Perseverance and suspense in tug-of-war

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A B S T R A C T

We study a tug-of-war game between two players using the lottery contest success function (CSF) and a quadratic cost (of effort) function. We construct a pure strategy symmetric Markov perfect equilibrium of this game, show that it is unique, and provide closed-form solutions for equilibrium strategies and values. In stark contrast to a model of tug-of-war with an all-pay auction CSF, players exert positive efforts until the very last battle in this equilibrium. We deliver a set of empirically appealing results on effort dynamics.

1. Introduction

In a multi-battle or dynamic contest efforts in an individual battle and the corresponding battle outcome may influence future efforts and battle outcomes. Such an influence may have implications for parties' behavior in earlier rounds (see Konrad, 2012). For instance, if losing the first battle makes the whole contest a write-off, then excessive efforts may be induced in the first battle. The presence of such dynamic linkages gives rise to some interesting questions such as “How do efforts vary across battles?”, “How do efforts vary with intensity of rivalry?”, and “Who exerts a greater effort: the leader or the follower?” (see Harris and Vickers, 1987).

The literature on contest games produced various models of dynamic contests such as race, tug-of-war, elimination contests, war of attrition, and repeated incumbency fights (see Konrad, 2012 for a review). Harris and Vickers (1987) argued that tug-of-war is possibly the simplest framework to address questions regarding effort dynamics since this model has a single state variable — a measure of the distance between players. Along these lines, the current paper is concerned with the questions mentioned above, and it focuses on the model of tug-of-war between two players.

A tug-of-war is a multi-battle contest game with a finite number of ordered states and potentially infinite number of battles.

Players start at an initial state (neutral or non-neutral) and simultaneously exert effort in each battle to win the contest. Winning a battle moves the state towards the winning player’s favorite terminal state. This game can be illustrated by a horizontal line with an interior point that represents the initial state and two end-points that represent respective players’ favorite terminal states, resembling the sports competition after which the model is named. Players win the prize/award if they have sufficiently many battle victories to pull the state to their terminal state. The winner of each battle is determined by a contest success function (CSF for short), which takes players’ efforts as input. As the above description of the game reveals, what matters in tug-of-war is not the absolute number of battle victories, but the difference between the two players' numbers of victories.

Two observations are worth mentioning here. First, we observe suspense and perseverance in real-life tug-of-war (see Deck and Sheremata, 2019 for recent experimental evidence) — an observation not completely captured by the theoretical literature. Second, we observe that exogenous noise is an essential feature of many real-life contests (see Thorngate and Carroll, 1987). Consequently, we study a tug-of-war game between two players, where the battle outcomes are determined by a lottery CSF (a special case of the Tullock CSF; see Tullock, 1980). In our model, (i) there are no intermediate prizes, (ii) the cost of effort is quadratic, and (iii) players do not discount the future.

We completely characterize the symmetric Markov perfect equilibrium of this game under a regularity assumption. The
equilibrium strategies are deterministic. Furthermore, this equilibrium is unique. We also offer a set of results on effort dynamics and some comparative statics. Our main results are as follows:

(i) equilibrium efforts in all interior (i.e., non-terminal) states are positive,
(ii) equilibrium values of interior nodes can be written as convex combinations of the winning prize and the losing prize,
(iii) the player who is closer to winning (advantaged player) exerts a higher effort than the other player,
(iv) the ratio of the advantaged player’s effort to the disadvantaged player’s effort increases as the former approaches his favorite terminal state,
(v) the sum of players’ efforts decreases as either player gets closer to winning,
(vi) players’ equilibrium efforts follow monotonic paths across interior nodes,
(vii) equilibrium effort levels at all interior nodes increase with the difference between the values of winner and loser prizes and decrease with the threshold level of victory difference, and
(viii) in two games with different numbers of nodes, players’ effort ratio is identical in any interior node common to both games.

Some of our results resemble some of those in Harris and Vickers (1987), who also provided comparative static results. For instance, (iii) and (vi) above can be deduced from Property 3.1 in Harris and Vickers (1987). Similarly, (iv) and (v) can be deduced from Property 3.2 in Harris and Vickers (1987). Note that (ii), (vii), and (viii) listed above are novel. On the other hand, our first result is in stark contrast with the equilibrium of tug-of-war with all-pay auction CSF (see Konrad and Kovenock, 2005). Also note that the positive equilibrium efforts in all interior states and the stochastic nature of the Tullock CSF imply that there will be swings back-and-forth (i.e., the advantage may change hands). Finally, (iii), (iv), and (v) show that a partial discouragement is still present.

