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ORIGINAL PAPER

Limit theorems for recursive delegation equilibria

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Abstract

Delegation games are studied in the context of a symmetric linear Cournotic duopoly where redelegation is permissible. In the absence of extraneous delegation costs, the following results are demonstrated:

(1) Each principal has an incentive to redelegate, increasing the length of his delegation chain.
(2) As the length of the delegation chain grows beyond bound,
   (i) total output at the (Cournot) equilibrium on the industry floor converges in monotonically increasing fashion to the socially efficient one, and
   (ii) the maximand delegated by each primal delegator converges in monotonically decreasing fashion to the (true) profit function.

As a consequence it is suggested that in a linear duopoly context socially efficient and truthful outcomes can be arbitrarily closely approximated by the use of Pretend-but-Perform Mechanisms of order sufficiently large.

This paper is included in this special issue in memory of Murat Sertel, whom we lost in 2003. Murat Sertel is among the founders of the area of economic and social design. His leading role is not only manifested in his original approach to design, but also in his impact upon the area on the organizational and educative planes. He is the founding editor of Review of Economic Design,the chief founder of the Society for Economic Design and the designer of the Bosphorus Workshop on Economic Design, which has been organized annually more than 40 years now. Murat Sertel's intellectual heritage is now represented by a broad class of researchers in the area of design consisting of his students and their academic descendants. His co-author in this paper also owes his contact with game theory and his entry to the area of design to Murat Sertel.
The 1989-version is kept unchanged as the main body of the paper, while it is related to the relevant literature in its aftermath in the Prologue and Epilogue. We are thankful to an anonymous referee for his/her detailed suggestions in that regard.

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Prologue

This paper based on the pretend-but-perform notion was written more than thirty years ago. The pretend-but-perform idea was originally introduced by Sertel as the dual of incentive compatibility, which duality is perhaps best reflected by Mevlana Celaleddin-i Rumi, who said already in the 13th century: "Either appear as you are or be as you appear". This prologue is meant to relate the paper to the relevant literature that came into being in the meantime without making any changes in the original version of the paper.

Delegation has been considered from mainly two different standpoints in the literature. From the viewpoint of positive economics, it reflects an attempt to explain the rationale behind real-life deviations of firms from profit maximization, while the design approach treats delegation as a means of regulating an oligopoly to improve social welfare. A comparison between these two approaches is naturally contingent upon the equilibrium notion employed. Here we consider the Cournot equilibrium based on quantity competition in the context of a symmetric linear duopoly and show that delegation intensifies competition between firms by creating a further ground for competing for a larger market share. It is also shown in Koray and Sertel (1992) that this fact extends to a symmetric linear oligopoly of n firms, where the n firms behave under the pretend-but-perform mechanism precisely as n^2 firms would do in the naked Cournot oligopoly.

The main focus of this paper is to illustrate the inadequacy of delegation to explain firms' deviations from profit maximization in a Cournot duopoly. This problem is, of course, essentially one of an empirical nature. Our consideration here, on the other hand, is based on some a priori theoretical reasons, which can be summarized as follows. The main result of the present paper is that each principal, i.e., each firm owner in the duopolistic context, has an incentive to redelegate, increasing the length of his delegation chain. Thus, if delegation is costless, then the principals are expected to increase the delegation chain length indefinitely. As delegation is costly in real life, the chain length they choose will be contingent upon delegation costs, which may or may not be equal to 1.

The only reason why the managers would come to a Cournot equilibrium under the assigned maximands is that they are instructed to do so by their principals, as the managers are in fact not recipients of the maxima they are to achieve. Thus, it must be the case that the firm owners have some reason to delegate the untrue profit functions of their firms by also instructing their managers to behave as Cournot duopolists under those maximands. It is shown, however, in Koray and Sertel (1989) that, given any pair of quantities $q = (q_1, q_2)$ yielding nonnegative profits to both firms, there is an equilibrium pair of maximands of the owners' delegation game inducing q as the Cournot-Nash equilibrium of the intermanagerial game. Thus, it is difficult to think of any reason why the owners should restrict the maximands they delegate to the form of a profit function, unless they are legally forced to do so.

The delegation setup, on the other hand, is meaningful as a pretend-but-perform regulation under which an improvement of social welfare is achieved. In case the delegation chain is of finite length, the increase in social welfare stays under the optimal level. The optimal level is achieved in the limit, when the delegation chain length is increased indefinitely.

The closest work in regulating a Cournot oligopoly is provided by Gradstein (1995)[1], which considers a standard Cournot oligopoly with n firms producing a homogeneous product. He constructs a mechanism to attain the social optimum in a Cournot oligopoly under the further restriction that each firm's message space consists of choosing its output level, which also reduces the outcome function of the mechanism to the identity function. The informational assumptions are that the inverse demand function is known to the designer, but the firms' cost functions are not, while all aspects of the environment are common knowledge among the firms. The standard regularity assumptions made about the inverse demand and the firms' cost functions by Gradstein (1995) are such that they also suffice for the existence of a Cournot equilibrium.

The regulator announces a balanced transfer vector as a function of the quantities chosen by the firms, which is enforceable as the output levels are observable. Gradstein (1995) shows that a balanced transfer vector inducing the social optimum exists if and only if the inverse demand function is a polynomial function of degree at most $n - 1$, where n is the number of firms in the oligopoly.

In our case the inverse demand function is assumed to be a polynomial function of degree 1, the highest degree possible for the characterization of Gradstein (1995) to apply to a duopoly. As for regulatory purposes, what our game cascades turn out to implement in the limit for a linear duopoly is achieved by him in one shot via balanced transfer vectors.

In this paper, however, our main focus is to model delegation and analyze the nature of recursive delegation equilibria rather than implementation. Implementation enters the picture via the question of whether the "meta-Cournotic" equilibria arising under delegation can be considered as part of positive or regulatory theory.

Ünver (1995) and Yıldırım (1995) also follow up the pretend-but-perform idea in the context of a symmetric duopolistic differentiated goods market when delegation is costless and redelegation is possible. Ünver (1995) considers the case where quantity is the strategic variable and thus Cournot equilibria are reached by the managers on the industry floor, while Yıldırım (1995) assumes that the managers on the industry floor compete in prices and end up with Bertrand equilibria. Their results parallel those obtained for a duopoly in which the two firms produce the same good, where redelegation drives the outcome towards the socially optimal one under quantity competition, while price competition creates forces that drive the equilibrium outcome closer to the collusive joint profit maximizing point when the delegation chain length is increased.

Kraekel (2005) considers a two-stage game in a duopolistic context, where the maximand delegated by each owner to his manager is a linear combination of profits and sales, while the two managers compete in a duopolistic tournament against each other. Unlike a Cournot or Bertrand duopoly, it turns out that there exist asymmetric equilibria, at which one owner puts a positive weight on sales and the other a negative one.

[1] We are thankful to an anonymous referee for having brought Gradstein (1995) to our awareness.

