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CONSTRUCTIVE APPROXIMATION

On Smoothness of the Green Function for the Complement of a Rarefied Cantor-Type Set

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Abstract Smoothness of the Green functions for the complement of rarefied Cantor-type sets is described in terms of the function $\varphi(\delta) = (1/\log \frac{1}{\delta})$ that gives the logarithmic measure of sets. Markov's constants of the corresponding sets are evaluated.

Keywords Green's function · Markov's inequality · Cantor-type sets

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1 Introduction

Let a compact set K be regular with respect to the Dirichlet problem. Then the Green function $g_{\mathbb{C}\setminus K}$ of $\mathbb{C} \setminus K$ with pole at infinity is continuous throughout $\mathbb{C}$. Related to polynomial inequalities and some other applications, the problem of smoothness of $g_{\mathbb{C}\setminus K}$ near the boundary of K has attracted the attention of many mathematicians (see, e.g., the survey [4] and the references given there). New incentive to analyze the problem has been provided by the monograph [17] by Totik, where the author characterized the smoothness of Green functions and harmonic measures in terms of the density $\Theta_K(t)$ (Theorems 2.1 and 2.2 in [17]). For the case $K \subset [0, 1]$ with $0 \in K$, which we will consider in what follows, the density at 0 is measured by the function $\Theta_K(t) = m([0, t] \setminus K)$, where m stands for the linear Lebesgue measure.

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The monotonicity of the Green function with respect to the set K implies $g_{\mathbb{C}\setminus K}(z) \ge g_{\mathbb{C}\setminus[0,1]}(z)$ for $z \in \mathbb{C}$. In this way, we get the optimal behavior ($Lip\frac{1}{2}$ smoothness) near the origin of the function $g_{\mathbb{C}\setminus K}$ for $K \subset [0,1]$. Various conditions for optimal smoothness of $g_{\mathbb{C}\setminus K}$ in terms of metric properties of the set K are suggested in [9, 17] and in papers of V. Andrievskii [2–4]. For example, the Green function corresponding to the classical Cantor set K_0 is Hölder continuous by [6], but is not optimal smooth, by Theorem 5.1 in [17]. A recent result on smoothness of $g_{\mathbb{C}\setminus K_0}$ can be found in [15].

Here we consider Cantor-type sets $K^{(\alpha)}$ with "lowest smoothness" of the corresponding Green function. Let $1 < \alpha, 0 < l_1 < \frac{1}{2}$, and $2l_1^{\alpha-1} < 1$. Then $K^{(\alpha)} = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0,1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,s}$ of length $l_s = l_{s-1}^{\alpha}$, and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,s}$ with $j = 1, 2, \dots, 2^s$. The set $K^{(\alpha)}$ is not polar if and only if $\alpha < 2$ ([8, Chap. IV, Theorem 3]). Also, by Pleśniak [13], in the case of the Cantor type set, the corresponding set is regular if and only if it is not polar. Thus, in the case $1 < \alpha < 2$, the Green function $g_{\mathbb{C}\setminus K^{(\alpha)}}$ is continuous. We show that its modulus of continuity can be estimated in terms of the function $\varphi(\delta) = (1/\log\frac{1}{\delta})$, which is used in the definition of the logarithmic measure (see, e.g., [12, Chap. V, 6]). Here and subsequently, log denotes the natural logarithm.

Since $\Theta_{K^{(\alpha)}}(t) = t$, neither the estimation from Theorem 2.2 in [17] nor the previous general bound of Green functions given by Tsuji [18, Theorem III, 67] can be applied to our case. Let Π_n denote the set of all polynomials of degree at most n, $\Pi = \bigcup_{n=0}^{\infty} \Pi_n$. Let $|f|_K := \sup_{x\in K} |f(x)|$. We use the representation

$$g_{\mathbb{C}\setminus K^{(\alpha)}}(z) = \sup\left\{ \frac{\log|P(z)|}{\deg P} : P \in \Pi, \deg P \ge 1, |P|_{K^{(\alpha)}} \le 1 \right\}, \tag{1}$$

which follows on one hand from the Bernstein–Walsh lemma ([19, p. 77]) and on the other hand by the possibility of approximating $\exp g_{\mathbb{C}\setminus K^{(\alpha)}}(z)$, for example by the sequence $(|\Phi_n(z)|^{1/n})$, where Φ_n denotes the normalized Fekete polynomial (see, e.g., [14, Theorem 11.1]).