This paper contributes to the theoretical literature on dynamic contests. Our main contribution is to give a closed-form solution for equilibrium efforts and values in tug-of-war with a lottery CSF and a quadratic cost function. To the best of our knowledge, ours is the first paper to do that. We deliver empirically appealing predictions on effort dynamics. Our results are also of interest from a design perspective: a contest-designer who values neck-to-neck competition or suspense (a desirable feature from audience’s perspective in sports competitions) should prefer lottery CSF to all-pay auction CSF in tug-of-war. Finally, we believe that our model will be of practical value for experimental economists studying dynamic contests due to the existence of pure strategy equilibrium (easier to interpret/identify empirically) and the rich set of testable hypotheses it produces.

2. Literature review

We focus on the literature on tug-of-war here. Theoretical work on tug-of-war mostly developed in the realm of economics. Harris and Vickers (1987) were the first to formally study it. They modeled an R&D competition between firms. The outcome of a battle was determined probabilistically and was a function of firms’ efforts. In fact, they used the same lottery CSF we use here. They proved the existence of a Markov perfect equilibrium, and even provided some comparative static results in a more general setting (compared to ours) where this generality is due to the fact that they did not assume an explicit cost function. They did not give a closed-form solution for equilibrium efforts and values though.

Konrad and Kovenock (2005) used an all-pay auction (without noise) as a CSF to study tug-of-war. They analytically solved for equilibrium and provided conditions for uniqueness. Since their CSF was deterministic, the equilibrium was in mixed strategies. Perhaps, the most striking result of the paper was the extreme discouragement effect (see Konrad, 2012 for a review) that emerges in equilibrium: players exert considerable effort in the first battle and zero efforts in all the remaining battles. Consequently, the player who wins the first battle wins the contest without exerting any further effort. The deterministic nature of the CSF employed was the major reason behind this result (see Konrad, 2010).

Later – building on McAfee (2000)’s analysis – Agastya and McAfee (2006) also investigated a model of tug-of-war using an all-pay auction CSF. Two major differences from Konrad and Kovenock (2005) were (i) the presence of a negative loser prize in Agastya and McAfee (2006) and (ii) the way their CSF broke ties. Now the disadvantaged party may have a reason to continue exerting effort: escaping from the negative loser prize. These authors showed that there exists two types of stationary equilibria with very different characteristics. In one of them, effort tends to rise as either player gets close to winning, whereas in the other one players remain in an interior state forever, not fighting against each other.

Moscarini and Smith (2007) extended Harris and Vickers (1987)’s model to a continuous–time and continuous state–space environment. In their model, a player continuously exerts effort at a quadratic cost to produce a flow output, and when a predetermined output difference is reached the game ends. A player’s effort controls (linearily) the drift of the Brownian motion, which governs his cumulative output. They showed that the optimal prize is finite and conjectured that the optimal scoring rule penalizes the leader so that the laggard does not give up, for which they provided numerical results.

Finally, Ewerhart and Teichgräber (2019) studied a class of dynamic contests using finite automata techniques. They restricted their attention to dynamic contests that could be defined using a finite state machine and satisfied three assumptions (exchangeability, monotonicity, and centeredness). Tug-of-war is of special interest. The alternative description of the problem and the methods they used allowed them to prove that tug-of-war admits a unique, symmetric, interior Markov perfect equilibrium under a general form of Tullock contest success function.

3. The model and results

We consider a tug-of-war contest, a multi-stage game with observed actions and potentially infinite horizon. Effort choices in each battle are simultaneous, but all past actions and battle outcomes are publicly observable. Players are denoted by $L$ (for left) and $R$ (for right). States (or nodes) refer to the difference between the number of wins by two players. We use the terms, state and node, interchangeably. In this context, (i) a win by Player $L$ results in a move towards the left; (ii) a win by Player $R$ results in a move towards the right. The probability of moving from one node to another is determined by the lottery CSF. More precisely, if players’ efforts at a (non-terminal) node $k$ are $(l_k, r_k) \in [0, \infty)^2$, the probability of a win by Player $L$ is given by

