To my grandmother Şövkat

A NOVEL CHARACTERIZATION OF NASH-IMPLEMENTABLE SOCIAL CHOICE RULES VIA NEUTRALITY

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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In this thesis, we study Nash implementability of social choice rules in relation with the neutrality notion. Several works in the literature provide a characterization of Nash-implementable social choice rules. However, they do not explicitly show the degree of neutrality property in the Nash equilibrium concept which is also existing in Nash-implementable rules. In this study, we define a weak version of the neutrality condition critical neutrality which is associated with the critical domain of a social choice rule. The critical neutrality notion when conjoined with Maskin monotonicity turns out to be equivalent to Nash implementability. Moreover, we propose an algorithm to obtain a maximal domain of preference profiles on which a specified social choice rule is Nash-implementable, by utilizing critical neutrality as a tool. The main result of the thesis is in support of the view that implementability on the full domain of preference profiles is highly related with implementability on the critical domain in Nash equilibrium and possibly, in other solution concepts.

Keywords: Implementation, Nash Equilibrium, Neutrality.

ÖZET

NASH-UYGULANABİLİR SOSYAL SEÇİM KURALLARININ NÖTRALLİK ARACIĞIYLA YENİ BİR KARAKTERİZASYONU

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Bu tezde, nötrallik kavramıyla ilgili olarak Nash uygulanabilirlik sosyal seçim kurallarını inceliyoruz. Literatürdeki bazı çalışmalarda Nash-uygulanabilir sosyal seçim kurallarının karakterizasyonları var. Ancak, Nash-uygulanabilir kurallarında da bulunan nötrallik özelliği derecesini Nash dengesi kavramında açıkça göstermezler. Bu çalışmada, bir sosyal seçim kuralının kritik tercih profiller alanı ile ilişkili nötrallik kavramının zayıf bir versiyonunu tanımladık. Maskin tekdüzelik ile birleştiğinde kritik nötrallik Nash uygulanabilirlikle eşdeğerdir. Ayrıca, kritik nötrallik kavramını bir araç olarak kullanarak belirli bir sosyal seçim kuralının Nash-uygulanabilir olduğu bir tercih profilleri alanı elde etmek için bir algoritma öneriyoruz. Tezin ana sonucu, Nash ve muhtemelen, diğer denge kavramlarında da tercih profillerinin tam alanı üzerindeki uygulanabilirlik ile kritik alanı üzerindeki uygulanabilirliyin yüksek oranda ilişkili olduğunu destekliyor.

Anahtar Kelimeler: Uygulama, Nash Dengesi, Nötrallik.

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CHAPTER 1

INTRODUCTION

Implementation theory implicitly assumes that individuals in a society have somehow unanimously agreed upon what alternatives from among the available ones are socially desirable contingent upon the society's preference profile. A social choice rule (SCR) summarizes this desirability by specifying the desirable alternatives for each possible preference profile of the individuals in the society. Now a natural question is how to obtain socially desirable alternatives as equilibrium outcomes under each societal preference profile. This is precisely the question which implementation theory deals with.

To obtain the correspondence between desirable alternatives and equilibria, a central authority (designer), without knowing the actual preferences of the individuals, proposes a mechanism. The mechanism is an institution through which the individuals interact. Initiated by Hurwicz (1972), it has become customary to use a game form as the mechanism in implementation theory. The designer, however, also needs to know the mode of behavior according to which the underlying society interacts. Thus, a mode of the interaction is assumed to be reflected by some game-theoretic solution concept. Thus, a game-form conjoined with the selected solution concept is designed to implement socially desirable alternatives as equilibrium outcomes.

This paper investigates implementation of a social choice rule (*SCR*) in Nash equilibrium. In the seminal work, Maskin (1977) defines a monotonicity condition, which is now referred to as *Maskin monotonicity*, and shows that monotonicity is necessary for *Nash implementation*. Moreover, he shows that $Maskin\ monotonicity$ together with the auxiliary condition $neutrality$ is sufficient for $Nash\ implementability$ in the presence of at least three individuals. However, it turns out that $Nash\ implementability$ of an SCR does not necessarily imply $neutrality$. Moore and Repullo (1990) give a full characterization of $Nash$-$implementable$ social choice rules via the existence of a particular system of sets satisfying certain conditions. Danilov (1992) singles out such system of sets the existence of which turns out to be equivalent to a particular monotonicity condition[1] being satisfied by the SCR considered, whose domain is assumed to be full, i.e., to include the entire set of preference profiles.

Koray et all (2001) define the notion of a *critical profile* of an *SCR*. Koray and Dogan (2007) show that given a *Maskin monotonic* social choice rule, knowing how this rule acts on a set of *critical profiles* is sufficient to predict the outcomes of the rule for the whole domain of linear order profiles. Based on $critical\ profiles$, they define the notion of a $self$-$monotonicity$ of an SCR.[2] Koray and Pasin (2009) consider a set-up in which they analyze $monotonicity$ of $Nash\ equilibrium$ solution concept itself, and show that it has a unique $self$-$monotonicity$.

The nature of the $Nash\ equilibrium$ concept is such that the naming of the alternatives does not matter. That is, as names of the alternatives change, the equilibrium outcomes change accordingly. Therefore, it is natural to ask whether there exists some $neutrality$ property of the $Nash\ equilibrium$ notion, which is also in-

[1] In the paper, he refers to it as $essential\ monotonicity$.

[2] One can refer to Koray and Erol (2009) for the further research on the $critical\ profiles$.

herited by $Nash$-$implementable$ social choice rules. This question is first asked and partially answered by Maskin (1977). This study fully answers this question, and characterizes $Nash\ implementability$ of social choice rules defined on the full domain of preference profiles utilizing a particular kind of neutrality. We define a weak kind of neutrality, which is associated with *critical profiles*, and refer to it as *critical neutrality*.