There is a strong connection between the smoothness of $g_{\mathbb{C}\setminus K}$ near the boundary of K and values of Markov's factors $M_n(K) = \sup_{P\in\Pi_n} \frac{|P'|_K}{|P|_K}$, which are well defined for any infinite set K. Indeed, suppose that for some increasing continuous function F we have $g_{\mathbb{C}\setminus K}(z) \le F(\delta)$ for $\operatorname{dist}(z, K) \le \delta$. Then for any $P \in \Pi_n$ the Bernstein–Walsh inequality gives $|P(z)| \le |P|_K \exp[n \cdot F(\delta)]$. Applying Cauchy's formula for P' on the circle with center at $\zeta \in K$ and of radius δ yields $|P'(\zeta)| \le \delta^{-1} \exp[n \cdot F(\delta)]|P|_K$. This gives the bound $M_n(K) \le \inf_\delta \delta^{-1} \exp[n \cdot F(\delta)]$. Particularly, if we choose δ with $F(\delta) = n^{-1}$, then $M_n(K) \le e \cdot [F_{-1}(n^{-1})]^{-1}$, where F_{-1} stands for the inverse to F function. For example, the Hölder continuity of the Green function $g_{\mathbb{C}\setminus K}$ implies Markov's property of the set K, which means that there are constants C, r such that $M_n(K) \le Cn^r$ for all n.

Here we give an asymptotic for $M_n(K^{(\alpha)})$ which is new compared to the previous results about Markov's constants of Cantor-type sets (see [10, Example 7], [16], and [7]).

As a method we employ local interpolations of functions that were used in [1] to present extension operators for the Whitney spaces $\mathcal{E}(K^{(\alpha)})$ and in [11] to construct topological bases in spaces $\mathcal{E}(K)$ for more general Cantor-type sets.

2 Results

Given $1 < \alpha < 2$, let $K^{(\alpha)}$ be the Cantor set defined in the introduction, $\varphi(\delta) = (\log \frac{1}{\delta})^{-1}$ for $0 < \delta < 1$ and $\gamma = \log \frac{2}{\alpha} / \log \alpha$.

Theorem 1 *For every* $0 < \varepsilon < \gamma$ *there exist constants* δ_0, C_0, *depending on* α *and* ε, *such that* $g_{\mathbb{C}\setminus K^{(\alpha)}}(z) \le C_0 \varphi^{\gamma-\varepsilon}(\delta)$ *for* $z \in \mathbb{C}$ *with* $\operatorname{dist}(z, K^{(\alpha)}) = \delta \le \delta_0$.

Theorem 2 *There are constants* δ_0, ε_0, *depending only on* α, *such that* $g_{\mathbb{C}\setminus K^{(\alpha)}}(-\delta) \ge \varepsilon_0 \varphi^{\gamma}(\delta)$ *for* $\delta \le \delta_0$.

Corollary 1 *If* $1 < \alpha < 2$, *then for every* $0 < \varepsilon < \gamma$ *there exists a constant* C_1 *such that* $M_n(K^{(\alpha)}) \le \exp[C_1 \cdot n^{(1+\varepsilon)\frac{\log \alpha}{\log 2}}]$ *for* $n \in \mathbb{N}$. *On the other hand, for each* $\alpha > 1$ *we have* $M_n(K^{(\alpha)}) > \exp[\alpha^{-2} \cdot n^{\frac{\log \alpha}{\log 2}}]$ *for* $n \in \mathbb{N}$.

3 Proof of Theorem 2

Let us first prove the more simple sharpness result.

