INTRINSIC ENTANGLEMENT OF PHOTONS

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Multipole radiation is treated both classically and also quantum mechanically. Dipole atom as a source of radiation is investigated within the Jaynes-Cummings model. Polarization properties of quantum multipole radiation are given. It is shown that multipole photons have all three components of polarization but we can perform a local transformation of radiation frame such that the new $z-$ axis corresponding to linear polarization becomes parallel to the Poynting vector. It is shown that the spin angular momentum and orbital angular momentum have the same operator structure, and in the far zone, they contribute equally to the total angular momentum. Hence in this regime, these two contributions are indistinguishable and they may differ from each other only by spatial dependence in the very vicinity of the source. Another aspect of the behavior in the far zone is that the longitudinal polarization of multipole photons vanish.

A variational approach to entanglement which is introduced recently based on analysis of dynamic symmetry of systems and quantum uncertainties, accompanying the measurement of mean value of basic observables is applied to investigate the intrinsic entanglement of electric dipole photons. The basic observables are defined in terms of an orthogonal basis of Lie Algebra, corresponding to the dynamic symmetry group of the system of interest. It is shown that electric dipole photons can carry entanglement with respect to its intrinsic degrees of freedom, namely the spin angular momentum and orbital angular momentum, each of which may be considered as a qubit.

Keywords: Quantum Optics, Quantum Multipole Radiation, Spin Angular Momentum, Orbital Angular Momentum, Quantum Entanglement.

ÖZET

FOTONLARIN İÇSEL DOLAŞIKLIĞI

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Bu çalışmada çok kutuplu ışınım hem klasik hem de kuvantum mekaniksel olarak ele alındı. Çift kutuplu atom kaynaklı ışınımın dinamikleri Jaynes-Cummings modeli icinde incelendi. Kuvantum çok kutuplu ışınımın kutuplaşma özellikleri verildi. Çok kutuplu fotonlarin üç kutuplaşma bileşenine de sahip oldukları ancak yerel dönüşümlerle doğrusal kutuplaşmaya karşılık gelen $z-$ekseninin ve Poynting vektörünün paralel hale getirilebileceği gösterildi. Spin ve yörüngesel açısal momentumların aynı işlemci yapısına sahip oldukları ve kaynaktan uzak bölgelerde toplam açısal momentuma katkılarının eşit olduğu gösterilmiştir. Dolayısıyla, kaynaktan uzak bölgelerde birbirlerinden ayrılamazlar. Uzak bolge davranışının bir diğer özelliği de çok kutuplu fotonların boyuna kutuplaşmasının yok olmasıdır.

Dolasıkliğa yakın zamanlarda getirilen, dinamik simetri gruplarına ve temel gözlenebilirlerin ortalama değerlerindeki kuvantum dalgalanmalarına dayanan yaklaşım, çift kutuplu elektrik fotonlarinin içkin dolasıkliğını incelemeye uygulandı. Temel gözlenebilirler incelenen sistemin dinamik simetri grubuna karşılık gelen Lie cebrinin dik bazlari cinsinden tanımlandı. Çift kutuplu elektrik fotonlarının her biri kübit gibi düsünülebilen spin ve yörüngesel açısal momentumlarına göre dolaşıklık taşıyabileceği gösterildi.

Anahtar sözcükler: Kuvantum Optiği, Çok Kutuplu Kuvantum Işinimi, Spin Açısal Momentumu, Yörüngesel Açısal Momentum, Dolaşıklık .

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Contents

Chapter 1

Introduction

Entanglement is one of the properties of quantum theory which caused Einstein and others to dislike the theory. In 1935, Einstein, Podolsky, and Rosen formulated the EPR paradox, demonstrating that entanglement makes quantum theory a non-local theory(see [20]). Einstein considered entanglement as "spooky action at a distance". On the other hand, quantum theory has been highly successful in producing correct experimental predictions, and the strong correlations associated with the phenomenon of quantum entanglement have in fact been observed(see [22, 23]). One apparent way to explain quantum entanglement is an approach known as hidden variable theory, in which unknown deterministic microscopic parameters would cause the correlations. However, in 1964 Bell showed that such a theory could not be local, the quantum entanglement predicted by quantum theory being experimentally distinguishable from a broad class of local hidden-variable theories(see [21]). Results of subsequent experiments have overwhelmingly supported quantum theory. It is known that there are a number of loopholes in these experiments, but these are generally considered to be of minor importance. Entanglement produces some interesting interactions with the principle of relativity that states that information cannot be transferred faster than the speed of light. Although two entangled systems can interact across large spatial separations, no useful information can be transmitted in this way, so causality can not be violated through entanglement. This occurs for two subtle

reasons: (i) Quantum mechanical measurements yield probabilistic results, and (ii) the no cloning theorem forbids the statistical inspection of entangled quantum states. Although no information can be transmitted through entanglement alone, it is possible to transmit information using a set of entangled states used in conjunction with a classical information channel(see [18]). This process is known as quantum teleportation. Despite its name, quantum teleportation cannot be used to transmit information faster than light, because a classical information channel is involved.

Many people think that atoms and ions, interacting with cavity photons, are basic building blocks of quantum information processing. At least, they represent a useful tool for testing quantum algorithms in communications, cryptography, and computing. It is well known that the interaction of the atoms and molecules with the field quantized in a cavity result in the emission of photons with well defined angular momentum and parity. Such photons are called multipole photons. Hence the problem of angular momentum of photons attracted a great deal of interest in the context of quantum computing recently(see [24, 25]). Conventional approach to quantum entanglement deals with the correlations in the spatially separated subsystems of a composite system. However, it has been shown recently that entanglement can be examined in connection with the quantum fluctuations(see [26]) and this allows single particles to be in an entangled state. The main objective of this work is to examine entanglement, within this new approach, of a single electric dipole photon with respect to its intrinsic degrees of freedom, namely its spin and orbital angular momentum.

The thesis is organized as follows:

First, we study multipole radiation both classically and quantum mechanically in Chapter 2. Types of radiations with respect to its parity and corresponding vector potentials are given in the classical picture. The fundamentals of analyzing the global system: fields+particles in the quantum picture is discussed and the Hamiltonian governing the evolution of a simple and an experimentally realizable system is derived within the framework of Jaynes-Cummings model.

In the third chapter, the polarization properties and Poynting vector of quantum multipole radiation are examined. Certain important features of the components of total angular momentum due to polarization and orbital motion, especially their behavior in the near zone and far zone, are verified.

In the fourth chapter, fundamentals of entanglement and how to quantify entanglement are given. Conventional approach to entanglement requires the system be composed of subsystems which are spatially separated. Single particle entanglement cannot be considered in this approach. A recent variational principle which allows single particle entanglement is explained.

In the fifth chapter, the variational principle given in Chapter 4 is applied to investigate the entanglement of a single $E1$ photon and it is shown that such photons can manifest entanglement.

Finally, in the last chapter, we summarize our results.

Chapter 2

Multipole Radiation

2.1 Classical Free Field

Sourceless electromagnetic field can be classically described by the vector potential, $\vec{A}(\vec{r},t)$, which obeys

$$\begin{aligned} \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0, \\ \nabla\cdot\vec{A} = 0. \end{aligned} \tag{2.1}$$

The first of the above equations is called the homogeneous wave equation and the second is the transversality condition. The fields in terms of the vector potential are:

$$\begin{aligned} \vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla}\times\vec{A}. \end{aligned} \tag{2.2}$$

The energy density of the electromagnetic field is

$$W(\vec{r},t) = \frac{1}{16\pi}(E^2+B^2), \tag{2.3}$$

the flux of energy is

$$\vec{S}(\vec{r},t) = \frac{1}{8\pi}\vec{E}\times\vec{B}, \tag{2.4}$$

and the angular momentum density of the field is

$$\vec{M}(\vec{r},t) = \frac{1}{4\pi c}\vec{r}\times(\vec{E}\times\vec{B}). \tag{2.5}$$

Consider the field to be contained in a spherical cavity of radius R at the center of which is a source distribution of very small spatial extension. Assume that the walls of the cavity are perfectly conducting which brings the boundary condition that electric field has no tangential component at the surface. Let us choose the origin of our coordinate system to be located at where the source is located, i.e. at the center of the cavity. Consider the following basis:

$$\vec{\chi}_{\pm} = \mp\frac{\vec{e}_x \pm i\vec{e}_x}{\sqrt{2}}, \quad \chi_0 = \vec{e}_z. \tag{2.6}$$

This basis is called the helicity basis and is introduced to establish the connection with the quantum picture. Indeed, the three vectors in this basis coincide with three states of a spin 1 photon and therefore we can interpret $\vec{\chi}_{\pm}$ as the unit vectors of circular polarization with either positive or negative helicity and $\vec{\chi}_0$ gives linear polarization in the $z-$direction. Within the sign at $\vec{\chi}_{\pm}$, the helicity basis coincides with the so-called polarization basis frequently used in optics. For our purposes, $\vec{A}$, like any other vector, can be expanded in this basis:

$$\vec{A} = \sum_{\mu=-1}^{\mu=1}(-1)^{\mu}\vec{\chi}_{-\mu}A_{\mu}. \tag{2.7}$$

The solution of (2.1) for $\vec{A}$, except at the origin, expressed in the helicity basis becomes

$$\vec{A}_{\lambda}(\vec{r},t) = \sum_{k}\sum_{\mu}\sum_{\jmath}\sum_{m=-\jmath}^{m=\jmath}(-1)^{\mu}\vec{\chi}_{-\mu}V_{\lambda k\jmath m\mu}(\vec{r})a_{\lambda k\jmath m}e^{-i\omega t} + c.c. \tag{2.8}$$

where c.c. means the complex conjugated of the first term. $\lambda = E, M$ denotes the type of radiation. The radiation is of two types with respect to its parity, electric or magnetic. The index $\jmath$ is related to the total angular momentum of the field. The electric type vector potential has parity $(-1)^{j+1}$ and magnetic type vector potential has parity $(-1)^{j}$. Another difference between them is that, in case of electric type radiation, the magnetic field is transverse to the the direction of propagation but the electric field has a radial component and vice versa for

magnetic type radiation, i.e. the electric field is transversal to the direction of propagation and the magnetic field has a longitudinal component. The first term in (2.8) is called the positive frequency part and its conjugated term is called the negative frequency part of the vector potential for obvious reasons. The mode functions are

$$\begin{aligned} V_{Ek\jmath m\mu} &= \gamma_{Ek\jmath}[\sqrt{\jmath}\langle 1, \jmath+1, \mu, m-\mu \mid \jmath, m\rangle f_{\jmath+1}(kr)Y_{\jmath+1,m-\mu}(\theta,\phi) \\ &\quad -\sqrt{\jmath+1}\langle 1, \jmath-1, \mu, m-\mu \mid \jmath, m\rangle f_{\jmath-1}(kr)Y_{\jmath-1,m-\mu}(\theta,\phi)], \\ V_{Mk\jmath m\mu} &= \gamma_{Mk\jmath}\sqrt{\jmath}\langle 1, \jmath, \mu, m-\mu \mid \jmath, m\rangle f_{\jmath}(kr)Y_{\jmath,m-\mu}(\theta,\phi). \end{aligned} \tag{2.9}$$