1 Introduction

Vickers (1985), Fershtman and Judd (1987) and Sklivas (1987) propose delegation equilibria as a compromise between Cournotic (1838) and sales-maximization (Simon (1964); Williamson (1964); Baumol (1977); Jensen and Meckling (1976)) solutions in industrial economics. This parallels certain equilibria of some of our games of pretension (Koray and Sertel 1983) and the outcome under certain types of pretend-but-perform regulation (Koray and Sertel 1987, 1988; Koray and Sertel 1990). The idea is that, in an oligopoly where the Cournot solution prevails, owners assign to their managers (not true profit but) a fictitious profit as maximand and, when the owners are themselves at Nash equilibrium in their choices of maximand to delegate to their respective managers, a new equilibrium is induced where industry output is higher and price lower than in the undelegated usual Cournot solution. Koray and Sertel (1989) give a critical and comparative account of the VFJS (Vickers (1985); Fershtman and Judd (1987); Sklivas (1987)) theory. Here we examine -albeit in a symmetric linear duopoly- what happens when the delegation chain is lengthened and recursively a maximand is delegated to a lower delegate at each link of the chain, finally arriving at a manager on the industry floor who then acts without further delegation and produces an output.

In the next section we first define a *game cascade* as a general framework and then, focusing attention on an oligopolistic setting, we derive from it and define a *linear duopoly cascade*. In Sect. 3 we analyze the *symmetric* linear duopoly cascade, in the context of which we obtain the following main results :

(1) At an equilibrium of any given chain length, as an owner it always pays to extend one's delegation chain by a further link.
(2) As the length of the delegation chain grows beyond bound,
    (i) total industrial output grows monotonically and converges to the socially efficient one, while
    (ii) the maximands delegated by the top delegators converge, monotonically declining, to their respective true profit functions.

Finally, Sect. 4 ends the study with closing remarks, suggesting also the use of Pretend-but-Perform Mechanisms of sufficiently high order to approximate socially efficient and truthful outcomes as closely as desired.

2 Basic notions

To focus better on the linear duopoly cascade which will occupy us specifically in this paper, it is useful to first acquire a glimpse of the general idea of a game cascade. Denote the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$.

Definition 2.1 Given any $m, k+1 \in \mathbb{N}$, by an m-person *game cascade* of order k we mean an ordered pair $\Gamma^k = \left(\{g^h\}_{h\in K}, \{A_i^{k+1}\}_{i\in M} \right)$, where for each $h \in K = \{0, 1, \ldots, k\}$

$$g^h = \left(\left\{A_i^h\right\}_{i\in M}, \left\{u_i^h\right\}_{i\in M}\right)$$

with $M = \{1, \ldots, m\}$ lists m nonempty sets A_i^h, for which we denote $A^h = \Pi_{j\in M} A_j^h$, and m real-valued functions $u_i^h : A_i^{h+1} \times A^h \to \mathbb{R}$, while $A_i^{k+1} = \{\alpha_i\}$ for each $i \in M$ is a singleton specifying a primitive parameter α_i.

We propose the idea of a game cascade as a form in modeling institutional settings where agents i at the $(h+1)th$ "echelon" each determine a parameter α_i^{h+1} fixing the maximands $u_i^h[\alpha_i^{h+1}] : A^h \to \mathbb{R}$ of their "subordinates" i at the next lower echelon h through the definition

$$u_i^h[\alpha_i^{h+1}](\alpha^h) = u_i^h(\alpha_i^{h+1}, \alpha^h) \quad (h \in K, \alpha^h \in A^h). \tag{1}$$

Note that a game cascade of order 0 gives us, through this formula (1), simply the *game* $g^0[\alpha_1] = \left(\left\{A_i^0\right\}_{i\in M}, \left\{u_i^0[\alpha_i^1]\right\}_{i\in M}\right)$, and so it is for order $k \geq 1$ that game *cascades* gain special interest.

It is from an oligopolistic context that we derive the particular game cascades which we will be examining throughout the rest of this paper. The first order ($k = 1$) instance of these is familiar as a simplest example of certain games of pretension (Koray and Sertel 1983) and appears in the work of Vickers (1985); Fershtman and Judd (1987); Sklivas (1987), as well as that of Koray and Sertel (1987, 1988); Koray and Sertel (1990), all of which is discussed by Koray and Sertel (1989). To pinpoint the games of interest to us here, henceforth we take $M = \{1, 2\}$, $X_1 = X_2 = [0, \infty)$ with $X = X_1 \times X_2$ and define, for each $i \in M$, the function

$$\pi_i : [0, \infty) \times X \to \mathbb{R} \tag{2.1}$$

at each $\xi \in [0, \infty)$ and $x = (x_1, x_2) \in X$ through

$$\pi_i(\xi, x) = \left(\xi - \sum_{j\in M} x_j\right) x_i \tag{2.2}$$

and for each $\xi \in [0, \infty)$ the function

$$\pi_i[\xi] : X \to \mathbb{R} \tag{3.1}$$

at each $x \in X$ through

$$\pi_i[\xi](x) = \pi_i(\xi, x). \tag{3.2}$$

Now we fix attention to the *"linear duopoly"*, by which we mean a game $g^0[\alpha]$, for some $\alpha = (\alpha_1, \alpha_2) \in [0, \infty)^2$, of the form

$$g^0[\alpha] = \left(\{X_i\}_{i\in M}, \{\pi_i[\alpha_i]\}_{i\in M}\right) \tag{4}$$

with reference to M and the X_i as above and (2.1-2),(3.1-2). Imagine duopolists $i \in M$ producing quantities $x_i \in X_i$ of a good at a constant marginal cost $c_i \in (0, a]$ and facing an inverse demand $p = a - (x_1 + x_2)$ with $\alpha_i = a - c_i \geq 0$. The profit of duopolist i is thus $\pi_i[\alpha_i]$ as given in (3) and (2.2). Notice that the linear duopoly corresponds trivially to the game cascade Γ^0 of order 0 defined by

$$\Gamma^0 = \left(g^0, \left\{\{\alpha_i^1\}\right\}_{i \in M}\right) \tag{5.1}$$

where

$$g^0 = \left(\{X_i\}_{i \in M}, \{\pi_i\}_{i \in M}\right) \tag{5.2}$$

(i.e., with reference to Definition 2.1 , $A_i^0 = X_i$ and $u_i^0 = \pi_i$ for each $i \in M$).[2]

A more interesting game cascade is Γ^1 of order 1, defined as

$$\Gamma^1 = \left(\left\{g^0, g^1\right\}, \left\{A_1^2, A_2^2\right\}\right) \tag{6.1}$$

with g^0 as in (5.2), $A_1^2 = \{\alpha_1^2\}$, $A_2^2 = \{\alpha_2^2\}$,

$$g^1 = \left(\left\{A_1^1, A_2^1\right\}, \left\{\pi_1^1, \pi_2^1\right\}\right) \tag{6.2}$$

where $A^1 = A_1^1 \times A_2^1 = [0, \infty)^2$ and $\pi_i^1 : A_i^2 \times A^1 \to \mathbb{R}$ is defined at each $\alpha^1 = (\alpha_1^1, \alpha_2^1) \in A^1$ through

$$\pi_i^1(\alpha_i^2, \alpha^1) = \pi_i[\alpha_i^2](\sigma(g^0[\alpha^1])), \tag{6.3}$$

by utilization of (3), (4) and the Cournot (1938)-Nash (1951) solution concept[3] σ. To see the interest of Γ^1, in the fashion of VFJS (Vickers (1985); Fershtman and Judd (1987); Sklivas (1987)), imagine that each firm i has an "efficiency" α_i^2 defining profit as $\pi_i[\alpha_i^2]$, but that it is to be operated by a manager who will maximize any $\pi_i[\alpha_i^1]$ which the owner of the firm can assign, by choice of $\alpha_i^1 \in A_i^1$ as managerial maximand. Assuming that the managers resolve their game $g^0[\alpha^1]$ at its (unique) Cournot solution for each $\alpha^1 = (\alpha_1^1, \alpha_2^1) \in A_1^1 \times A_2^1$, a game $g^1[\alpha^2] = \left(\left\{A_1^1, A_2^1\right\}, \left\{\pi_1^1[\alpha_1^2], \pi_2^1[\alpha_2^2]\right\}\right)$ is defined among the owners, where $\pi_i^1[\alpha_i^2](\alpha^1) = \pi_i^1(\alpha_i^2, \alpha^1)$ at each $\alpha^1 \in A^1$ (and each owner i's strategy space is $A_i^1 = [0, \infty)$ from which he selects a parameter α_i^1 to fix his manager's maximand). As an alternative scenario consider the pretend-but-perform regulation of KS (Koray and Sertel (1987, 1988); Koray and Sertel (1990)), where the authorities require each owner i to commit to henceforth have his firm act so as to maximize a (possibly fictitious) profit $\pi_i[\alpha_i^1]$ with "efficiency" parameter