The rest of the paper is organized as follows. We give the set-up and preliminaries in section 2. The new characterization of $Nash$-$implementable$ social choice rules via $critical\ neutrality$ is presented in section 3. In section 4, related with $critical\ neutrality$, we propose an algorithm to obtain a maximal domain of linear order profiles on which an SCR is $Nash$-$implementable$. Section 5 closes the paper with concluding remarks and further research perspectives.

CHAPTER 2

PRELIMINARIES

A denotes a finite nonempty set of alternatives and N a finite nonempty set $\{1, 2, \ldots ,N\}$ of individuals in a society. A linear order is a complete, transitive and antisymmetric binary relation. $\mathscr{L}$(A) is the set of all linear orders on A. For each $i \in N$, u_i stands for the linear order of i on A. One can also refer to u_i as the preference of i on A. Any u = (u_1, u_2, ... ,u_N) $\in \mathscr{L}$(A)^N is referred to as a **linear order profile** or **preference profile** on A. A **social choice rule** (SCR) $F : \mathscr{L}$(A)$^N \to A$ is a correspondence which associates a subset of A with each linear order profile on A. The image $Im(F)$ of F is defined by $Im(F) \equiv \{$a $\in A$: $a \in F(v)$ for some $v \in \mathscr{L}$(A)$^N\}$. $Gr(F) \equiv \{(a, u)$: $a \in F(u)\}$. Let $a \in A$, $i \in N$ and $u_i \in \mathscr{L}$(A). The **lower contour set** of a **at** u_i is $L(a,u_i) = \{b \in A : a\ u_i\ b\}$. Let B be a nonempty subset of A. Then $\mu(B, u_i)$ represents the **set** of **maximal alternatives in** B **with respect to** u_i. Since u_i is a linear order, $\mu(B, u_i)$ is indeed a singleton set.

A function $\tau : A \to A$ is said to be a **permutation** on A if τ is one-to-one and onto. Denote the set of all permutations on A by T_A. Given $u \in \mathscr{L}$(A)^N and $\tau \in T_A$, we define $u^\tau \in \mathscr{L}$(A)^N as follows: Given any a, $b \in A$ and $i \in N$, we let $a\ u_i^\tau\ b$ if and only if $\tau(a)\ u_i\ \tau(b)$. Let a, $b \in A$. A permutation $\tau : A \to A$ with $\tau(a) = b$, $\tau(b) = a$, and for any $x \in A\backslash\{a, b\}$, $\tau(x) = x$, is referred to as the $(a$,$b)$-**transposition** on A.

For $i \in N$, denote an abstract nonempty **strategy set** by M_i. The set $M = \prod_{i \in N} M_i$ is called a **joint strategy space**. Let $g : M \rightarrow A$ be an **outcome function**. We call G = $(M$, $g)$ a **mechanism**.

Given a mechanism G = $(M$, $g)$ and a linear order profile $u \in \mathscr{L}(\mathsf{A})^N$, the relation $\succsim^u$ on M is defined such that for any $i \in N$ and any $m, m' \in M$, $m \succeq_i^u m'$ if and only if $g(m)\ u_i\ g(m')$. Note that $\succsim^u$ will be a complete preorder on M. Now G^u = $(N$, M, $\succeq^u)$ is referred to as the **strategic game-form induced by** u **under** G. Finally, let σ be a **solution concept**, that associates G^u with a subset of M for each $u \in \mathscr{L}(\mathsf{A})^N$.

Given an SCR F and a solution concept σ, a mechanism G = $(M$, $g)$ is said to σ-**implement** F if, for any $u \in \mathscr{L}(\mathsf{A})^N$, $F(u) = g(\sigma(\mathrm{G}^u))$. An SCR F is said to be σ-**implementable** if there exists a mechanism G = $(M$, $g)$ which σ-implements F.

CHAPTER 3

NASH IMPLEMENTATION

In this section, we show a novel characterization of $Nash$-$implementable$ social choice rules via a neutrality notion. In the characterization, we consider social choice rules which are defined on the full domain of preference profiles. Initially, we give a few definitions, which are helpful to present the result.

Definition: For any u, $u' \in \mathscr{L}(\mathsf{A})^N$ and $a \in A$, u' is an a-**improvement** of u if for all $i \in N$: $L(a,u_i) \subset L(a,u_i')$.

Definition: For any u, $u' \in \mathscr{L}(\mathsf{A})^N$ and $a \in A$, u' is a **strict** a-**improvement** of u if for all $i \in N$: $L(a,u_i) \subset L(a,u_i')$ and there exists some $j \in N$: $L(a,u_j) \subsetneq L(a,u_j')$.

Definition: For any u, $u' \in \mathscr{L}(\mathsf{A})^N$ and $a \in A$, u' is an a-**refinement** of u if for all $i \in N$: $L(a,u_i') \subset L(a,u_i)$.

Definition: For any u, $u' \in \mathscr{L}(\mathsf{A})^N$ and $a \in A$, u' is a **strict** a-**refinement** of u if for all $i \in N$: $L(a,u_i') \subset L(a,u_i)$ and there exists some $j \in N$: $L(a,u_j') \subsetneq L(a,u_j)$.

Now, consider a preference profile u such that $a \in F(u)$. Via a strict a-refinement, we obtain u' from u, and check whether a is chosen by F at u'. If $a \in F(u')$, then we obtain a strict a-refinement u'' of u' and check whether a is chosen by F at u''. If we continue this process, we would eventually reach a preference profile, say u^*, such that $a \in F(u^*)$, but any strict a-refinement prevents a from being chosen by F. Such preference profile is referred to as an a-$critical\ profile$ in the sense that the rankings of a at u^* are "minimally sufficient" to have a in $F(u^*)$.