$$p_l (l_k, r_k) = \begin{cases} \frac{l_k}{l_k + r_k} & \text{if max} |l_k, r_k| > 0, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$^1$ Konrad and Kovenock (2005) assumed that the discount factor is strictly less than 1. Later, Vojnović (2015) proved that the same result holds for the no-discounting case, as well.
and the probability of a win by Player R is given by \(1 - p \left( l_k, r_k \right)\). The cost of effort is given by a quadratic cost function. In particular, the cost of spending effort \(e\) is \(C(e) = \frac{e^2}{2}\). The game ends when one of the players wins sufficient number of battles to pull the state of the game to its favored terminal state. Corresponding winning/losing prizes are awarded to the winner/loser. There are no intermediate prizes, and the players do not discount the future. We denote this game by \(\Gamma\).

### 3.1. Equilibrium

Harris and Vickers (1987) proved the existence of a Markov perfect equilibrium of the two player tug-of-war contest with a lottery CSF. They showed that there exists a symmetric Markov perfect equilibrium if the game is symmetric. In this section we go a step further and provide a closed form solution for equilibrium efforts and values when the cost function has a quadratic form. In our proof of the main result we multiply the first-order conditions with each other to solve for the equilibrium efforts. It would be fair to say that this method works thanks to the quadratic form of the cost function.

We solve for the pure strategy equilibrium for an odd number of nodes. The analysis of even number of nodes is similar and relegated to the appendix of the working paper version (see Karagözlu et al., 2020).

Suppose that we have \(2n + 1\) nodes and the set of states is given by \(s^n = \{-n, -n + 1, \ldots, 0, n - 1, n\}\), where \(n\) is any positive integer. We denote the set of terminal states by \(s^n = \{-n, n\}\). A winning (losing) prize is awarded to Player L (R) if the terminal state \(-n\) is reached and, vice versa if the terminal state \(n\) is reached (see Fig. 1).

We restrict our attention to Markov strategies that end the game in finite time with probability one. In particular, when choosing their efforts at a given state, the players do not distinguish among different histories that may lead them to that state (see Konrad, 2012, page 9). In other words, a player’s strategy depends only on the nodes visited and not on the paths by which they are reached. We write the strategies for Player L and Player R as finite dimensional vectors \(l = (l_k)_{k \in s^n} / s^n\) and \(r = (r_k)_{k \in s^n} / s^n\), respectively.

A strategy profile \((l, r)\) is called symmetric if \(l_k = r_{-k}\) for all \(k \in s^n\). Given such a strategy profile, let \(V_k\) be the value of reaching state \(k\) for Player L and reaching state \(-k\) for Player R. The strategy profile is a symmetric Markov perfect equilibrium if

\[
V_k = \max_l \left\{ p \left(l, r_k\right) V_{-k} + (1 - p \left(l, r_k\right)) V_{k+1} - C \left(l\right) \right\}
\]

and

\[
V_{-k} = \max_r \left\{ p \left(l_k, r\right) V_{-(k+1)} + (1 - p \left(l_k, r\right)) V_{-(k-1)} - C \left(r\right) \right\}.
\]

Considering the unconstrained optimization problems in (1) and (2), and the associated first-order conditions for optimality, we obtain

\[
\frac{r_k}{l_k + r_k}\left(V_{k-1} - V_{k+1}\right) = l_k,
\]

\[
\frac{l_k}{l_k + r_k}\left(V_{-(k+1)} - V_{-(k-1)}\right) = r_k.
\]

We have unconstrained optimization problems with strictly concave objective functions (due to the quadratic cost functions). Hence, the second-order conditions are always met, implying an interior solution. Now using (3) and (4), we obtain

\[
r_k = \left( \frac{V_{-(k+1)} - V_{-(k-1)}}{V_{k-1} - V_{k+1}} \right)^{1/2},
\]

\[
l_k = \left( \frac{V_{-(k+1)} - V_{-(k-1)}}{V_{k-1} - V_{k+1}} \right)^{1/2}.
\]

Now, let \(t_k = \frac{V_{-(k+1)} - V_{-(k-1)}}{V_{k-1} - V_{k+1}}\) and \(\pi_k = \frac{V_{-(k+1)} - V_{-(k-1)}}{V_{k-1} - V_{k+1}}\). Note that \(t_k\) represents the relative efforts of the players at node \(k\) and \(\pi_k\) represents the ratio of the changes in the valuations of the players when Player L wins at node \(k\). Then, for each decision state \(k\), we have

\[
r_k = \frac{\pi_k}{1 + \pi_k},
\]

\[
l_k = \frac{1}{1 + \pi_k}.
\]