[2] Strictly speaking, we should take u_i^0 to be the restriction of π_i to $\left\{\alpha_i^1\right\} \times X$, rather than π_i itself. To avoid notational complications, however, we will feel free to sometimes neglect emphasizing the distinction between a function and its restrictions to various subsets of its domain.

[3] Note that $\sigma(g^0[\alpha^1])$ here is always a singleton, and thus, π_i^1 is a well-defined function.

$\alpha_i^1 = a - c_i^1$, where a is the price intercept of a publicly known (affine) inverse demand while (constant) marginal costs are private data.

Whichever scenario one may take, the mathematical analysis is the same and can be found (a few times over) in VFJS or KS. For a summary, it is best to refer to Koray and Sertel (1988), who treat the case of asymmetric true efficiencies (α_1^2 not necessarily equal to α_2^2) as well. For every $\alpha^2 \in [O, \infty)^2$ the (pretension or delegation) game $g^1[\alpha^2]$ among owners has a Nash equilibrium, and at any such equilibrium $\overline{\alpha}^1$, the Cournot equilibrium $\overline{x}[\overline{\alpha}^1]$ of $g^0[\overline{\alpha}^1]$ always has greater industry output and social welfare (sum of consumers' surplus and (true) profits), at the expense of some total profit, when compared with the naked Cournot equilibrium $\overline{x}[\alpha^2]$ where the firms act so as to maximize true profits $\pi_i[\alpha_i^2]$.

This should suffice to indicate why we advocate our "pretend-but-perform" methods of regulation for an oligopoly. (Koray and Sertel (1990) derive similar desirable properties at von Stackelberg solutions of the game $g^1[\alpha^2]$ among owners.) But there are serious a priori reasons of pure theory why the outcome $\overline{x}[\overline{\alpha}^1]$ induced by a Nash equilibrium $\overline{\alpha}^1$ of $g^1[\alpha^2]$ is unsuitable to serve as an equilibrium in the sense of positive economic theory, i.e., as an explanatory construct, as VFJS propose it. Koray and Sertel (1989) discuss these reasons at some length, but here we wish to concentrate on demonstrating an instability result regarding the "chain length" of a "delegation equilibrium" and two limit theorems regarding what happens to the maximands delegated by the top delegators and the total industrial output as the delegation chain grows beyond bound.

To give a precise formulation to these results we first need to construct the linear duopoly cascade Γ^k of order k for any $k \in \mathbb{N}$, starting with our linear duopoly Γ^0 defined through (5.1-2). Our construction will, of course, be such that for the particular case where $k = 1$ we will obtain the game cascade Γ^1 already defined in (6.1-3). To extend the definition of Γ^0 and Γ^1 to a game cascade Γ^k of order $k \in \mathbb{N}$, we will now introduce some auxiliary apparatus and prove a lemma which will not only enable us to inductively define Γ^k but also constitute a central result of the paper from which all of our main theorems will be derived.

At each two-person game $g = (\{A_1, A_2\}, \{u_1, u_2\})$ in normal form where $A_1 = A_2 = [0, \infty)$, we define a refinement $\overline{\sigma}$ of the Cournot-Nash solution concept σ through

$$\overline{\sigma}(g) = \left\{ x \in \sigma(g) | x_i = \max \left\{ x_i' \in A_i | \begin{array}{l} (x_i', x_j') \in \sigma(g) \text{ for some} \\ x_j' \in A_j \text{ where } \{i, j\} = \{1, 2\} \end{array} \right\} \text{ for each } i \in \{1, 2\} \right\} \quad (7)$$

Lemma 2.2 *Let $k \in \mathbb{N}$, $K = \{0, 1, \ldots, k\}$ and $M = \{1, 2\}$. Assume that a game cascade $\Gamma^k = \left\{ \left\{g^h\right\}_{h \in K}, \left\{A_i^{k+1}\right\}_{i \in M} \right\}$ satisfies the following conditions :*

(1) g^0 is defined as in (5.2).

(2) $A_i^{k+1} = \left\{\alpha_i^{k+1}\right\} \subset [0, \infty)$ for each $i \in M$.

(3) For each $h \in K \setminus \{0\}$, we have $g^h = \left(\left\{A_i^h\right\}_{i \in M}, \left\{\pi_i^h\right\}_{i \in M}\right)$ such that

(a) $A_i^h = [0, \infty)$ for all $i \in M$;

(b) $\overline{\sigma}(g^h[\alpha^{h+1}])$ *is a singleton for all* $\alpha^{h+1} \in A^{h+1}$ *whenever* $h \neq k$, *where* $g^h[\alpha^{h+1}] = \left(\left\{A_i^h\right\}_{i\in M}, \left\{\pi_i^h[\alpha_i^{h+1}]\right\}_{i\in M}\right)$;

(c) $\pi_i^h[\alpha_i^{h+1}](\alpha^h) = \pi_i^{h-1}[\alpha_i^{h+1}](\overline{\sigma}(g^{h-1}[\alpha^h]))$ *for all* $\alpha_i^{h+1} \in A_i^{h+1}$ *and all* $\alpha^h \in A^h$.

Then we have, for each $h \in K\setminus\{0\}$,

$$\pi_i^h[\alpha_i^{h+1}](\alpha^h) = \begin{cases} 0 & , \text{if } \alpha_i^h \leq \frac{h}{h+1}\alpha_j^h \\ \frac{h}{2h+1}(\alpha_1^{h+1} - \frac{h}{2h+1}(\alpha_i^h + \alpha_j^h)) & , \text{if } \frac{h}{h+1}\alpha_j^h \leq \alpha_i^h \leq \frac{h+1}{h}\alpha_j^h \\ ((h+1)\alpha_i^h - h\alpha_j^h) & \\ \frac{h}{h+1}\alpha_i^h(\alpha_i^{h+1} - \frac{h}{h+1}\alpha_i^h) & , \text{if } \frac{h+1}{h}\alpha_j^h \leq \alpha_i^h \end{cases} \tag{8}$$

at all $(\alpha_i^{h+1}, \alpha^h) \in A_i^{h+1} \times A^h$ *where* $\{i, j\} = \{1, 2\}$. *Furthermore,* $\overline{\sigma}(g^k[\alpha^{k+1}])$ *is also a singleton.*

Proof : See Appendix.

We are now ready to define the linear duopoly cascade Γ^k of order k for any $k \in \mathbb{N}$.