$\langle .... \mid \jmath, m\rangle$ is the Clebsch-Gordon coefficient and Y_{lm}'s are the spherical harmonics. The complex field amplitudes, a's, are dependent upon the properties of the source, distributions of charges, currents, polarization and magnetization but are independent of position(see [3]). The radial parts in the mode functions are proportional to

$$f_\ell(kr) = \begin{cases} h_\ell^{(1)}(kr), \text{ outgoing spherical wave} \\ h_\ell^{(2)}(kr), \text{ incoming spherical wave} \\ j_\ell(kr), \text{ standing spherical wave} \end{cases} \tag{2.10}$$

where $h_\ell^{(1,2)}$ denote the spherical Hankel functions and j_ℓ denotes the spherical Bessel functions. The proportionality constant is obtained from the condition

$$\int_0^R f_\jmath(kr) f_\jmath(k'r) r^2 dr = \frac{4\pi R^3}{3}\delta_{kk'} = V\delta_{kk'}. \tag{2.11}$$

Index $\jmath$ and m takes values $\jmath = 1, 2, 3, ...$ and $m = -\jmath, ..., \jmath$. The discrete set of variables k are determined by the condition $f_\jmath(kR) = 0$ and $\omega = ck$. In the case of standing waves, the normalization constants in (2.9), $\gamma_{Ek\jmath}$ and $\gamma_{Mk\jmath}$, are

$$\begin{aligned} \gamma_{Ek\jmath} &= \sqrt{\frac{2\pi\hbar c}{kV(2\jmath+1)}}, \\ \gamma_{Mk\jmath} &= \sqrt{\frac{2\pi\hbar c}{kV}} \end{aligned} \tag{2.12}$$

2.2 Quantization of Free Field

The global system of the electromagnetic field and the (nonrelativistic) charged particles can be shown to be equivalent to a set of mutually interacting oscillators and charged particles. In the special case of the free field, the oscillators are completely decoupled(see[19]). All these are formally verified in the classical picture and the quantum picture is based on this result. The simplest idea which can be put forth for quantizing the free field is to quantize the oscillators in the known way as introduced by Dirac(see [28]). This seems to be a heuristic argument and more will be said about the justification of this approach in the next section. The independent oscillators for the system described in the first section are labelled by $\lambda, k, \jmath, m$ and the creation and annihilation operators of oscillators are the field amplitudes, $a_{\lambda k\jmath m}$. The field amplitudes turn into operators which act in a Hilbert space, the space of the field states which is the tensor product of the state space of independent oscillators. There is the following commutation relation, the so-called bosonic commutation relation:

$$[\hat{a}_{\lambda k\jmath m}, \hat{a}^{\dagger}_{\lambda' k' \jmath' m'}] = \delta_{kk'}\delta_{\lambda\lambda'}\delta_{\jmath\jmath'}\delta_{mm'}. \tag{2.13}$$

$\hat{a}$ is called the photon annihilation operator and $\hat{a}^{\dagger}$ is called the photon creation operator. The operators corresponding to the field observables in terms of the creation and annihilation operators are obtained by replacing the field amplitudes with the corresponding photon operators, $a_{\lambda k\jmath m} \rightarrow \hat{a}_{\lambda k\jmath m}, a^{*}_{\lambda k\jmath m} \rightarrow \hat{a}^{\dagger}_{\lambda k\jmath m}$. The vector potential, hence all the other field observables, now become operators acting on the state space of the field and the commutation relation imposed on the photon operators specify the spectrum of these observables. Here it is important that position is not an operator but a parameter like time. Then the positive frequency part of the vector potential operator of the multipole radiation of a given type λ is given by

$$\hat{\vec{A}}_{\lambda}(\vec{r}) = \sum_{k}\sum_{\mu}\sum_{\jmath}\sum_{m=-\jmath}^{m=\jmath}(-1)^{\mu}\vec{\chi}_{-\mu}V_{\lambda k\jmath m}(\vec{r})\hat{a}_{\lambda k\jmath m} \tag{2.14}$$

where harmonic time dependence is included in the creation and annihilation operators. The energy is given by

$$\hat{H}_{spherical} = \sum_{k}\hbar\omega_{k}\sum_{\lambda,\jmath,m}\left(\hat{a}^{\dagger}_{\lambda\jmath m}\hat{a}_{\lambda\jmath m} + \frac{1}{2}\right). \tag{2.15}$$

The vacuum state $|0\rangle$ is such a state that

$$\hat{a}_{\lambda k \jmath m}|0_{spherical}\rangle = 0, \quad \forall \lambda, k, \jmath, m. \tag{2.16}$$

The number states are defined as follows:

$$|n_{\lambda k \jmath m}\rangle = \frac{(\hat{a}^{\dagger}_{\lambda k \jmath m})^n}{\sqrt{n!}}|0_{spherical}\rangle. \tag{2.17}$$

The number states form a basis for the state space of the field but this is only one of an infinite number of possible choices. Coherent states which are overcomplete can be used to construct another basis. The expectation value of the Hamiltonian operator in the vacuum state gives the vacuum contribution:

$$H^{vac}_{spherical} = \sum_k \hbar\omega_k \sum_{\lambda,\jmath,m} \frac{1}{2} = \sum_k \hbar\omega_k \left(\sum_{\jmath} (2\jmath + 1) \right). \tag{2.18}$$

The energies of states other than the vacuum state are even greater and we have infinite energy. Dirac has an explanation for this phenomena. Practically, this is not important because of the following reason. The measuring detector measures averages over finite volumes and finite duration of measurement. Also such measurement filtrates photons of certain frequencies or types. Photons of other frequencies and types are not sensed.

Because of the spherical symmetry, the Hamiltonian commutes with total angular momentum and we can obtain a basis consisting of common eigenstates of both operators. The spherical wave representation corresponds to states of radiation with given angular momentum. The Clebsch-Gordon coefficients in the radial parts of the mode functions represent the addition of the spin and orbital parts of the total angular momentum of the field.

2.3 Dipole Atom as the Source of Radiation

Consider a system of charged particles α with masses m_α and charges q_α. The state space of the global system:fields+particles, ε, is the tensor product of the state space of the particles, ε_P in which the operators like $\hat{\vec{r}}_\alpha, \hat{\vec{p}}_\alpha, ...$ act, and the

state space of the radiation field, ε_R in which the operators like $\hat{a}, \hat{a}^\dagger, \hat{\vec{A}}, \hat{\vec{E}}, \hat{\vec{B}}, ...$ act.

$$\varepsilon = \varepsilon_P \otimes \varepsilon_R. \tag{2.19}$$

ε_R is itself a tensor product of the state spaces of the oscillators associated with the modes of the field which are in general coupled. The oscillators are countable in number and can be indexed. Let ε_i be the state space of the various oscillators i.

$$\varepsilon_R = \varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3 \otimes ... \otimes \varepsilon_i \otimes ... \tag{2.20}$$

One possible orthonormal basis of ε_i is $\{\mid n_i\rangle\}, i = 0, 1, 2, 3, ...$, the number states. If $\{\mid s\rangle\}$ is an orthonormal basis of ε_P, then the tensor product of these two bases provides a basis for the global system and the sate of the global system at any time, $\mid \psi(t)\rangle$, can be specified by giving its components in this final basis. The equation governing the evolution of the global system is

$$i\hbar \frac{d}{dt} \mid \psi(t)\rangle = \hat{H} \mid \psi(t)\rangle \tag{2.21}$$

where $\hat{H}$ is the Hamiltonian operator for the global system. We can split this Hamiltonian into three parts:

$$\hat{H} = \hat{H}_P + \hat{H}_R + \hat{H}_I \tag{2.22}$$

where $\hat{H}_P$(particle Hamiltonian) depends on the variables $\hat{\vec{r}}_\alpha$ and $\hat{\vec{p}}_\alpha$ of the particles. $\hat{H}_R$(radiation Hamiltonian) depends on the photon creation and annihilation operators. The rest is written for the interaction and depends on $\hat{\vec{r}}_\alpha, \hat{\vec{p}}_\alpha, \hat{a}, \hat{a}^\dagger$. $\hat{H}_R$ is that operator given in Section 2. The particle Hamiltonian is

$$\hat{H}_P = \sum_\alpha \frac{\hat{\overrightarrow{p}}_\alpha^{\,2}}{2m_\alpha} + \sum_{\alpha \neq \beta} \frac{q_\alpha q_\beta}{\mid \hat{\overrightarrow{r}}_\alpha - \hat{\overrightarrow{r}}_\beta \mid} \tag{2.23}$$