Definition: Let F : $\mathscr{L}(\mathsf{A})^N \to A$ be an SCR and $a \in A$. We say that $u \in \mathscr{L}(\mathsf{A})^N$ is a**-critical** if $a \in F(u)$, but $a \notin F(u')$ for any strict a-refinement u' of u.

We denote the set of all a-*critical profiles* of F by $C_a(F)$. Obviously, for some a' $\in A$, $C_{a'}(F)$ is empty if and only if $a' \notin Im(F)$. The set of all *critical profiles* of F is denoted by $C(F) = \bigcup_{a \in Im(F)} C_a(F)$. We refer $C(F)$ as the $critical\ domain$ of F. At any a-$critical\ profile$ u, if an alternative b, which is lower-ranked than a by some i, jumps over a in i's ranking, a stops to be chosen by the social choice rule, since the new preference profile is a strict a-refinement of u. Such alternative b is called an a-$critical\ element$ for i at preference profile u. The precise definition follows.

Definition: Let F : $\mathscr{L}(\mathsf{A})^N \to A$ be an SCR. Let $a \in Im(F)$, $u \in C_a(F)$, $i \in N$ and $b \in L(a,u_i)$. We refer to b as an a-$critical\ element$ for i at u.

The definition of $Maskin\ monotonicity$, which is a necessary condition for $Nash$ $implementability$ of a social choice rule, in general, is given below.

Definition: Let F : $\mathscr{L}(\mathsf{A})^N \to A$ be an SCR. F is **Maskin monotonic** if for any u, $u' \in \mathscr{L}(\mathsf{A})^N$ and $a \in F(u)$, we have $a \in F(u')$ whenever for each $i \in N$, $L(a,u_i)$ $\subset L(a,u'_i)$.

Assume that the society finds a acceptable at some linear order profile u, i.e., $a \in F(u)$, and u changes such that the relative ranking of a at u with respect to the any other alternative does not get worse from the viewpoint of any individual, i.e., an a-improvement happens. $Maskin\ monotonicity$ simply means that the society continues to find a acceptable. That is, a is chosen by F at the new preference profile as well.

Before defining $critical\ neutrality$, the conjunction of which with $Maskin\ monotonicity$ turns out to be equivalent to $Nash\ implementability$, we present two neutrality notions from the literature.

Definition: Let $F : \mathscr{L}(\mathsf{A})^N \to A$ be an SCR. We say that F is **neutral** if for any $u \in \mathscr{L}(\mathsf{A})^N$ and any $\tau \in T_A$: $F(u^\tau) = \tau^{-1}(F(u))$.

$Neutrality$ of an SCR implies that if some alternatives are chosen at a preference profile u, then at a new profile obtained via renaming (permuting) alternatives, the same alternatives should be chosen under their new names. If we constraint renaming to an interchange of names between two alternatives, we obtain $transposition\ neutrality$.

Definition: Let $F : \mathscr{L}(\mathsf{A})^N \to A$ be an SCR. Let a, $b \in Im(F)$. We say that F is (a,b)-**transposition neutral** if for any $u \in \mathscr{L}(\mathsf{A})^N$ and the (a,b)-**transposition** τ, $F(u^\tau) = \tau^{-1}(F(u))$. If F satisfies (a',b')-**transposition neutrality** for any (a',b') $\in Im(F) \times Im(F)$, then we say F is **transposition neutral**.

Based on $critical\ profiles$ and $critical\ elements$, we next define the novel condition of $critical\ neutrality$, and then, compare it with the other neutrality notions.

Definition: Let $F : \mathscr{L}(\mathsf{A})^N \to A$ be an SCR, $a \in Im(F)$, $u \in C_a(F)$, $i \in N$ and b

$\in L(a,u_i)$. Let $\tau : A \to A$ be the (a,b)-$transposition$. We say that F satisfies a**-critical neutrality relative to** u**,** i **and** b if for any $\bar{u} \in \mathscr{L}(\mathsf{A})^N$ such that for i, $\bar{u}_i = u_i$ and for any $j \in N\backslash\{i\}$, $L(a,\bar{u}_j) = Im(F)$ and for $\bar{u}^\tau \in \mathscr{L}(\mathsf{A})^N$, we have $b \in F(\bar{u}^\tau)$ whenever $a \in F(\bar{u})$.

We say that F satisfies a**-critical neutrality** if F satisfies a*-critical neutrality relative to* any $u' \in C_a(F)$, $i' \in N$ and $b' \in L(a,u'_{i'})$. Finally, F satisfies **critical neutrality** if F satisfies a'-$critical\ neutrality$ for any $a' \in Im(F)$.

$Critical\ neutrality$ is a very restricted kind of neutrality. Firstly, $critical\ neutrality$ individualizes neutrality criterion. Then, it only requires neutrality of an alternative in $Im(F)$ with its one of critical elements. More than this, $critical\ neutrality$ is not only restricted to transpositions, but to some special transpositions at profiles of a particular structure.

Clearly, $neutrality$ implies $critical\ neutrality$. However, the converse is not true. We can illustrate this point in the following example.