Definition 2.3 Let $k+1 \in \mathbb{N}$, $K = \{0, 1, \ldots, k\}$ and $M = \{1, 2\}$. By a linear duopoly cascade of order k we mean a game cascade $\Gamma^k = \left(\left\{g^h\right\}_{h\in K}, \left\{A_i^{k+1}\right\}_{i\in M}\right)$ with $g^h = \left(\left\{A_i^h\right\}_{i\in M}, \left\{\pi_i^h\right\}_{i\in M}\right)$ for each $h \in K$, where

(1) g^0 is as in (5.2);
(2) $A_i^h = [0, \infty)$ for all $i \in M$ and all $h \in K\setminus\{0\}$;
(3) $A_i^{k+1} = \left\{\alpha_i^{k+1}\right\} \subset [0, \infty)$ for each $i \in M$;
(4) for each $h \in K \setminus \{0\}$, $i \in M$ and $\alpha_i^{h+1} \in A^{h+1}$, we have $\pi_i^h[\alpha_i^{h+1}](\alpha^h) = \pi_i^{h-1}[\alpha_i^{h+1}](\overline{\sigma}(g^{h-1}[\alpha^h]))$at all $\alpha^h \in A^h$.[4]

[4] The reason why we use $\overline{\sigma}$ instead of σ here is that $\pi_i^{h-1}[\alpha_i^{h+1}]$ need not be constant on $\sigma(g^{h-1}[\alpha^h])$ if this set contains more than one member. To choose $\overline{\sigma}$ instead of σ as the solution concept according to which the $(h-1)th$ level delegates are assumed to resolve their game $g^{h-1}[\alpha^h]$ $(h-1 \in K)$ may seem to be rather artificial. This artificiality will cause no trouble, of course, if we regard a linear duopoly cascade Γ^k as a regulatory mechanism, for a regulatory mechanism is an artificial design by its very nature. But if one tries to use Γ^k in the spirit of VFJS for explanatory purposes, then it is natural to ask why the $(h-1)th$ level delegates should resolve their game $g^{h-1}[\alpha^h]$ according to $\overline{\sigma}$ and not according to some other solution concept. The same question would still remain valid, however, if one could utilize σ instead of $\overline{\sigma}$ in the construction of Γ^k. What would guarantee that managers (delegates) will resolve their game à la Cournot and not according to some other solution concept? This, in fact, is one of the three major problems which Koray and Sertel (1989) identify with the VFJS theory.
There is actually a rather natural scenario underlying the solution concept $\overline{\sigma}$ in the particular context of a linear duopoly cascade. Assume that all the delegates (except for the top ones) are endowed with the following lexicographical preference: If a delegate is indifferent between two outcomes according to the maximand delegated to him from above, then he prefers that outcome from among the two which leads to a higher consumers' surplus on the industry floor. One can check that the Cournot-Nash solution of a game among the delegates at the $(h-1)th$ level carrying these lexicographical preferences is nothing but $\overline{\sigma}(g^{h-1}[\alpha^h])$.
Finally, let us also note that when we come to the proofs of our main results in this paper, which are all about symmetric linear duopoly cascades Γ^k (i.e., those with $A_1^{k+1} = A_2^{k+1}$), it will become clear that $\overline{\sigma}$ is but one of a large variety of refinements of σ which we could just as well have used in the construction of a linear duopoly cascade without affecting our results.

Note that *Lemma* 2.2 above not only guarantees that the linear duopoly cascade of order k is a well-defined game cascade for each $n \in \mathbb{N}$, but also gives a complete characterization of it which we record as our next proposition.

Proposition 2.4 *In the linear duopoly cascade* Γ^k *of order* $k \in \mathbb{N}$, $\pi_i^h[\alpha_i^{h+1}](\alpha^h)$ *is given through (8) for each* $i \in \{1, 2\}$, $k \in K\setminus\{0\}$ *and* $(\alpha_i^{h+1}, \alpha^h) \in A_i^{h+1} \times A^h$.

The linear duopoly cascade Γ^k of order k can again be regarded as a common framework allowing to extend both the VFJS and the KS theories to the case where the delegation chain is arbitrarily long. In terms of VFJS, we imagine a linear duopoly where the owners of the two firms with true efficiencies $\alpha_1^{k+1}, \alpha_2^{k+1} \in [0, \infty)$, resp., are top delegators placed at the kth level (i.e., players of the game $g^k[\alpha^{k+1}]$) and each owner $i \in M$ hands down a maximand (completely determined by the choice of a possibly fictitious efficiency $\alpha_i^k \in A_i^k$) to his $(k-1)th$ level manager, defining a game $g^{k-1}[\alpha^k]$ among the $(k-1)th$ level managers. These managers are counted on to resolve the game $g^{k-1}[\alpha^k]$ according to the solution concept $\overline{\sigma}$ and to delegate maximands (again entirely determined once the parameters $\alpha_i^{k-1} \in A_i^{k-1} (i = 1, 2)$ are chosen) to their subordinates at the $(k-2)th$ level in due fashion, now defining a game $g^{k-2}[\alpha^{k-1}]$ among the $(k-2)th$ level managers. This recursive process where a maximand is delegated to a lower delegate at each link finally arrives at managers on the industry floor who then act, without further delegation, resolving their game $g^0[\alpha^1]$ à la Cournot. The top delegators (owners) are, of course, assumed to know in advance how many different levels of subordinates they have and reckon that their subordinates at each level will behave according to the rationale of the solution concept $\overline{\sigma}$, the lth level subordinates acting as if they were lth level bosses with true efficiencies $\alpha_1^{l+1}, \alpha_2^{l+1}$, resp., where $\alpha_1^{l+1}, \alpha_2^{l+1}$ are the parameters handed down to the lth level managers from the $(l+1)th$ level. Also note that the game $g^k[\alpha^{k+1}] = \left(\left\{A_i^k\right\}_{i \in M}, \left\{\pi_i^k[\alpha_i^{k+1}]\right\}_{i \in M}\right)$ defined in definition 2.3 and characterized in proposition 2.4 is such that the utility functions $\pi_i^k[\alpha_i^{k+1}] (i \in M)$ actually take the overall effect of the recursive delegation process into account.

In terms of the KS theory, on the other hand, the linear duopoly cascade Γ^k of order k can be thought of as a regulatory mechanism applied to a linear duopoly where the two firms have true efficiencies $\alpha_1^{k+1}, \alpha_2^{k+1}$, resp.. Here we imagine that each owner $i \in M$ with true efficiency α_i^{k+1} is allowed to pretend to have a (possibly fictitious) $\alpha_i^k \in [0, \infty)$, but is required to behave accordingly thereafter, i.e., to have his firm produce that quantity which would be produced by "his manager" on the industry floor in a linear duopoly cascade Γ^{k-1} of order $k-1$ with $A^k = \left\{\alpha^k\right\}$.

Although the mathematical analysis is the same (namely, that of the linear duopoly cascade Γ^k of order $k \in \mathbb{N}$) whichever scenario one takes, the results this analysis leads to in the case of a "symmetric" duopoly render Γ^k unsuitable as a device of positive theory in the fashion utilized by VFJS[5] while, on the other hand, they seem to indicate that Γ^k is a promising tool for purposes of economic design.

[5] For a detailed discussion of problems with the VFJS theory, see Koray and Sertel (1989).

