DYNAMIC IMPLICATIONS OF THE WEALTH-LEISURE NEXUS

A Master's Thesis

by

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İhsan Doğramacı Bilkent University
Ankara
June 2022

To my family

DYNAMIC IMPLICATIONS OF THE WEALTH-LEISURE NEXUS

The Graduate School of Economics and Social Sciences
of
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by

ARLINDO SKËNDERAJ

In Partial Fulfillment of the Requirements for the Degree of
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By Arlindo Skënderaj

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

DYNAMIC IMPLICATIONS OF THE WEALTH-LEISURE NEXUS

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This thesis analyses a one-sector optimal growth model in which wealth affects the utility obtained from leisure. We consider that an increase in wealth increases the propensity to consume leisure goods and services and hence affects how the instantaneous utility depends on leisure time. We prove the existence of the optimal path and characterize the dynamics and the properties of equilibria. We provide the conditions under which the model has unique or multiple steady states. The intensity of wealth in the utility obtained from leisure and the output elasticity of physical capital play an important role in the number of steady states and in the monotonicity of the optimal path of physical capital. We find that the optimal path of physical capital is monotonic and converges to the unique steady state, provided that the output elasticity of capital is higher than the intensity of wealth in the utility obtained from leisure.

Keywords: Bifurcation, Leisure, Optimal Growth, Steady State, Wealth

ÖZET

ZENGİNLİK-BOŞ ZAMAN BAĞINTISININ DİNAMİK SONUÇLARI.

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Bu tez, zenginliğin boş zamandan elde edilen faydayı etkilediği tek sektörlü bir optimal büyüme modelini analiz etmektedir. Zenginlikteki artışın boş zamana özgü mal ve hizmetleri tüketme eğilimini artırdığını ve dolayısıyla anlık fayda ile boş zaman arasındaki ilişkiyi etkilediğini dikkate alıyoruz. Optimal çözümün varlığını kanıtlıyoruz ve dengelerin dinamiklerini ve özelliklerini karakterize ediyoruz. Modelin bir veya birden çok kararlı dengeye sahip olduğu koşulları ortaya koyuyoruz. Boş zamandan elde edilen faydadaki zenginliğin yoğunluğu ve fiziksel sermayenin çıktı esnekliği, kararlı dengelerin sayısı ve optimal çözümde sermayenin tekdüzeliği (monotonluğu) üzerinde önemli bir rol oynar. Sermayenin çıktı esnekliğinin boş zamandan elde edilen faydadaki zenginliğin yoğunluğundan daha yüksek olması koşulu altında, sermayenin optimal çözümde tekdüze olduğunu ve yegane kararlı dengeye yakınsadığını bulduk.

Anahtar Kelimeler: Çatallanma, Boş Zaman, Optimal Büyüme, Kararlı Denge, Zenginlik

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CHAPTER 1

INTRODUCTION

In many growth models with elastic labor supply, leisure is considered solely as non-work time or even as free or wasted time (see Psarianos (2007) and Ortigueira (2000)). Several studies have looked at leisure time allocation and inequality, and its consequences on welfare. Aguiar and Hurst (2006) and Boppart and Ngai (2021) highlight the importance of leisure by finding that increasing leisure inequality offsets the welfare effects of consumption and income. Moreover, studies have analyzed how technological innovations have shaped how people allocate their leisure time. Aguiar et al. (2021) investigates the impact of video games on the labor supply of young men and finds that innovation in leisure technology has reduced the hours worked by young men. Yet, many studies have ignored that leisure time is perceived differently by people with different income levels. Along with wealth and technological innovations, many other factors, such as social status and education, might affect how people value leisure time. Incorporating these elements into economic models provides a better perspective for discussing issues of growth and welfare.

Unlike in the past, today, people have the option to benefit from numerous leisure goods and services. The leisure economy continues to attract people to engage in leisure activities more through technological innovation, globalization, ability to travel easily, etc. Nevertheless, to benefit from leisure goods and services people must pay for them. So, it appears that the rich have the ability to consume such goods, whereas the poor don't. As a result, the rich get more satisfaction from their leisure time, not necessarily because they have more free time, but because of the way they are spending it.

The increasing gap in hours worked among people of different income levels is of considerable interest nowadays. Recent data show that there is a link between income and working hours. Figure 1 shows the link between income and working hours across countries, whereas Figure 2 shows this link for individuals countries through time (1950-2017). In both situations, increasing wealth is associated with a decrease in working hours. Moreover, Aguiar and Hurst (2006) document a significant increase in leisure time over the last five decades in the United States and suggest that higher income implies higher leisure. People in rich countries such as the United States, the Netherlands, France, Sweden, and Germany earn a lot while working less. In contrast, people in poor countries such as Cambodia, Bangladesh, and Nigeria earn less and still work more than ten times as much. Moreover, if we look at the same countries across time, we observe that as income has increased, working hours have decreased in developed countries. The driving factors of this occurrence can be industrialization, technology, and labor productivity improvement as a country becomes more prosperous. However, it is also worth considering that the preferences for leisure time among people of different income levels might be an important factor causing this trend in working hours. Therefore, taking into account the wealth effect in the utility obtained from leisure will be able to explain the decreasing trend of hours worked in countries with high-income levels.

In this thesis, we investigate the relationship between wealth and leisure inequality by considering a one-sector optimal growth model with infinitely lived households where leisure preferences depend on wealth. The focus of including the wealth in leisure utility brings this thesis close to Kurz (1968) and Majumdar and Mitra (1994) who show that the presence of consumption and physical

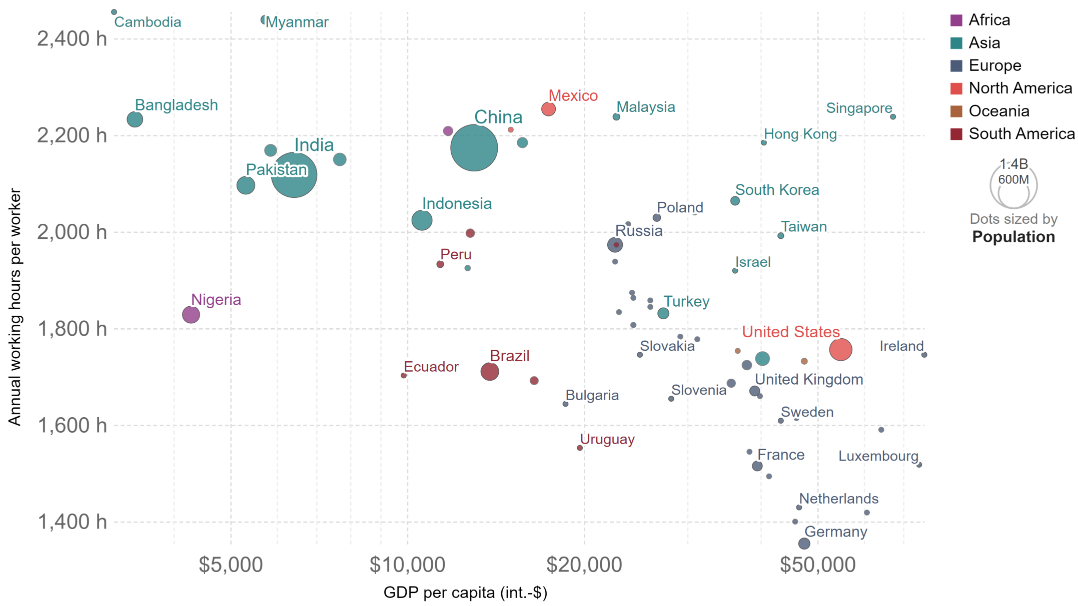

Figure 1: Cross-countries differences in leisure time and wealth (2017). **Source:** Our World in Data, June 2022: https://ourworldindata.org/working-hours.

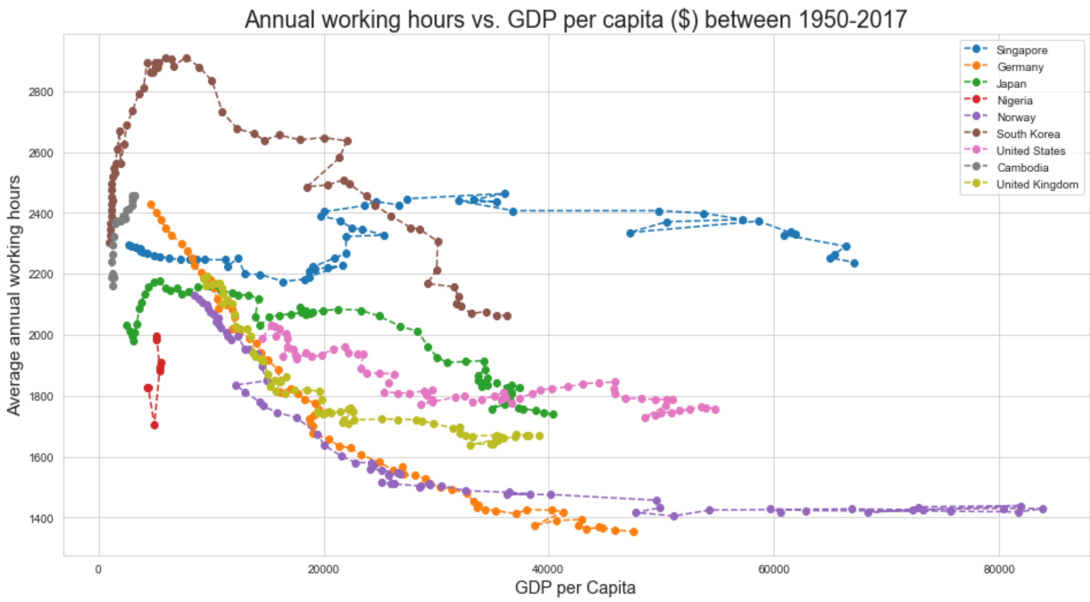

Figure 2: Working hours and GDP/capita for individual countries over time. **Source:** Our World in Data, June 2022: https://ourworldindata.org/working-hours.

capital in the utility function can generate multiple steady states. De Hek (1998) considers a setting with consumption and leisure and shows that preferences play an important role in determining the steady state to which the economy converges. He shows that if consumption and leisure are substitutes, the model can generate multiple steady states, whereas if they are complements, the optimal path may turn out to be cyclical. Psarianos (2007) introduces an endogenous growth model with leisure as a choice variable in the utility function and argues that although leisure reduces the economy's growth rate, agents are still willing to accept a lower rate of growth of income in exchange for more leisure time. Kamihigashi (2015) considers the Ramsey model with endogenous labor supply and shows that the presence of leisure can lead to multiple steady states (when leisure is not a normal good), a continuum of steady states, or no steady state. Sorger (2018) shows that period solutions and chaos can emerge due to strong income effects. Iwasa and Sorger (2018) extends the model with elastic labor supply and show that period solutions may occur only if leisure is not a normal good. Dufourt et al. (2015) considers a two-sector Real Business-Cycle model with productive externalities and Greenwood-Hercovitz-Huffman preferences. They show that local indeterminacy might emerge through Hopf and flip bifurcations for different values of the elasticity of intertemporal substitution in consumption.