Example.[3] Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Let an SCR $F : \mathscr{L}(\mathsf{A})^N \to A$ be defined as follows:

For any $u \in \mathscr{L}(\mathsf{A})^N$,

$$F(u) = \begin{cases} \{a, b, c\}, & \text{if } u \text{ is such that for all } i \in N\text{: } a\ u_i\ b \\ \{b, c\}, & \text{otherwise.} \end{cases}$$

Observe that all c-$critical\ profiles$ of F are such that c is bottom-ranked by

[3] One may find this example odd or unrealistic. However, it is both simple and rich enough to serve our aim well.

each $i \in N$. Now, take any $c\text{-}critical\ profile$ $v \in C_c(F)$. Fix some $j \in N$, say $j = 1$. Consider $\bar{v} \in \mathscr{L}(\mathsf{A})^N$ at which $\bar{v}_1 = v_1$ and for $j' \in \{2, 3\}$: $L(c,\bar{v}_{j'}) = \{a, b, c\} = A$. By the definition of F, we know that $c \in F(\bar{v})$ and thus, F satisfies $c\text{-}critical\ neutrality$ relative to v, 1 and c, as $L(c,v_1)$ contains only {c}. Thus, F trivially satisfies $c\text{-}critical\ neutrality$. As the structure of $b\text{-}critical\ profiles$ is the same as that of $c\text{-}critical\ profiles$, i.e., at $b\text{-}critical\ profiles$, b is bottom-ranked by every individual in N, we conclude that F satisfies $b\text{-}critical\ neutrality$ as well. Finally, at all $a\text{-}critical\ profiles$ of F, c is top-ranked, a is middle-ranked and b is bottom-ranked by everyone. Then, if each individual, but not one, brings a to the top, a continues to be chosen. After the (a,b)-$transposition$, b is chosen by F. Thus, one concludes that F also satisfies $a\text{-}critical\ neutrality$. Hence, F satisfies $critical\ neutrality$.

However, F does not satisfy *neutrality*. For the example, the choice of the specific $c\text{-}critical$ profile v and alternative a is important as a is not a $c\text{-}critical\ element$ for 1 at v. If we exchange the names of a and c at $\bar{v}$, then a is not preferred to b for 1 and thus, is not chosen by F.

Furthermore, *transposition neutrality*, which means $neutrality$ only between any two alternatives, is sufficient for $Nash\ implementation$ of a $Maskin\ monotonic\ SCR$. Nevertheless, *transposition neutrality* is not a necessary condition either. Since *critical neutrality* considers transpositions of two specific alternatives occupying prescribed positions in preference profiles of a special form, it is even weaker than *transposition neutrality*. The above example illustrates this point as well.

The Moore and Repullo (1990) conditions characterizing $implementability$ of an SCR in $Nash\ equilibrium$ can very well be summarized as the conjunction of $Maskin\ monotonicity$ and a weakened version of $no\ veto\ power$. Hence, the relation between $critical\ neutrality$ and “weakened no veto power” notion is interesting to be analyzed. To reflect the essence of the “weakened no veto power” in condition (ii) in Moore and Repullo (1990), we adopt the following version of this notion, which is the strongest possible form that weak $no\ veto\ power$ can take in Moore and Repullo’s (1990) context.

Definition: Let F : $\mathscr{L}$(A)$^N \to A$ be an SCR. Now F satisfies $restricted\ no\ veto\ power$ if, for each $(a,\ u) \in Gr(F)$ and $u' \in \mathscr{L}$(A)^N, one has $b \in F(u')$ whenever $b \in \mu(L(a,\ u_i),\ u'_i)$ for some $i \in N$ and $b \in \mu(Im(F),\ u'_j)$ for each $j \in N\backslash\{i\}$.

Take any $(a,\ u) \in Gr(F)$ and $i \in N$. Then i can use veto power against the alternatives which are not in $L(a,\ u_i)$. Equivalently, $no\ veto\ power$ of i is $restricted$ to $L(a,\ u_i)$. Thus, it is a weaker condition than $no\ veto\ power$. We next claim and show that $critical\ neutrality$ is even weaker than $restricted\ no\ veto\ power$, which is not unexpected as $restricted\ no\ veto\ power$ is the “strongest possible” form that weakened $no\ veto\ power$ can take.

Proposition 1: Let F : $\mathscr{L}$(A)$^N \to A$ be an SCR. If F satisfies $restricted\ no\ veto\ power$, then F satisfies $critical\ neutrality$.

Proof: Assume that F satisifes $restricted\ no\ veto\ power$. Take any $a \in Im(F)$, $u \in C_a(F)$, $i \in N$ and $b \in L(a,u_i)$. Let τ : $A \to A$ be the (a,b)-$transposition$. Assume that $\bar{u} \in \mathscr{L}$(A)^N is such that for i, $\bar{u}_i$ = u_i and for any $j \in N\backslash\{i\}$, $L(a,\bar{u}_j)$ = $Im(F)$. Assume also that $a \in F(\bar{u})$. At preference profile $\bar{u}$, a is a maximal element in $L(a,u_i)$ and $Im(F)$ for i and for any $j \in N\backslash\{i\}$, respectively. Then, at preference profile $\bar{u}^\tau$, which is obtained from $\bar{u}$ via the $(a,\ b)$-$transposition$, b is a maximal element in $L(a,u_i)$ and $Im(F)$ for i and for any $j \in N\backslash\{i\}$, respectively. Thus, $restricted\ no\ veto\ power$ guarantees that F chooses b at the new profile $\bar{u}^\tau$. Since a, u, i and b are arbitrary, we say F satisfies $critical$

$neutrality$. $\square$

With the help of the example given before, we show that the converse is not true, i.e., F satisfying $critical\ neutrality$ does not necessarily imply $restricted\ no\ veto\ power$. Let u be the preference profile at which b is top-ranked for 1 while c is middle-ranked for 1 and top-ranked for the other two agents. By the definition, c is in $F(u)$. Now, consider the profile obtained from u via the (a, c)-$transposition$. $Restricted\ no\ veto\ power$ implies that a should be chosen by F at this new profile. However, since b is better-ranked than a for 1, we know that a is not chosen by F. Even though it is previously shown that the SCR in the example satisfies $critical\ neutrality$, it does not satisfy $restricted\ no\ veto\ power$. Thus, we conclude that $critical\ neutrality$ is a weaker condition than $restricted\ no\ veto\ power$.