3 Main results

In this section we will first show that the delegatory institution as modeled in the linear duopoly cascade Γ^k of order k, for any fixed $k \in \mathbb{N}$, is unstable in the sense that it gives incentives for redelegation. In other words, at any Cournot-Nash equilibrium of the owners in the game $g^k[\alpha^{k+1}]$ it will pay an owner to increase the length of his delegation chain by redelegation one more time, meaning, of course, that the original delegation chains cannot be considered to be institutionally at equilibrium if each owner has the right to choose the length of his own delegation chain. To prove this result we will first formalize what we mean by an owner's lengthening his delegation chain in a linear duopoly cascade Γ^k by a further link.[6]

Definition 3.1 Let $\Gamma^k = \left(\{g^h\}_{h\in K}, \{A_i^{k+1}\}_{i\in M}\right)$ with $g^h = \left(\{A_i^h\}_{i\in M}, \{\pi_i^h\}_{i\in M}\right)$ for each $h \in K$ be a linear duopoly cascade of order $k \in \mathbb{N}$. Given $l \in M$, we mean by the l-*extension* of Γ^k the game cascade $\overline{\Gamma}^{k+1} = \left(\{\overline{g}^h\}_{h\in K\cup\{k+1\}}, \{\overline{A}_i^{k+2}\}_{i\in M}\right)$ of order $k+1$ satisfying the following conditions:

(3.1.1) $\overline{A}_i^{k+2} = A_i^{k+1}$ for each $i \in M$;
(3.1.2) $\overline{g}^h = g^h$ for each $h \in K$;
(3.1.3) $\overline{g}^{k+1} = \left(\{\overline{A}_i^{k+1}\}_{i\in M}, \{\overline{\pi}_i^{k+1}\}_{i\in M}\right)$, where $\overline{A}_l^{k+1} = [0,\infty)$, $\overline{A}_j^{k+1} = A_j^{k+1}$ with $\{j,l\} = M$ and $\overline{\pi}_i^{k+1}(\overline{\alpha}_i^{k+2}, \overline{\alpha}^{k+1}) = \pi_i^k[\overline{\alpha}_i^{k+2}](\overline{\sigma}(\overline{g}^k[\overline{\alpha}^{k+1}]))$ for all $(\overline{\alpha}_i^{k+2}, \overline{\alpha}^{k+1}) \in \overline{A}_i^{k+2} \times \overline{A}^{k+1}$.

Henceforth we will assume that our linear duopoly cascade Γ^k is a *symmetric* one in the sense that $A_1^{k+1} = A_2^{k+1}$. Also it is clear that the following proposition expressed in terms of the l-extension of a symmetric linear duopoly cascade Γ^k is equivalent to saying that it pays an owner to lengthen his delegation chain in Γ^k by a further link.

Proposition 3.2 *Let* $\Gamma^k = \left(\{g^h\}_{h\in K}, \{A_i^{k+1}\}_{i\in M}\right)$ *be a symmetric linear duopoly cascade with* $A_1^{k+1} = A_2^{k+1} = \{\alpha\}$ *and let* $\overline{\Gamma}^{k+1} = \left(\{\overline{g}^h\}_{h\in K\cup\{k+1\}}, \{\overline{A}_i^{k+2}\}_{i\in M}\right)$ *be the* l*-extension of* Γ^k *for some* $l \in M$. *Then*

$$\overline{\pi}_l^{k+1}[\alpha](\sigma(\overline{g}^{k+1}[(\alpha,\alpha)])) > \pi_l^k[\alpha](\sigma(g^k[(\alpha,\alpha)]))$$

for all $\alpha \in (0,\infty)$.

Proof One easily obtains from (A5) in the Appendix that

$$\sigma(g^k[(\alpha,\alpha)]) = \left\{\left(\frac{(k+1)(2k+1)}{k(2k+3)}\alpha, \frac{(k+1)(2k+1)}{k(2k+3)}\alpha\right)\right\},$$

[6] Another way in which the delegation chain can be extended is by one of the managers in the delegation chain himself appointing a delegate - imagine the manager at the bottom of the chain appointing his bottom manager, for instance. Formally, we are not analyzing this, but the logic certainly seems to be the same.

and thus

$$\pi_l^k[\alpha](\sigma(g^k[(\alpha,\alpha)])) = \frac{(k+1)\alpha^2}{(2k+3)^2}$$

To find $\sigma(\overline{g}^{k+1}[(\alpha,\alpha)])$, we first compute $\overline{\pi}_l^{k+1}[\alpha] : \overline{A}^{k+1} \to \mathbb{R}$ utilizing the fact that $\overline{\pi}_l^{k+1}[\alpha](\alpha_l^{k+1},\alpha) = \overline{\pi}_l^k[\alpha](\overline{\sigma}(\overline{g}^k[(\alpha_l^{k+1},\alpha)]))$ for each $\alpha_l^{k+1} \in \overline{A}_l^{k+1} = [0,\infty)$, and we get

$$\overline{\pi}_l^{k+1}[\alpha](\alpha_l^{k+1},\alpha) = \begin{cases} 0, \text{ if } \alpha_l^{k+1} \le \frac{k+1}{k+2}\alpha \\ \frac{k+1}{(2k+3)^2}((k+2)\alpha - (k+1)\alpha_l^{k+1})((k+2)\,\alpha_l^{k+1} - (k+1)\alpha), \\ \qquad \text{if } \frac{k+1}{k+2}\alpha \le \alpha_l^{k+1} \le \frac{k+2}{k+1}\alpha \\ \frac{k+1}{k+2}\alpha_l^{k+1}(\alpha - \frac{k+1}{k+2}\alpha_l^{k+1}), \text{ if } \frac{k+2}{k+1}\alpha \le \alpha_l^{k+1} \end{cases}.$$

Now note that

$$\max_{\alpha_l^{k+1}\in A_l^{k+1}} \overline{\pi}_l^{k+1}[\alpha](\alpha_l^{k+1},\alpha) = \frac{\alpha^2}{4(k+2)},$$

and

$$\overline{\pi}_l^{k+1}[\alpha](\tilde{\alpha}_l^{k+1},\alpha) = \frac{\alpha^2}{4(k+1)} \text{ iff } \tilde{\alpha}_l^{k+1} = \frac{1}{2}\frac{(k+1)^2+(k+2)^2}{(k+1)(k+2)}\alpha.$$

So,

$$\sigma(\overline{g}^{k+1}[(\alpha,\alpha)]) = \left\{\left(\tilde{\alpha}_l^{k+1},\alpha\right)\right\},$$

and thus,

$$\overline{\pi}_l^{k+1}[\alpha](\sigma(\overline{g}^{k+1}[(\alpha,\alpha)])) = \frac{\alpha^2}{4(k+2)}.$$

Now it is straightforward to check that

$$\frac{\alpha^2}{4(k+2)} > \frac{(k+1)\alpha^2}{(2k+3)^2},$$

i.e.,

$$\overline{\pi}_l^{k+1}[\alpha](\sigma(\overline{g}^{k+1}[(\alpha,\alpha)])) > \pi_l^k[\alpha](\sigma(g^k[(\alpha,\alpha)])).$$

□

Note that in the trivial case where $\alpha = 0$, the inequality in the proposition above will collapse to equality, i.e., the owners will be indifferent about the length of their delegation chains since their profits will be zero in any case.

Finally, we will examine now what happens when the length of the delegation chain grows beyond bound in a symmetric linear duopoly cascade. More specifically, we will examine the effect of increasing the length of the delegation chain on

(i) the maximands delegated by the top delegators (owners), and

(ii) the total industrial output.