Within the theory of endogenous growth, Uzawa-Lucas [1] is one of the first models to consider the time allocation for economic development. Ladrón-de Guevara et al. (1999) extends the Uzawa-Lucas model by introducing leisure as a choice variable and shows that the presence of leisure generates multiple balanced growth paths. Candela et al. (2016) extend the model of Leadron-de-Guevara and shift the focus to how total time is allocated between work, education, purely free time, and time spent on leisure services. They focus on time-consuming

[1] See Uzawa (1965) and Lucas (1988)

activities and show that possible multiple balanced growth paths may emerge where the higher growth is characterized by more time spent on educational activities.

In this thesis, we prove the existence of the optimal path and characterize the dynamics and the properties of equilibria. We find that multiple steady states may emerge for low values of the output elasticity of physical capital, large values of the intensity of wealth in the utility obtained from leisure, provided that the labor supply is sufficiently elastic. Moreover, we show that the optimal path of physical capital is monotonic, provided that the output elasticity of capital is greater than the parameter accounting for the wealth intensity in the utility obtained from leisure.

The rest of the thesis is organized as follows: Chapter 2 states the assumptions, formulates the model, and provides the conditions under which the optimal path of physical capital stock is monotonic; Chapter 3 provides the conditions under which the model has unique or multiple steady states and performs a local analysis around the steady state; Chapter 4 concludes.

CHAPTER 2

THE MODEL

We consider a discrete-time optimal growth model with endogenous labor supply where the preferences for leisure time depend on the level of wealth. The agent derives utility from consumption, physical capital, and leisure time. Throughout this thesis k_t, c_t, ℓ_t and n_t will denote the period-t factor inputs of physical capital, consumption, leisure and labor, respectively. The function $f(k_t, n_t)$ consists of the output produced at time t plus the non-depreciated capital. We maintain the following assumptions for f. [1]

Assumption 1 *(i) The function* $f : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ *is strictly concave, homogeneous of degree 1, and continuously differentiable on* $\mathbb{R}_{++} \times (0,1]$.

(ii) For each $n \in (0,1]$, $f(k,n)$ *is increasing and strictly concave with respect to* $k \in \mathbb{R}_+$. *Moreover,* $\lim_{k\to 0} f_1(k,n) = +\infty$ *and* $\lim_{k\to\infty} f_1(k,n) = 0$.

(iii) For each $k > 0$, $f(k,n)$ *is increasing and strictly concave with respect to* $n \in [0,1]$. *Moreover,* $\lim_{n\to 0} f_2(k,n) = +\infty$ *and* $\lim_{n\to\infty} f_2(k,n) = 0$.

We assume that the physical capital is accumulated by saving some of the output produced and investing it to build up new capital stock. The equation for physical

[1] The partial derivatives will be denoted by subscripts. For example, $f_1(k,n)$ is the partial derivative of f with respect to its first argument, in this case k, at the point (k,n). Similarly, $f_2(k,n)$ is the partial derivative of f with respect to its second argument, in our case n, at the point (k,n).

capital accumulation is given by

$$k_{t+1} = f(k_t, n_t) - c_t. \tag{2.1}$$

Every agent is endowed with a unit of time that he chooses to spend on leisure or labor, therefore we have $\ell_t = 1 - n_t$, for all $t \geq 0$. Henceforth we will consider a representative agent whose preferences are given by the following functional

$$\mathcal{U} = \sum_{t=0}^{\infty} \beta^t \big(u(c_t) + \psi v(k_t, \ell_t) \big),$$

where u is the instantaneous utility function of consumption, v is the utility obtained from leisure – which is augmented by the level of wealth – and, $\beta \in (0, 1)$ is the discount rate. We maintain the following set of assumptions on u and v:

Assumption 2 *The function* $u : \mathbb{R}_+ \to \mathbb{R}$ *is strictly concave, increasing and twice continuously differentiable on* $\mathbb{R}_{++}$. *Moreover* $\lim_{c \to 0} u'(c) = +\infty$ *and* $\lim_{c \to +\infty} u'(c) = 0$.

Assumption 3 *(i) The function* $v : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ *is strictly concave, increasing and twice continuously differentiable in* $\mathbb{R}_{++} \times (0, 1)$.

(ii) For all $k > 0$ *it holds that* $v(k, \ell)$ *is strictly concave and increasing with respect to* $\ell \in [0, 1]$.

(iii) For all $\ell \in [0, 1]$ *it holds that* $v(k, \ell)$ *is strictly concave and increasing with respect to* $k \in \mathbb{R}_+$.

(iv) For all $(k, \ell) \in \mathbb{R}_+ \times [0, 1]$ *it holds that* $v_{12}(k, \ell) \geq 0$.

The last condition of Assumption 3 will be sufficient in proving that leisure is a normal good. That condition suggests that the agent's additional disutility of working one-hour longer increases with wealth. The latter is associated with the fact that the disutility of labor – which incorporates the opportunity cost of being unable to consume – increases with income. In other words, when the agent has a higher level of wealth, his disutility of working an extra hour is higher since he is able to benefit from a more luxurious and qualitative leisure time.

Next, we will show that the last condition of Assumption 3 is sufficient for leisure to be a normal good. Consider the static maximization problem:

$$\begin{aligned} \max_{c,\ell} \quad & u(c) + \mu(I)\nu(\ell) \\ \text{s.t.} \quad & 0 \leq c, \ 0 \leq \ell \leq 1, \ c + q\ell \leq I, \end{aligned} \tag{$\mathcal{S}$}$$

where I denotes the (nonlabor) income and q the price of leisure in terms of consumption. Note that the function form $\mu(I)\nu(\ell)$ corresponds to $v(I, l)$. By taking this form of the utility function we highlight the fact that the parameter accounting for the importance of leisure, in our case μ, is endogenous and characterized as a function of wealth. Analogously to Assumption 3, we have:

Assumption 4 *(i) The function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly concave, increasing and twice continuously differentiable in $\mathbb{R}_+$. Moreover, for all $(I, \ell) \in \mathbb{R}_+ \times [0, 1]$ it holds that $\mu(I)\nu'(\ell) \geq 0$.*

(ii) The function $\nu : [0, 1] \to \mathbb{R}$ is strictly concave, increasing and twice continuously differentiable in $(0, 1)$. Moreover, for all $(I, \ell) \in \mathbb{R}_+ \times [0, 1]$ it holds that $\mu'(I)\nu(\ell) \geq 0$.

(iv) For all $(I, \ell) \in \mathbb{R}_+ \times [0, 1]$ it holds that $\mu'(I)\nu'(\ell) \geq 0$.

For any $q > 0$ and for any $I > 0$, the objective function is a strictly concave function, and the budget constraint is a convex set, hence this problem has a unique solution $(c^*(q, I), \ell^*(q, I))$. Moreover, this unique solution satisfies the following first order condition

$$u'(c) = \frac{\mu(I)\nu'(\ell)}{q}, \tag{2.2}$$

and the budget constraint

$$c + q\ell = I, \tag{2.3}$$

which holds with equality in this case.

Definition 1 *Leisure is a normal good if $\ell^*(I, q)$ is a non-decreasing function of I, for all $q > 0$.*

Proposition 1 *Under Assumption 4, leisure is a normal good.*

Proof. *By contradiction, assume that leisure is not a normal good. That is, there exist I and I' with $I' > I$, such that $\ell^*(I', q) < \ell^*(I, q)$, for some $q > 0$. Note that from the fourth condition of Assumption 4, it holds that*

$$\mu(I')\nu'(\ell^*(I, q)) > \mu(I)\nu'(\ell^*(I, q)).$$

Moreover, from our supposition that $\ell^*(I', q) < \ell^*(I, q)$ it holds that $\nu'(\ell^*(I', q) > \nu'(\ell^*(I, q))$. The first condition of Assumption 4 gives

$$\mu(I')\nu'(\ell^*(I', q)) > \mu(I')\nu'(\ell^*(I, q)).$$

From equation 2.2 we have

$$u'(c^*(I', q)) = \frac{\mu(I')\nu'(\ell^*(I', q))}{q} > \frac{\mu(I)\nu'(\ell^*(I, q))}{q} = u'(c^*(I, q)).$$

From Assumption 2, we get $c^*(I', q) < c^*(I, q)$. Then from 2.3 it follows that

$$I' = c^*(I', q) + q\ell^*(I', q) < c^*(I, q) + q\ell^*(I, q) = I,$$

contradicting the fact that $I' > I$. *Thus,* $\ell^*(I,q)$ *must be non-decreasing in* I, *implying that leisure is a normal good.* ■

In the next section, we will formalize the social planner's problem and we will define the equilibrium path of the economy.

2.1 The Social Planner's Problem

The social planner's problem can be formalised as follows:

$$\begin{aligned}
\max_{\{c_t,\ell_t,k_t\}_{t\geq 0}} \quad & \sum_{t=0}^{\infty} \beta^t \big(u(c_t) + \psi v(k_t,\ell_t)\big) \\
\text{s.t.} \quad & k_{t+1} = f(k_t, n_t) - c_t, \\
& n_t + \ell_t = 1, \\
& c_t \geq 0,\ n_t \geq 0,\ \ell_t \geq 0\,,\ \forall t \geq 0, \\
& k_0 \geq 0,\ \text{given.}
\end{aligned} \tag{$\mathcal{P}$}$$

To solve this problem we employ the discrete Hamiltonian approach. The Hamiltonian function for problem ($\mathcal{P}$) is given by:

$$\mathcal{H}_t(k_t, c_t, \ell_t) = \beta^t \big(u(c_t) + \psi v(k_t,\ell_t)\big) + \lambda_{t+1}\big(f(k_t, 1-\ell_t) - c_t\big),$$

where λ_{t+1} represents the shadow price of physical capital.