By substituting the sum of the players’ efforts given by (6), in (7) and (8), the optimal efforts can be written as

\[
r_k = \frac{\pi_k^{1/2}}{1 + \pi_k},
\]

\[
l_k = \frac{1}{1 + \pi_k}.
\]

Our first proposition concerns the existence and uniqueness of a symmetric Markov perfect equilibrium, and characterizes the equilibrium value of each non-terminal node as a convex combination of the values of the terminal nodes. These weights (i.e., \(b_k\) and \(1 - b_k\)) will be derived in a constructive fashion in the proof of Proposition 1 and we will be able to show that \(V_{-n} > V_{-(n-1)} > \cdots > V_0 > \cdots > V_n\) in equilibrium (see Harris and Vickers, 1987 for a similar result). Note that since we consider Markov strategies that end the game in finite time with probability 1, payoffs are continuous at infinity, which allows us to use the one-deviation property in our analysis.

**Proposition 1.** There exists a unique symmetric Markov perfect equilibrium of \(\Gamma\). In this equilibrium, for any decision state \(k\),

\[
V_k = b_k V_{-n} + (1 - b_k) V_n,
\]

where the closed-form solution for \(b_k\) explicitly constructed below satisfies \(0 < b_{k+1} < b_k < 1\).

**Proof.** Rewriting (1) and (2), using (9) and (10) respectively, we obtain

\[
V_k = \frac{1}{1 + t_k} V_{-k} + \frac{t_k}{1 + t_k} V_{k+1} - \frac{\left(V_{k-1} - V_{k+1}\right) t_k}{2(1 + t_k)^2},
\]

\[
V_{-k} = \frac{1}{1 + t_k} V_{-(k+1)} + \frac{t_k}{1 + t_k} V_{-(k-1)} - \frac{\left(V_{-(k+1)} - V_{-(k-1)}\right) t_k}{2(1 + t_k)^2}.
\]

After rearranging these expressions, we have

\[
\frac{V_{k-1} - V_k}{V_{k-1} - V_{k+1}} = \frac{(2k + 3) t_k}{2(1 + t_k)^2},
\]

\[
\frac{V_{-(k+1)} - V_{-(k-1)}}{V_{-(k+1)} - V_{-(k+1)}} = \frac{(2k + 1) t_k}{2(1 + t_k)^2}.
\]

We deduce from (11) and (12) that

\[
t_k^2 = \frac{2 t_k + 3}{x_k} \frac{2 t_k + 3}{x_k},
\]

and hence

\[
x_{k+1} = x_{k} + \frac{2 t_k + 3}{2 t_{k+1} + 1}.
\]
Now using (11), we have
\[ 1 - \frac{V_{k-1} - V_k}{V_k - V_{k+1}} = 1 - \frac{(2k + 3)t_k}{2(1 + t_k)^2}. \]
or equivalently,
\[ \frac{V_k - V_{k+1}}{V_k - V_{k-1}} = \frac{t_k + 2}{2(1 + t_k)^2}. \]  
(14)
Similarly, using (12) we also have
\[ \frac{V_{(k+1)} - V_k}{V_k - V_{(k-1)}} = \frac{3t_k + 2}{2(1 + t_k)^2}. \]  
(15)
Dividing both sides of (15) by those of (14) yields
\[ \frac{t_k^2}{x_{k+1}} = \frac{t_k + 2}{3t_k + 2}. \]  
(16)

By combining (13) and (16), we can get
\[ \frac{(3t_k + 2)t_k^2}{t_k + 2} = x_{k+1} = \frac{(2t_k + 1 + 1)t_k^2}{2t_k + 3}. \]  
(17)

Now, define \( f(t) = \frac{(3t+2)t^2}{t+2} \) and \( g(t) = \frac{(2t+1)t^2}{2t+3} \). Notice that \( f(0) = g(0) = 0 \) and \( f(\infty) = g(\infty) = \infty \). Moreover,
\[ f'(t) = \frac{6t^2 + 20t^2 + 8t}{(t+2)^2}. \]

So, for all \( t > 0, f'(t) > 0 \). Similarly, it can be shown that for all \( t > 0, g'(t) > 0 \). Now, note that we have \( f(t_k) = g(t_{k+1}) \). Moreover, since \( g(t) \) is continuous and increasing in \( t \) for all \( t \in (0, \infty) \), it is one-to-one; thus \( t_{k+1} = g^{-1}(f(t_k)) \). Similarly, \( t_k = f^{-1}(g(t_{k+1})) \).

