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A mixture representation of the Linnik distribution

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Abstract

Linnik distribution with the characteristic function

$$\varphi_\alpha(t) = 1/(1 + |t|^\alpha), \quad 0 < \alpha < 2,$$

is shown to possess the following property.

Let X_α, X_β be random variables possessing the Linnik distribution with parameters α and β respectively $(0 < \alpha < \beta \leqslant 2)$. Denote by $Y_{\alpha\beta}$ an independent of X_β non-negative random variable with the density

$$g(s;\alpha,\beta) = \left(\frac{\beta}{\pi}\sin\frac{\pi\alpha}{\beta}\right)\frac{s^{\alpha-1}}{1 + s^{2\alpha} + 2s^\alpha\cos\frac{\pi\alpha}{\beta}}, \quad 0 < s < \infty.$$

Then

$$X_\alpha \doteq X_\beta Y_{\alpha\beta},$$

where $\doteq$ denotes the equality in the sense of distributions.

Infinite divisibility of mixtures of Linnik distributions with respect to the parameter α and scale is obtained as a corollary.

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1. Introduction and statement of the theorem

Recently, the Linnik distribution – originally introduced by Ju.V. Linnik in 1953 (Linnik, 1963) – has attracted attention of a number of researchers (see e.g. Arnold, 1973; Devroye, 1986, 1990; Anderson, 1992;

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Anderson et al., 1993; Devroye, 1993). Although the characteristic function of this distribution is of a simple form, a general expression of the distribution is not easily attainable.[1] In this connection, any properties of the distribution such as a mixture representation which facilitates generation of Linnik random variables ought to be of interest. One such property is proved in the present note.

Recall that the generic definition of a Linnik random variable is given in terms of the characteristic function

$$\varphi_\alpha(t) = 1/(1 + |t|^\alpha), \quad 0 < \alpha < 2.$$

We shall denote the corresponding density by $p_\alpha(x)$. This density can be viewed as a generalization of the well-known Laplace (double exponential) density $p_2(x) = c^{-|x|}/2$ for the case $\alpha = 2$ (see, e.g. Johnson and Kotz, 1970). The main result of the paper is the following theorem.

Theorem. *For any* $0 < \alpha < \beta \leqslant 2$, *the following equality is valid*

$$\varphi_\alpha(t) = \int_0^\infty \varphi_\beta(t/s) g(s; \alpha, \beta)\, \mathrm{d}s, \quad -\infty < t < \infty, \tag{1}$$

where

$$g(s; \alpha, \beta) = \left(\frac{\beta}{\pi} \sin\frac{\pi\alpha}{\beta}\right) \frac{s^{\alpha-1}}{1 + s^{2\alpha} + 2s^\alpha \cos\frac{\pi\alpha}{\beta}}.$$

Noting that the equality (1) is equivalent to the following

$$p_\alpha(x) = \int_0^\infty p_\beta(sx) g(s; \alpha, \beta)\, \mathrm{d}s, \quad -\infty < x < \infty,$$

and taking into account that $g(s; \alpha, \beta)$ is a genuine density function, we arrive at the representation of the form

$$X_\alpha \doteq X_\beta Y_{\alpha\beta},$$

as stipulated in the abstract. This representation allows us to generate Linnik variables of different parameters starting from convenient base, e.g. from the Laplace distribution corresponding to $\beta = 2$. That is, our theorem yields immediately the following corollary.

Corollary 1. *For any* $\alpha \in (0, 2)$, *Linnik distribution with characteristic function* $\varphi_\alpha(t)$ *is a scale mixture of Laplace distributions with characteristic functions* $\varphi_2(t/s) = s^2/(s^2 + t^2)$, $0 < s < \infty$.

By Steutel's theorem (Steutel, 1970), any scale mixture of Laplace distributions is infinitely divisible. Therefore, Corollary 1 yields infinite divisibility of Linnik distributions. This fact is not new and was proved in Devroye (1990). However, noting that any mixture of scale mixtures of Laplace distributions is again a scale mixture of Laplace distributions, we obtain a stronger result: any mixture of Linnik distributions is infinitely divisible. More precisely, the following result is valid.

Corollary 2. *Let* P *be a probability measure on the half-strip* $S = \{(\alpha, s)\colon 0 < \alpha \leqslant 2,\ 0 \leqslant s < \infty\}$. *Then the distribution with the characteristic function*

$$\varphi(t) = \int_S \varphi_\alpha(st) P(\mathrm{d}\alpha\, \mathrm{d}s)$$

is infinitely divisible.

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[1] See Hayfavi, A., S. Kotz and I.V. Ostrovskii (1994), Analytic and asymptotic properties of Linnik's probability densities, *C.R. Acad. Sci. Paris*, Série I, **319**, 985–990.

2. Proof of the theorem

Note, that the equality (1) is equivalent to the following one:

$$\frac{1}{1+t^\alpha} = \int_0^\infty \frac{s^\beta}{s^\beta + t^\beta} g(s;\alpha,\beta)\,\mathrm{d}s, \quad 0<\alpha<\beta\leqslant 2,\ t\geqslant 0.$$

To prove the latter, we denote

$$I = \frac{\pi}{\beta\sin\frac{\pi\alpha}{\beta}} \int_0^\infty \frac{s^\beta}{s^\beta + t^\beta} g(s;\alpha,\beta)\,\mathrm{d}s = \int_0^\infty \frac{s^{\beta+\alpha-1}\,\mathrm{d}s}{(s^\beta+t^\beta)(1+s^{2\alpha}+2s^\alpha\cos\frac{\pi\alpha}{\beta})}.$$

It is sufficient to establish that

$$I = \frac{\pi}{\beta\sin\frac{\pi\alpha}{\beta}}\cdot\frac{1}{1+t^\alpha}. \tag{2}$$