An interior solution is then characterized by the following first order conditions of optimally:

$$\beta^t u'(c_t) = \lambda_{t+1}, \tag{2.4}$$

$$\beta^t \psi v_1(k_t,\ell_t) + \lambda_{t+1} f_1(k_t,n_t) = \lambda_t, \tag{2.5}$$

$$\beta^t \psi v_2(k_t, \ell_t) = \lambda_{t+1} f_2(k_t, n_t), \tag{2.6}$$

and the transversality condition: [2]

$$\lim_{k\to\infty} \beta^t u'(c_t) f_1(k_t, n_t) k_{t+1} = 0.$$

Plugging 2.4 in 2.5 and 2.6 yields

$$u'(c_t) = \beta u'(c_{t+1}) f_1(k_{t+1}, n_{t+1}) + \beta \psi v_1(k_{t+1}, \ell_{t+1}), \tag{2.7}$$

and

$$\frac{\psi v_2(k_t, \ell_t)}{u'(c_t)} = f_2(k_t, n_t). \tag{2.8}$$

Definition 2 *An equilibrium path consists of an allocation* $\{c^*, k^*, l^*, n^*\} \in \ell_+^\infty \times \ell_+^\infty \times [0,1]^\infty \times [0,1]^\infty$ *and an initial capital stock* $k_0 \geq 0$ *that satisfy the following equations:*

$$u'(c_t) = \beta u'(c_{t+1}) f_1 k_{t+1}, \ell_{t+1}) + \beta \psi v_1(k_{t+1}, \ell_{t+1}), \tag{2.9}$$

$$\frac{\psi v_2(k_t, \ell_t)}{u'(c_t)} = f_2(k_t, n_t), \tag{2.10}$$

$$k_{t+1} = f(k_t, n_t) - c_t, \tag{2.11}$$

as well as the transversality condition:

$$\lim_{t\to\infty} \beta^t u'(c_t) f_1(k_t, n_t) k_{t+1}. \tag{2.12}$$

[2] For this transversality condition see Kamihigashi (2001), Michel (1982) and Le Van et al. (2007). See also the transversality condition in Schumacher (2011), a one sector optimal growth with wealth dependent discounting.

2.2 Properties of the Value Function and the Bellman Equation

In this section we introduce the indirect utility function and a modified optimization problem which is equivalent to our initial model. Using the modified problem we will then study the properties of the optimal paths.

Let (k, k') be such that $0 \leq k' \leq f(k, 1)$. The indirect utility function $V : \mathbb{R}^2_+ \to \mathbb{R}$ is given by:

$$\begin{aligned} V(k, k') = \max_{\{c,l\}} \quad & u(c) + v(k, \ell) \\ \text{s.t.} \quad & c + k' \leq f(k, 1 - \ell), \\ & c \geq 0,\ 0 \leq \ell \leq 1. \end{aligned} \tag{$\mathcal{S}'$}$$

Consider the following problem:

$$\begin{aligned} \max_{\{c_t, \ell_t, k_t\}_{t \geq 0}} \quad & \sum_{t=0}^{\infty} \beta^t V(k_t, k_{t+1}) \\ \text{s.t.} \quad & 0 \leq k_{t+1} \leq f(k_t, 1), \quad \forall t \geq 0, \\ & k_0 \geq 0, \text{ given.} \end{aligned} \tag{$\mathcal{Q}$}$$

Next, we will analyze the properties of the indirect utility function. We will later see that the second partial derivatives of the indirect utility function will be crucial in proving the monotonicity of the optimal paths of the physical capital.

Proposition 2 *The indirect utility function V is strictly concave, increasing with respect to the first argument and decreasing with respect to the second one.*

Proof. *Obvious.* ■

Note that the objective function in $\mathcal{S}'$ is strictly concave and the constraint set is convex. Hence, for all $(k, y) \in \mathbb{R}^2_+$, there exist a unique $(c, \ell) \in \mathbb{R}_+ \times [0, 1]$ such that $V(k, y) = u(c) + v(k, \ell)$.

The following proposition proves that problem $\mathcal{Q}$ is equivalent to our main model $\mathcal{P}$. Moreover, problems $\mathcal{Q}$ and $\mathcal{P}$ have the same optimal path of capital stocks.

Proposition 3 *(i) If $(c_t^*, \ell_t^*, k_t^*)_{t\geq 0}$ is a solution to problem ($\mathcal{P}$), then $(k_t^*)_{t\geq 0}$ is a solution to problem ($\mathcal{Q}$).*

(ii) If $(k_t^*)_{t\geq 0}$ is a solution to problem ($\mathcal{Q}$), then there exist a sequence of consumption $(c_t^*)_{t=0}^{\infty}$ and a sequence of leisure time $\{\ell_t\}_{t=0}^{\infty}$ such that $(c_t^*, \ell_t^*, k_t^*)_{t\geq 0}$ is a solution to problem ($\mathcal{P}$).

Proof. *(i) By contradiction, assume that $(c_t^*, \ell_t^*, k_t^*)_{t\geq 0}$ is a solution to problem ($\mathcal{P}$), but $(k_t^*)_{t\geq 0}$ is not a solution to problem ($\mathcal{Q}$). Then, there exist a sequence $(k_t')_{t\geq 0}$ with $0 \leq k_{t+1}' \leq f(k_t', 1)$ for all $t \geq 0$, such that*

$$\sum_{t=0}^{+\infty} \beta^t V(k_t', k_{t+1}') > \sum_{t=0}^{+\infty} \beta^t V(k_t^*, k_{t+1}^*). \tag{2.13}$$

From the property of V and from the fact that $c_t^* + k_{t+1}^* \leq f(k_t^*, 1-\ell_t^*)$ for all $t \geq 0$, we have $V(k_t^*, k_{t+1}^*) \geq u(c_t^*, \ell_t^*, k_t^*)$ for all $t \geq 0$. Hence, we have

$$\sum_{t=0}^{+\infty} \beta^t V(k_t^*, k_{t+1}^*) \geq \sum_{t=0}^{+\infty} u(c_t^*, k_t^*, \ell_t^*). \tag{2.14}$$

On the other hand, since $0 \leq k_{t+1}' \leq f(k_t', 1)$ for all $t \geq 0$, and since problem S' has a unique solution, there exists a sequence $\left(c_t', \ell_t'\right)_{t\geq 0}$ with $c_t' + k_{t+1}' \leq f(k_t', 1-\ell_t')$ and $\ell_t' \in [0, 1]$ for all $t \geq 0$ such that

$$V(k_t', k_{t+1}') = u(c_t', k_t', \ell_t'), \quad \textit{for all } t \geq 0.$$

But then we will have

$$\begin{aligned}\sum_{t=0}^{+\infty}\beta^t u(c'_t,k'_t,\ell'_t) &= \sum_{t=0}^{+\infty}\beta^t V(k'_t,k'_{t+1}) \\ &> \sum_{t=0}^{+\infty}\beta^t V(k^*_t,k^*_{t+1}) \qquad (\text{from } 2.13) \\ &\geq \sum_{t=0}^{+\infty}\beta^t u(c^*_t,k^*_t,\ell^*_t) \quad (\text{from } 2.14).\end{aligned}$$

This contradicts the fact that $\left(c^*_t,\ell^*,k^*_t\right)_{t\geq 0}$ *is a solution to problem* ($\mathcal{P}$).

(ii) Assume that $(k^*_t)_{t\geq 0}$ *is a solution to problem* ($\mathcal{Q}$). *For any* $t\geq 0$*, since* $0\leq k^*_{t+1}\leq f(k^*_t,1)$*, and since problem* $\mathcal{S}'$ *has a unique solution, there exist a sequence* $(c^*_t,\ell^*_t)_{t\geq 0}$ *such that* $c^*_t+k^*_{t+1}\leq f(k^*_t,1-\ell^*_t)$ *and* $V(k^*_t,k^*_{t+1})=u(c^*_t,k^*_t,\ell^*_t)$.

Claim: $\left(c^*_t,\ell^*_t,k^*_t\right)_{t\geq 0}$ *is a solution to problem* ($\mathcal{P}$).

By contradiction, assume that $\left(c^*_t,\ell^*_t,k^*_t\right)_{t\geq 0}$ *is not a solution to problem* ($\mathcal{P}$). *That is, there exists* $\left(c'_t,\ell'_t,k'_t\right)_{t\geq 0}$ *with* $c'_t+k'_{t+1}\leq f(k'_t,1-\ell'_t)$ *and* $\ell'_t\in[0,1]$ *for all* $t\geq 0$ *such that*

$$\sum_{t=0}^{+\infty}\beta^t u(c^*_t,\ell^*_t,k^*_t) < \sum_{t=0}^{+\infty}\beta^t u(c'_t,\ell'_t,k'_t)$$

Now observe that

$$\begin{aligned}\sum_{t=0}^{+\infty}\beta^t u(c'_t,\ell'_t,k'_t) &> \sum_{t=0}^{+\infty}\beta^t u(c^*_t,\ell^*_t,k^*_t) \\ &= \sum_{t=0}^{+\infty}\beta^t V(k^*_t,k^*_{t+1}) \\ &\geq \sum_{t=0}^{+\infty}\beta^t V(k'_t,k'_{t+1}) \quad (\text{since } (k^*_t)_{t\geq 0} \text{ is a solution to } \mathcal{Q}) \\ &\geq \sum_{t=0}^{+\infty}\beta^t u(c'_t,\ell'_t,k'_t) \quad (\text{since } c'_t+k'_{t+1}\leq f(k'_t,1-\ell'_t)).\end{aligned}$$

Contradiction! Thus, (k^*_t,c^*_t,ℓ^*_t) *is a solution to* ($\mathcal{P}$). ■

Note that the value function W:

$$W(k_0) = \max\{V(k_0, k_1) + \beta W(k_1) \mid 0 \leq k_1 \leq f(k_0, 1)\},$$

which solves the Bellman equation, is the same for the two problems. Moreover a feasible path $\{k_t^*\}_{t=0}^{\infty}$ is optimal if and only if

$$W(k_t^*) = V(k_t^*, k_{t+1}^*) + \beta W(k_{t+1}^*),$$

for any $t \geq 0$. This information is useful while performing a numerical analysis for the value function iteration.

Observe that the Lagrangian for the static optimization problem ($\mathcal{S}'$) is given by:

$$\mathcal{L} = u(c) + v(k, \ell) + \lambda(f(k, 1 - \ell) - c - k') + \mu(1 - \ell)$$

Proposition 4 *For all* $(k, k') \in \mathbb{R}_+ \times \mathbb{R}_+$*,* V *is differentiable at* (k, k') *and*

$$\begin{aligned} \frac{\partial V(k, k')}{\partial k} &= v_1(k, \ell) + \lambda f_1(k, 1 - \ell), \\ \frac{\partial V(k, k')}{\partial k'} &= -\lambda. \end{aligned} \tag{2.15}$$

Moreover, if

$$\frac{v_{12}}{v_2} < \frac{f_{12}}{f_2}, \tag{2.16}$$

then

$$\frac{\partial^2 V(k, k')}{\partial k' \partial k} > 0. \tag{2.17}$$

Proof. *See the Appendix.* ■

From Amir (1996), since V is strictly super-modular, the optimal policy function is increasing. Hence, any optimal path of physical capitals is monotonic. Now we will show that the optimal path of physical capital does not converge to zero.