The interaction Hamiltonian is

$$\hat{H}_I = \sum_\alpha \frac{q_\alpha}{2m_\alpha c} \left(\hat{\overrightarrow{p}} \cdot \hat{\overrightarrow{A}} + \hat{\overrightarrow{A}} \cdot \hat{\overrightarrow{p}} \right). \tag{2.24}$$

Indeed, there is a term proportional to $\hat{\overrightarrow{A}}^{\,2}$ which is neglected in the rotating wave approximation and another term due to the interaction of the spin magnetic moments of the particles with the radiation. We assume that the particles

are spinless. Consequently, the interaction of the spin magnetic moments with the radiation is neglected and in the space of the states of particles, there is no spin consideration. The above Hamiltonian is in the Coulomb gauge and it can be formally reached within the Lagrangian formulations for continuous systems and fields. This Hamiltonian is justified and the normal variables are identified as conjugate variables, a result which justifies the commutation relations, in the Lagrangian formulation(see [19]). It is possible to get other equivalent descriptions of electrodynamics, being adapted to this or that type of problem, either by changing the gauge or by adding the standard lagrangian the total derivative of the generalized coordinates of the system, or else by directly performing a unitary transformation on the Coulomb gauge Hamiltonian(see [19]). All these descriptions predict the same physical results as expected. We also assume that the kinetic energies of the charged particles are small compared to their rest energies and that their number is invariant and also that the modes of the field are not relativistic, i.e. the photons have low frequencies. This is sufficient for low energy domain. In addition, the choice of the Coulomb gauge, which explicitly yields the Coulomb interaction between particles which is predominant at low energy, is very convenient for the study of bound states of charged particles, such as atoms or molecules. A quantum relativistic description of particles requires that one considers them as elementary excitations of a relativistic matter field, such as the Dirac field for electrons and positrons. This consideration is used in Dirac's explanation for the infinite energy of the vacuum. Nonrelativistic Hamiltonians are effective Hamiltonians acting inside manifolds with a fixed number of particles derived from the Hamiltonian of relativistic quantum electrodynamics, in which the number of particles like the number of photons is indeterminate(see [2]).

Even though, it is mathematically allowed that there may exist photons without any charge at all(see [28]), there is no known photon that had not been created by a source. The simplest quantum source of photons is the atomic transition, creating according to the selection rules, photons. Consider an atom at the center of an ideal spherical cavity under the influence of a laser field. Such systems can be described by models in which the atom is considered to be a

two-level one interacting with one(or very few) modes of the cavity field. The fact is that using lasers as sources of electromagnetic radiation, we can act on atoms with field having frequency very close to the transition frequency between any pair of levels. In this case, the influence of the other levels can be ignored and we can consider the atom to be two-level one. On the other hand, the use of high quality cavities has the consequence that in such a cavity the atom interacts with one(or a few) modes of the field quantized in the cavity. Transitions in the atom occur between states of well defined angular momentum and parity. Indeed many representations such as states of photons with definite linear momentum, the so-called plane wave representation, may also be used but since $[\hat{\vec{J}}, \hat{\vec{H}}] = 0$, the most convenient representation is provided by photons with definite angular momentum and parity which are spherical photons. The branch of quantum optics studying the process of interaction of one or few atoms with the quantized cavity modes is called cavity QED. The theoretical concept of cavity QED are based in the first place on the investigation of Jaynes-Cummings model and its generalizations. The reason is that the model describes the process fairly well and admits exact solutions. In the usual formulation of Jaynes-Cummings model, the atom is considered as though it consists of two nondegenerate levels. However, in real atoms the radiative transitions occur between states with given angular momentum quantum numbers $|\jmath, m\rangle \rightarrow |\jmath', m'\rangle$ such that $j > j' \geq 0$. This means that at least the upper level is degenerated with respect to the quantum number $m(-\jmath \leq m \leq \jmath)$. In the case of electric dipole transitions between the states $|\jmath = 1, m = 0, \pm 1\rangle$ and $|\jmath' = 0, m' = 0\rangle$, the excited state is triple degenerate. But Jaynes-Cummings model can be generalized to such situations.

Consider the electric dipole transition between the triple degenerated excited atomic state with $\jmath = 1$ and the nondegenerated ground state with $\jmath = 0$. The atom is supposed to be located at the center of an ideal spherical cavity. We denote the mass of the electron by m_e and its charge by e. $\vec{p}$ is the momentum operator for the electron and $\vec{A}$ is the vector potential operator for the free field. The coupling constant of the atom-field interaction can be found by calculating the matrix element

$$-\frac{e}{2m_e c}\langle \jmath = 0, m = 0|\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}|\jmath = 1, m\rangle \tag{2.25}$$

obtained form the Hamiltonian (2.24). We may denote the atomic states simply as $|\jmath = 0, m = 0\rangle \equiv |g\rangle$ and $|\jmath = 1, m\rangle \equiv |m\rangle$. Using the fact that $\vec{E} = ik\vec{A}, \omega = ck$ with ω being the transition frequency and $\vec{p}\cdot\vec{A}$ is equivalent to the $\vec{r}\cdot\vec{E}$, we can write the coupling constant as

$$ik\langle g|\vec{d}\cdot\vec{A}|m\rangle \tag{2.26}$$

where $d = e\vec{r}$ is the dipole moment of the atomic transition. We can represent the atomic states under consideration with the wavefunctions

$$\begin{aligned}\langle r,\theta,\phi|m\rangle &= R_{exc}(r)Y_{1m}(\theta,\phi),\\ \langle r,\theta,\phi|g\rangle &= R_{grnd}(r)Y_{00}(\theta,\phi).\end{aligned} \tag{2.27}$$

In the $\{\chi_\mu\}$ basis

$$\begin{aligned}\vec{r} &= \frac{r}{\sqrt{2}}\sin\theta e^{i\phi}\vec{\chi}_{-1} + r\cos\theta\vec{\chi}_0 - \frac{r}{\sqrt{2}}\sin\theta e^{i\phi}\vec{\chi}_1,\\ \vec{A}\cdot\vec{B} &= \sum_{\mu=-1}^{\mu=1} A_\mu B_{-\mu}.\end{aligned} \tag{2.28}$$

and the coupling constant, for the case when the excited atomic state is taken to have the projection quantum number m, g_m, is

$$g_m = k\langle g|\vec{d}\cdot\vec{A}|m\rangle = \begin{cases} -\frac{\gamma k}{\sqrt{6\pi}}(\frac{7}{10}D_2 + D_0), & \text{if m=}\pm 1\\ \frac{\gamma k}{\sqrt{6\pi}}(\frac{1}{5}D_2 - D_0), & \text{if m=0}\end{cases} \tag{2.29}$$

where γ is the normalization constant and

$$D_\ell = \int_0^{r_a} r^3 R^\star_{exc}(r)R_{grnd}(r)f_\ell(kr)d^3r \tag{2.30}$$

Above r_a denotes the atomic radius and f_ℓ's are the radial parts of the mode functions. In the case of standing waves $f_\ell = j_\ell$. Owing to the structure of the mode functions, only the below have contributions to the vector potential:

$$\begin{aligned}j_0(kr) &= \frac{\sin kr}{kr},\\ j_2(kr) &= \frac{3-(kr)^2}{(kr)^3}\sin kr - \frac{3\cos kr}{(kr)^2}.\end{aligned} \tag{2.31}$$

Assuming that the atom is very small in dimensions compared to the wavelength of the radiation field, the limiting behavior as $\vec{r}$ approaches the origin is a good

approximation. In the near zone, $kr \ll 1$, $f_0(kr) \approx 1, f_2(kr) \approx 0$. In this limit, we get $V_{Ek1m} = -\delta_{m\mu}$ and

$$\vec{A}_{Ek1}(0) = -\sqrt{\frac{\hbar c}{3kV}} \sum_{m=-1}^{m=1} (-1)^{\mu} \chi_{-\mu} a_{Ek1m} \delta_{m\mu}. \tag{2.32}$$

This means that the electric dipole transition $|m\rangle \rightarrow |g\rangle$creates a photon with spin state(polarization) $\mu = m$. However, the picture of polarization changes with distance from the atom because of the position dependence of the mode functions. Another consequence of using this limit is that the coupling constants become equal to each other for all m which we can be written as

$$g = -\frac{k}{\sqrt{6\pi}} D\gamma \text{ for m=}\pm 1,0. \tag{2.33}$$

The model Hamiltonian can be represented as

$$\begin{aligned} \hat{H} &= \hat{H}_P + \hat{H}_R + \hat{H}_I = \hat{H}_0 + \hat{H}_I, \\ \hat{H}_0 &= \hbar \sum_{m=-1}^{m=1} \omega a_m^{\dagger} a_m + \omega_0 \hat{R}_{mm}, \\ \hat{H}_I &= \sum_{m=-1}^{m=1} ig \hat{R}_{mg} \hat{a}_m + H.c. \end{aligned} \tag{2.34}$$

where ω and ω_0 are the cavity and transition frequencies, respectively, and the atomic operators are

$$\begin{aligned} \hat{R}_{mg} &= |m\rangle\langle g|, \\ \hat{R}_{mm'} &= |m\rangle\langle m'|. \end{aligned} \tag{2.35}$$

