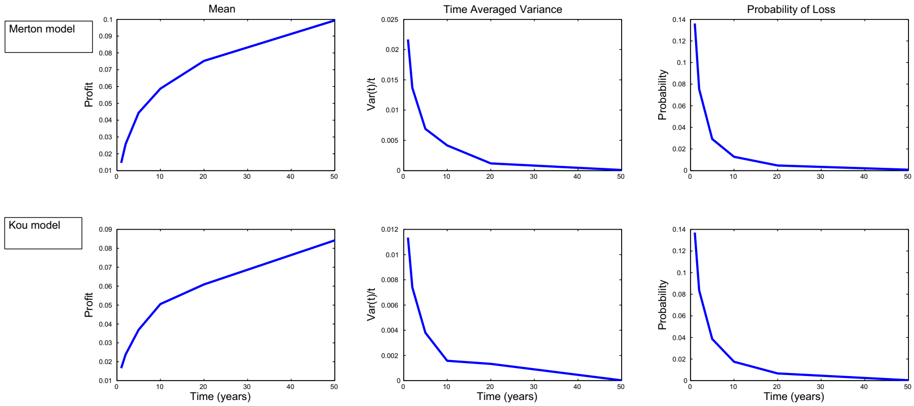


**Fig. 4** Evolution of mean, time averaged variance and probability of loss for the short until barrier strategy. Investment horizons considered: 1, 2, 5, 10, 20, 50 years

## 5.2 Double barrier strategies

Alternative to the case of single barrier statistical arbitrage strategies discussed above, statistical arbitrage strategies can be designed utilizing variations of similar trading strategies. In the previous strategies, the trader enters a single trade at time zero and there are no other trades until the position is closed. However, a more realistic case is where the trader sets different levels to open positions at multiple times in order to reduce the overall position costs. For example, if the stock price is expected to go up, the trader opens a long position at time zero and if the stock price goes down reaching a lower price target level (i.e., a lower barrier), a second position is added. Therefore, this class of strategies can be represented with multiple barrier levels to open long or short positions to smooth the cost of position openings with multiple target levels.

Assume that the condition given in Eq. (22) holds. Then, following the previous notation, let us denote the upper barrier level as $B_t = S_0(1+\gamma)e^{r_f t}$ and denote the lower barrier level as $B_t^l = S_0(1-\beta)e^{r_f t}$. The trading strategy is as follows: At time zero, the investor borrows

$S_0$ dollars from the bank at the risk-free rate and invests in one stock as in the previous example. However, if the stock price hits the lower barrier level $B_t^l < S_0$, then the investors borrows an additional $S_{\tau_{B^l}}$ from the bank and purchases one additional unit of the same stock. In other words, the stock price hitting the lower barrier for the first time is considered as a re-investment opportunity to buy one more unit of the stock. Similar to the first invested stock, the second holding of the stock is closed if the stock price reaches the barrier level $B_t^{(u^*)} = S_{\tau_{B^l}}(1+\gamma)e^{r_f t}$.

The discounted cumulative profits of this strategy is given as

$$v^*(t) = \begin{cases} S_0(\gamma+\beta) & \text{, if } \tau_B \in [0,t] \text{ and } \tau_{B^l} \notin [0,t] \\ 2S_t e^{-r_f t} - S_0 - S_{\tau_{B^l}} e^{-r_f \tau_{B^{u*}}} & \text{, if } \tau_B \notin [0,t] \text{ and } \tau_{B^l} \in [0,t] \\ S_t e^{-r_f t} - S_0 & \text{, if } \tau_B \notin [0,t] \text{ and } \tau_{B^l} \notin [0,t]. \end{cases} \tag{28}$$

It can be seen that investing additional amounts in the same stock does not change the existence of the statistical arbitrage since the stock price still exhibits the finite first passage time to the upper barrier level $S_{\tau_{B^l}}(1+\gamma)e^{r_f t}$ based on the time $\tau_{B^l}$ when the second unit of the share is purchased.