Consider the following proposition:[3]

Proposition 5 *Let $(k_t)_{t\geq 0}$ denote the solution to problem $(\mathcal{Q})$. For any initial condition $k_0 > 0$, the optimal path of physical capital starting from k_0 cannot converge to zero.*

Proof. *By contradiction, assume that $k_0 > 0$ and $(k_t)_{t\geq 0}$ is the optimal path of physical capital, but $k_t \to 0$. From the Inada conditions, since $\lim_{k\to 0} f_1(k,1) > 1$ we have $f(k,1) > k$ for sufficiently small k. Then for sufficiently large t we have $0 < k_{t+1} < f(k_{t+1}, 1)$. From Proposition 2, there exist sequences $(c_t)_{t\geq 0}$ and $(\ell_t)_{t\geq 0}$ for consumption and leisure such that $(c_t, \ell_t, k_t)_{t\geq 0}$ is a solution to problem $(\mathcal{P})$. Moreover, since $c_t \leq f(k_t, 1)$ and $f(0,1) = 0$, it follows that $c_t \to 0$.*

Let $\left(\frac{k_{t_n}}{1-\ell_{t_n}}\right)_{n\geq 0}$ be a subsequence of $\left(\frac{k_t}{1-\ell_t}\right)_{t\geq 0}$ such that

$$\limsup_{t\geq 0} \frac{k_t}{1-\ell_t} = \lim_{n\to+\infty} \frac{k_{t_n}}{1-\ell_{t_n}}. \tag{2.18}$$

Claim: $\left(\frac{k_{t_n}}{1-\ell_{t_n}}\right)_{t\geq 0}$ converges to zero.

We can choose $(\ell_{t_n})_{n\geq 0}$ such that it converges to some $\ell \in [0,1]$. If $\ell \in [0,1)$, then it follows that $\frac{k_{t_n}}{1-l_{t_n}} \to 0$.

Now consider the case where $\ell = 1$. The first order condition in 2.8 implies that

$$f_2(k_t, 1-l_t) = \frac{\psi v_2(k_t, l_t)}{u'(c_t)}. \tag{2.19}$$

Since $v_2(0,0) = 0$, and $\lim_{c\to 0} u'(c) = +\infty$, it follows that

$$\lim_{n\to+\infty} f_2(k_{t_n}, 1-l_{t_n}) = 0. \tag{2.20}$$

[3] See Le Van and Vailakis (2005)

Since f is homogeneous of degree one, we have

$$f_2(k_{t_n}, 1-l_{t_n}) = f\left(\frac{k_{t_n}}{1-l_{t_n}}, 1\right) - \frac{k_{t_n}}{1-l_{t_n}} f_1\left(\frac{k_{t_n}}{1-l_{t_n}}, 1\right) = F\left(\frac{k_{t_n}}{1-l_{t_n}}\right) \quad (2.21)$$

Denote $y_{t_n} = \frac{k_{t_n}}{1-l_{t_n}}$. Next we show that y_{t_n} is bounded from above. Note that F is an increasing function. If y_{t_n} were unbounded from above, then that would contradict 2.20. Thus, y_{t_n} is a bounded sequence. This implies that there exists $\left(\frac{k_{t_{nm}}}{1-\ell_{t_{nm}}}\right)_{m\geq 0}$ such that

$$\limsup_{n\to+\infty} \frac{k_{t_n}}{1-\ell_{t_n}} = \lim_{m\to+\infty} \frac{k_{t_{nm}}}{1-\ell_{t_{nm}}} = y < +\infty, \quad (2.22)$$

for some $y \geq 0$, since y_{t_m} is bounded. It is left to be shown that $y = 0$. By contradiction, assume that $y > 0$ (y_t cannot take negative values). Then from 2.20, since F is continuous, it follows that $F(y_{t_{mp}}) \to F(y) = 0$.

From our supposition that $y > 0$, and from the fact that f is strictly concave we have

$$f(y,1) = f(y,1) - f(0,1) > yf_1(y,1) \quad \textit{or} \quad f(y,1) - yf_1(y,1) = F(y) > 0$$

contradiction! Thus, $y = 0$. Moreover, since $\left(\frac{k_{t_n}}{1-\ell_{t_n}}\right)_{n\geq 0}$ is a non-negative sequence, from 2.22 it follows that $\left(\frac{k_{t_n}}{1-\ell_{t_n}}\right)_{n\geq 0}$ converges to zero. Furthermore, following the same argument, from 2.18 we have

$$\frac{k_t}{1-\ell_t} \longrightarrow 0. \quad (2.23)$$

From 2.5 we have

$$\frac{\lambda_t}{\lambda_{t+1}} = f_1(k_{t+1}, 1-l_{t+1}) + \psi\beta \frac{v_1(k_{t+1}, l_{t+1})}{\lambda_{t+1}} \geq f_1(k_t, 1-l_t) = f_1\left(\frac{k_t}{1-l_t}, 1\right)$$

The last equality comes from the fact that f_1 is homogeneous of degree zero. From the Inada conditions and 2.23, it holds that $\lambda_t/\lambda_{t+1} > 1/\beta$, for sufficiently large t. That is, $u'(c_t)/u'(c_{t+1}) > 1$ for large t, implying that $c_{t+1}/c_t > 1$ for large n. That *is,* $c_{t+1} > c_t$*, for sufficiently large* t*, contradicting that fact that* $c_t \to 0$*. Therefore, the optimal sequence of physical capital starting from* $k_0 > 0$ *does not converge to zero.* ■

From the Inada conditions we know that the optimal path of physical capital stocks is bounded. Moreover, the optimal path of physical capital stock is a monotonic sequence on a compact set, hence it is convergent. In the next chapter, we will provide the conditions under which the model has a unique nontrivial steady state. Since we have shown that the sequence of physical capital does not converge to zero, we can therefore deduce that the optimal path of physical capital converges monotonically to the nontrivial steady state.

CHAPTER 3

THE DYNAMICS

In this chapter, we analyze the dynamics and provide the conditions under which the model has multiple steady states and a unique steady state. We adopt the following functional forms for the utility function:

$$\begin{aligned} u(c) &= \ln c, \\ v(k,l) &= k^{\gamma}\left(\varphi - \frac{(1-l)^{1+\eta}}{1+\eta}\right), \end{aligned} \tag{3.1}$$

where $\eta > 0$ is the inverse of the Frisch elasticity of labor supply, $\gamma > 0$ is the intensity of wealth in the utility obtained from leisure, and $\varphi > 0$ is a scale parameter ensuring that the function v satisfies all of the conditions in Assumption 3. We consider a Cobb-Douglas production function with technology parameter $A > 0$, and the capital share $\alpha \in (0,1)$. That is, the total amount of output at period t is $f(k_t, n_t) = Ak_t^{\alpha}n_t^{1-\alpha} + (1-\delta)k_t$.

Proposition 6 *Under the functional forms in 3.1, the Hamiltonian function is concave in (c,k,l) if and only if $\gamma > 1$, and $\eta\left(\varphi - \frac{1}{1+\eta}\right) \geq \frac{\gamma}{1-\gamma}$.*

Proof. *See the Appendix.* ■

Next, we derive the dynamical system in physical capital and consumption. From equation 2.10 we have

$$(1-\ell_t) = \left(\frac{(1-\alpha)A}{\psi}\frac{1}{c_t}k_t^{\alpha-\gamma}\right)^{\frac{1}{\alpha+\eta}} \quad, \forall t \geq 0. \tag{3.2}$$

From this relationship we observe that when $\gamma > \alpha$, leisure and wealth are directly proportional, whereas when $\gamma < \alpha$, leisure and wealth are inversely proportional. Note that under the functional forms in 3.1, the condition that guarantees the monotonicity of the optimal paths of physical capital in Proposition 4 becomes $\gamma < \alpha$. That is, if the output elasticity of physical capital is greater than the parameter accounting for the wealth intensity in the utility obtained from leisure, then the optimal path of physical capital is monotonic.
Moreover, using the functional forms in 3.1 in equations 2.9 and 2.11 we get

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}}\left(\alpha A k_{t+1}^{\alpha-1}(1-\ell_{t+1})^{1-\alpha} + 1 - \delta\right) - \beta\psi\gamma k_{t+1}^{\gamma-1}\left(\varphi - \frac{(1-\ell_{t+1})^{1+\eta}}{1+\eta}\right) \tag{3.3}$$

$$k_{t+1} = Ak_t^{\alpha}(1-\ell_t)^{1-\alpha} + (1-\delta)k_t - c_t \tag{3.4}$$

for all $t \geq 0$.
We now plug 3.2 into 3.3 and 3.4 to obtain the dynamic system in physical capital and consumption. Equations 3.3 and 3.4 become

$$k_{t+1} = Ak_t^{\alpha}\left(\frac{(1-\alpha)A}{\psi} \cdot k_t^{\alpha-\gamma}c_t^{-1}\right)^{\frac{1-\alpha}{\alpha+\eta}} - c_t + (1-\delta)k_t, \tag{3.5}$$

and

$$\begin{aligned}\frac{1}{c_t} = \beta\frac{1}{c_{t+1}}&\left(\alpha A k_{t+1}^{\alpha-1}\left(\frac{(1-\alpha)A}{\psi}\cdot k_{t+1}^{\alpha-\gamma}c_{t+1}^{-1}\right)^{\frac{1-\alpha}{\alpha+\eta}} + 1 - \delta\right)\\ &- \frac{\beta\psi\gamma}{1+\eta}\left(\frac{1-\alpha}{\psi}A\right)^{\frac{1+\eta}{\alpha+\eta}} c_{t+1}^{-\frac{1+\eta}{\alpha+\eta}} k_{t+1}^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}} + \beta\psi\varphi\gamma k_{t+1}^{\gamma-1},\end{aligned} \tag{3.6}$$

respectively.

After rearranging equation 3.6 we can write the system of dynamic equations as:

$$\begin{cases} \frac{1}{c_t} = \beta\Big(\alpha - \frac{(1-\alpha)\gamma}{1+\eta}\Big)\Big(\frac{1-\alpha}{\psi}\Big)^{\frac{1-\alpha}{\alpha+\eta}} A^{\frac{1+\eta}{\alpha+\eta}} k_{t+1}^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}} c_{t+1}^{-\frac{1+\eta}{\alpha+\eta}} + \frac{\beta(1-\delta)}{c_{t+1}} + \beta\psi\varphi\gamma k_{t+1}^{\gamma-1} \\ k_{t+1} = \Big(\frac{1-\alpha}{\psi}\Big)^{\frac{1-\alpha}{\alpha+\eta}} A^{\frac{1+\eta}{\alpha+\eta}} k_t^{\frac{\alpha(1+\eta)-(1-\alpha)\gamma}{\alpha+\eta}} c_t^{\frac{\alpha-1}{\alpha+\eta}} + (1-\delta)k_t - c_t \end{cases} . \tag{3.7}$$

In the next section, we will show the existence of the steady state(s), and analyze its/their properties.

3.1 Steady State

Several studies that have considered models with endogenous labor supply show that preferences play an important role in the quality of the dynamics (see Kamihigashi (2015) and De Hek (1998)). In this section, we prove the existence of the steady states, and characterize the conditions under which the model has multiple steady states or a unique steady state.