Without loss of generality we can suppose that $l_1 = e^{-1}$, so $l_s = \exp(-\alpha^{s-1})$. If $l_{q+1} < \delta \le l_q$, then $\alpha^{-q} < \varphi(\delta) \le \alpha^{-q+1}$. Since $\alpha^{-\gamma} = \alpha/2$, we have

$$\left(\frac{\alpha}{2}\right)^q < \varphi^{\gamma}(\delta) \le \left(\frac{\alpha}{2}\right)^{q-1}. \tag{2}$$

Let us fix q_0 with $(\alpha/2)^{q_0-1} \le (\alpha-1)/2$, $\delta_0 = l_{q_0}$, and $\varepsilon_0 = \frac{\alpha}{8}\frac{\alpha-1}{2-\alpha}$. In view of (1) and (2), it is enough for given $l_{q+1} < \delta \le l_q \le l_{q_0}$ to find a polynomial $P \in \Pi_n$ with $|P|_{K^{(\alpha)}} \le 1$ such that

$$\frac{\log |P(-\delta)|}{n} \ge \frac{1}{4}\frac{\alpha-1}{2-\alpha}\left(\frac{\alpha}{2}\right)^q. \tag{3}$$

For fixed $m \in \mathbb{N}$ let $(x_k)_{k=1}^{2^m}$ be the set of all endpoints of the basic intervals $I_{j,m-1}$ with $j = 1, 2, \dots, 2^{m-1}$. We arrange them in increasing order, so $x_1 = 0, x_2 = l_{m-1}, x_3 = l_{m-2} - l_{m-1}, \dots, x_{2^k} = l_{m-k}, \dots, x_{2^m} = 1$. Set $\omega(z) = \prod_{k=1}^{2^m}(z - x_k)$. Then the fundamental Lagrange polynomial $L_1(z) = (\omega(z)/z \cdot \omega'(0))$ has the norm $|L_1|_{K^{(\alpha)}}$ equal 1, as is easy to check. Indeed, if $x \in K^{(\alpha)} \cap I_{1,m-1}$, then $|L_1(x)| \le |L_1(0)| = 1$, by the monotonicity of $\omega(z)/z$ there. Otherwise, $x \in K^{(\alpha)} \cap I_{j,m-1}$ with $j = 2, \dots, 2^{m-1}$, and $|\omega(x)| \le l_m \cdot x_2 \cdot x_3 \cdots x_{2^m}$, so $|L_1(x)| \le l_m/x < 1$.

Now for given q we take $m=2q, n=4^q-1$ and $P=L_1\in\Pi_n$. Then

$$\left|P(-\delta)\right|=\prod_{k=2}^{2^{q-1}}\frac{x_k+\delta}{x_k}\prod_{k=2^{q-1}+1}^{4^q}\left(1+\frac{\delta}{x_k}\right).$$

We disregard the second product, which exceeds 1, and x_k in the numerator of the first product. For its denominator we have $\prod_{k=2}^{2^{q-1}}x_k<l_{2q-1}\cdot l_{2q-2}^2\cdots l_{q+1}^{2^{q-2}}=l_{q+1}^{\varkappa}$, where $\varkappa=2^{q-2}+\alpha\cdot 2^{q-3}+\cdots+\alpha^{q-3}\cdot 2+\alpha^{q-2}=(2-\alpha)^{-1}(2^{q-1}-\alpha^{q-1})$. Therefore, $|P(-\delta)|>l_{q+1}^{2^{q-1}-1-\varkappa}=\exp[(\varkappa-2^{q-1}+1)\alpha^q]$. Here, $\varkappa-2^{q-1}=2^{q-1}[\frac{\alpha-1}{2-\alpha}-\frac{1}{2-\alpha}(\frac{\alpha}{2})^{q-1}]>2^{q-2}\frac{\alpha-1}{2-\alpha}$, due to the choice of q_0. Thus, $\log|P(-\delta)|>2^{q-2}\alpha^q\frac{\alpha-1}{2-\alpha}$. This gives the desired bound (3), since $n<4^q$. □

4 Proof of Theorem 1

Let us fix ε with $0<\varepsilon<\gamma$. As above, we suppose that $l_1=e^{-1}$.