We know \( t_0 = 1 \). Thus, by the equations above, we can find the value of \( t_k \) for any decision state \( k \). As \( g(t) < t^2 < f(t) \) for all \( t > 0 \), we have \( t_{k+1} = g^{-1}(f(t_k)) > g^{-1}(g(t_k)) = t_k \).

We can now determine the value of \( x_k \) for any decision state \( k \) as well. By definition, we know that
\[ \frac{V_{(k+1)} - V_k}{V_k - V_{(k-1)}} + \frac{V_k - V_{(k-1)}}{V_{(k-1)} - V_k} = t_k^2. \]
Since
\[ x_k = \frac{V_k - V_{(k-1)}}{V_{(k-1)} - V_k} \text{ and } x_{k+1} = \frac{V_{(k-1)} - V_k}{V_k - V_{(k+1)}}, \]
we can get
\[ (V_{(k+1)} - V_k) + (V_k - V_{(k-1)}) = \frac{t_k^2}{x_{k+1}} (V_{(k-1)} - V_k) + \frac{t_k^2}{x_k} (V_k - V_{(k-1)}). \]
Thus,
\[ \frac{V_{(k+1)} - V_k}{V_k - V_{(k-1)}} + \frac{V_k - V_{(k-1)}}{V_{(k-1)} - V_k} = \frac{t_k^2}{x_{k+1}} - \frac{1}{1 - \frac{x_{k+1}}{x_k}} = \frac{3t_k + 2}{2(t_k + 1)t_k} = h(t_k). \]

Similarly, we know that
\[ \frac{V_k - V_{(k+1)}}{V_{(k+1)} - V_{(k-1)}} + \frac{V_{(k-1)} - V_{(k+1)}}{V_{(k-1)} - V_k} = t_k^2. \]

Note that \( t_k = 1/t_k \) as we consider symmetric Markov strategies. Knowing that
\[ x_k = \frac{V_k - V_{(k+1)}}{V_{(k+1)} - V_k} \text{ and } x_{(k-1)} = \frac{V_{(k-1)} - V_k}{V_k - V_{(k-1)}}, \]
we can obtain
\[ \frac{V_k - V_{k+1}}{V_{k-1} - V_k} = \frac{t_k^2 - \frac{1}{x_{(k-1)}}}{1 - \frac{x_{(k+1)}}{x_k}} = \frac{(2t_k + 1)t_k}{3t_k + 2}. \]

Thus,
\[ \frac{V_k - V_{k+1}}{V_{k-1} - V_k} = \frac{t_k + 2}{2t_k + 3}. \]

Now, let us define \( a_j, j = 1, 2, \ldots, 2n \) according to
\[ a_j = V_-(n-j) - V_-(n-j). \]
(19)
It is clear that
\[ \sum_{j=1}^{2n} a_j = V_n - V_0. \]
(20)

Note from (18) that \( \frac{a_j}{n} = h(t_{n-j}) \) for all \( j = 1, 2, \ldots, 2n - 1 \), which implies
\[ a_j = \sum_{j=1}^{n} h(t_{n-j}), \quad \forall j = 1, 2, \ldots, 2n - 1. \]

From (20), we have
\[ a_1 = \frac{V_n - V_0}{h(2n)}. \]

where \( h(s) = \sum_{j=1}^{n} \frac{1}{h(t_{n+j})} \), for all \( s = 1, 2, \ldots, 2n \).

From (19), for any decision state \( k \), we deduce that
\[ V_k = V_n - \sum_{j=1}^{n} a_j. \]
Equivalently,
\[ V_k = V_n - a_1 h(n + k) = V_n - \sum_{j=1}^{n} V_{n-j} h(n + k). \]
Hence,
\[ V_k = \left( 1 - \frac{h(n+k)}{h(2n)} \right) V_{n+k} + \left( \frac{h(n+k)}{h(2n)} \right) V_0. \]

Notice that \( 0 < b_{k+1} < b_k < 1 \) follows from (i) \( k < n \), (ii) \( h(t_k) \) is positive for all \( k \) (it follows from (18) and the fact that \( t_k \geq 1 \)), and (iii) the only difference between the numerator and the denominator in \( b_k \) is that the summation in the numerator runs through \( n + k \), whereas the summation in the denominator runs through \( 2n \).