The expected discounted profits from this strategy can be written as

$$E[v(t)] = E[v(t)|\tau_{B^l} \notin [0,t]]P(\tau_{B^l} \notin [0,t]) + E[v(t)+v^*(t)|\tau_{B^l} \notin [0,t]]P(\tau_{B^l} \in [0,t]). \tag{29}$$

The limit of the discounted cumulative trading profits of the alternative strategy can be written in terms of the previous strategy $v(t)$. Therefore, the limiting expected discounted profits are guaranteed to be positive similar to the case of the previous trading strategy.

Similarly, if the drift of the stock price process satisfies the condition $\mu + \lambda E[Y] < 0$, then the double barrier strategy can be implemented with an initial short position at time zero. In the previous short until barrier strategy, we open a short position at time zero and close the position whenever the stock price hits the lower barrier level $B_t = S_0(1-\gamma)e^{r_f t}$. We extend this strategy by introducing a second barrier that is higher than the initial barrier. If the stock price reaches the upper barrier level $B_t^u = S_0(1+\beta)e^{r_f t}$, for $\beta > 0$, then a second short position is opened in addition to the initial short position.

The discounted cumulative profits of the double barrier short strategy is given as

$$v^*(t) = \begin{cases} S_0(\gamma+\beta) & \text{, if } \tau_B \in [0,t] \text{ and } \tau_{B^u} \notin [0,t] \\ S_0 - S_t e^{-r_f t} + S_{\tau_{B^u}} - S_t e^{-r_f \tau_{B^u}} & \text{, if } \tau_B \notin [0,t] \text{ and } \tau_{B^u} \in [0,t] \\ S_0 - S_t e^{-r_f t} & \text{, if } \tau_B \notin [0,t] \text{ and } \tau_{B^u} \notin [0,t]. \end{cases} \tag{30}$$

The same arguments for the long strategy follow similarly for the double barrier short strategy, and thus, create statistical arbitrage trading strategies. Next, we present the Monte Carlo simulation results for the validity of the statistical arbitrage definition for the double barrier strategies.

*Numerical examples for the double barrier long and short strategies* In order to verify that our double barrier strategies satisfy the statistical arbitrage definition numerically, we consider the Monte Carlo simulation of stock price processes under the Merton and Kou jump-diffusion models. Similar to the numerical examples for the single barrier case, we simulate 10,000 stock price paths at investment horizons of 1, 2, 5, 10, 20, and 50 years. We assume that $r_f = 0.05$, $\gamma = 0.05$, $\beta = 0.05$, $\alpha = 0.1$, $\sigma = 0.2$, $\lambda = 3$, and $S_0 = 1$. For the Merton jump-distribution model, we assume that $\mu_y = 0$, $\sigma_y = 0.1$, whereas for the Kou jump-diffusion model, we assume $p = q = 0.5$ (symmetric case), $1/\eta_1 = 1/\eta_2 = 2\%$. In this

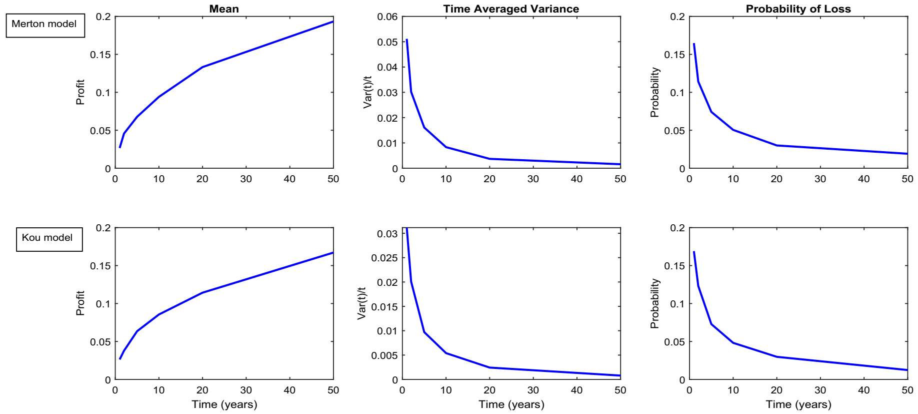


**Fig. 5** Evolution of mean, time averaged variance and probability of loss for the hold until barrier strategy. Investment horizons considered: 1, 2, 5, 10, 20, 50 years