We want to find q_0 and C_0 such that if $\operatorname{dist}(z,K^{(\alpha)})=\delta\in(l_{q+1},l_q]$ with $q\ge q_0$, then

$$g_{\mathbb{C}\setminus K^{(\alpha)}}(z)\le C_0\left(\frac{\alpha}{2}\right)^q\alpha^{q\varepsilon}. \tag{4}$$

We set $Q_\alpha:=\frac{\alpha}{\alpha-1}\log\frac{2}{\alpha}$ and choose q_0 so large that for $q\ge q_0$ the following conditions hold:

$$Q_\alpha q<\alpha^{q-1}, \tag{5}$$

$$\log(Q_\alpha q)<q\varepsilon\log^2\alpha, \tag{6}$$

$$\left(\frac{\alpha}{2}\right)^q<\frac{1}{4}. \tag{7}$$

Now for fixed $q\ge q_0$ we take $k=[\frac{\log(Q_\alpha q)}{\log\alpha}]+1$, where $[x]$ denotes the greatest integer in x. Due to the choice of Q_α, we have

$$l_k^{\alpha-1}<\left(\frac{\alpha}{2}\right)^q\le l_{k-1}^{\alpha-1}, \tag{8}$$

that is, $k=\min\{j:l_j^{\alpha-1}<(\frac{\alpha}{2})^q\}$. Since (5) is equivalent to $l_{q-1}^{\alpha-1}<(\frac{\alpha}{2})^q$, we get $k<q$. Also (6) implies

$$2^k<2\alpha^{q\varepsilon}, \tag{9}$$

as is easy to check.

Arguing as in the proof of Theorem 2.2 in [17], we see that it is enough to consider (4) only for $z=-\delta$. Let us fix any polynomial P with $|P|_{K^{(\alpha)}}\le 1$. Let $m\in\mathbb{N}$ be such that $2^{m-1}\le\deg P<2^m$. In view of (1), we can reduce (4) to

$$\log\left|P(-\delta)\right|\le C_0 2^{m-q}\alpha^{q(1+\varepsilon)}, \tag{10}$$

and what is more, since polynomials in the representation (1) can be of arbitrary large degree, we can suppose without loss of generality that $m \geq 2q$.

We interpolate P on the interval $I_{1,k}$ at 2^m endpoints of $I_{j,k+m-1}$ with $j = 1, 2, \dots, 2^{m-1}$. Thus, $x_1 = 0, x_2 = l_{k+m-1}, x_3 = l_{k+m-2} - l_{k+m-1}, \dots, x_{2^i} = l_{k+m-i}, \dots, x_{2^m} = l_k$. Here $\omega(z) = \prod_{i=1}^{2^m}(z - x_i)$ and $L_j(z) = \frac{\omega(z)}{(z-x_j)\cdot\omega'(x_j)}$ for $1 \leq j \leq 2^m$. Since $\deg P < 2^m$, the interpolating polynomial $\mathcal{L}_{2^m-1} = \sum_{j=1}^{2^m} P(x_j)L_j$ coincides with P. In our case $|P(x_j)| \leq 1$. Therefore,