Chapter 3

Polarization Properties and Poynting Vector

3.1 Polarization Properties

Consider sourceless classical electromagnetic field contained in a cubical box of edge length L and satisfying periodic boundary conditions at the sides of the cube. The field can be written as a superposition of plane waves with allowed wave vectors. A plane wave with wave vector $\vec{k} = k\hat{e}_z$ can be described, in the coulomb gauge, by the vector potential

$$\vec{A}(\vec{r},t) = \sum_{\sigma=x,y} \hat{e}_\sigma A_\sigma(\vec{r}) e^{-i\omega t} + c.c \tag{3.1}$$

where

$$A_\sigma = \gamma e^{i\vec{k}\cdot\vec{r}} \tag{3.2}$$

and γ is the normalization constant given by

$$\gamma = \sqrt{\frac{\hbar c}{2\pi k L^3}}. \tag{3.3}$$

Then by equations (2.2),

$$\vec{E}\cdot\vec{B} \;=\; 0,$$

$$\begin{aligned} B_x &= -E_y, \\ B_y &= E_x. \end{aligned} \tag{3.4}$$

The transversal anisotropy can be specified by either $\vec{E}$ or $\vec{B}$ but conventionally electric field is used. The polarization or coherence matrix is

$$P_{plane} = \begin{pmatrix} E_x^\star E_x & E_x^\star E_y \\ E_y^\star E_x & E_y^\star E_y \end{pmatrix}. \tag{3.5}$$

Above, it is the positive requency parts of the fields. From here on in this chapter, unless otherwise stated, we work with the positive frequancy parts of the fields. We now turn to the classical monochromatic $\jmath$-pole radiation. In the helicity basis, the positive frequency part of the vector potential has the form

$$\vec{A}_\lambda(\vec{r},t) = \sum_{\mu=-1}^{\mu=1} (-1)^\mu \vec{\chi}_{-\mu} A_{\lambda\mu}(\vec{r}) e^{-i\omega t}. \tag{3.6}$$

We interpret the $A_{\lambda\mu}$ as the component of the vector potential with given polarization $\vec{\chi}_\mu$ at a given point $\vec{r}$ and λ specifies the type of radiation. $A_{\lambda\mu}$ are given in the first section of chapter 1. Again with (2.2), for electric multipole radiation,

$$\begin{aligned} \vec{E}\cdot\vec{r} &\neq 0, \\ \vec{B}\cdot\vec{r} &= 0, \end{aligned} \tag{3.7}$$

while for magnetic multipole radiation

$$\begin{aligned} \vec{B}\cdot\vec{r} &\neq 0, \\ \vec{E}\cdot\vec{r} &= 0. \end{aligned} \tag{3.8}$$

In electric multipole radiation, magnetic field is transverse but electric field has a longitudinal component and in magnetic multipole radiation, electric field is transverse but magnetic field has a longitudinal component. The polarization matrix of electric multipole radiation is

$$P(\vec{r}) = k^2 \begin{pmatrix} A_{E+}^\star A_{E+} & A_{E+}^\star A_{E0} & A_{E+}^\star A_{E-} \\ A_{E0}^\star A_{E+} & A_{E0}^\star A_{E0} & A_{E0}^\star A_{E-} \\ A_{E-}^\star A_{E+} & A_{E-}^\star A_{E0} & A_{E-}^\star A_{E-} \end{pmatrix} \tag{3.9}$$

where we use $\vec{E} = ik\vec{A}$. Using the reciprocity relation

$$\vec{B}_M = \vec{E}_E = ik\vec{A}_E, \tag{3.10}$$

it can be shown that the polarization of magnetic multipole radiation is also described in the same way. In plane waves, the polarization matrix is independent of position, but in spherical waves polarization is dependent upon $\vec{r}$.

Conventional description of polarization is based on the use of the polarization matrix, as mentioned above. The quantum counterpart of the classical relations are obtained by the substitution of photon operators in places of field amplitudes. That is, polarization is represented by a matrix of operators as its entries which are

$$\hat{P}_{\mu\mu'} = \hat{\vec{E}}^{\dagger}_{\mu}\hat{\vec{E}}_{\mu'}. \tag{3.11}$$

In contrast to the conventional plane wave of photons, $E1$ photons, photons emitted in electric type transitions with $\jmath = 1$, may have three polarizations because of the nonparaxial nature of spherical waves of photons. In particular, the direction of the Poynting vector does not coincide with the the radial direction. We need the following about $E1$ photons which can be obtained immediately from the analysis in chapter 1. The positive frequency part of the vector potential can be written as

$$\hat{\vec{A}}(\vec{r},t) = \sum_{\mu=0,\pm1}\sum_{m=0,\pm1}(-1)^{\mu}\vec{\chi}_{-\mu}\mathcal{A}_{\mu m}(\vec{r})\hat{a}_m e^{-i\omega t} \tag{3.12}$$

where the mode functions have the form

$$\begin{aligned}\mathcal{A}_{\mu m}(\vec{r}) = \gamma_E[&j_2(kr)\langle 1,2,\mu,m-\mu \mid 1,m\rangle Y_{2,m-\mu}(\theta,\phi)\\ &-\sqrt{2}j_0(kr)\langle 1,0,\mu,m-\mu \mid 1,m\rangle Y_{0,m-\mu}(\theta,\phi)].\end{aligned} \tag{3.13}$$

γ_E being the normalization constant. From (2.12) is

$$\gamma_E = \sqrt{\frac{2\pi\hbar c}{3kV}} \tag{3.14}$$

The positive frequency part of the electric field operator is

$$\hat{\vec{E}} = ik\hat{\vec{A}}. \tag{3.15}$$

Since

$$[\hat{a}_m, \hat{a}^\dagger_{m'}] = \delta_{mm'}\mathbf{1}_{id}, \tag{3.16}$$

the difference between the antinormal and normal operators of polarization,

$$\begin{aligned}\hat{P}^{an}_{\mu\mu'} &= k^2 \sum_{m,m'} \mathcal{A}^*_{\mu m}\mathcal{A}_{\mu' m'}\hat{a}_{m'}\hat{a}^\dagger_m, \\ \hat{P}^{n}_{\mu\mu'} &= k^2 \sum_{m,m'} \mathcal{A}^*_{\mu m}\mathcal{A}_{\mu' m'}\hat{a}^\dagger_m\hat{a}_{m'},\end{aligned} \tag{3.17}$$

gives the vacuum polarization

$$\begin{aligned}\hat{P}^{vac}_{\mu\mu'}(\vec{r}) &= \hat{P}^{an}_{\mu\mu'} - \hat{P}^{n}_{\mu\mu'} \\ &= k^2 \sum_{m,m'} \mathcal{A}^*_{\mu m}\mathcal{A}_{\mu' m'}\hat{\mathbf{1}}_{id}\end{aligned} \tag{3.18}$$

at any point $\vec{r}$ of space surrounding the atom. It can be easily seen that these matrix elements of coincide with the commutators

$$[\hat{A}_{\mu m}, \hat{A}^\dagger_{\mu m}] \tag{3.19}$$

where the operator $\hat{A}_{\mu m} = \mathcal{A}_{\mu m}\hat{a}_m$. It is clear that the spatial distribution of polarization should depend only on the distance from the source, r. In other words, $\hat{P}^{vac}_{\mu\mu'}(\vec{r})$ should have the same value at all positions $\vec{r}$ having the same spherical angles θ and ϕ. Consider the direction $\theta = 0$. Since

$$Y_{\jmath\pm1,\mu-m}(0,\phi) = 4\pi\sqrt{2(\jmath\pm1)+1}\delta_{\mu m}, \forall\phi, \tag{3.20}$$

we get

$$\hat{P}_{\mu\mu'}(r,0,\phi) = k^2 \mid \mathcal{A}_{\mu\mu}(r,0,\phi) \mid^2 \delta_{\mu\mu'}\hat{\mathbf{1}}_{id}. \tag{3.21}$$

It is seen from (3.14) that

$$\begin{aligned}\mathcal{A}(r,0,\phi) = \sqrt{\frac{\hbar c}{6kV}}[&\sqrt{5}j_2(kr)\langle 1,2,\mu,0 \mid 1,\mu\rangle \\ &-\sqrt{2}j_0(kr)\langle 1,0,\mu,0 \mid 1,\mu\rangle]\end{aligned} \tag{3.22}$$

which is independent of ϕ as well. In the local frame with $\vec{\chi}_0 = \vec{r}/r$ that can be obtained from the helicity basis by a rotation, the vacuum polarization takes the diagonal form

$$\begin{pmatrix} \hat{P}_T(r) & 0 & 0 \\ 0 & \hat{P}_L(r) & 0 \\ 0 & 0 & \hat{P}_T(r) \end{pmatrix} \tag{3.23}$$

where

$$\begin{aligned}\hat{P}_T(r) &= k^2 \mid \mathcal{A}_{\pm 1 \pm 1} \mid^2 \\ &= \sqrt{\frac{\hbar c}{6kV}}[\frac{1}{\sqrt{2}} j_2(kr) - \sqrt{2} j_0(kr)] \end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\hat{P}_L(r) &= k^2 \mid \mathcal{A}_{\pm 1 \pm 1} \mid^2 \\ &= \sqrt{\frac{\hbar c}{6kV}}[-\sqrt{2} j_2(kr) - j_0(kr)]. \end{aligned} \tag{3.25}$$

As we get away from the source, the longitudinal polarization vanishes and the spherical waves becomes close to plane waves. For the explicit form of the transformation matrix U(see [27]). Through the use of the same rotation, we can transform the normal ordered polarization matrix into the following from

$$\hat{\mathcal{P}}^{vac}_{\mu\mu'}(\vec{r}) = k^2 \hat{L}^{\dagger}_{\mu}(\vec{r}) \hat{L}_{\mu}(\vec{r}) \tag{3.26}$$

where

$$\hat{L}_{\mu}(\vec{r}) = \sum_{\mu'} U^{*}_{\mu\mu'} \sum_{m} \mathcal{A}_{\mu' m}(\vec{r}) \hat{a}_m. \tag{3.27}$$

$U_{\mu\mu'}$ denotes the entries of the transformation matrix. It is now clear that

$$[\hat{L}_{\mu}(\vec{r}), \hat{L}^{\dagger}_{\mu}(\vec{r})] = \delta_{\mu\mu'} \begin{cases} \hat{P}_T(r) \text{ at } \mu = \pm 1 \\ \hat{P}_L(r) \text{ at } \mu{=}0 \end{cases} \tag{3.28}$$