The fact that any intermediate node has an equilibrium value which can be expressed as a convex combination of the values of terminal nodes has important implications for equilibrium effort levels. In particular, it implies that any non-terminal node has a positive value, which leads to positive equilibrium efforts by both players in any such node. This shows that the extreme discouragement (e.g., losing the first battle makes the whole contest a write-off) in Konrad and Kovenock (2005) is not present in our model, and every node in the game is reached with a positive probability.
3.2. Effort dynamics

Now, we investigate the changes in individual efforts and the sum of individual efforts across different states. Note that by (9) and (10), we can compute the equilibrium effort levels of both players. In the following proposition, we show that (i) the player who is closer to winning (i.e., the advantaged player) exerts a higher effort than the other (disadvantaged) player, and the ratio of the advantaged player’s effort to the disadvantaged player’s effort increases as the advantaged player approaches his favorite terminal node, (ii) the sum of players’ efforts decreases as the advantaged player approaches his favorite terminal node.

Proposition 2. Given $V_{-n}$ and $V_n$, the equilibrium effort choices of the two players in $G$ satisfy

(i) $\forall k \in \{1, 2, \ldots, n-1\}$, $r_k > l_k$ and, moreover, $\frac{a_{l_k}}{a_{r_k}}$ increases in $k$.
(ii) $\forall k \in \{1, 2, \ldots, n-1\}$, $l_{k-1} + r_{k+1} > l_k + r_k$.

Proof. (i) Recall that $\frac{a_{l_k}}{a_{r_k}} = t_k$ for all $k$ and $t_0 = 1$. As $t_{k+1} > t_k$ for all $k$ (see the proof of Proposition 1), the result immediately follows.

(ii) Using (6) and (19), we have for all $k \in \{1, 2, \ldots, n-1\}$,

$L_{k-1} + R_{k+1} = l_k + r_k = (a_{n-k} + a_{n-k+1})^{1/4} (a_{n+k} + a_{n+k+1})^{1/4}.$

Also recall from (18) that $\frac{a_{l_j}}{a_{r_j}} = h(t_{n-j})$, $j = 1, 2, \ldots, 2n$.

These allow us to write

$$ \frac{l_{k-1} + r_{k+1}}{l_k + r_k} = \left( \frac{a_{n-k+1} + a_{n-k+2}}{a_{n-k} + a_{n-k+1}} \right)^{1/4} \left( \frac{a_{n+k} + a_{n+k+1}}{a_{n+k+2}} \right)^{1/4} \left( h(t_k) a_{n-k} + h(t_{k-1}) a_{n-k} \right)^{1/4} \left( h(t_k) a_{n+k+2} + h(t_{k+1}) a_{n+k+1} \right)^{1/4}$$

$$ > \left( \frac{h(t_{k-1})}{h(t_k)} \right)^{1/4}, \forall k \in \{1, 2, \ldots, n-1\},$$

as $t_{k+1} > t_k$ for all $k$ and $h$ is a strictly decreasing function. Noting that $\frac{h(t_{k+1})}{h(t_{k-1})} = 1$ for $k = 1$ and $\frac{h(t_{k})}{h(t_{k+1})}$ is increasing in $k$ ends the proof.

Proposition 2 shows that partial discouragement is still present in our game: the disadvantaged player exerts a lower effort (than the advantaged player), and the gap between their efforts (measured as a ratio) widens as the advantaged player approaches his favorite terminal state. That said, the advantage can still change hands since the disadvantaged player keeps exerting a positive effort (till the very end) and as long as he does so, he has a chance to win.