$$\left|P(-\delta)\right| \leq \sum_{j=1}^{2^m}\left|L_j(-\delta)\right|.$$

Let us fix any $1 \leq j \leq 2^m$ and estimate $|L_j(-\delta)|$ from above. We have $|\omega(-\delta)| < l_q \cdot (l_q + l_{k+m-1}) \cdot (l_q + l_{k+m-2})^2 \cdots (l_q + l_k)^{2^{m-1}} = l_q^{2^{k+m-q}} \cdot 2^{2^{k+m-q-1}} \cdot l_{q-1}^{2^{k+m-q}} \cdots l_k^{2^{m-1}} \cdot B$, where $B = (1 + \frac{l_{k+m-1}}{l_q})(1 + \frac{l_{k+m-2}}{l_q})^2 \cdots (1 + \frac{l_{q+1}}{l_q})^{2^{k+m-q-2}}(1 + \frac{l_q}{l_{q-1}})^{2^{k+m-q}} \cdots (1 + \frac{l_q}{l_k})^{2^{m-1}}$. On the other hand, by the structure of the set $K^{(\alpha)}$, $|\omega'(x_j)| > l_{k+m-1} \cdot h_{k+m-2}^2 \cdot h_{k+m-3}^4 \cdots h_k^{2^{m-1}} = l_{k+m-1} \cdot l_{k+m-2}^2 \cdots l_k^{2^{m-1}} \cdot \beta$ with $\beta = (1 - 2\frac{l_{k+m-1}}{l_{k+m-2}})^2 \cdots (1 - 2\frac{l_{k+1}}{l_k})^{2^{m-1}}$. Also, $|-\delta - x_j| \geq l_{q+1}$. Therefore,

$$\left|L_j(-\delta)\right| \leq \frac{B}{\beta} 2^{2^{k+m-q-1}} l_q^{\varkappa},$$

with $\varkappa = 2^{k+m-q-1} - \alpha - [\alpha^{k+m-q-1} + 2\alpha^{k+m-q-2} + \cdots + 2^{k+m-q-2}\alpha] = 2^{k+m-q-1} - \alpha - 2^{k+m-q-1} \cdot \frac{\alpha}{2-\alpha} + \frac{\alpha^{k+m-q}}{2-\alpha} = -2^{k+m-q} \cdot \frac{\alpha-1}{2-\alpha} + \frac{\alpha^{k+m-q}}{2-\alpha} - \alpha$.

From this, $|P(-\delta)| \leq 2^m \frac{B}{\beta} 2^{2^{k+m-q-1}} l_q^{\varkappa}$, and the desired inequality (10) is analogous to

$$m \log 2 + \log B - \log\beta + 2^{k+m-q-1}\log 2 + \varkappa \log l_q \leq C_0 2^{m-q}\alpha^{q(1+\varepsilon)}.$$

Here, $\varkappa \log l_q = 2^{k+m-q} \cdot \alpha^{q-1} \cdot \frac{\alpha-1}{2-\alpha} - \frac{\alpha^{k+m-1}}{2-\alpha} + \alpha^q$. Since $k + m > 2q$, the sum of the last two terms is negative. We neglect this sum. Thus it is enough to show

$$m + 2^{k+m-q-1} + \log B - \log\beta + 2^{k+m-q} \cdot \alpha^{q-1} \cdot \frac{\alpha - 1}{2 - \alpha} \leq C_0 2^{m-q}\alpha^{q(1+\varepsilon)}. \quad (11)$$

Each of the 5 summands on the left will be estimated separately from above by $R := 2^{m-q}\alpha^{q(1+\varepsilon)}$.

S_1 $:= m$. Since $m \geq 2q$, we have $2^{m-q} \geq 2^{m/2} \geq m/2$, so $S_1 \leq 2R$.

S_2 $:= 2^{k+m-q-1} < \alpha^{q\varepsilon}2^{m-q}$, by (9). Thus, $S_2 < R$.

S_3 $:= \log B$. Clearly, $l_q/l_{q-1} > l_{q+1}/l_q$, since $l_{q-1}^{(\alpha-1)^2} < 1$. Therefore, $B < (1 + \frac{l_q}{l_{q-1}})^{2^m}$ and $\log B < 2^m l_{q-1}^{\alpha-1} < 2^m(\frac{\alpha}{2})^q$, by (5). Therefore, $S_3 < R$.