Apart from the normalization, this expression coincides with bosonic commutation relations, Thus, we can introduce local photon opertaors with given polarization

$$\hat{b}_{\mu}(\vec{r}) = \frac{\hat{L}_{\mu}(\vec{r})}{\sqrt{P_{\mu}(\vec{r})}} \tag{3.29}$$

In terms of the above local operators, the operator polarization matrix at any point $\vec{r}$ can be rewritten in the bare operator form

$$\mathbf{P}_{\mu\mu'}(\vec{r}) = \hat{b}^{\dagger}_{\mu}(\vec{r}) \hat{b}_{\mu}(\vec{r}). \tag{3.30}$$

3.2 Operator Poynting Vector

Time independent part of the operator poynting vector is given by

$$\hat{\vec{S}}(\vec{r}) = \frac{1}{8\pi}(\hat{\vec{E}}^{\dagger} \times \hat{\vec{B}} + \hat{\vec{E}} \times \hat{\vec{B}}^{\dagger}). \tag{3.31}$$

Above, by $\hat{\vec{E}}$ and $\hat{\vec{B}}$, we mean only the positive frequency parts. Obviously, all three components of the operator Poynting vector, $\hat{S}_x(\vec{r}), \hat{S}_y(\vec{r}), \hat{S}_z(\vec{r})$, have real vector coefficients, $\vec{S}_x(\vec{r}), \vec{S}_y(\vec{r}), \vec{S}_z(\vec{r})$. The positive frequency part of the magnetic field operator for $E1$ phtons is given by equation (2.2):

$$\hat{\vec{B}}(\vec{r}, t) = -ik \sum_{\mu} \sum_{m} (-1)^{\mu} \vec{\chi}_{-\mu} \mathcal{B}_{\mu m}(\vec{r}) \hat{a}_m e^{-i\omega t} \tag{3.32}$$

where

$$\mathcal{B}_{\mu m}(\vec{r}) = \sqrt{\frac{2\pi\hbar c}{kV}} j_1(kr) \langle 1, 1, \mu, m-\mu \mid 1, m \rangle Y_{1,m-\mu}(\theta, \phi). \tag{3.33}$$

The above operator Poynting vector may not be pointing in the radial direction. Similar to what is done in the previous section, we can prepare a local frame by shift of the origin to the point $\vec{r}$ and by a rotation of the axis to put he z'-direction in the direction of the Poynting vector. The rotation is given by the rotation matrix

$$\mathcal{U}(\vec{r}) = \begin{pmatrix} \frac{1+\cos\theta_S}{2} e^{-i\phi_S} & \frac{1-\cos\theta_S}{2} e^{i\phi_S} & \frac{\sin\theta_S}{\sqrt{2}} \\ \frac{1-\cos\theta_S}{2} e^{-i\phi_S} & \frac{1+\cos\theta_S}{2} e^{i\phi_S} & -\frac{\sin\theta_S}{\sqrt{2}} \\ -\frac{\sin\theta_S}{\sqrt{2}} e^{-i\phi_S} & \frac{\sin\theta_S}{\sqrt{2}} e^{i\phi_S} & \cos\theta_S \end{pmatrix} \tag{3.34}$$

where θ_S and ϕ_S specify the direction of S. This transformation kills the middle row and middle column in the polarization matrix, entries involving longitudinal polarization. In this local frame, polarization is purely transverse. That is, this transformation matrix reduces the (3x3) polarization matrix into the local (2x2) polarization matrix of the form

$$\begin{pmatrix} \tilde{A}_+^{\dagger}\tilde{A}_+ & 0 & \tilde{A}_+^{\dagger}\tilde{A}_- \\ 0 & 0 & 0 \\ \tilde{A}_-^{\dagger}\tilde{A}_+ & 0 & \tilde{A}_-^{\dagger}\tilde{A}_- \end{pmatrix}. \tag{3.35}$$

Here

$$\tilde{A}_+^{\dagger}(\vec{r}) = \sum_{\mu'} \mathcal{U}_{\mu\mu'} A_{\mu'}^{\dagger}(\vec{r}). \tag{3.36}$$

For the case of $E1$ photons,with projection $m = 1$,

$$\begin{aligned} S_z &= 0, \\ S_x + iS_y &= kj_1(kr)\left(2j_0(kr) + 3j_2(kr)\right)e^{i(\frac{\pi}{2}-\phi)}. \end{aligned} \tag{3.37}$$

where k is a positive real constatnt. Using

$$\begin{aligned} \theta_S(\vec{r}) &= \arccos\left(\frac{\mid \vec{S}_z \mid}{\mid \vec{S}_x \mid^2 + \mid \vec{S}_y \mid^2 + \mid \vec{S}_z \mid^2}\right), \\ \phi_S(\vec{r}) &= Arg(\vec{S}_x + i\vec{S}_y), \end{aligned} \tag{3.38}$$

where Arg denotes the arguemnt of the complex vector $\vec{S}_x + i\vec{S}_y$, gives the transformation matrix as

$$U(\vec{r}) = \begin{pmatrix} \frac{1}{2}e^{-i(\frac{\pi}{2}-\phi)} & \frac{1}{2}e^{i(\frac{\pi}{2}-\phi)} & \frac{1}{\sqrt{2}} \\ \frac{1}{2}e^{-i(\frac{\pi}{2}-\phi)} & \frac{1}{2}e^{i(\frac{\pi}{2}-\phi)} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}e^{-i(\frac{\pi}{2}-\phi)} & \frac{1}{\sqrt{2}}e^{-i(\frac{\pi}{2}-\phi)} & 0 \end{pmatrix}. \tag{3.39}$$

3.3 Polarization Component of Angular Momentum

It is also classically verified that the total angular momentum of the field has two components, one of them being independent of the choice of the origin and the other being dependent upon the choice of the origin. In the quantum picture, the component independent of the origin is associated with spin(or polarization) and the other is associated with the orbital angular angular momentum(see[8]). The vector potential operator for the $E1$ type radiation is given in Eq (3.14). It can be shown that the angular momentum of an $E1$ photon at a distance r from the atom can be decomposed as(see [15])

$$\hat{\vec{S}}(\vec{r}) = f_S(kr)\hat{\vec{J}}, \;\; \hat{\vec{L}}(\vec{r}) = f_L(kr)\hat{\vec{J}}. \tag{3.40}$$

Here $\hat{\vec{J}}$ denotes the the operator of total angular momentum in the whole volume of quantization V and have the strucuture given in Eq(5.5) and satisfy $[J_\alpha, J_\beta] =$

$i\epsilon_{\alpha\beta\gamma}J_\gamma$. The distance dependent functions are

$$\begin{aligned} f_S(kr) &= \frac{\hbar}{3V}[2j_0^2(kr) - \frac{1}{2}j_2^2(kr)], \\ f_L(kr) &= \frac{\hbar}{3V}\frac{3}{2}j_2^2(kr). \end{aligned} \tag{3.41}$$

From the normalization condition (2.10)

$$\int_0^R f_S(kr)r^2dr = \int_0^R f_L(kr)r^2dr = \frac{\hbar}{2} \tag{3.42}$$

so that the total angular momentum is of the form

$$\hat{\vec{M}} = \hbar\hat{\vec{J}}. \tag{3.43}$$

as one can expect from an $E1$ photon with total angular momentum equal to one. To specify the spin and orbital angular momentum at a distance r from the atom, we take into account that the photon localization appears in a natural way in the form of wavefront. Therefore, we need to perform an integration over the spherical shell. The total spin and orbital angular momentums have the form

$$\hat{\vec{S}} = \hat{\vec{L}} = \frac{\hbar}{2}\hat{\vec{J}}. \tag{3.44}$$

An important deduction from above is that component due to polarization and orbital motion of the total angular momentum have the same operator structure. It is seen that spin and orbital angular monetum density operators have different spatial dependence at short distances from the atom. Since

$$\lim_{x\to 0} j_l(x) = \begin{cases} 1, \text{ if } l = 0 \\ 0, \text{ otherwise} \end{cases} \tag{3.45}$$

$f_L(kr)$ vanishes as $kr \to 0$. Thus, at the very vicinity of the atom, the photon has only spin, while the orbital angular momentum arises in the process of propagation. A more detailed invastigation shows that the spin density strongly prevails over orbital angular momentum density at $r < 0.1\lambda$ where $\lambda = 2\pi/k$ is the wavelength. Since the maximum of $f_S(kr)$ corresponds to $kr = 0$, it is possible to say that the atom creates the photon with spin alone and without orbital angular momentum. In turn, orbital angular momentum achieves maximum at $r \sim \lambda/2$ (intermediate zone). It is also seen that the main contribution to the

total angular momentum comes from the near zone in contrast to the energy that derives its main contribution from the wave zone.

At far distances we have

$$j_l(x) \sim \frac{1}{x}\sin(x - \frac{l\pi}{2}), x = kr \gg l \tag{3.46}$$

so that

$$\hat{\vec{S}}(kr) = \hat{\vec{L}}(kr) \sim \frac{\hbar}{2V}\frac{\sin^2(kr)}{(kr)^2}\hat{\vec{J}}. \tag{3.47}$$

Thus the spin and orbital angular momentum densities contribute equally into the total angular momentum of a monochromatic $E1$ photon in the wavezone. Because they have the same operator structure, it is impossible to distinguish betweeen the spin and orbital angular momentum parts by any measurement in the wavezone. This reflects the well known fact that the total angular momentum of $E1$ photon cannot be divided into spin and orbital angular momentum parts(see [1]).