Proposition 2 makes a point about the relative values and the sums of individual efforts, but does not pin down the dynamics of individual efforts, separately. Proposition 3 completes the picture by analytically showing that – except around the central node – players’ individual efforts follow monotonic paths (across nodes) in equilibrium.3

Proposition 3. The equilibrium effort choices of the advantaged player in $G$ follow a monotonic path, i.e.,

$L_1 > l_2 > \cdots > l_{n-1} \text{ or } r_1 > r_2 > \cdots > r_{n-1}.$

The equilibrium effort choices of the disadvantaged player also follow a monotonic path, i.e.,

$l_1 > l_2 > \cdots > l_{n-1} \text{ or } r_1 > r_2 > \cdots > r_{n-1}.$

However, $l_1 > l_0$ and $r_0 < r_1$.

Proof. By (9), we have

$$ \frac{r_{k+1}}{r_k} = \left( \frac{V_{-(k+1)} - V_{-(k-1)}}{V_{-(k+2)} - V_{-k}} \right)^{1/2} \phi(t_k),$$

where $\phi(t_k) = \frac{t_k}{1 + t_k}$ for $k \in \{0, 1, 2, \ldots, n - 1\}$. It follows from (18) and (19) that

$$ \frac{V_{-(k+1)} - V_{-(k-1)}}{V_{-(k+2)} - V_{-k}} = \frac{a_{n-k} + a_{n-k+1}}{a_{n-k-1} + a_{n-k}} = \frac{a_{n-k+1} (1 + h(t_k))}{a_{n-k} (1 + h(t_{k+1}))}$$

$$ = \frac{1}{h(t_k) (1 + h(t_{k+1}))}.$$

Accordingly, we can write $\frac{r_{k+1}}{r_k} = \Phi_k$ for $k \in \{0, 1, 2, \ldots, n - 1\}$, where

$$ \Phi_k = \left( \frac{h(t_k)}{h(t_{k+1})} \right)^{1/2} \phi(t_k).$$

From the proof of Proposition 1, recall that $t_{k+1} = g^{-1}(f(t_k))$ for all $k$. Thus, we can find the value of $t_k$ for any state $k$:

$t_1 = 1.57196, t_2 = 2.48995, t_3 = 3.99793, t_4 = 6.52672, t_5 = 10.8317, \ldots, etc.$

Note that $t_k > 1$ implies $\phi'(t_k) < 0$, which further implies $\frac{\phi(t_k)}{\phi(t_{k+1})} > 1$. Further, note from (18) that $h$ is a strictly decreasing function and $h(t_k) h(t_{k+1})$ is a strictly decreasing function for all $k \geq 1$. Using (18) and the values of $t_1$ and $t_2$ calculated above, we obtain $h(t_1) h(t_2) = (1.03098)(0.636001) = 0.655704 \leq 1$. Hence, $h(t_k) h(t_{k+1}) < 1$ for all $k \geq 1$, which implies that

$$ \left( \frac{1}{h(t_k) (1 + h(t_{k+1}))} \right)^{1/2} > 1.$$  

Thus, $\Phi_k > 1$ for all $k \geq 1$. Indeed, we can obtain the value of each $\Phi_k$ for all $k$:

$\Phi_0 = 0.91037, \Phi_1 = 1.18309, \Phi_2 = 1.53796, \Phi_3 = 2.00374, \Phi_4 = 2.61808, \Phi_5 = 3.42945, \ldots, etc.$

As $\Phi_0 < 1$ and $\Phi_k > 1$ for all $k \geq 1$, we conclude that $r_0 < r_1$ and $r_n > r_{n+1}$ for all $k \geq 1$. As $r_n = l_k$ for all $k$, we can also conclude that $l_0 < l_1$ and $l_k > l_{k+1}$ for all $k \geq 1$.

The fact that $r_{k-1} > r_{-(k-1)}$ or $l_k > l_{k+1}$, $\forall k \in \{1, 2, \ldots, n - 1\}$ can also be shown in a similar way. □

3.3. Comparative statics

In this subsection we investigate how the players’ equilibrium efforts respond to changes in (i) the difference between winning and losing prizes and (ii) the required victory threshold. More precisely, in the next proposition, we show that an increase in the difference between the winning prize and the consolation (or losing) prize encourages players to exert more effort in every interior node. We also show that an increase in the required winning threshold decreases the players’ efforts at every interior node.