Now remember that we have as our underlying structure a symmetric linear duopoly where each firm $i \in M$ produces the quantity $x_i \in X_i = [0, \infty)$ of a good at a constant marginal cost $c_i \in (0, a]$ with $c_1 = c_2 = c$ (or equivalently with an efficiency $\alpha^i = a - c_i \in [0, a)$ where $\alpha^1 = \alpha^2 = \alpha = a - c$) and faces an inverse demand $p = a - (x_1 + x_2)$. In a linear duopoly cascade Γ^k of order k constructed upon this symmetric linear duopoly we have

$$A_1^{k+1} = A_2^{k+1} = \{\alpha\}\,. \tag{9}$$

While (8) allows use to compute that for any $k \in \mathbb{N}$

$$\sigma(g^k[(\alpha, \alpha)]) = \left\{\left(\frac{(k+1)(2k+1)}{k(2k+3)}\alpha, \frac{(k+1)(2k+1)}{k(2k+3)}\alpha\right)\right\} \tag{10}$$

represents the unique Cournot-Nash equilibrium pair of maximands to be handed down by the top delegators (owners) (expressed, of course, in terms of "pretended" efficiencies). Furthermore, this equilibrium pair of maximands sent down by the owners uniquely determines a total output $x(\alpha; k)$ to be actually produced on the industry floor which is given through the formula[7]

$$x(\alpha; k) = \sum_{i \in M} \sigma_i(g^0[\ldots \sigma(g^{k-1}[\sigma(g^k[(\alpha, \alpha)])]) \ldots]). \tag{11}$$

We will now prove that, as the length of the delegation chain grows beyond bound (i.e., as $k \to \infty$), the maximands delegated by the owners converge in a monotonically decreasing manner to the true profit function (thus, $\lim_{k\to\infty} \sigma(g^k[(\alpha, \alpha)]) = (\alpha, \alpha)$), and the total industrial output converges in a monotonically increasing manner to the socially efficient one (thus,$\lim_{k\to\infty} x(\alpha; k) = a - c = \alpha$).

Theorem 3.3 *If, for each $k \in \mathbb{N}$, $\Gamma^k = \left(\{g^h\}_{h \in K}, \{A_i^{k+1}\}_{i \in M}\right)$ is a symmetric linear duopoly cascade with $A_1^{k+1} = A_2^{k+1} = \{\alpha\}$, then as $k \to \infty$*

(i) $\sigma(g^k[(\alpha, \alpha)])$ converges in monotonically decreasing fashion to (α, α), and

(ii) $x(\alpha; k)$ converges in monotonically increasing fashion to α.

[7] Note that it makes no difference whether we apply σ or $\overline{\sigma}$ here, since all Cournot-Nash solutions occurring in this formula are singletons, a fact which follows directly from the symmetry present in the linear duopoly cascade considered.

Proof (i) For each $k \in \mathbb{N}$, $\sigma(g^k[(\alpha,\alpha)])$ is given through (10). It is now obvious that $\left\{\sigma\left(g^k[(\alpha,\alpha)]\right)\right\}_{k\in\mathbb{N}}$ is a monotonically decreasing sequence and $\lim_{k\to\infty}\sigma(g^k[(\alpha,\alpha)]) = (\alpha,\alpha)$.

(ii) One can easily compute that, for each $k \in \mathbb{N}$,

$$\sigma(g^0[\ldots\sigma(g^{k-1}[\sigma(g^k[(\alpha,\alpha)])])\ldots]) = \left\{\left(\frac{k+1}{2k+3}\alpha, \frac{k+1}{2k+3}\alpha\right)\right\}$$

and thus,

$$x(\alpha;k) = \frac{2(k+1)}{2k+3}\alpha.$$

Now it is clear that $\{x\,(\alpha;k)\}_{k\in\mathbb{N}}$ is a monotonically increasing sequence and $\lim_{k\to\infty} x(\alpha;k) = \alpha$. □

Note that, for any fixed $k \in \mathbb{N}$, each owner exaggerates the efficiency of his firm when he sends down a maximand to his immediate subordinate, for the parameter $\frac{(k+1)(2k+1)}{k(2k+3)}\alpha$ which he delegates to his immediate subordinate, (i.e., his "pretended" efficiency) is greater than his firm's true efficiency α (except for the trivial case where $\alpha = 0$, which we will ignore in what follows). Moreover, the efficiency is further exaggerated at each link of the delegation chain, as the parameter $\frac{(k+1)(2k+1)}{k(2k+3)}\alpha^{h+1}$ is greater than α^{h+1} whenever $h \in K\setminus\{0\}$, where α^{h+1} is the parameter which the hth level delegate receives from the $(h+1)th$ level delegate. So total industrial output corresponds to that at the ordinary Cournot equilibrium of a symmetric linear duopoly with an exaggerated efficiency and is thus greater than total output at the ordinary Cournot equilibrium of the actually existing symmetric linear duopoly whose true efficiency is α.

Now it may seem paradoxical that as the length of the delegation chain gets larger, the owners exaggerate their efficiency less, yet total industrial output becomes greater. The explanation is, of course, that although the efficiency delegated by the owners to their respective immediate subordinates gets smaller as k increases, the efficiency figure which the delegates on the industry floor receive from above grows, due to the fact that the efficiency is exaggerated at a larger number of links as k gets bigger. In fact, the efficiency handed down to the delegates on the industry floor is equal to $\frac{3(k+1)}{2k+3}\alpha$ which is a monotonically increasing function of k. Also note that as k grows beyond bound this quantity converges to $\frac{3}{2}\alpha$, in consistency with the fact that output at the ordinary Cournot equilibrium of a symmetric linear duopoly with efficiency $\frac{3}{2}\alpha$ is equal to the socially efficient output of a symmetric linear duopoly with efficiency α.

4 Closing remarks

Here we have introduced an institutional setting which forms a common framework for both the VFJS and KS theories but is also much more general than that. The concept of a linear duopoly cascade allows us to extend our different kinds of pretend-but-perform mechanism (Koray and Sertel 1987, 1988; Koray and Sertel 1990), which are nothing but particular game cascades of order 1, to pretend-but-perform mechanisms with delegation chains of arbitrary length. Doing so, furthermore, we obtain a regulatory mechanism which is both efficient and incentive compatible in the limit. That is, by considering a linear duopoly cascade Γ^k of order k as a regulatory mechanism and choosing k sufficiently large, total industrial output can be made as close as one wishes to the socially efficient one, while the declarations of owners about their efficiencies approximate arbitrarily closely their true efficiency.

Regarding the VFJS theory, however, our result 3.2 about the instability of Γ^k for any fixed $k \in \mathbb{N}$ (along with other results discussed in Koray and Sertel (1989)) tells us that this construct is not suitable for purposes of an explanatory theory in the sense in which VFJS propose it. For if the owners have incentives for lengthening their respective delegation chains, what would prevent them from doing so? This problem with the VFJS theory does not apply when Γ^k is regarded as a regulatory mechanism, of course, since the regulator's mechanism is so designed that redelegation simply will not occur there.