Critical neutrality is a necessary condition for $Nash\ implementability$ of an SCR. Moreover, when there are at least three individuals, *critical neutrality* coupled with *Maskin monotonicity*, turns out to be sufficient for *Nash implementation*. We restate this result in the following theorem.

Theorem 1: *Let* $|N| \geq 3$ *and* $F : \mathscr{L}(\mathsf{A})^N \to A$ *be an SCR. Now* F *is Nash-implementable if and only if* F *satisfies Maskin monotonicity and critical neutrality.*

Proof: Assume that F is *Nash-implementable*. Then, there exists a mechanism G = (M, g) which *Nash-implements* F. It is well-known that *Maskin monotonicity* is necessary for *Nash implementability* of an SCR F.[4]

To prove that *critical neutrality* is a necessary condition, take any $a \in Im(F)$,

[4]For the proof, one can refer to Maskin (1977).

$u \in C_a(F)$, $i \in N$ and $b \in L(a,u_i)$. Let $\tau : A \to A$ be the (a,b)-$transposition$. Assume that $\bar{u} \in \mathscr{L}(\mathsf{A})^N$ is such that for some $i \in N$, $\bar{u}_i = u_i$ and for any $j \in N\backslash\{i\}$, $L(a,\bar{u}_j) = Im(F)$. Assume also that $a \in F(\bar{u})$. Since G *Nash-implements* F, there is some $m \in \sigma_{NE}(\mathsf{G}^{\bar{u}})$ such that $g(m) = a$. Let u' be obtained from u by interchanging the roles of a and b for only i. Since u was an a-*critical profile*, $a \notin F(u')$. Then, this implies that $m \notin \sigma_{NE}(\mathsf{G}^{u'})$. As we only changed the preference of i by interchanging a and b, i has an incentive to unilaterally deviate from m to obtain b as the outcome under g. Thus, there exits $m'_i \in M_i$ such that $g(m'_i,m_{-i}) = b \; u'_i \; a = g(m)$. Now consider the transposed profile $\bar{u}^\tau$, which is obtained from $\bar{u}$ via the given transposition τ above. It is obvious that at $\bar{u}^\tau$, no individual has an incentive to unilaterally deviate from (m'_i,m_{-i}). So, $(m'_i,m_{-i}) \in \sigma_{NE}(\mathsf{G}^{\bar{u}^\tau})$. Since F is *Nash-implementable* via G, $b \in F(\bar{u}^\tau)$. As we arbitrarily chose $a \in Im(F)$, $u \in C_a(F)$, $i \in N$ and $b \in L(a,u_i)$, we conclude that F satisfies *critical neutrality*.

Conversely, assume that F is $Maskin\ monotonic$ and satisfies *critical neutrality*. For each $(a,\, u) \in Gr(F)$, we obtain an a-$critical\ profile$ from u and fix it. If u is an a-$critical\ profile$, then u itself is fixed. Now, we use the following mechanism G = $(M,\, g)$. For each $i \in N$, the strategy set M_i is such that i announces $a^i \in Im(F)$, u^i with $a^i \in F(u^i)$ and a number $k^i \in \mathbb{N}$, i.e., $M_i = \{(a^i,\, u^i,\, k^i) \in Im(F) \times \mathscr{L}(\mathsf{A})^N \times \mathbb{N}$: $a^i \in F(u^i)\}$. We define the outcome function $g : M \to A$ as follows:

$Case\ 1$: If all individuals play the same strategy $(a,\, u^0,\, 0)$, then the outcome is a.

$Case\ 2$: All individuals, except i, play the same strategy $(a,\, u^0,\, 0)$. Now, if $a^i \in L(a,u_i^{0*})$), where u^{0*} is the fixed a-$critical\ profile$ obtained from u^0, the outcome is a^i. Otherwise, the outcome is a.

$Case\ 3$: In all other situations, the outcome is a^l, where l = Min N' with $N' = \{i$

$\in N$: k^i = $\text{Max}_{j \in N}$ $k^j\}$.

Now, we show that for each $u \in \mathscr{L}(\mathsf{A})^N$, $F(u)$ = $g(\sigma_{NE}(\mathsf{G}^u))$, where σ_{NE} stands for the $Nash\ equilibrium$ concept. Let $u \in \mathscr{L}(\mathsf{A})^N$. Firstly, assume that $a \in F(u)$. Let $m \in M$ be such that all individuals play the same strategy $(a,\ u,\ 0)$. Then, the outcome is a, i.e., $g(m)$ = a. Any i can only change the outcome to some other alternative in $L(a{,}u_i^*)$, where u^* is the fixed a-$critical\ profile$ obtained from u, but does not have an incentive to do so. Thus, m is a $Nash\ equilibrium$ of G^u, i.e., $m \in \sigma_{NE}(\mathsf{G}^u)$. So, $a \in g(\sigma_{NE}(\mathsf{G}^u))$. Thus, we conclude that for any $u \in \mathscr{L}(\mathsf{A})^N$, $F(u) \subset g(\sigma_{NE}(\mathsf{G}^u))$.

Conversely, assume that $m \in \sigma_{NE}(\mathsf{G}^u)$ is such that $g(m)$ = a. We show that $a \in F(u)$.