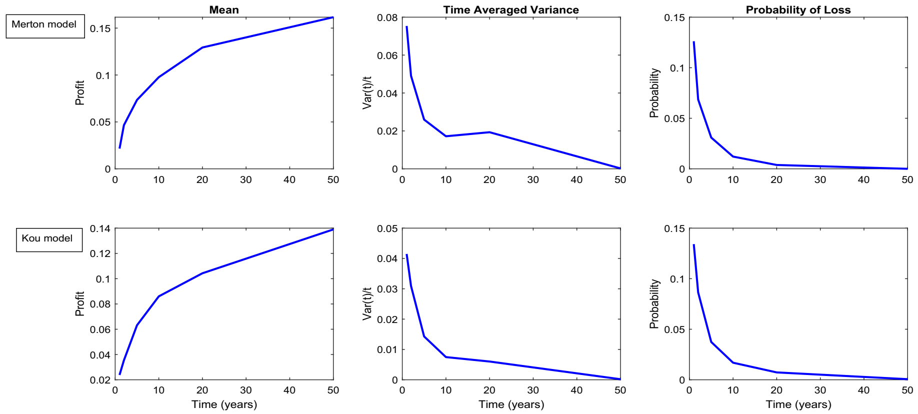


**Fig. 6** Evolution of mean, time averaged variance and probability of loss for the short until barrier strategy. Investment horizons considered: 1, 2, 5, 10, 20, 50 years

case, the condition $\alpha - r_f + \lambda k > 0$ is satisfied for both models which is consistent with the results shown in Fig. 5.

As a second example for double barrier strategies, we verify the existence of statistical arbitrage under the short until barrier type of strategy. We consider an under-performing stock with $\alpha < r_f$. In this case, we change the value of $\alpha$ to 0.01, the value of $\mu_y$ to -0.01, and the value of $p$ to 0.45, while keeping the same values for the other parameters. In Fig. 6, we observe the average trading profits, time averaged variance and the probability of loss.

## 6 Conclusion

In this paper, we prove that there exists statistical arbitrage opportunities in the sense of the statistical arbitrage definition given in Hogan et al. (2004) for a widely used class of jump-diffusion models with a compound Poisson process. Two well-known special cases of this class are the Merton and Kou jump-diffusion models. We first consider the buy-and-hold type of strategies and prove that these strategies do not satisfy the statistical arbitrage conditions. Then, we introduce a deterministic boundary to terminate the trading strategy, i.e. we lock-in a certain level of profit to close the risky asset position in finite time. Since the jump-diffusion models do not guarantee no-arbitrage pricing, our results can be considered intuitive.

Realization of the statistical arbitrage profits depends on the ability of a trader to identify high-performing or under-performing stocks in the market. This implies that at least the violation of the strong form of market efficiency is needed to exploit the statistical arbitrage opportunities. In the general framework of Lévy processes for stock prices, the existence of statistical arbitrage opportunities depends on the finiteness of the first passage time to the stopping boundary that we define in our trading strategies. We show that given a deterministic selling (buying) condition for long until barrier strategy (short until barrier), the trading strategy we introduce terminates almost surely in finite time. Therefore, the time average of the variance of the discounted trading profits do not diverge both under single and double barrier strategies.

Our results uncover the simple fact that physical drift and physical measure of the stock price process are crucial in order to exploit statistical arbitrage opportunities. In this sense, a better understanding of the physical measure and drift of a given stock price process allows for riskless profit opportunities in the limit.

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