Indeed, our result 3.3 promotes the design and institution of regulatory mechanisms utilizing the kth order "pretend-but-perform" spirit of linear duopoly cascades Γ^k, as these promise to arbitrarily closely approximate socially efficient and - whatever the virtue - truthful outcomes by choice of $k \in \mathbb{N}$ sufficiently large. The case of order $k = 1$ is our old (1988) pretend-but-perform mechanism (PPM) where each duopolist[8] bids to produce henceforth with a Cournotic reaction function parametrized by his declared efficiency. The Nash equilibrium in their efficiency pretensions leads the duopolists, at the Cournot equilibrium under their pretended efficiencies, to greater output and social efficiency. In the $2nd$ order PPM the duopolists are now faced with Γ^2, each pretending to have an efficiency according to which he will react in efficiency pretension to the efficiency declaration (pretension) of his opponent in the context of a first order PPM. And so on for PPMs of higher order. Note that PPMs need only to be instituted for $m = 2$ players, i.e. for duopolies alone, since this suffices to approximate socially efficient results so long as k is chosen large enough.

The constancy of marginal cost in our setup plays a more important role than that of simplifying our computations. Without it, players in a first order PPM context have to be allowed to ask for compensations depending on the output vector (see Koray and Sertel (1987)) in order to commit exhibiting their chosen Cournotic reactions, for otherwise they cannot be sure to come out without a loss. The analysis becomes significantly more involved in this case.

The symmetric linear duopoly cascade considered here is only a particular and modest example of general game cascades which seem to offer a promising tool for the analysis of hierarchical structures in general.[9]

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[8] Koray and Sertel (1987) may also be consulted for the case of PPMs for m oligopolists.

[9] It is noteworthy that Ichiishi's (1989) analysis of stability (having the core non-empty for every preference profile which can be ascribed to the players) in the context of games in effective function form leads to a very particular hierarchical("authority") structure which readily admits of a game cascade formulation as well.

Epilogue

The modesty of our symmetric linear duopoly cascade is also reflected in the form we assume about the inverse demand, which we take as a polynomial function of degree 1. Although whether our results can be extended to duopolistic markets with inverse demands of a more complicated form still seems to be an open problem, the chances seem not to be promising in that regard in the light of the results found by Gradstein (1995), as we will come across with compensation problems similar to those with nonlinear cost functions.

In closing we wish to note that in our model the duopolistic firms can only redelegate further, but are not allowed to shorten the length of the delegation chain. Our result is simply that each of the two firm owners prefers to lengthen his delegation chain given that the other firm keeps it fixed. An interesting problem that is not dealt with in this paper concerns the"delegation chain length choosing" game. It is intuitively clear that the lengths of the delegation chains will not grow indefinitely if the firm owners are also equipped with the possibility to shorten it. The equilibria of a game in which the length of the delegation chain is the strategic variable for both principals are expected to be informative about at what length of the chain the gains from naked quantity competition start to overweigh the gains from a larger market share.

Appendix

Proof of Lemma 2.2: We first consider the case where $k = 1$. From Koray and Sertel (1988) we know that

$$\pi_i^1[\alpha_i^2](\alpha^1) = \begin{cases} 0, \text{ if } \alpha_i^1 \leq \frac{1}{2}\alpha_j^1 \\ \frac{1}{3}(\alpha_i^2 - \frac{1}{3}(\alpha_i^1 + \alpha_j^1))(2\alpha_i^1 - \alpha_j^1), \text{ if } \frac{1}{2}\alpha_j^1 \leq \alpha_i^1 \leq 2\alpha_j^1 \\ \frac{1}{2}\alpha_i^1(\alpha_i^2 - \frac{1}{2}\alpha_i^1), \text{ if } 2\alpha_j^1 \leq \alpha_i^1. \end{cases} \tag{A1}$$

for all $(\alpha_i^2, \alpha^1) \in A_i^2 \times A^1$, where $\{i, j\} = \{1, 2\}$, and that

$$\sigma(g^1[\alpha^2]) = \begin{cases} \{(\alpha_1, \alpha_2^2) \in A^1 | \alpha_1 \in [0, \frac{1}{2}\alpha_2^2]\} \cup \\ \{(\alpha_1, \alpha_2) \in A^1 | \alpha_2 = 2\alpha_1 \text{ and } \alpha_1 \in [\frac{1}{2}\alpha_2^2, \frac{2}{3}\alpha_2^2]\}, & \text{if } \alpha_1^2 \leq \frac{1}{2}\alpha_2^2 \\ \{(\alpha_1, \alpha_2) \in A^1 | \alpha_2 = 2\alpha_1 \text{ and } \alpha_1 \in [\alpha_1^2, \frac{2}{3}\alpha_2^2]\}, & \text{if } \frac{1}{2}\alpha_2^2 < \alpha_1^2 \leq \frac{2}{3}\alpha_2^2 \\ \{(\frac{2}{5}(4\alpha_1^2 - \alpha_2^2), \frac{2}{5}(4\alpha_2^2 - \alpha_1^2))\}, & \text{if } \frac{2}{3}\alpha_2^2 < \alpha_1^2 < \frac{3}{2}\alpha_2^2 \\ \{(\alpha_1, \alpha_2) \in A^1 | \alpha_1 = 2\alpha_2 \text{ and } \alpha_2 \in [\alpha_2^2, \frac{2}{3}\alpha_1^2]\}, & \text{if } \frac{3}{2}\alpha_2^2 \leq \alpha_1^2 < 2\alpha_2^2 \\ \{(\alpha_1^2, \alpha_2) \in A^1 | \alpha_2 \in [0, \frac{1}{2}\alpha_1^2]\} \cup \\ \{(\alpha_1, \alpha_2) \in A^1 | \alpha_1 = 2\alpha_2 \text{ and } \alpha_2 \in [\frac{1}{2}\alpha_1^2, \frac{2}{3}\alpha_1^2]\}, & \text{if } 2\alpha_2^2 \leq \alpha_1^2 \end{cases} \tag{A2}$$

for any $\alpha^2 \in [0, \infty)^2$. From (A2) one easily computes

$$\overline{\sigma}(g^1[\alpha^2]) = \begin{cases} \left\{\left(\frac{2}{3}\alpha_2^2,\ \frac{4}{3}\alpha_2^2\right)\right\}, & \text{if } \alpha_1^2 \leq \frac{2}{3}\alpha_2^2 \\ \left\{\left(\frac{2}{5}(4\alpha_1^2 - \alpha_2^2),\ \frac{2}{5}(4\alpha_2^2 - \alpha_1^2)\right)\right\}, & \text{if } \frac{2}{3}\alpha_2^2 < \alpha_1^2 < \frac{3}{2}\alpha_2^2 \\ \left\{\left(\frac{4}{3}\alpha_1^2,\ \frac{2}{3}\alpha_1^2\right)\right\}, & \text{if } \frac{3}{2}\alpha_2^2 \leq \alpha_1^2 \end{cases} \tag{A3}$$

Thus, $\overline{\sigma}(g^1[\alpha^2])$ is a singleton for any $\alpha^2 \in [0, \infty)^2$ and, so, our lemma holds if $k = 1$.