$Case\ 1$: m is such that all individuals play the same strategy $(a,\ u^0,\ 0)$. Since $m \in \sigma_{NE}(\mathsf{G}^u)$, no individual has a reason to unilaterally deviate from m. Therefore, for each $i \in N$, $L(a{,}u_i^{0*}) \subset L(a{,}u_i)$, where u^{0*} is the fixed a-$critical\ profile$ obtained from u^0. Then, as F is $Maskin\ monotonic$ and $a \in F(u^{0*})$, we have $a \in F(u)$.

$Case\ 2$: m is such that all individuals, except i, play the same strategy $(a,\ u^0,\ 0)$. Therefore, at the preference profile u, for i, $g(m)$ is maximal in $L(a{,}u_i^{0*})$ and for each $j \in N \setminus \{i\}$, $g(m)$ is maximal in $Im(F)$. Since $g(m)$ is either a or any other alternative in $L(a{,}u_i^{0*})$, by $critical\ neutrality$, we have $g(m) \in F(u)$.

$Case\ 3$: In all other situations, since $m \in \sigma_{NE}(\mathsf{G}^u)$, at the linear order profile u, for each $i \in N$, $g(m)$ is maximal in $Im(F)$. Then, by $Maskin\ monotonicity$, $g(m) \in F(u)$.

Thus, we conclude that for any $u \in \mathscr{L}(\mathsf{A})^N$, $g(\sigma_{NE}(\mathsf{G}^u)) \subset F(u)$. Hence, F is $Nash\text{-}implementable$. $\square$

This characterization result applies for social choice rules defined on the unrestricted domain of preference profiles. It is clear that this result can be extended to social choice rules whose domain includes all linear order profiles to render the notion of $critical\ neutrality$ well-defined.

Danilov (1992) characterizes $Nash\text{-}implementable$ social choice rules via a particular monotonicity condition. As a point of departure, in this thesis work, we weaken the neutrality notion such that we obtain a characterization of $Nash\ implementability$ of a $Maskin\ monotonic\ SCR$. Thus, the equivalence of "Danilov monotonicity" to the conjunction of $Maskin\ monotonicity$ and $critical\ neutrality$ is directly shown below.

Proof. Assume that F is $Maskin\ monotonic$ and satisfies $critical\ neutrality$. Take any $(a, u) \in Gr(F)$ and $u' \in \mathscr{L}(\mathsf{A})^N$. Assume also that for each $i \in N$, $Ess(i, L_i(a, u))$[5] $\subset L_i(a, u')$. Obtain an $a\text{-}critical\ profile$ u^a from u. By $critical\ neutrality$, for each $i \in N$, $Ess(i, L_i(a, u^a)) = L_i(a, u^a)$. Since the operator Ess preserves the set inclusion, we know that for each $i \in N$, $L_i(a, u^a) = Ess(i, L_i(a, u^a)) \subset Ess(i, L_i(a, u)) \subset L_i(a, u')$. Then, by $Maskin\ monotonicity$, a is in $F(u')$. Hence, F is "Danilov monotonic". The converse implication is straightforward. $\square$

We next present examples of social choice rules, and see whether they are $Nash\text{-}implementable$ via checking for $Maskin\ monotonicity$ and $critical\ neutrality$. Let N consist of at least three individuals.

Example 1. Consider a constant SCR F^a, i.e., the SCR which chooses the same

[5]For the definition, please refer to Danilov (1992).

alternative a at any preference profile in $\mathscr{L}(\mathsf{A})^N$. This constant rule is $Maskin$ $mono$-$tonic$. For any b in A, if $b \neq a$, $C_b(F^a)$ is empty while $C_a(F^a)$ is non-empty. For any $u^a \in C_a(F^a)$, a is bottom-ranked by each individual. Take any a-$critical$ $profile$ u^a of F^a. Since a itself is the only a-$critical$ $element$ for any i at u^a, F^a trivially satisfies $critical$ $neutrality$. Thus, it is $Nash$-$implementable$.

Example 2. Let F^{PO} be the Pareto SCR, i.e., for each $u \in \mathscr{L}(\mathsf{A})^N$, $F^{PO}(u)$ = $\{a \in A$: $\bigcup_{i\in N} L(a, u_i)$ = $A\}$. Pareto rule is $Maskin$ $monotonic$. Take any $a \in Im(F^{PO})$ = A. Note that for each $u^a \in C_a(F^{PO})$, any alternative other than a appears in only one of the individuals' lower counter set. Take any $i \in N$, and obtain $\bar{u}$ such that for $i \in N$, $\bar{u}_i$ = u^a_i and for any $j \in N\backslash\{i\}$, $L(a,\bar{u}_j)$ = $Im(F)$ = A. By the definition, Pareto rule chooses a at preference profile $\bar{u}$. If we do a $renaming$ between a and any $critical$ $element$ of a for i at u^a, the same alternative with the new name is chosen by F^{PO}. Thus, Pareto rule satisfies $critical$ $neutrality$. Hence, it is $Nash$-$implementable$.

Example 3. Let F^{IR} be Individually Rational SCR relative to some $b \in A$, i.e., for each $u \in \mathscr{L}(\mathsf{A})^N$, $F^{IR}(u)$ = $\{a \in A$: a u_i b for each $i \in N\}$. The Individually Rational SCR is $Maskin$ $monotonic$. Take any $a \in Im(F)$ = A. Note that for each $u^a \in C_a(F^{IR})$, if a = b, b is bottom-ranked by all i. Otherwise, b is bottom-ranked by all i while a is ranked just above b for all i. By definition, F^{IR} chooses a at preference profile $\bar{u}$, where for some $i \in N$, $\bar{u}_i$ = u^a_i and for any $j \in N\backslash\{i\}$, $L(a,\bar{u}_j)$ = $Im(F)$ = A. So, if we exchange the names of a and b at $\bar{u}$, b is chosen by the Individually Rational rule at the new profile. Thus, F^{IR} satisfies $critical$ $neutrality$. Hence, it is $Nash$-$implementable$.