Definition 3 *A stock of physical capital $k > 0$, a stock of consumption $c > 0$, a level of leisure $l \in [0,1]$ constitute a steady state if the associated stationary path is optimal, i.e., they are solutions to the following equations:*

$$k = \left(\frac{1-\alpha}{\psi}\right)^{\frac{1-\alpha}{\alpha+\eta}} A^{\frac{1+\eta}{\alpha+\eta}} k^{\frac{\alpha(1+\eta)-(1-\alpha)\gamma}{\alpha+\eta}} c^{\frac{\alpha-1}{\alpha+\eta}} + (1-\delta)k - c, \tag{3.8}$$

$$\frac{1}{c} = \beta\left(\alpha - \frac{(1-\alpha)\gamma}{1+\eta}\right)\left(\frac{1-\alpha}{\psi}\right)^{\frac{1-\alpha}{\alpha+\eta}} A^{\frac{1+\eta}{\alpha+\eta}} k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}} c^{-\frac{1+\eta}{\alpha+\eta}} + \frac{\beta(1-\delta)}{c} + \beta\psi\varphi\gamma k^{\gamma-1}, \tag{3.9}$$

$$(1-l) = \left(\frac{(1-\alpha)A}{\psi}\frac{1}{c}k^{\alpha-\gamma}\right)^{\frac{1}{\alpha+\eta}}. \tag{3.10}$$

Equation 3.8 can be written as

$$\delta k = \left(\frac{1-\alpha}{\psi}\right)^{\frac{1-\alpha}{\alpha+\eta}} A^{\frac{1+\eta}{\alpha+\eta}} k^{\frac{\alpha(1+\eta)-(1-\alpha)\gamma}{\alpha+\eta}} c^{\frac{\alpha-1}{\alpha+\eta}} - c. \tag{3.11}$$

Multiply equation 3.9 by c and we have

$$\beta\left(\alpha-\frac{(1-\alpha)\gamma}{1+\eta}\right)\left(\frac{1-\alpha}{\psi}\right)^{\frac{1-\alpha}{\alpha+\eta}}A^{\frac{1+\eta}{\alpha+\eta}}k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}}c^{\frac{\alpha-1}{\alpha+\eta}}+\beta\psi\varphi\gamma k^{\gamma-1}c=1-\beta(1-\delta). \tag{3.12}$$

Next, we use equations 3.11 and 3.12 to come up with an equation in physical capital. Equation 3.11 can be written as

$$\left(\frac{1-\alpha}{\psi}\right)^{\frac{1-\alpha}{\alpha+\eta}}A^{\frac{1+\eta}{\alpha+\eta}}c^{\frac{\alpha-1}{\alpha+\eta}}=(\delta k+c)k^{\frac{(1-\alpha)\gamma-\alpha(1+\eta)}{\alpha+\eta}}. \tag{3.13}$$

We then then plug it into 3.12 and have

$$\beta\left(\alpha-\frac{(1-\alpha)\gamma}{1+\eta}\right)k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}}(\delta k+c)k^{\frac{(1-\alpha)\gamma-\alpha(1+\eta)}{\alpha+\eta}}+\beta\psi\varphi\gamma k^{\gamma-1}c=1-\beta(1-\delta). \tag{3.14}$$

Define

$$\chi:=\alpha-\frac{(1-\alpha)\gamma}{1+\eta},$$

and observe that χ can take both positive and negative value. In fact, $\chi\in(-1,1)$. We will later see that the sign of χ will be crucial for determining the number of steady states. Note that if $\alpha>0.5$, then χ is always positive. The parameter χ takes negative values only when the output elasticity of physical capital α is small, the intensity of wealth in the utility obtained from leisure is large, and when the labor is sufficiently elastic.

Equation 3.14 can be written as

$$\beta\chi\frac{1}{k}(\delta k+c)+\beta\psi\varphi\gamma k^{\gamma-1}c=1-\beta(1-\delta). \tag{3.15}$$

Now from equation 3.15, which can be written as

$$1-\beta(1-\delta)-\beta\delta\chi=\beta\chi\frac{c}{k}+\beta\psi\varphi\gamma k^{\gamma-1}c, \tag{3.16}$$

we will be able to write consumption in terms of physical capital. From 3.16 we see an explicit relationship between consumption and physical capital, given by

$$c = \frac{\phi k}{\psi \varphi \gamma k^{\gamma} + \chi}, \tag{3.17}$$

where

$$\phi := \frac{1 - \beta(1-\delta) - \beta\delta\chi}{\beta}.$$

After plugging 3.17 into 3.12 we can finally obtain an equation in physical capital, given by

$$1 - \beta(1-\delta) = \beta\mathcal{E}\chi\left(\frac{\phi k^{1+\gamma+\eta}}{\psi\varphi\gamma k^{\gamma} + \chi}\right)^{\frac{\alpha-1}{\alpha+\eta}} + \frac{\beta\psi\varphi\gamma\phi k^{\gamma}}{\psi\varphi\gamma k^{\gamma} + \chi}, \tag{3.18}$$

where

$$\mathcal{E} := \left(\frac{1-\alpha}{\psi}\right)^{\frac{1-\alpha}{\alpha+\eta}} A^{\frac{1+\eta}{\alpha+\eta}}.$$

We can rearrange equation 3.18, so that we can have an equation where k is the only variable.

$$\frac{1-\beta(1-\delta)}{1-\beta(1-\delta)-\beta\delta\chi} = \mathcal{E}\chi\phi^{\frac{-1-\eta}{\alpha+\eta}}\left(\frac{\psi\varphi\gamma k^{\gamma} + \chi}{k^{1+\gamma+\eta}}\right)^{\frac{1-\alpha}{\alpha+\eta}} + \frac{\psi\varphi\gamma k^{\gamma}}{\psi\varphi\gamma k^{\gamma} + \chi} \tag{3.19}$$

To make our analysis simpler we define the followings:

$$LHS := \frac{1-\beta(1-\delta)}{1-\beta(1-\delta)-\beta\delta\chi}$$

$$g(k) := \frac{\psi\varphi\gamma k^{\gamma}}{\psi\varphi\gamma k^{\gamma} + \chi}$$

$$h(k) := \mathcal{E}\chi\phi^{\frac{-1-\eta}{\alpha+\eta}}\left(\frac{\psi\varphi\gamma k^{\gamma} + \chi}{k^{1+\gamma+\eta}}\right)^{\frac{1-\alpha}{\alpha+\eta}}$$

We can write 3.19 as

$$LHS = g(k) + h(k) \tag{3.20}$$

We check that

$$\frac{\mathrm{d}g(k)}{\mathrm{d}k} = \frac{\psi\varphi\gamma^2\chi k^\gamma}{(\psi\varphi\gamma k^\gamma + \chi)^2} = \frac{\gamma\chi}{\psi\varphi\gamma k^\gamma + \chi}\frac{f_1(k)}{k}, \tag{3.21}$$

and

$$\begin{aligned}\frac{\mathrm{d}h(k)}{\mathrm{d}k} &= \frac{1-\alpha}{\alpha+\eta}\frac{f_2(k)}{k}\left(\frac{\psi\varphi\gamma^2k^\gamma - (1+\eta+\gamma)\left(\psi\varphi\gamma k^\gamma + \chi\right)}{\psi\varphi\gamma k^\gamma + \chi}\right) \\ &= -\frac{1-\alpha}{\alpha+\eta}\frac{f_2(k)}{k}\left(1+\eta+\frac{\gamma\chi}{\psi\varphi\gamma k^\gamma + \chi}\right),\end{aligned} \tag{3.22}$$

The number of the steady states will depend on the sign of $\chi = \alpha - \frac{(1-\alpha)\gamma}{1+\eta}$. For this we consider the two cases seprately.

Case 1: $\chi > 0$

Assume that $\chi = \alpha - \frac{(1-\alpha)\gamma}{1+\eta} > 0$ i.e., $\frac{\alpha}{1-\alpha} > \frac{\gamma}{1+\eta}$. Then we have $LHS > 1$. Note that

$$\lim_{k\to 0^+} g(k) = 0,$$
$$\lim_{k\to +\infty} g(k) = 1,$$

and $g(k)$ is a monotonically increasing function in k since $\frac{\mathrm{d}g(k)}{\mathrm{d}k} > 0$, for all $k > 0$. Note also that

$$\lim_{k\to 0^+} h(k) = +\infty,$$
$$\lim_{k\to +\infty} h(k) = 0,$$

and $h(k)$ is a monotonically decreasing function in k since $\frac{\mathrm{d}h(k)}{\mathrm{d}k} < 0$, for all $k > 0$.

Since h is continuous and monotonically decreasing, there exist unique $k_m, k_M \in \mathbb{R}_+$ such that $h(k_m) = LHS$ and $h(k_M) = LHS - 1$. The following proposition will provide the conditions for the existence and uniqueness of a steady-state.

Proposition 7 (Existence and Uniqueness) *Assume that* $\alpha - \frac{(1-\alpha)\gamma}{1+\eta} > 0$*. Then the followings hold:*

(i) There exists a steady state $k^{ss} \in (k_m, k_M)$*. There is no steady state outside this region.*

(ii) If $g(k_M) < \frac{1-\alpha}{\alpha+\eta} h(k_M)$*, then the steady state* k^{ss} *is unique.*

Proof. *(i) First of all, note that* $k_m < k_M$ *since* h *is a monotonically decreasing function and* $0 < LHS - 1 < LHS < \infty$*. Moreover, note that* $g(k_m) + h(k_m) > LHS$*, since* $g(k_m) > 0$ *and* $g(k_M) + h(k_M) = LHS - 1 + h(k_M) < LHS$*, since* $h(k_M) < 1$*. Thus, there exists a steady state* $k^{ss} \in (k_m, k_M)$*. For all* $k < k_m$*, we have* $g(k) + h(k) > f_1(k) > LHS$ *and for* $k > k_M$*, we have* $g(k) + h(k) < (LHS - 1) + 1 = LHS$*. Thus, there is no steady state outside this region.*

(ii) Assume that $g(k_M) < \frac{1-\alpha}{\alpha+\eta} h(k_M)$*. From (3.21) and (3.22), we have*

$$\begin{aligned}\frac{\mathrm{d}g(k)}{\mathrm{d}k} + \frac{\mathrm{d}h(k)}{\mathrm{d}k} &= \frac{\gamma\chi}{\psi\varphi\gamma k^{\gamma} + \chi}\frac{g(k)}{k} - \frac{1-\alpha}{\alpha+\eta}\left(1 + \eta + \frac{\gamma\chi}{\psi\varphi\gamma k^{\gamma} + \chi}\right)\frac{h(k)}{k} \\ &= -\frac{1-\alpha}{\alpha+\eta}(1+\eta)\frac{h(k)}{k} + \left(g(k) - \frac{1-\alpha}{\alpha+\eta}h(k)\right)\left(\frac{\gamma\chi}{\psi\varphi\gamma k^{\gamma} + \chi}\right)\frac{1}{k} < 0,\end{aligned}$$

for $k < k_M$*, since for all* $k < k_M$*, we have* $f_1(k) < f_1(k_M) < \frac{1-\alpha}{\alpha+\eta} h(k_M) < \frac{1-\alpha}{\alpha+\eta} h(k)$*. Thus, the right hand side of 3.21 is a monotonically decreasing function. Thus, the steady state is unique.* ■