Now assume that the lemma is true for all game cascades Γ^k of order k for some $k \in \mathbb{N}$, and take a game cascade $\Gamma^{k+1} = \left(\{g^h\}_{h \in K \cup \{k+1\}}, \{A_i^{k+2}\}_{i \in M}\right)$ with $g^h = \left(\{A_i^h\}_{i \in M}, \{\pi_i^h\}_{i \in M}\right)$ satisfying 2.2.1-3 for each $h \in K \cup \{k+1\}$. For any $\alpha^{k+1} \in A^{k+1}$, we now define a game cascade $\overline{\Gamma}^k[\alpha^{k+1}] = \left(\{\overline{g}^h\}_{h \in K}, \{\overline{A}_i^{k+1}\}_{i \in M}\right)$ by letting $\overline{g}^h = g^h$ for each $h \in K$ and $\overline{A_i}^{k+1} = \{\alpha_i^{k+1}\}$ for each $i \in M$. It is clear that $\overline{\Gamma}^k$ also satisfies 2.2.1-3 and hence, by our induction assumption, we have for each $h \in K \setminus \{0\}$,

$$\pi_i^h[\alpha_i^{h+1}](\alpha^h) = \begin{cases} 0, & \text{if } \alpha_i^h \leq \frac{h}{h+1}\alpha_j^h \\ \frac{h}{2h+1}(\alpha_i^{h+1} - \frac{h}{2h+1}(\alpha_i^h + \alpha_j^h))((h+1)\alpha_i^h - h\alpha_j^h), & \text{if } \frac{h}{h+1}\alpha_j^h \leq \alpha_i^h \leq \frac{h+1}{h}\alpha_j^h \\ \frac{h}{h+1}\alpha_i^h(\alpha_i^{h+1} - \frac{h}{h+1}\alpha_i^h), & \text{if } \frac{h+1}{h}\alpha_j^h \leq \alpha_i^h \end{cases} \tag{A4}$$

at each $(\alpha_i^{h+1}, \alpha^h) \in A_i^{h+1} \times A^h$, where $\{i, j\} = \{1, 2\}$. Also $\overline{\sigma}(g^k[\alpha^{k+1}])$ is a singleton. In fact, utilizing (A4), we compute that

$$\sigma(g^k[\alpha^{k+1}]) = \begin{cases} \left\{\left(\alpha_1^k,\ \frac{k+1}{2k}\alpha_2^{k+1}\right) \in A^k | \alpha_1^k \in [0,\ \frac{1}{2}\alpha_2^{k+1}]\right\} \cup & \\ \left\{\left(\alpha_1^k, \alpha_2^k\right) \in A^k | \alpha_2^k = \frac{k+1}{k}\alpha_1^k \text{ and } \alpha_1^k \in [\frac{1}{2}\alpha_2^{k+1},\ \frac{k+1}{k+2}\alpha_2^{k+1}]\right\} & \text{if } \alpha_1^{k+1} \leq \frac{1}{2}\alpha_2^{k+1} \\ \left\{\left(\alpha_1^k, \alpha_2^k\right) \in A^k | \alpha_2^k = \frac{k+1}{k}\alpha_1^k \text{ and } \alpha_1^k \in [\alpha_1^{k+1},\ \frac{k+1}{k+2}\alpha_2^{k+1}]\right\} & \text{if } \frac{1}{2}\alpha_2^{k+1} < \alpha_1^{k+1} \leq \frac{k+1}{k+2}\alpha_2^{k+1} \\ \left\{\left(\frac{k+1}{(2k+3)k}(2(k+1)\alpha_1^{k+1} - \alpha_2^{k+1}),\ \frac{k+1}{(2k+3)k}(2(k+1)\alpha_2^{k+1} - \alpha_1^{k+1})\right)\right\} & \text{if } \frac{k+1}{k+2}\alpha_2^{k+1} < \alpha_1^{k+1} < \frac{k+2}{k+1}\alpha_2^{k+1} \\ \left\{\left(\alpha_1^k, \alpha_2^k\right) \in A^k | \alpha_1^k = \frac{k+1}{k}\alpha_2^k \text{ and } \alpha_2^k \in [\alpha_2^{k+1},\ \frac{k+1}{k+2}\alpha_1^{k+1}]\right\} & \text{if } \frac{k+2}{k+1}\alpha_2^{k+1} \leq \alpha_1^{k+1} < 2\alpha_2^{k+1} \\ \left\{\left(\frac{k+1}{2k}\alpha_1^{k+1}, \alpha_2^k\right) \in A^k | \alpha_2^k \in [0,\ \frac{1}{2}\alpha_1^{k+1}]\right\} \cup & \\ \left\{\left(\alpha_1^k, \alpha_2^k\right) \in A^k | \alpha_1^k = \frac{k+1}{k}\alpha_2^k \text{ and } \alpha_2^k \in [\frac{1}{2}\alpha_1^{k+1},\ \frac{k+1}{k+2}\alpha_1^{k+1}]\right\} & \text{if } 2\alpha_2^{k+1} \leq \alpha_1^{k+1} \end{cases} \tag{A5}$$

But now it easily follows from (A5) that

$$\overline{\sigma}(g^k[\alpha^{k+1}]) = \begin{cases} \left\{\left(\frac{k+1}{k+2}\alpha_2^{k+1},\ \frac{(k+1)^2}{k(k+2)}\alpha_2^{k+1}\right)\right\} & \text{if } \alpha_1^{k+1} \leq \frac{k+1}{k+2}\alpha_2^{k+1} \\ \left\{\left(\frac{k+1}{(2k+3)k}(2(k+1)\alpha_1^{k+1} - \alpha_2^{k+1}),\ \frac{k+1}{(2k+3)k}(2(k+1)\alpha_2^{k+1} - \alpha_1^{k+1})\right)\right\} & \text{if } \frac{k+1}{k+2}\alpha_2^{k+1} < \alpha_1^{k+1} < \frac{k+2}{k+1}\alpha_2^{k+1} \\ \left\{\left(\frac{(k+1)^2}{k(k+2)}\alpha_1^{k+1},\ \frac{k+1}{k+2}\alpha_2^{k+1}\right)\right\} & \text{if } \frac{k+2}{k+1}\alpha_2^{k+1} \leq \alpha_1^{k+1} \end{cases} \tag{A6}$$

Now we get for each $\alpha^{k+1} \in A^{k+1}$ that

$$\pi_i^{k+1}[\alpha_i^{k+2}](\alpha^{k+1}) = \pi_i^k[\alpha_i^{k+2}](\overline{\sigma}(g^k[\alpha^{k+1}]))$$
$$= \begin{cases} 0 & \text{if } \alpha_i^{k+1} \leq \frac{k+1}{k+2}\alpha_j^{k+1} \\ \frac{k+1}{2k+2}(\alpha_i^{k+2} - \frac{k+1}{2k+3}(\alpha_i^{k+1} + \alpha_j^{k+1}))((k+2)\alpha_i^{k+1} - (k+1)\alpha_j^{k+1}) & \text{if } \frac{k+1}{k+2}\alpha_j^{k+1} \leq \alpha_i^{k+1} \leq \frac{k+2}{k+1}\alpha_j^{k+1} \\ \frac{k+1}{k+2}\alpha_i^{k+1}(\alpha_i^{k+2} - \frac{k+1}{k+2}\alpha_i^{k+1}) & \text{if } \frac{k+2}{k+1}\alpha_j^{k+1} \leq \alpha_i^{k+1} \end{cases} \quad \text{(A7)}$$

where $\left\{\alpha_i^{k+2}\right\} = A_i^{k+2}$ for each $i \in M$ and $\{i, j\} = \{1, 2\}$. Now in view of (A6) and (A7), we also conclude that $\overline{\sigma}(g^{k+1}[\alpha^{k+2}])$ is obtained from formula (A6) by substituting $k + 1$ for k everywhere in (A6). Hence, $\overline{\sigma}(g^{k+1}[\alpha^{k+2}])$ is a singleton for any $\alpha^{k+2} \in [0, \infty)^2$, completing the proof of our lemma. □

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