Example 4. Let N = $\{1, 2, 3\}$ and A = $\{a, b, c\}$. Let an SCR $F : \mathscr{L}(\mathsf{A})^N \to A$ be defined as follows:

For any $u \in \mathscr{L}(\mathrm{A})^N$,

$$F(u) = \begin{cases} \{a, b, c\}, & \text{if } u \text{ is such that for all } i \in N\text{: } a\ u_i\ b \text{ and } b\ u_i\ c \\ \{a, c\}, & \text{if } u \text{ is such that for all } i \in N\text{: } a\ u_i\ b \\ \{b, c\}, & \text{if } u \text{ is such that for all } i \in N\text{: } b\ u_i\ c \\ \{c\}, & \text{otherwise.} \end{cases}$$

One can easily check that F satisfies $Maskin\ monotonicity$. Now, we show that F does not satisfy $critical\ neutrality$. Note that there is the unique a-$critical\ profile$ u^a of F such that c is top-ranked, a is middle-ranked while b is bottom-ranked by all i. Thus, b is the only a-$critical\ element$ (other than a itself) for any i at u^a. Take this profile u^a and any $i \in N$, say 1. Then, obtain $\bar{u}$, where for 1, $\bar{u_1}$ = u_1^a and for any $j \in N\backslash\{1\}$, $L(a,\bar{u_j})$ = $Im(F)$ = A. By definition, a is in $F(\bar{u})$. However, if the names of a and b are interchanged at $\bar{u}$, F does not choose b at the new profile since b is lower-ranked than c for 1. Thus, F does not satisfy $critical\ neutrality$. Hence, it is not $Nash$-$implementable$.

One observes that the message space of the mechanism that is used in the sufficiency proof of the above theorem, would be constrained to the $critical\ domain$ of an SCR. Via such a restriction, we obtain the relation between $Nash\ implementability$ on the full domain and on the critical domain of preference profiles.

Let G = (M, g) be the mechanism above. Define G′ = (M', g') where M' = $\prod_{i \in N} M_i'$ with M_i' = $\{(a^i, u^i, k^i) \in Im(F) \times C(F) \times \mathbb{N}\text{: } a^i \in F(u^i)\}$ and $g' : M' \to A$ is such that for any $m \in M'$, $g'(m)$ = $g(m)$.

Corollary 1: *Let* $|N| \geq 3$ *and* $F : \mathscr{L}(\mathrm{A})^N \to A$ *be an SCR, which satisfies Maskin monotonicity and critical neutrality.* $Now\ F$ is *Nash-implementable via the mech-*

anism G′ = (M', g').

Proof: Assume that F satisfies $Maskin\ monotonicity$ and $critical\ neutrality$. One can easily reproduce the sufficiency proof via the mechanism G′. □

Let $F' : C(F) \to A$ be an SCR such that for any $u \in C(F)$, $F'(u) = F(u)$. The corollary shows that given F satisfies $Maskin\ monotonicity$ and $critical\ neutrality$, the $implementability$ of F' is equivalent to the $implementability$ on the full domain of preference profiles. Moreover, the mechanism G′ which can be used for the $implementation$ of F', $Nash$-$implements$ F as well. Hence, this corollary confirms our conjecture that $critical\ neutrality$ is a condition imposed on the structure of the $critical\ domain$.

CHAPTER 4

NASH IMPLEMENTATION ON MAXIMAL DOMAIN

The previous section shows that given a $Maskin\ monotonic\ SCR\ F$, $critical\ neutrality$ guarantees $Nash$-$implementability$ of F and moreover, relates this with the $implementability$ on the $critical\ domain$. Therefore, $critical\ neutrality$ is considered as the condition imposed on the structure of the $critical\ domain$. Now the natural question that arises is whether using $critical\ neutrality$ as a tool, one can obtain a maximal domain of preference profiles on which F is $Nash$-$implementable$. So, this section deals with the above-posed question and proposes an algorithm to obtain such a maximal domain.

Let $F : \mathscr{L}(\mathsf{A})^N \to A$ be a $Maskin\ monotonic\ SCR$. One can always restrict the domain to one preference profile, and trivially has F to be $Nash$-$implementable$. Therefore, we are interested to find a maximal domain D of preference profiles on which F is $Nash$-$implementable$. We use the $critical\ domain$ as the benchmark set to produce D via the following algorithm:

$Stage\ 1$: Recall that $critical\ domain$ $C(F) = \bigcup_{a \in Im(F)} C_a(F)$. Let $a \in Im(F)$. Take any $u \in C_a(F)$. Check whether F satisfies a-$critical\ neutrality$ relative to u for any

$i \in N$ and any $b \in L(a, u_i)$. If yes, let u be an element of the set D_a^0. If no, proceed with other a-$critical\ profile$. Continue this process for any a-$critical\ profile$. As $C_a(F)$ is the finite set, the process ends after the finite number of steps. We obtain the set D_a^0 containing all a-$critical\ profiles$ relative to which F satisfies a-$critical$ $neutrality$.

$Stage\ 2$: Next, for any $a \in Im(F)$, we produce D_a from D_a^0. For any u in D_a^0, one has $u \in D_a$. Then, take any $u \in D_a^0$. Check whether its a-improvement preference profile u' is a $critical\ profile$ for some alternative in $Im(F)$. If no, then $u' \in D_a$. If yes, assume it is a c-$critical\ profile$ for some $c \in Im(F)$. Then, check whether u' is in D_c^0. If yes, then $u' \in D_a$ as well. If no, then u' and c-improvement preference profiles of u' are excluded from D_a. This exclusion is due to the fact that at $stage\ 1$, u' is not included in D_c^0. Continue this process for any $u \in D_a^0$ and its any a-improvement preference profile. Iteratively, we construct D_a for any $a \in Im(F)$.