S_4 $:= -\log\beta$. Here, $\frac{l_{k+1}}{l_k} > \frac{l_{j+1}}{l_j}$ for $j = k + 1, \dots, k + m - 2$. Consequently, $\beta > (1 - 2\frac{l_{k+1}}{l_k})^{2^m}$ and $\log\beta > -2^{m+2}l_k^{\alpha-1}$. Here we use the inequality $\log(1 - x) >$

$-2x$, which is valid for $0 < x < 1/2$. In our case $x = 2l_k^{\alpha-1} < 1/2$, by (8) and (7). Additionally, (8) implies $S_4 < 2^{m+2}(\frac{\alpha}{2})^q < 4R$.

Finally, $\mathbf{S_5} := 2^{k+m-q} \cdot \alpha^{q-1} \cdot \frac{\alpha-1}{2-\alpha} < \frac{2}{\alpha} \cdot \frac{\alpha-1}{2-\alpha} \cdot R$, by (9). This gives (11) with $C_0 = 8 + \frac{2}{\alpha} \cdot \frac{\alpha-1}{2-\alpha}$ and completes the proof of Theorem 1. □

5 Markov's Factors

By the arguments given in the introduction,

$$M_n\big(K^{(\alpha)}\big) \le \inf_{\delta \le \delta_0} \delta^{-1} \exp\big[n \cdot C_0 \varphi^{\gamma-\varepsilon}(\delta)\big].$$

Let us take $\delta = \exp(-n^{\frac{\log\alpha}{\log 2}})$. Then $\varphi(\delta) = n^{-\frac{\log\alpha}{\log 2}}$ and $n \cdot \varphi^{\gamma}(\delta) = n^{\frac{\log\alpha}{\log 2}}$. Therefore, $M_n(K^{(\alpha)}) \le \exp[(C_0+1) \cdot n^{(1+\varepsilon)\frac{\log\alpha}{\log 2}}]$ for large enough n. By increasing the constant, if necessary, we have the first bound in corollary.

Of course these arguments cannot be used for the case of polar sets $K^{(\alpha)}$ with $\alpha \ge 2$. But the lower bound of $M_n(K^{(\alpha)})$ can be presented easily for any $\alpha > 1$. Indeed, let us fix $n \in \mathbb{N}$. Let $2^m \le n < 2^{m+1}$. We take the same 2^m points $(x_k)_{k=1}^{2^m}$ as in Sect. 3 and $P(z) = \prod_{k=1}^{2^m}(z - x_k)$. Then $|P|_{K^{(\alpha)}} = |P(l_m)| = l_m(x_2 - l_m)\cdots(1 - l_m) < l_m \cdot \prod_{k=2}^{2^m} x_k$. On the other hand, $|P'(0)| = \prod_{k=2}^{2^m} x_k$. By definition, the sequence (M_n) is not decreasing. Therefore, $M_n(K^{(\alpha)}) \ge M_{2^m}(K^{(\alpha)}) \ge |P'(0)|/|P|_{K^{(\alpha)}} > l_m^{-1} = \exp\alpha^{m-1} = \exp[\alpha^{-2} \cdot (2^{m+1})^{\frac{\log\alpha}{\log 2}}] > \exp(\alpha^{-2} \cdot n^{\frac{\log\alpha}{\log 2}})$.

6 Remarks

1. The function $\varphi(\delta) = (1/\log\frac{1}{\delta})$ was used in [5] to define the *logarithmic dimension* of compact sets, as the Hausdorff dimension corresponding to the function φ. In particular, the logarithmic dimension of $K^{(\alpha)}$ is $\frac{\log 2}{\log\alpha}$.
2. We conjecture that the genuine modulus of continuity of $g_{\mathbb{C}\setminus K^{(\alpha)}}(z)$ is given by $\varphi^{\gamma}(\mathrm{dist}(z, K^{(\alpha)}))$, that is, $-\varepsilon$ in the upper bound can be removed by another distribution of interpolating nodes, which will be closer to the distribution of the Fekete points on the set $K^{(\alpha)} \cap I_{1,k}$. This will mean that $\exp(\alpha^{-2} \cdot n^{\frac{\log\alpha}{\log 2}}) < M_n(K^{(\alpha)}) < \exp(C \cdot n^{\frac{\log\alpha}{\log 2}})$ for some constant C.

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