Chapter 4

Entanglement

4.1 Fundamentals of Entanglement

The fundamental unit of classical information theory is a bit. Every computational task is a manipulation of a bit or of a string of bits. Information processing capabilities of a device are not independent of the physical system being used to perform these tasks. Rolph Landauer stated(see [18]):

"*Information is physical.*"

Quantum computers, if they were able to be built, making use of physical systems whose behavior cannot be approximated classically as in today's computers, will allow computational tasks which are definitely impossible in classical computers. It offers superior power in cryptography and communication technologies, too. In quantum information theory, it is a qubit that corresponds to a bit in classical information theory. One of the differences between a classical bit and a qubit is that although classical bit can only take values either 0 or 1, a quantum bit can take both with some probability. Qutrits and in general qunits, which have no classical counterpart, play important roles in quantum computation, too. There are many radical differences between classical and, yet not realized, quantum computers. Physically, qubit is a representation of the state of a two level system.

The most general state of a qubit system, i.e. a two level system, is represented by a vector in a two dimensional Hilbert space over the field of complex numbers, $\mathbf{C}$, which we denote by $\mathbf{H_2}$. We denote the vectors in a certain basis of this space as $|0\rangle$ and $|1\rangle$ and they usually represent the eigenstates of some known observable of the system. More precisely,

$$|\psi\rangle = a|0\rangle + b|1\rangle \tag{4.1}$$

where we must have $a^2 + b^2 = 1$. This constraint follows from the conservation of probabilities. For an N-qubit system, the space associated, $\mathbf{H}$, is the tensor product of the spaces for each individual qubit, $\mathbf{H} = \otimes_{i=1}^{N} \mathbf{H_2}$ and the product states form a basis for $\mathbf{H}$.

For reasons not fully understood, states with a certain physical property, called *entanglement*, play a crucial role in quantum computation and quantum information. In general, it is a property of multi-party systems, like an N-qubit system. We can simply define an entangled state as one which cannot be represented as the product of states of individual subsystems. Consider the case of a two-qubit system. The most general two-qubit state is represented by a vector in $\mathbf{H_2} \otimes \mathbf{H_2}$ and can be written as

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle. \tag{4.2}$$

Depending on the coefficients a, b, c, d, this vector may represent either an entangled or an unentangled state. Consider situation 1 where $c = d = 0$. The state vector $|\psi_1\rangle = a|00\rangle + b|01\rangle$ can be written as the tensor product of the states in which one subsystem is in state $|0\rangle$ and the other is in the state $a|0\rangle + b|1\rangle$. This means that, in this case, we do not have any entanglement. However, if we consider situation 2 where $a = d = 0, b = -c = \frac{1}{\sqrt{2}}$, then the state of the whole system is

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle \tag{4.3}$$

and cannot be written as a tensor product of its constituents. This means that there is some entanglement. A system in the entangled state $|\psi_2\rangle$ has certain physical properties which were considered preposterous by many important physicists. The famous EPR paper by Albert Einstein, Boris Podolsky and Nathan Rosen

was about entangled states(see [20]). Another famous paper by John Bell is intimately connected to the EPR paper and also to the concept of entanglement(see [21]). EPR paper was designed to prove, purely on theoretical grounds, that the outcome of measurements can be predicted with certainty although quantum theory provides only a probability distribution for outcomes. The fundamental assumption on which EPR argument rests is that no influence can propagate faster than light, the principle known as locality. However, entangled states force us to question locality. Hence these states can be used to oppose quantum theory, as is done in the EPR paper. Based on locality, the authors of EPR paper showed that description of physical phenomena must be deterministic and hence quantum theory should be supplemented with some additional information known as the hidden variable. John Bell considered the implications of a local hidden variable theory suggested in the EPR paper and derived his well known Bell inequalities as a consequence of such a theory. These inequalities must hold for any local hidden variable theory. This important paper proved something much more radical than its authors have imagined: predictions of quantum theory are incompatible with Bell inequalities. This means that quantum theory cannot be supplemented with additional complements so that it becomes both local and also deterministic. In other words, no theory which is both local and deterministic can contain whole of quantum theory. So either we have to give up the quantum theory or we have to accept that nature is not locally realistic. Experimental results agree with the predictions of quantum theory and violate Bell inequalities. This shows the nonexistence of a local hidden variable theory. There is still room for a nonlocal hidden variable theory to which Bell inequalities do not apply. Now is the question of finding where the authors of EPR paper might have been mistaken. After all, it is well accepted that c is an upper limit for the speed of physical influences but its implications such as a local hidden variable theory are not in accordance with the results of the experiments testing Bell inequalities(or quantum theory). Here we need to loosen the statement of relativity and we are led to distinguish between two types of influences: the causal ones which produce actual changes in some physical property, detectable by measurements on the influenced subsystem alone and an "etheral" kind which do not transmit energy or information and for which the only evidence is a correlation in the data taken on the influencing

and influenced subsystems. Causal influences cannot propagate faster than light but etheral ones can be superluminal. The influence mentioned the EPR paper, namely the collapse of the wavefunction upon measurement, is not a causal one.

For the most general form of the two qubit state in Eq(4.2), the amount of entanglement can be quantified in terms of how much the Bell inequalities are violated(see [30]). The more these inequalities are violated for a certain state, the more entanglement that state contains. This provides us a way of comparing the amount of entanglement of different states or finding the most entangled state(s). The state $|\psi_2\rangle$ provides the most possible entanglement among two qubit states and is called a maximally entangled state. $|\psi_2\rangle$ is not the only maximally entangled two qubit state. There exists some other states with the same amount of entanglement but no states with more entanglement. A unique quantitative measure of entanglement does not exist. We discuss two approaches here and the second one will show the intrinsic entanglement of dipole photons. First one is the so-called concurrence and the second one is a variational principle. Concurrence makes use of the eigenvalues of a certain matrix deduced from the density matrix. Indeed, the use of the density matrix is not strictly necessary to find the concurrence of a two qubit state but this both simplifies the calculations and also density matrix is useful in many other topics in quantum computation, notably in the understanding of quantum noise and quantum error correcting codes. We will first introduce the density matrix and define concurrence. The second measure of entanglement makes use of the following fact. For maximally entangled states, the amount of quantum fluctuations become maximum and on the opposite side, for coherent states which are the most classical states, quantum fluctuations becomes minimum. Concurrence is meaningful only for two qubit states while the second approach can be applied to systems other than two qubit systems. The relation between the second measure and concurrence can be used to define concurrence to states other than two qubit states.

4.2 Density Matrix

A quantum system whose state is known exactly is said to be in a pure state. Unfortunately, this is not always the case and we say that the system is in a mixed state. The density operator language provides a convenient means for describing quantum systems whose states are not completely known. More precisely, suppose a quantum system is in one of a number of states $|\psi_i\rangle$, where i is just an index, with respective probabilities p_i. We shall call $\{|\psi_i\rangle, p_i\}$ an ensemble. The density operator, $\hat{\rho}$, for the system is defined by the equation

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{4.4}$$

The matrix representation of the density operator is referred to as the density matrix and these two terms are sometimes used interchangeably. It can be shown easily that:

- the density matrix has trace equal to 1,
- the density matrix is Hermitian and hence all its eigenvalues are real, and
- all the eigenvalues of the density matrix are nonnegative real numbers.

These properties are independent of the basis we use. Different bases are connected through unitary transformations which have no effect on the traces, determinants, eigenvalues of the matrix representations of operators. If the system is in a pure state $|\psi\rangle$, then the density operator is simply $|\psi\rangle\langle\psi|$. It can also be shown that a pure state satisfies $Tr(\rho^2) = 1$ while a mixed state satisfies $Tr(\rho^2) < 1$ and the converses of these statements are also true. Consider a qubit system which must be in state $|0\rangle$ with probability 3/4 and in state $|1\rangle$ with probability 1/4. The density operator is

$$\hat{\rho} = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|. \tag{4.5}$$

Now suppose that the system is prepared in state $|a\rangle$ with probability 1/2 and in state $|b\rangle$ with again probability 1/2 where the states $|a\rangle$ and $|b\rangle$ are

$$|a\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle,$$

$$|b\rangle = \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle. \tag{4.6}$$

The density operator turns out to be the same as in the previous situation. The question of what class of ensembles give rise to a particular density matrix is extremely important in quantum computation and in quantum information. $\hat{\rho} = \sum_i p_i|\psi_i\rangle\langle\psi_i| = \sum_j q_j|\varphi_j\rangle\langle\varphi_j|$ for normalized states $|\psi_i\rangle$, $|\varphi_j\rangle$ and the probability distributions p_i and q_j if and only if $\sqrt{p_i}|\psi_i\rangle = \sum_j u_{i,j}\sqrt{q_j}|\varphi_j\rangle$ for some unitary matrix $u_{i,j}$. If the ensembles are not of the same size, we may pad the smaller ensemble with entries having zero probability to make the ensembles of the same size.