Case 2: $\chi < 0$

Now assume that $\chi = \alpha - \frac{(1-\alpha)\gamma}{1+\eta} < 0$ i.e., $\frac{\alpha}{1-\alpha} < \frac{\gamma}{1+\eta}$. Observe that $LHS < 1$. To ensure that $c > 0$ we should have $\psi\varphi\gamma k^{\gamma} + \chi > 0$ (see 3.17). Hence it holds that

$$k > \overline{k} := \left(\frac{-\chi}{\psi\varphi\gamma}\right)^{\frac{1}{\gamma}} > 0.$$

In this light, we have

$$\lim_{k \to \bar{k}^+} g(k) = +\infty,$$
$$\lim_{k \to \infty} g(k) = 1,$$

and

$$\lim_{k \to \bar{k}^+} h(k) = 0,$$
$$\lim_{k \to \infty} h(k) = 0.$$

Moreover $g(k)$ is a monotonically decreasing function. Moreover, $g(k) > 1$ for all $k > \bar{k}$ (since $\frac{\mathrm{d}g(k)}{\mathrm{d}k} < 0$). On the other hand, $h(k) < 0$ for all $k > \bar{k}$. Define $k_p := \left(-\frac{1}{\psi\varphi\gamma}\frac{1+\eta+\gamma}{1+\eta}\chi\right)^{\frac{1}{\gamma}} > \bar{k}$, and see that

$$\frac{\mathrm{d}h(k)}{\mathrm{d}k}\begin{cases} \leq 0, & \text{for } k \in (\bar{k}, k_p] \\ > 0, & \text{for } k \in (k_p, \infty) \end{cases}$$

Note that under this case, we do not need to have a steady state and yet the following condition on the parameters ensures that there are multiple steady states.

Let's define $k_{max} \in (k_p, +\infty]$ such that $h(k_{max}) = LHS - 1$. Consider the following propositions:

Proposition 8 *When $\chi < 0$ there is no steady state outside of $[\overline{k}, k_{max})$. Moreover, if $h(k_p) \geq LHS - 1$, then there exists no steady state.*

Proof. *For all $k \geq k_{max}$ we have $g(k) + h(k) > g(k) + LHS - 1 > LHS$ (since $g(k) \geq 1$ for all $k \geq \overline{k}$). So, there is no steady state in the complement of $[k, k_{max})$.*

Now assume that $h(k_p) \geq LHS - 1$. Then $g(k) + h(k) \geq 1 + h(k) \geq 1 + h(k_p) \geq 1 + (LHS - 1) = LHS$. That is, $g(k) + h(k) > LHS$ for all $k \geq \overline{k}$, implying that there is no steady state. ■

Proposition 9 (Multiple Steady States) *If $g(k_p) + h(k_p) \leq LHS$, then there exist multiple steady states. In particular, there exists a steady state $k_{low} \in (\overline{k}, k_p)$, unique in this region, and at least one another $k_{high} \in (k_p, k_{max})$. Moreover, if $LHS = g(k_p) + h(k_p)$, then $k_{low} = k_p$.*

Proof. *Assume that $g(k_p) + h(k_p) \leq LHS$. Then $g(k) + h(k)$ will be strictly increasing in $[\overline{k}, k_p]$. Since $\lim_{k \to k_+} \big(g(k) + h(k)\big) = +\infty$, from the Intermediate Value Theorem (IVT), there exists a unique solution in $(\overline{k}, k_p)]$. Moreover, since $\lim_{k \to +\infty}(g + h)(k) = 1 > LHS$, from IVT, we conclude that there is as least a solution in $(k_p, +\infty)$. Note that when $g(k_p) + h(k_p) = LHS$, we have $k_{low} = k_p$.* ■

3.2 Local Stability

We linearize the following dynamical system

$$\frac{1}{c_t} = \beta \mathcal{E}\Big(\alpha - \frac{(1-\alpha)\gamma}{1+\eta}\Big) k_{t+1}^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}} c_{t+1}^{-\frac{1+\eta}{\alpha+\eta}} + \frac{\beta(1-\delta)}{c_{t+1}} + \beta\psi\varphi\gamma k_{t+1}^{\gamma-1}$$

$$k_{t+1} = \mathcal{E} k_t^{\frac{\alpha(1+\eta)-(1-\alpha)\gamma}{\alpha+\eta}} c_t^{\frac{\alpha-1}{\alpha+\eta}} + (1-\delta)k_t - c_t$$

around its steady state(s) to obtain the following linearized system:

$$\begin{bmatrix} \mathrm{d}k_{t+1} \\ \mathrm{d}c_{t+1} \end{bmatrix} = \mathcal{J} \cdot \begin{bmatrix} \mathrm{d}k_t \\ \mathrm{d}c_t \end{bmatrix}$$

where $\mathcal{J}$ is the 2×2 Jacobian matrix whose entries are:

$$J_{11} = \frac{(1+\eta)\chi}{\alpha+\eta}\left(\frac{c}{k} + \delta\right) + 1 - \delta$$

$$J_{12} = -\frac{(1-\alpha)\delta}{\alpha+\eta}\frac{k}{c} - \frac{1+\eta}{\alpha+\eta}$$

$$J_{21} = -\frac{(1-\alpha)(\gamma+\eta)\chi}{\alpha+\eta}\left(\frac{c^2}{k^2} + \delta\frac{c}{k}\right) + \psi\varphi\gamma(\gamma-1)k^{\gamma-2}c^2$$

$$J_{22} = \frac{J_{12} \cdot J_{21}}{J_{11}} + \frac{1}{\beta J_{11}}$$

where

$$\chi = \alpha - \frac{(1-\alpha)}{1+\eta}.$$

Observe that the determinant of the Jacobian matrix is $1/\beta$, which is greater than 1. Thus, we know that if a steady state exists, it can either be saddle path stable or unstable. Next, we will provide the conditions under which the unique steady state is saddle-path stable i.e., |Trace $(\mathcal{J}) - 1| \geq$ Det$(\mathcal{J})$. First, consider the following lemma.

Lemma 1 *If $J_{11} \geq \frac{1}{\beta}$, then $J_{11} + \frac{1}{\beta J_{11}} - 1 \geq \frac{1}{\beta}$.*

Proof. *Consider the equation*

$$x + \frac{1}{\beta x} - 1 = \frac{1}{\beta},$$

which can also be written as

$$x^2 - \left(\frac{1}{\beta} + 1\right)x + \frac{1}{\beta} = 0.$$

From here it is straightforward to check that when $x \geq \frac{1}{\beta}$, we have

$$x^2 - \left(\frac{1}{\beta} + 1\right)x + \frac{1}{\beta} \geq 0,$$

or equivalently,

$$x + \frac{1}{\beta x} - 1 \geq \frac{1}{\beta},$$

for all $x \geq \frac{1}{\beta}$. ■

Proposition 10 *The unique steady state k^{ss} is saddle-path stable, if*

$$\frac{\alpha(1+\eta) - (1-\alpha)\gamma}{\alpha + \eta} \geq \frac{1-\beta}{\beta\delta}$$

Proof. *To show that the steady state is saddle-path stable it enough to show that*

$$|Trace(\mathcal{J}) - 1| > Det(\mathcal{J})$$

Note that if

$$\frac{\alpha(1+\eta) - (1-\alpha)\gamma}{\alpha + \eta} \geq \frac{1-\beta}{\beta\delta},$$

then

$$J_{11} > \frac{1}{\beta}.$$

From Lemma 1, it follows that $J_{11} + \frac{1}{\beta J_{11}} - 1 > \frac{1}{\beta}$. *We have*

$$Trace(\mathcal{J}) - 1 = J_{11} + \frac{J_{12}J_{21}}{J_{11}} + \frac{1}{\beta J_{11}} \geq J_{11} + \frac{1}{\beta J_{11}} \geq \frac{1}{\beta} = Det(\mathcal{J}),$$

since $\frac{J_{12}J_{21}}{J_{11}} \geq 0$. *Therefore, the unique steady state level of physical capital is saddle-path stable.* ■

3.3 Comparative Statics

In this section, we will illustrate (a) how the elasticity of labor supply and intensity of wealth in leisure preferences affect the level of physical capital at the unique steady-state and (b) how the elasticity of labor supply and output elasticity of physical capital affect the influence that wealth intensity in leisure preferences has on the steady-state level of physical capital and (c) how wealth intensity in leisure preferences affects the influence that the elasticity of labor supply has on the steady-state level of physical capital.

The complicated nature of our model does not allow us to conduct the comparative statics analytically. For this reason, we will conduct our analysis by using the following plausible values for the parameters:

$$A = 1 \ ; \ \beta = 0.98 \ ; \ \psi = 1 \ ; \ \delta = 0.12 \ ; \ \varphi = 20.$$

Moreover, unless otherwise mentioned we will use the following: $\eta = 1$, $\gamma = 0.5$, and $\alpha = 0.6$.