$Stage\ 3$: We set $D = \bigcup_{a \in Im(F)} D_a$.

This algorithm starts with the $critical\ domain$, and obtains a set of preference profiles relative to which an $SCR\ F$ meet both $Maskin\ monotonicity$ and the $critical$ $neutrality$ condition by throwing $critical\ profiles$, with respect to which F does not satisfy $critical\ neutrality$, and their improvement profiles. This set of preference profiles is a maximal domain on which $Nash$-$implementability$ of F is attained. Via $example\ 4$ from the previous section, we illustrate how the algorithm works and why the obtained set is one of the maximal domains.

Recall the example. Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Let an SCR $F : \mathscr{L}(\mathsf{A})^N$ $\rightarrow A$ be defined as follows:

For any $u \in \mathscr{L}(\mathsf{A})^N$,

$$F(u) = \begin{cases} \{a, b, c\}, & \text{if } u \text{ is such that for all } i \in N\text{: } a\ u_i\ b \text{ and } b\ u_i\ c \\ \{a, c\}, & \text{if } u \text{ is such that for all } i \in N\text{: } a\ u_i\ b \\ \{b, c\}, & \text{if } u \text{ is such that for all } i \in N\text{: } b\ u_i\ c \\ \{c\}, & \text{otherwise.} \end{cases}$$

Let us firstly see the structure of the $critical\ profiles$ of F. $C_a(F)$ consists of the unique profile u^a at which c is top-ranked, a is middle-ranked and b is bottom-ranked by all i while $C_b(F)$ consists of the unique profile u^b at which a is top-ranked, b is middle-ranked and c is bottom-ranked by all i. However, $C_c(F)$ has several preference profiles at which c is bottom ranked by all i while the relative ranking of a and b is different at each profile. Observe that u^b is an element of $C_c(F)$ as well i.e., u^b is a c-$critical\ profile$ of F.

As $stage\ 1$ of the algorithm proposes, we obtain $D^0_{a'}$ for any $a' \in Im(F)$ = A. We already know that F does not satisfy a-$critical\ neutrality$ relative to u^a. Hence, D^0_a = $\varnothing$. Check that F satisfies b-$critical\ neutrality$ relative to u^b. Thus, D^0_b = $\{u^b\}$. Since F trivially satisfies c-$critical\ neutrality$, D^0_c = $C_c(F)$.

Next, for any $a' \in Im(F)$ = A, we produce $D_{a'}$ from $D^0_{a'}$. Since D^0_a = $\varnothing$, we have D_a = $\varnothing$ as well. Then, $u^b \in D_b$. As any b-improvement profile of u^b is an element of the set D^0_c, D_b contains u^b and its all b-improvement profiles. Finally, let $D^0_c \subset D_c$. Note that u^a is a c-improvement profile of some c-$critical\ profile$ in D^0_c. Since u^a is not included in D^0_a, this preference profile is excluded from D_c. Moreover, any a-improvement profile of u^a is also excluded from D_c as it is c-improvement of some c-$critical\ profile$ in D^0_c. Thus, D_c is comprised of the c-$critical\ profiles$ and their other c-improvement profiles.

The union of sets D_a, D_b and D_c define the set D, and it is clear that D is a maximal domain on which F is $Nash\text{-}implementable$. Observe that D is simply the set of all preference profiles at which a is not chosen by F. If instead of the $c\text{-}critical\ profile$ whose c-improvement profile is a-improvement of u^a, we include its any one of c-improvement profiles to D_c, we obtain another maximal set of preference profiles. Therefore, this algorithm leads to one of the maximal domains to guarantee $Nash\text{-}implementability$ of F. Nevertheless, we consider D as the more "intuitive" maximal domain among all the possible maximal sets of preference profiles since elements of the $critical\ domain$ are included in the set D, when they meet the requirement of the $critical\ neutrality$ notion.

CHAPTER 5

CONCLUSION

Implementation theory is concerned with implementing socially desirable alternatives as equilibrium outcomes of a mechanism in different game-theoretic solution concepts. This study concentrates on $implementability$ of social choice rules in $Nash\ equilibrium$. We analyze $Nash\ implementability$ to see to what extent the neutrality property is inherited by the $Nash\ equilibrium$ concept and thus, by $Nash$-$implementable$ social choice rules. Associated with $critical\ profiles$ and $critical\ elements$, we define a restricted kind of $neutrality$ which we refer to as $critical\ neutrality$. Via $critical\ neutrality$, a novel characterization of $Nash\ implementability$ of a $Maskin\ monotonic$ social choice rule is presented. This characterization covers social choice rules defined on the full domain of preference profiles. The result confirms our conjecture that the $implementability$ on the full domain of preference profiles is highly related with the $implementability$ on the "critical domain". Moreover, we utilize $critical\ neutrality$ as a tool to obtain a maximal domain of preference profiles on which an SCR is $Nash$-$implementable$. This domain is considered to be "intuitive" as it includes elements of the $critical\ domain$ when they satisfy the condition of $critical\ neutrality$.

The characterization result that we reach in this study has the following possible extension. Via modifying definitions of $critical\ profiles$, $critical\ elements$ and hence, $critical\ neutrality$ with respect to a given set of preference profiles, one can characterize $Nash$-$implementability$ on any domain of preference profiles. Additionally, as another future work, we intend to obtain new characterizations for $implementability$ in refinements of $Nash\ equilibrium$, particularly in $strong$ and $subgame\ perfect\ Nash$, via conditions imposed on the "critical domain". The aim is to have a better understanding of different $implementation$ problems through the unified approach.

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