4.3 Measures of Entanglement

It is only in recent years that consideration has been given to finding methods to quantify entanglement. Historically, Bell inequalities were seen as a means of determining whether a two qubit system is entangled. It was known that the larger the Bell inequality is violated, the more the entanglement is present in the system. In 1994, it was discovered that not all entangled two qubit states violate Bell inequality(see [31]). A two qubit state, called the Werner state, which is a mixture of the maximally entangled state and the maximally mixed state can be entangled (inseparable) and yet still not violate the conventional two qubit Bell inequality(see [32]). However, the situation is clearer for pure two qubit states. It is true for all pure two qubit states that the state is entangled if and only if it violates the two qubit Bell inequality and also that the amount of entanglement increases with the amount of violation. That the violation of the Bell inequality quantifies the amount of entanglement for pure two qubit states verifies our conclusion in section 1 that the pure two qubit state $|\psi_2\rangle$ is maximally entangled. It is customary to use another quantity, called concurrence, which also increases with the amount of violation of the Bell inequality and hence may be used as a measure of entanglement for two qubit states. A two qubit system described by the density matrix ρ has the concurrence

$$C = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{4.7}$$

where λ_i's are the square roots of the eigenvalues of the matrix $\rho\tilde{\rho}$ in descending order,i.e. $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and $\tilde{\rho} = (\sigma_y^1 \otimes \sigma_y^2)\rho^*(\sigma_y^1 \otimes \sigma_y^2)$. $\sigma_y^{1,2}$ is the well known Pauli matrix and the superscripts denote the different single qubit subsystems on which the operators act. ρ^* is the complex conjugated of the density matrix. Here it is important that the the density matrix is evaluated using the computational basis. This definition of concurrence can be applied to mixed states where $Tr(\rho^2) < 1$. For the most general pure two qubit state, Eq(4.2),

$$(\sigma_y^1 \otimes \sigma_y^2)|\psi\rangle = -d|00\rangle + c|01\rangle + b|10\rangle - a|11\rangle \tag{4.8}$$

and

$$\rho\tilde{\rho} = 2(ad - bc)\begin{pmatrix} ad & -ac & -ab & a^2 \\ bd & -bc & -b^2 & ab \\ cd & -c^2 & -bc & ac \\ d^2 & -cd & -bd & ad \end{pmatrix} \tag{4.9}$$

which gives the concurrence as $C = 2 \mid ad - bc \mid$. For seperable states, the concurrence is 0 and for maximally entangled states, like $|\psi_2\rangle$ in section 1, the concurrence is 1. Another measure of entanglement, called the entanglement of formation can be formulated as a function of concurrence as follows:

$$E = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \tag{4.10}$$

where h is Shannon's entropy function given by

$$h(x) = -x\log(x) - (1 - x)\log(1 - x). \tag{4.11}$$

Yet another simple measure is the tangle, denoted by τ, which is simply the concurrence squared, $\tau = C^2$. The degree of violation of the two qubit Bell inequality in a two qubit state with tangle τ is $2\sqrt{1 + \tau^2}$. This connects the concurrence and how much the two qubit Bell inequality is violated. The entanglement of distillation may be a much more useful practical measure but it is difficult to calculate in practice. In general, the entanglement of distillation is smaller than entanglement of formation(see [30]).

Now we give another measure of entanglement which will be used in the investigation of the intrinsic entanglement of photons. Every physical system

has a dynamic symmetry group associated, G, with it and every such group is in one to one correspondence with a Lie algebra of observables, $\mathcal{L}$. A basis for the Lie algebra form a set of fundamental observables. Since observables are represented by Hermitian matrices, it may be necessary to comlexify the Lie algebra, that is, to use $\mathcal{L}^c = \mathcal{L} \otimes \mathcal{C}$ instead of $\mathcal{L}$. In case of a qubit system, the dynamical symmetry group is $SU(2)$ and the observables are represented by the Pauli matrices. They form an infinitesimal representation of $SL(2, C)$ which is the complexification of the $SU(2)$. The amount of entanglement can be described physically as the amount of correlation between distinct subsystems which cannot be created by local actions on each subsystem separately. Consequently, seperable states cannot contain any entanglement. Another physical definition is in terms of the uncertainties in the fundamental set of observables of the system. It was shown recently that maximally entangled states manifest the maximal amount of quantum fluctuations of the fundamental set of observables(see [26]). This is a common way in quantum optics. Coherent and squeezed states provide important examples. In particular, it has been recognized recently that coherent states can in general be associated with the unentangled states.

Let $\{\hat{\mathcal{O}}_i, i = 1, 2, ...N\}$ denote the set of fundamental observables. The quantum fluctuation of an observable $\hat{\mathcal{O}}_i$ in a pure state $|\psi\rangle$ is

$$V_i(\psi) = \langle\psi|\hat{\mathcal{O}}_i^2|\psi\rangle - \langle\psi|\hat{\mathcal{O}}_i|\psi\rangle^2 \tag{4.12}$$

and that in a mixed state with the density matrix ρ is

$$V_i(\rho) = Tr(\rho\mathcal{O}_i^2) - Tr(\rho\mathcal{O}_i)^2. \tag{4.13}$$

The total amount of fluctuations in a given state is

$$V_{tot} = \sum_i V_i. \tag{4.14}$$

The maximum entanglement corresponds to the maximum of total fluctuations. For maximally entangled states $|\psi_{ME}\rangle$ or ρ_{ME}

$$V_{tot}(\psi_{ME}) = \max_{\psi\in\mathbf{H}} V_{tot}(\psi) \tag{4.15}$$

and

$$V_{tot}(\rho_{ME}) = \max_{\{\rho\}} V_{tot}(\rho). \tag{4.16}$$

where **H** is the Hilbert space associated with the system $\{\rho\}$ denote the set of all density matrices. This condition expresses a variational principle, defining the ME states in the similar way with the equilibrium states in quantum statistical mechanics (principle of the maximum entropy). It is a general property of the Lie algebra that the observables in its basis form a Casimir operator. That is,

$$\sum_i \hat{\mathcal{O}}_i^2 = \hat{C} \tag{4.17}$$

where $\hat{C} = C\mathbf{1}_{id}$ is a constant times the identity operator. For instance, in the case of a single qubit system, the squares of the Pauli operators add up to 3 times the identity matrix. This means that the maximum of total variations is C and this maximum is achieved when

$$\langle \psi_{ME} \mid \hat{\mathcal{O}}_i \mid \psi_{ME} \rangle = 0, \forall i \tag{4.18}$$

and

$$Tr(\rho_{ME}\mathcal{O}_i) = 0, \forall i. \tag{4.19}$$

These last conditions are very useful and operational compared to the variation principle. It is the definition of maximally entangled states in terms of what can be measured.

Consider the case of a two qubit system. The dynamic group of symmetry is $SU(2) \otimes SU(2)$ and there are a total of six fundamental observables which are represented by the three Pauli matrices for each qubit. The most general two qubit state is as given in Eq(4.2). The conditions in Eq(4.17) and Eq(4.18) imply that

$$\begin{aligned} ac^* + bd^\star &= 0, \\ ab^* + cd^* &= 0, \\ \mid a \mid^2 &= \mid d \mid^2, \\ \mid b \mid^2 &= \mid c \mid^2 . \end{aligned} \tag{4.20}$$

The EPR state in section 1 is a solution of these equations. There are infinitely many states which satisfy the above equations and all of them are maximally

entangled. The relation between concurrence and this variational principle is given by

$$C = \sqrt{\frac{V_{tot} - V_{min}}{V_{max} - V_{min}}} \tag{4.21}$$

This equality can be used to define concurrence for systems other than two qubit systems.

Entangled states are equivalent to the maximum entangled states to within a certain local transformation such as stochastic local transformations assisted by classical communications. As soon as maximally entangled states are defined, all other entangled states can be obtained from maximally entangled states by means of SLOCC(Stochastic Local Operations assisted by Classical Communication). SLOCC operations cannot create or destroy entanglement but can change the amount of entanglement. SLOCC corresponds to transformations in the complexified algebra. Consider $g = e^{i\alpha\sigma_\alpha^{1,2}}$ where $\alpha = x, y, z$. It can be shown that

$$e^{i\alpha\sigma_\alpha^{1,2}} = \cos(x)1_{id} + i\sin(x)\sigma_\alpha^{1,2}. \tag{4.22}$$

The state in Eq (4.3) is a maximally entangled state. When we act on this state by g, the resultant states can be shown to have nonzero concurrence no matter what value α assumes.

Chapter 5

Intrinsic entanglement of a single photon

5.1 Single-particle entanglement

Conventional picture assumes entanglement of multipartite systems with two or more spatially separated parts, which is caused by specific quantum correlations between the parties. A mathematical theory of entanglement based on the dynamic symmetry approach makes it possible to consider correlations between intrinsic degrees of freedom of a single particle, leading to entanglement (see [17]). An electric-dipole (E1) photon with total angular momentum $J = 1$, that consists of spin S and orbital L parts (see Chapter 2), represents an easy example of a single particle that can manifest entanglement.

It was indicated in the previous Chapter that the two-qubit state

$$|S\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{5.1}$$

manifests complete entanglement. In conventional treatment, the two qubits, forming the state $|\ell\ell'\rangle = |\ell\rangle \otimes |\ell'\rangle$ are supposed to be spatially separated. Now we assume that they correspond to the helicity (two allowed polarizations) and orbital angular momentum $L = 0, 2$ of the same E1 photon.

The single-particle interpretation of these two qubits comes directly from the Clebsch-Gordon decomposition of the Hilbert space of two qubits:

$$\mathcal{H}_2 \otimes \mathcal{H}_2 = \mathcal{H}_3 \oplus \mathcal{H}_0. \tag{5.2}$$

Here $\mathcal{H}_2$ denotes the two-dimensional space of states of a single qubits, $\mathcal{H}_3$ is the symmetric part of the two-qubit space spanned by the vectors (1) and

$$|00\rangle, \quad |11\rangle,$$

while $\mathcal{H}_0$ denotes singlet (scalar) part associated with the antisymmetric state

$$|A\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

By construction, the three-dimensional space $\mathcal{H}_3$ has the symmetry with respect to the group SU(2), so that a physical system defined in $\mathcal{H}_3$ should be associated with a "spin-1" system. Angular momentum of a single E1 photon represents just this object. Then, the basic states can be associated with the spin-projection states as follows

$$\left\{ \begin{array}{rcl} |+1\rangle & = & |00\rangle, \\ |\mathbf{0}\rangle & = & \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right), \\ |-1\rangle & = & |11\rangle. \end{array} \right.$$

In other words, states of a single "spin-1" object are equivalent to the symmetric states of two qubits.

It is now clear that a single E1 photon can manifest entanglement. For example, the state $|\mathbf{0}\rangle$ is completely entangled.