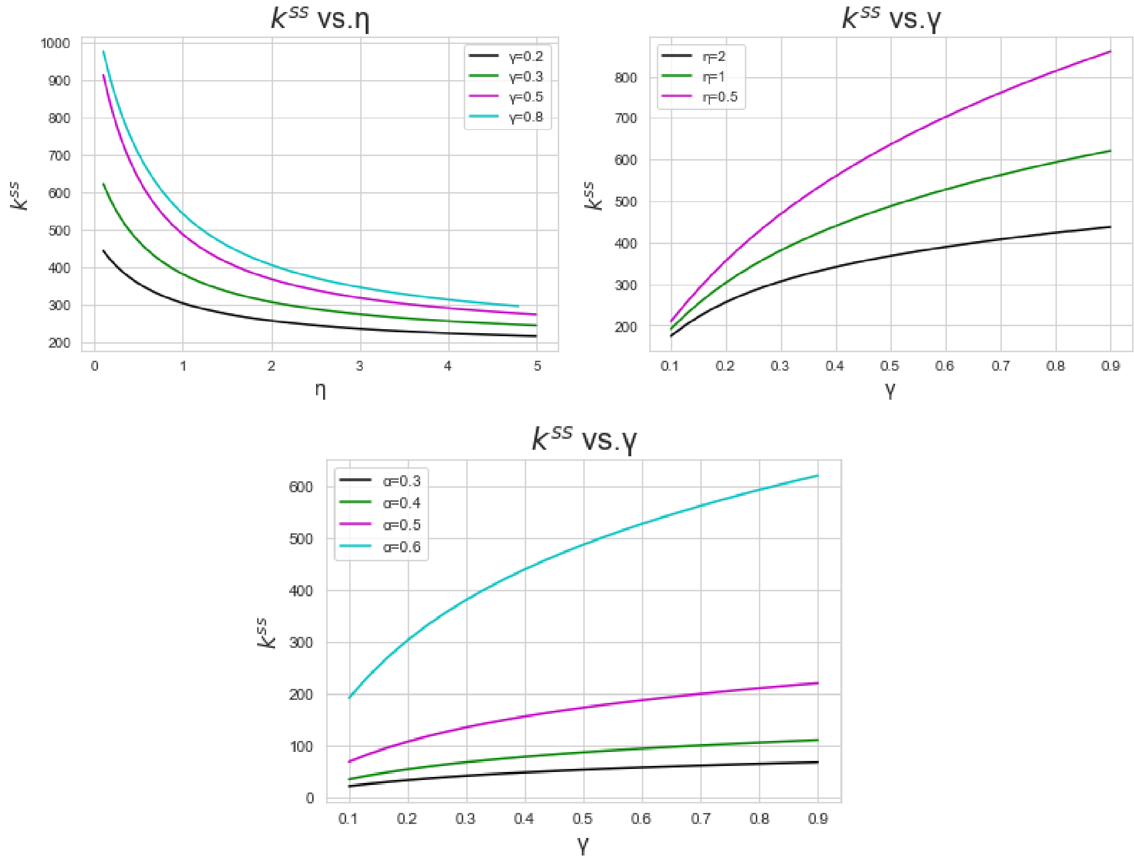

Figure 3: Comparative statics

Notes: The first graph shows the relationship between the labor supply elasticity and the steady-state level of physical capital for different values of the parameter accounting for the intensity of wealth in the utility obtained from leisure. The second graph shows the relationships between the intensity of wealth in the utility obtained from leisure and the steady-state level of physical capital for different values of the labor supply elasticity. The last graph shows the relationship between the intensity of wealth in the utility obtained from leisure and the steady-state level of physical capital for different values of the output elasticity of physical capital.

Recall that η is the inverse of the elasticity of labor supply. The graphs above show that the steady-state level of physical capital increases as the labor supply becomes more elastic. What's more, the influence that the elasticity of labor supply has on the steady-state level of physical capital increases as the intensity of wealth in the utility from leisure increases.

Moreover, the steady-state level of physical capital increases as the intensity of wealth in the utility obtained from leisure increases. The influence that the intensity of wealth in leisure preferences has on the steady-state level of physical capital increases as labor supply becomes more elastic. Moreover, the output elasticity of physical capital increases the influence that the intensity of wealth in the utility obtained from leisure has on the steady-state level of physical capital.

CHAPTER 4

CONCLUSION

In this thesis, we have analyzed a one-sector optimal growth model in which wealth affects the utility obtained from leisure. We have shown the existence of the optimal path and characterized the dynamics and the properties of equilibria. We consider that wealth increases the propensity to consume leisure goods and services and hence affects how the instantaneous utility depends on leisure time. We have built on the fact that the disutility of labor – which incorporates the opportunity cost of being unable to consume – increases with wealth. The latter comes from the fact that as individuals' wealth raises, so does their ability to enjoy more luxurious and qualitative leisure time.

We have found that the intensity of wealth in the preferences for leisure time and the output elasticity of physical capital play an important role in the number of steady states and in the monotonicity of the optimal path of physical capital. In particular, we have found that: multiple steady states can emerge for low values of the output elasticity of physical capital, high values of the intensity of wealth in the utility obtained from leisure, provided that labor supply is sufficiently elastic; and the steady state may be unique, provided that the output elasticity of physical capital is sufficiently high. Furthermore, we have shown that the optimal path of physical capital is monotonic, provided that the output elasticity of capital is greater than the parameter accounting for the wealth intensity in the utility obtained from leisure.

We have performed a local analysis around the unique steady state, and have found that the presence of wealth in the utility obtained from leisure increases the steady state level of physical capital. Moreover, the influence that the intensity of wealth in the utility obtained from leisure has on the steady state level of physical capital increases as the labor becomes more elastic. Furthermore, the influence that the intensity of wealth in the utility obtained from leisure has on the level of physical capital at its steady state increases as the output elasticity of physical capital increases.

For future work, we can analyze the multiple steady state case. It can be shown that under plausible values of the parameters, the steady states are saddle-path stable. We can investigate the optimal path of physical capital for whether we have global indeterminacy, and hence threshold dynamics. Moreover, this may allow us to explain the fact that countries that start from the same initial wealth, may experience different development paths if they start from different initial values of consumption and leisure. Other models can also be considered. This thesis characterizes wealth in the form of physical capital; however, we can also think of wealth as a non-labor income (e.g., bequest). We can also extend the model by investigating the effect that investment in leisure goods and services has on preferences for leisure time.

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APPENDIX

Proof of Proposition 4

Consider

$$V(k,k') = \max_{\{c,l\}} \quad u(c) + v(k,\ell)$$
$$\text{s.t.} \quad c + k' \leq f(k, 1-\ell),$$
$$c \geq 0, \; 0 \leq \ell \leq 1.$$

The Lagrangian is given by:

$$\mathcal{L} = u(c) + v(k,\ell) + \lambda(f(k,1-\ell) - c - k') + \mu(1-\ell)$$

Kuhn-Tucker first order conditions are:

$$u'(c) - \lambda = 0 \tag{4.1}$$

$$v_2(k,\ell) - \lambda f_2(k,1-\ell) - \mu = 0 \tag{4.2}$$

$$\mu(1-l) = 0, \mu \geq 0 \tag{4.3}$$

$$\lambda(f(k,1-\ell) - c - k') = 0, \lambda \geq 0$$

Note that $\ell = 1$ is not optimal for this problem (since $f_2(k,0) = -\infty$). So $\mu = 0$. Also $\lambda > 0$, since $u'(c) > 0$.

Kuhn-Tucker first order conditions will become:

$$u'(c) - \lambda = 0 \tag{4.4}$$

$$v_2(k, \ell) - \lambda f_2(k, 1 - \ell) = 0 \tag{4.5}$$

$$c + k' - f(k, 1 - \ell) = 0 \tag{4.6}$$

Taking the total derivative of the both sides in each of the equations above gives

$$u_{11}\mathrm{d}c - \mathrm{d}\lambda = 0 \tag{4.7}$$

$$v_{12}\mathrm{d}k + v_{22}\mathrm{d}\ell - \lambda(f_{12}\mathrm{d}k - f_{22}\mathrm{d}\ell) - f_2\mathrm{d}\lambda = 0 \tag{4.8}$$

$$\mathrm{d}c - f_1\mathrm{d}k + f_2\mathrm{d}\ell = 0 \tag{4.9}$$

Write the last three equations in matrix form:

$$\underbrace{\begin{bmatrix} u_{11} & 0 & -1 \\ 0 & v_{22} + \lambda f_{22} & -f_2 \\ 1 & f_2 & 0 \end{bmatrix}}_{A} \cdot \begin{bmatrix} \mathrm{d}c \\ \mathrm{d}\ell \\ \mathrm{d}\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda f_{12} - v_{12} \\ f_1 \end{bmatrix} \mathrm{d}k$$

Determinant of A is: $u_{11}f_2^2 + v_{22} + \lambda f_{22} \neq 0$. By Implicit function theorem we know that $c(k, k'), l(k, k')$ and $\lambda(k, k')$ are continuously differentiable. From the Envelope Theorem we have:

$$\frac{\partial V(k, k')}{\partial k} = \frac{\partial \mathcal{L}}{\partial k} = v_1(k, \ell) + \lambda f_1(k, 1 - \ell),$$
$$\frac{\partial V(k, k')}{\partial k'} = -\lambda.$$

To find $\frac{\partial^2 V(k,k')}{\partial k' \partial k}$, it is enough to find $\frac{\partial \lambda}{\partial k}$.

The inverse of A is:

$$A^{-1} = \frac{1}{u_{11}f_2^2+v_{22}+\lambda f_{22}} \begin{bmatrix} f_2^2 & -f_2 & v_{22}+\lambda f_{22} \\ -f_2 & 1 & f_2 u_{11} \\ -v_{22}-\lambda f_{22} & -u_{11}f_2 & u_{11}(v_{22}+\lambda f_{22}) \end{bmatrix}$$

From the Implicit Function Theorem we have:

$$\begin{bmatrix} \frac{\mathrm{d}c}{\mathrm{d}k} \\ \frac{\mathrm{d}\ell}{\mathrm{d}k} \\ \frac{\mathrm{d}\lambda}{\mathrm{d}k} \end{bmatrix} = \frac{1}{u_{11}f_2^2+v_{22}+\lambda f_{22}} \begin{bmatrix} f_2^2 & -f_2 & v_{22}+\lambda f_{22} \\ -f_2 & 1 & f_2 u_{11} \\ -v_{22}-\lambda f_{22} & -u_{11}f_2 & u_{11}(v_{22}+\lambda f_{22}) \end{bmatrix} \begin{bmatrix} 0 \\ \lambda f_{12}-v_{12} \\ f_1 \end{bmatrix}$$

Hence, it holds that

$$\begin{aligned} \frac{\partial\lambda}{\partial k} &= \frac{-u_{11}f_2(\lambda f_{12}-v_{12})+u_{11}f_1(v_{22}+\lambda f_{22})}{u_{11}f_2^2+v_{22}+\lambda f_{22}} \\ &= \frac{u_{11}\big(-f_2(\lambda f_{12}-v_{12})+f_1(v_{22}+\lambda f_{22})\big)}{u_{11}f_2^2+v_{22}+\lambda f_{22}}. \end{aligned}$$

From equation 4.5 we can write $\lambda = \frac{v_2}{f_2}$. So, we have

$$\frac{\partial\lambda}{\partial k} = \frac{u_{11}\big(-v_2 f_{12}+f_2 v_{12}+f_1(v_{22}+\lambda f_{22})\big)}{u_{11}f_2^2+v_{22}+\lambda f_{22}}$$

Note that u_{11} and the denominator are both negative. Moreover, $f_{22}<0$, $v_{22}<0$, and $\lambda>0$. So, $f_1(v_{22}+\lambda f_{22})$ is negative. Observe that

$$\frac{\partial^2 V(k,k')}{\partial k'\partial k} = -\frac{\partial\lambda}{\partial k}.$$

Now the following holds:

If

$$\frac{v_{12}}{v_2} < \frac{f_{12}}{f_2},$$

then

$$\frac{\partial^2 V(k,k')}{\partial k'\partial k} = -\frac{\partial\lambda}{\partial k} > 0.$$

Proof of Proposition 6.