Transforming the integral I by means of the change of variables $\tau = t^\beta s^{-\beta}$, we have

$$I = \frac{t^\alpha}{\beta}\int_0^\infty \frac{\tau^{(\alpha/\beta)-1}\,\mathrm{d}\tau}{(1+\tau)(\tau^{2\alpha/\beta}+t^{2\alpha}+2\tau^{\alpha/\beta}t^\alpha\cos\frac{\pi\alpha}{\beta})}.$$

Utilizing the identity

$$\frac{1}{\tau^{2\alpha/\beta}+t^{2\alpha}+2\tau^{\alpha/\beta}t^\alpha\cos\frac{\pi\alpha}{\beta}} = \frac{1}{t^\alpha 2\mathrm{i}\sin\frac{\pi\alpha}{\beta}}\left\{\frac{1}{\tau^{\alpha/\beta}+t^\alpha\mathrm{e}^{-\mathrm{i}\pi\alpha/\beta}} - \frac{1}{\tau^{\alpha/\beta}+t^\alpha\mathrm{e}^{\mathrm{i}\pi\alpha/\beta}}\right\},$$

we have

$$I = \frac{1}{2\mathrm{i}\beta\sin\frac{\pi\alpha}{\beta}}\left\{\int_0^\infty \frac{\tau^{(\alpha/\beta)-1}\,\mathrm{d}\tau}{(1+\tau)(\tau^{\alpha/\beta}+t^\alpha\mathrm{e}^{-\mathrm{i}\pi\alpha/\beta})} - \int_0^\infty \frac{\tau^{(\alpha/\beta)-1}\,\mathrm{d}\tau}{(1+\tau)(\tau^{\alpha/\beta}+t^\alpha\mathrm{e}^{\mathrm{i}\pi\alpha/\beta})}\right\}. \tag{3}$$

Consider the function

$$q(\tau) = \frac{1}{(1+\tau)(\tau^{\alpha/\beta}+t^\alpha\mathrm{e}^{\mathrm{i}\pi\alpha/\beta})}, \tag{4}$$

in the complex τ-plane cut along the positive ray. Define the branch of $\tau^{\alpha/\beta}$ in (4) by the conditions

$$\tau^{\alpha/\beta} = r^{\alpha/\beta}\mathrm{e}^{\mathrm{i}\varphi\alpha/\beta}, \quad \tau = r\mathrm{e}^{\mathrm{i}\varphi}, \qquad 0<\varphi<2\pi.$$

Since

$$\left|\frac{\alpha}{\beta}(\pi-\varphi)\right| < \pi$$

when $0<\varphi<2\pi$, we have

$$\tau^{\alpha/\beta} + t^\alpha\mathrm{e}^{\mathrm{i}\pi\alpha/\beta} = \mathrm{e}^{\mathrm{i}\varphi\alpha/\beta}(r^{\alpha/\beta}+t^\alpha\mathrm{e}^{\mathrm{i}\,\alpha/\beta(\pi-\varphi)}) \neq 0.$$

Therefore the function $q(\tau)$ is analytic in the cut out plane. Denote by $G_{R,\varepsilon}$ the simply connected region

$$G_{R,\varepsilon} = \{\tau\colon \varepsilon<|\tau|<R\}\backslash\{\tau\colon \varepsilon<\tau<R\},\ 0<\varepsilon<R(>1)$$

and denote by $\partial G_{R,\varepsilon}$ its boundary traversed in the direction which leaves $G_{R,\varepsilon}$ from the left (the line interval $\{\tau\colon \varepsilon<\tau<R\}$ is being traversed twice in the opposite directions). By the Cauchy Residue Theorem, we have

$$\oint_{\partial G_{R,\varepsilon}} q(\tau)\,\mathrm{d}\tau = 2\pi\mathrm{i}\,(\text{Residue of } q(\tau) \text{ at } \tau=-1) = -\frac{2\pi\mathrm{i}}{1+t^\alpha}.$$

Taking the limit of the last integral as $R \to \infty$, $\varepsilon \to 0$, we arrive at

$$\int_0^\infty \frac{\tau^{(\alpha/\beta)-1}\,\mathrm{d}\tau}{(1+\tau)(\tau^{\alpha/\beta}+t^\alpha \mathrm{e}^{\mathrm{i}\pi\alpha/\beta})} - \mathrm{e}^{2\pi\mathrm{i}[(\alpha/\beta)-1]}\int_0^\infty \frac{\tau^{(\alpha/\beta)-1}\,\mathrm{d}\tau}{(1+\tau)(\tau^{\alpha/\beta}\mathrm{e}^{2\pi\mathrm{i}\alpha/\beta}+t^\alpha \mathrm{e}^{\mathrm{i}\pi\alpha/\beta})} = -\frac{2\pi\mathrm{i}}{1+t^\alpha}.$$

It is evident that this equality can be rewritten in the form

$$\int_0^\infty \frac{\tau^{(\alpha/\beta)-1}\,\mathrm{d}\tau}{(1+\tau)(\tau^{\alpha/\beta}+t^\alpha \mathrm{e}^{\mathrm{i}\pi\alpha/\beta})} - \int_0^\infty \frac{\tau^{(\alpha/\beta)-1}\,\mathrm{d}\tau}{(1+\tau)(\tau^{\alpha/\beta}+t^\alpha \mathrm{e}^{-\mathrm{i}\pi\alpha/\beta})} = -\frac{2\pi\mathrm{i}}{1+t^\alpha}.$$

Thus, the difference of the integrals appearing in the braces of (3) has been calculated to be equal to $2\pi\mathrm{i}/(1+t^\alpha)$. Substituting this value into (3), we arrive at (2). The theorem is thus proved.

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