Concerning the basic observables, we know that for each two-qubit subsystem they are given by the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ that have the form

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the basis $|0\rangle, |1\rangle$. Their representation in the whole four-dimensional Hilbert space $\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}}$ for the A and B parties of the system of two qubits has the form

$$\sigma_\ell^A = \sigma_\ell \otimes \mathbf{1}, \quad \sigma_\ell^B = \mathbf{1} \otimes \sigma_\ell, \quad \ell = x, y, z.$$

Changing the basis

$$|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle,$$

by the basis $\{|00\rangle, |S\rangle, |11\rangle|A\rangle\}$, for the Pauli operators with $\ell = x$ in parties A and B we get

$$\sigma_x^A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \; \sigma_x^B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

It is seen that the only difference between σ_x^A and σ_x^B consists in the form of the column and row, corresponding to the antisymmetric state $|A\rangle$, while the 3×3 principle submatrices coincide. Turning now to the three-dimensional symmetric subspace of $\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}}$, we should discard the row and column, corresponding to the antisymmetric state $|A\rangle$. This reduces the four-dimensional Pauli operators $\sigma_x^{(A)}$ and $\sigma_x^{(B)}$ to the x-component of spin-1 operator

$$\sigma_x^{(A,B)} \to S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.3}$$

For the y and z Pauli operators we get in the same way

$$\sigma_y^{(A,B)} \to S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_z^{(A,B)} \to S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5.4}$$

Thus, entanglement of a single "spin-1" object can be examined in terms of the basic observables provided by the spin-1 operators (3) and (4) and acting in the three-dimensional Hilbert space $\mathcal{H}_3$.

5.2 E1 photon entangled states

It can be easily seen that the spin operators (3) and (4) can be represented in terms of E1 photon operators with given projection m of total angular momentum

as follows(see [15])

$$\left\{\begin{array}{lcl} S_x &=& \frac{1}{\sqrt{2}}[a^+_{k0}(a_{k+}+a_{k-})+H.c.], \\ S_y &=& \frac{i}{\sqrt{2}}[a^+_{k0}(a_{k+}-a_{k-})-H.c.], \\ S_z &=& a^+_{k+}a_{k+}-a^+_{k-}a_{k-} \end{array}\right. . \tag{5.5}$$

Then, in the single-photon sector, the spin-projection states $|+1\rangle, |\mathbf{0}\rangle, |-1\rangle$ can be associated with the states

$$\left\{\begin{array}{lcl} |+1\rangle &=& |1_{k+},0,0\rangle, \\ |\mathbf{0}\rangle &=& |0,1_{k0},0\rangle, \\ |-1\rangle &=& |0,0,1_{k-}\rangle \end{array}\right. . \tag{5.6}$$

Thus, completely entangled state of the form of (1) corresponds to a single photon emitted by the atomic transition

$$|J=1,m=0\rangle \rightarrow |J'=0,m'=0\rangle.$$

In the near zone, such a photon has no orbital momentum and polarization along the radial direction that does not coincide with the direction of Poynting vector(see Chapter 2).

There is also another possibility to realize completely entangled symmetric states

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|1_{k+},0,0\rangle \pm |0,0,1_{k-}\rangle). \tag{5.7}$$

This assumes emission from an atomic electric dipole transition prepared in a coherent mixture of the states with projection of the total angular momentum $m=\pm 1$. Such a mixture may be realized in alkali atoms due to the influence of the nuclear field (see [1]).

Consider now quantum fluctuations that, as was shown in Chapter 4, can be used to quantify entanglement carried by a given state. Let us begin with the states (6). We get

$$V(S_{x,y}) = \left\{\begin{array}{ll} 1, & \text{if } m=0 \\ \frac{1}{2}, & \text{otherwise} \end{array}\right. \tag{5.8}$$

Besides that, $V(S_z) = 0$ for all states (6). Thus, the completely entangled state $|0, 1_{k0}, 0\rangle$ manifests maximum of quantum uncertainty as all one can expect from the operational definition of complete entanglement has been discussed in Chapter 4.

Similar result can also be obtained for the state (7) in comparison with the states $|1_{k+}, 0, 0\rangle$ and $|0, 0, 1_{k-}\rangle$. If the right-hand side of Eq. (7) does not represent an equiprobable mixture of two projections of the angular momentum $m = \pm 1$, so that instead of (7) we have the state

$$x|1_{k+}, 0, 0\rangle + \sqrt{1 - x^2}|0, 0, 1_{k-}\rangle,$$

the use of Eq. (4.20) gives concurrence of the form

$$C = 2|x\sqrt{1 - x^2}|.$$

The above results were obtained in the approximation of monochromatic photons. In reality, any excited atomic state has a finite lifetime even in a cavity, which leads to a certain line broadening (natural line breadth in the case of emission in empty space). To take this effect into account, let us use conventional Weiskopf-Wigner theory(see [33]) and introduce the time-dependent wave function of the system atom + radiation in the following way

$$|\psi(t)\rangle = C(t)|\psi_0\rangle + \int B(k, t)|\psi_k\rangle dk \tag{5.9}$$

The first term here describes the excited atomic state and vacuum state of all photon modes. The second term corresponds to the ground atomic state and a single emitted E1 photon. Employing Markov approximation then gives (see [16])

$$\begin{aligned} C(t) &= e^{-i\omega_0 t - \Gamma t}, \\ B(k, t) &= -\frac{k^{3/2}}{\omega_k - \omega_0 + i\Gamma}\left(1 - e^{-i(\omega_k - \omega_0)t - \Gamma t}\right). \end{aligned}$$

Here ω_0 denotes the atomic transition frequency, ω_k is the photon frequency, and Γ denotes the radiative decay width.

Through the averaging of angular momentum operators (5) over the state (9), we get

$$S_\alpha = \hbar(1 - e^{-2\Gamma t}), \quad \alpha = x, y, z. \tag{5.10}$$

Since the Markov approximation corresponds to the long-time scale $t \geq \Gamma^{-1}$, the above result is valid at the distances $r \geq c/\Gamma \gg c\omega_0$, which corresponds to the far zone. Same result is valid for the spin and orbital parts of the angular momentum as well. Each of them contributes exactly one half of $\hbar(1 - e^{-2\Gamma t})$ into the right-hand side of Eq. (10). Thus, the spin and orbital momentum contributions are indistinguishable from each other at far distances. This can be interpreted as the physical manifestation of entanglement of a single E1 photons expressed in terms of two qubits (helicity and orbital angular momentum). The analysis that has been performed in Chapter 2 shows that only in the near zone, where spin contribution prevails over orbital one, we can distinguish between polarization and orbital angular momentum.

Chapter 6

Conclusion

Let us briefly summarize our results. We have studied multipole radiation in the classical and quantum picture. They are the spherical waves of photons with well defined angular momentum and parity rather than the plane waves of photons with well defined linear momentum that are emitted during multipole radiation. The transitions in atoms occur between states with well defined angular momentum and parity and spherical waves of photons rather than plane waves should be considered. Dynamics of the process of a dipole atom in an ideal spherical cavity interacting with a single mode of the field quantized in the cavity can be described by a (2x2) Hamiltonian within the framework of Jaynes-Cummings model. The angular momentum of a photon consists of two parts which are spin angular momentum and orbital angular momentum with the spin part being associated with the polarization. Conventional picture of polarization of $E1$ photons in the radiation frame connected with the atom gives three polarizations. It is known that photon has spin 1 but can be observed only in two spin states(helicities or polarization states) because of the requirement of Poincaré invariance on the light cone. This picture does not contradict with the existence of two helicities because we can perform a local transformation of the radiation frame such that the new z-axis is parallel to the Poynting vector. The effect of this transformation on the polarization matrix evaluated in the radiation frame is to kill the entries involving longitudinal polarization. The transformation matrix for the case of radiation

emitted when the excited state of the atom has projection $m = 1$ is evaluated. Another important feature of multipole radiation is that the polarization matrix is dependent upon position with respect to the source which is different from the polarization of plane waves in which case polarization is position independent. Another transformation which diagonalizes the polarization matrix in the radiation frame allows us to introduce local photon operators describing the photons with given polarization at given position with respect to the source. The polarization matrix can be represented in a bare operator form using these new local photon operators. The diagonalized form of the polarization matrix gives us the longitudinal and transverse polarizations as in Chapter 2. The longitudinal polarization of multipole photons vanish in the far zone where they become close to plane wave photons. At far distances, multipole photons can be well approximated by plane wave photons. Another property of multipole photons in the far zone is that the contributions to the angular momentum from the spin and orbital parts become identical. Their contributions to the total angular momentum cannot be distinguished in the far zone. In the near zone, spin is dominant over orbital part.

That polarization forms a qubit is a well known phenomena. In case the total angular momentum quantum number is 1, orbital angular momentum can assume two values, 0 and 2, forming another qubit. This reflects itself in the Clebsch-Gordon coefficients involved in the field variables for $E1$ type radiation. Hence an $E1$ photon nay be considered as consisting of two qubits, one qubit due to spin angular momentum and another qubit due to orbital angular momentum and a photon can be intrinsically entangled. There is not a unique measure of entanglement. Concurrence is a common measure but is not useful in all cases. A recent approach uses the fact that the more entangled a state is, the more is the total amount of quantum fluctuations in the basic observables. The basic observables for a system are obtained from a basis of the Lie algebra corresponding to the dynamic symmetry group of the system. This leads to a simple equation for maximally entangled states which is given in Chapter 4 and also allows the definition of concurrence to be extended to states other than two qubit states. All entangled states can be obtained from any entangled state by means of SLOCC(Stochastic

Local Operations assisted by Classical Communication), explained in Chapter 4. Such transformations cannot create or destroy entanglement but can change the amount of entanglement. It is shown in Chapter 5 that an $E1$ photon can manifest entanglement with respect to its intrinsic degrees of freedom, namely the spin angular momentum and orbital angular momentum. In particular, $E1$ photons emitted in transitions when the atom falls from the state with projection $m = 0$ are maximally entangled. We can have maximally entangled photons when the excited state is a certain mixture of the states with projections $m = \pm 1$, too. Such a mixture can be realized in alkali atoms due to nuclear field. It is not known at this stage how such a single photon entanglement can be observed and this may be future work.

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