To find the conditions for concavity we check the Hessian matrix of the Hamiltonian.

$$\mathcal{H}(c,k,l) = \ln c + \psi k^{\gamma}\left(\varphi - \frac{(1-l)^{1+\eta}}{1+\eta}\right) + \lambda\big(Ak^{\alpha}(1-l)^{1-\alpha} + (1-\delta)k - c\big)$$

First note that for the period utility function to be increasing in k we must have

$$\varphi - \frac{1}{1+\eta} > 0$$

The Hessian matrix is given by

$$\begin{bmatrix} \frac{-1}{c^2} & 0 & 0 \\ 0 & -\psi\gamma(1-\gamma)k^{\gamma-2}\left(\varphi - \frac{(1-l)^{1+\eta}}{1+\eta}\right) - \lambda A\alpha(1-\alpha)k^{\alpha-2}(1-l)^{1-\alpha} & \psi\gamma k^{\gamma-1}(1-l)^{\eta} - \lambda A\alpha(1-\alpha)k^{\alpha-1}(1-l)^{-\alpha} \\ 0 & \psi\gamma k^{\gamma-1}(1-l)^{\eta} - \lambda A\alpha(1-\alpha)k^{\alpha-1}(1-l)^{-\alpha} & -\eta\psi k^{\gamma}(1-l)^{\eta-1} - \lambda A\alpha(1-\alpha)k^{\alpha}(1-l)^{-\alpha-1} \end{bmatrix}$$

The sufficient and necessary condition for the hessian to be negative definite is: $\forall k \in \{1,2,3\}$ we must have $(-1)^k \cdot \det H_k > 0$, where H_k is the k^{th} principal matrix obtained by retaining *only* the first k rows and the first k columns.

1- $H_1 = 1/c^2$, so $-1 \cdot \text{Det} H_1 = 1/c^2 > 0$

2- $H_2 = \begin{bmatrix} -1/c^2 & 0 \\ 0 & -\psi\gamma(1-\gamma)k^{\gamma-2}\left(\varphi - \frac{(1-l)^{1+\eta}}{1+\eta}\right) - \lambda A\alpha(1-\alpha)k^{\alpha-2}(1-l)^{1-\alpha} \end{bmatrix}$

For $\gamma \in (0,1)$, it is clear that Det$H_2 > 0$.

3- Observe that $H_2 = H$. Above we show that $(-1)^3 \cdot$ Det $H_3 > 0$, or equivalently Det$(H) < 0$. Note that

$$\text{Det}(H) = (-1/c^2)(\mathcal{H}_{kk} \cdot \mathcal{H}_{ll} - \mathcal{H}_{kl}^2) < 0$$

Find the conditions for which $\mathcal{H}_{kk} \cdot \mathcal{H}_{ll} - \mathcal{H}_{kl}^2 > 0$. We see that

$$\mathcal{H}_{kl}^2 = \psi^2\gamma^2 k^{2\gamma-2}(1-l)^{2\eta} - 2\psi\gamma\lambda A\alpha(1-\alpha)k^{\gamma+\alpha-2}(1-l)^{\eta-\alpha} + \lambda^2 A^2\alpha^2(1-\alpha)^2 k^{2\alpha-2}(1-l)^{-2\alpha} \tag{4.10}$$

and

$$\mathcal{H}_{kk} \cdot \mathcal{H}_{ll} = \psi^2 \eta\gamma(1-\gamma)k^{2\gamma-2}(1-l)^{\eta-1}\left(\varphi - \frac{(1-l)^{1+\eta}}{1+\eta}\right)$$
$$+ \psi\gamma(1-\gamma)\lambda A\alpha(1-\alpha)k^{\alpha+\gamma-2}(1-l)^{-\alpha-1}\left(\varphi - \frac{(1-l)^{1+\eta}}{1+\eta}\right)$$
$$+ \lambda A\alpha(1-\alpha)\eta\psi k^{\alpha+\gamma-2}(1-l)^{\eta-\alpha} + \lambda^2 A^2\alpha^2(\alpha-1)^2 k^{2\alpha-2}(1-l)^{-2\alpha} \tag{4.11}$$

From (1) and (2)

$$\mathcal{H}_{kk}\mathcal{H}_{ll} - \mathcal{H}_{kl}^2 = \psi^2\gamma k^{2\gamma-2}(1-l)^{2\eta}\big(\eta(1-\gamma)(1-l)^{-1-\eta}\big(\varphi - \frac{(1-l)^{1+\eta}}{1+\eta}\big) - \gamma\big)$$
$$+ \psi\lambda A\alpha(1-\alpha)k^{\alpha+\gamma-2}(1-l)^{\eta-\alpha}\big(\gamma(1-\gamma)(1-l)^{-1-\eta}\big(\varphi - \frac{(1-l)^{1+\eta}}{1+\eta}\Big)$$
$$+ \eta + 2\gamma\big)$$

For $\gamma \in (0,1)$, second term of the RHS is always positive. The sign of the RHS depends on the sign of the first term of the RHS, since it should hold even for small λ. It is enough to check the sign of

$$\eta(1-\gamma)(1-l)^{-1-\eta}\left(\varphi - \frac{(1-l)^{1+\eta}}{1+\eta}\right) - \gamma$$

which can be written as:

$$\varphi\eta(1-\gamma)(1-l)^{-1-\eta} - \frac{\eta(1-\gamma)}{1+\eta} - \gamma$$

For it to be positive for all $l \in (0,1)$ we must have

$$\eta\left(\varphi - \frac{1}{1+\eta}\right) > \frac{\gamma}{1-\gamma}$$

The sufficient and necessary conditions for the Hamiltonian to be concave in (c,k,l) when $\gamma \in (0,1)$ is : $\eta\left(\varphi - \frac{1}{1+\eta}\right) > \frac{\gamma}{1-\gamma}$.

The Jacobian Matrix

The entries of matrix A are denoted by a_{ij}, where i and j are the row and column index, respectively. The entries of B matrix are denoted by b_{ij} using the same notation. We denote the entries of A^{-1} by a_{ij}^{-1}.
The entries of matrix A are given by:

$$
\begin{aligned}
a_{11} &= 1, \\
a_{12} &= 0, \\
a_{21} &= \beta\mathcal{E}\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}\chi k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}-1}c^{\frac{-(1+\eta)}{\alpha+\eta}} + \beta\psi\varphi\gamma(\gamma-1)k^{\gamma-2}, \\
a_{22} &= -\beta\mathcal{E}\frac{1+\eta}{\alpha+\eta}\chi k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}}c^{\frac{-(1+\eta)}{\alpha+\eta}-1} - \beta(1-\delta)c^{-2}
\end{aligned}
$$

Compute the A inverse matrix. The entries of matrix A^{-1} are given by:

$$
\begin{aligned}
a_{11}^{-1} &= 1, \\
a_{12}^{-1} &= 0, \\
a_{21}^{-1} &= \frac{\mathcal{E}\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}\chi k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}-1}c^{\frac{-(1+\eta)}{\alpha+\eta}} + \psi\varphi\gamma(\gamma-1)k^{\gamma-2}}{c^{-2}\left(\mathcal{E}\frac{1+\eta}{\alpha+\eta}\chi k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}}c^{\frac{\alpha-1}{\alpha+\eta}} + (1-\delta)\right)}, \\
a_{22}^{-1} &= \frac{-c^2}{\beta\mathcal{E}\frac{1+\eta}{\alpha+\eta}\chi k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}}c^{\frac{\alpha-1}{\alpha+\eta}} + \beta(1-\delta)}
\end{aligned}
$$

The entries of matrix B are given by:

$$
\begin{aligned}
b_{11} &= \mathcal{E}\frac{\alpha(1+\eta)-(1-\alpha)\gamma}{\alpha+\eta}k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}}c^{\frac{\alpha-1}{\alpha+\eta}} + 1 - \delta, \\
b_{12} &= \mathcal{E}\frac{\alpha-1}{\alpha+\eta}k^{\frac{\alpha(1+\eta)-(1-\alpha)\gamma}{\alpha+\eta}}c^{\frac{-(1+\eta)}{\alpha+\eta}} - 1, \\
b_{21} &= 0, \\
b_{22} &= -c^{-2}
\end{aligned}
$$

From the equation 3.11 we have

$$\mathcal{E}k^{\frac{\alpha(1+\eta)-(1-\alpha)\gamma}{\alpha+\eta}}c^{\frac{\alpha-1}{\alpha+\eta}} = \delta k + c \tag{4.12}$$

Divide both sides by k we have

$$\mathcal{E}k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}}c^{\frac{\alpha-1}{\alpha+\eta}} = \delta + \frac{c}{k} \tag{4.13}$$

Divide both sides of equation 4.12 by c and we have

$$\mathcal{E}k^{\frac{(\alpha(1+\eta)-(1-\alpha)\gamma}{\alpha+\eta}}c^{\frac{-1-\eta}{\alpha+\eta}} = \delta\frac{k}{c} + 1 \tag{4.14}$$

Divide both sides of equation 4.12 by ck^2 and we have

$$\mathcal{E}k^{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}}c^{\frac{-1-\eta}{\alpha+\eta}} = \frac{\delta}{ck} + \frac{1}{k^2} \tag{4.15}$$

So using the last three equations the matrices A^{-1} and B can be written as

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}\left(\frac{\delta}{ck}+\frac{1}{k^2}\right)\chi+\psi\varphi\gamma(\gamma-1)k^{\gamma-2}}{c^{-2}\left(\left(\delta+\frac{c}{k}\right)\frac{1+\eta}{\alpha+\eta}\chi+1-\delta\right)} & \frac{-c^2}{\beta\left(\left(\delta+\frac{c}{k}\right)\frac{1+\eta}{\alpha+\eta}\chi+1-\delta\right)} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1+\eta}{\alpha+\eta}\chi\left(\delta+\frac{c}{k}\right)+1-\delta & \frac{\alpha-1}{\alpha+\eta}\left(\delta\frac{k}{c}+1\right)-1 \\ 0 & -c^{-2} \end{bmatrix}$$

Finally we can obtain the Jacobian matrix $\mathcal{J} = A^{-1}B$

$$\begin{bmatrix} \frac{1+\eta}{\alpha+\eta}\chi\left(\delta+\frac{c}{k}\right)+1-\delta & \frac{\alpha-1}{\alpha+\eta}\left(\delta\frac{k}{c}+1\right)-1 \\ \frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}\frac{c}{k}\left(\delta+\frac{c}{k}\right)\chi+\psi\varphi\gamma(\gamma-1)k^{\gamma-2}c^2 & \frac{c^2\left(\frac{(\alpha-1)(\gamma+\eta)}{\alpha+\eta}\left(\frac{\delta}{ck}+\frac{1}{k^2}\right)\chi+\psi\varphi\gamma(\gamma-1)k^{\gamma-2}\right)\left(\frac{\alpha-1}{\alpha+\eta}\left(\delta\frac{k}{c}+1\right)-1\right)+1}{\left(\delta+\frac{c}{k}\right)\frac{1+\eta}{\alpha+\eta}\chi+1-\delta} \end{bmatrix}$$