Proposition 4. Players’ equilibrium efforts at any interior node $(i)$ increase with $V_{-n} - V_n$ and $(ii)$ decrease with the number of nodes.
Proof. (i) Note from (17) that the values of $r_k$ and $x_k$ for all $k$ are both independent of $V_{-n}$ and $V_n$. Then, Eq. (21) shows that the $b_i$'s are also independent of $V_{-n}$ and $V_n$. Recalling Eq. (9) and applying simple algebra, we obtain

$$r_k = \left( (b_{-(k+1)} - b_{-(k+1)}) (V_{-n} - V_n) \right)^{1/2} \frac{1}{(1 + t_k)},$$

which shows that $r_k$ is strictly increasing in $V_{-n} - V_n$. The fact that $l_k$ is strictly increasing in $V_{-n} - V_n$ can be shown in a similar way. (ii) We will now prove that the equilibrium effort choices of the players decrease with the number of nodes. To do so, consider the tug-of-war contest under two different sets of states: $\mathcal{X}_n = \{-n, -(n - 1), \ldots, 0, \ldots, (n - 1), n\}$ and $\mathcal{X}_n = \{-n, -(n - 1), \ldots, 0, (n - 1), n\}$, where $n, \tilde{n} \in \mathbb{Z}_{++}$. Without loss of generality, assume that $\tilde{n} > n$. Let the corresponding equilibrium values of the nodes be denoted by $V_i$, $i \in \mathcal{X}_n$ and $V_i$, $i \in \mathcal{X}_n$, since the winning and losing prizes are not altered, we have

$$\tilde{V}_{-n} < V_{-n} = V_{-n}$$

Since

$$V_{-n} < \tilde{V}_{-n} < V_{-n} - V_n.$$

This implies

$$\tilde{V}_{-n} - V_{-n} < V_{-n} - V_n.$$

As we have already shown in (i) that the equilibrium effort choices of the players increase with $V_{-n} - V_n$, this ends the proof. $\square$

We state an immediate consequence of Proposition 4 for the sum of efforts.

Corollary 1. The sum of the equilibrium individual efforts at any interior node (i) increases with $V_{-n} - V_n$ and (ii) decreases with the number of nodes.

Proof. Follows directly from Proposition 4. $\square$

We finish this subsection by stating another consequence of Proposition 4.

Corollary 2. Consider two different sets of states,

$$\mathcal{X}_n = \{-n, -(n - 1), \ldots, 0, \ldots, (n - 1), n\},$$

$$\mathcal{X}_n = \{-n, -(n - 1), \ldots, 0, (n - 1), n\},$$

where $n, \tilde{n} \in \mathbb{Z}_{++}$. For any node $k < \min(n, \tilde{n})$, the ratio $\frac{b_k}{b_{\tilde{k}}}$ is the same in both games.

Proof. Follows from (5) and (21), and Proposition 4. $\square$

This result has an interesting implication: the ratio of players' efforts – the density of competition – in a sense, depends on how far the state in question is from the initial state but not on how far it is from the terminal states. In other words, the density of competition has a backward-focus rather than a forward-focus, which is an interesting theoretical result that calls for an empirical test.

3.4. Examples: 3, 5, and 7 nodes

We present numerical illustrations for 3, 5, and 7 nodes. It is worthwhile emphasizing that the numbers we provide are calculations based on our characterization of the equilibrium, not approximations or simulation outcomes. Substituting the values of $t_k$ for all $k$ into (21) allows us to write the equilibrium valuations of players at each interior node as a convex combination of $V_{-n}$ (the prize of winning the contest) and $V_n$ (the consolation prize) for all $n$. The equilibrium effort choices of players at any interior node can then be determined by (9) and (10). To do so, let us consider $V_{-n} = W$ and $V_n = L$ for all $n$. Denote the equilibrium effort of Player $R$ at node $k$ in an $2n + 1$-node game by $r_{2n+1}^k$. Since $r_{2n+1}^k = r_{2n+1}^k$ for all $k$ and $n$, we only list the effort levels for Player $R$.

Table 1 consolidates all effort choices, effort ratios, and sums of efforts. We rounded all numbers after the third decimal for expositional simplicity.

Below, we present some observations from these examples along with the theoretical results to which they are related:

- Values of all interior nodes are convex combinations of $W$ and $L$ (Proposition 1).
- $r_1^3 > r_1^6 > r_2^6 > r_2^3$ (Proposition 2).
- $r_3^1 > r_1^6 > r_1^5 > r_1^6 > r_1^5 > r_1^6$ (Proposition 2).
- $l_2^3 + r_2^3 + r_1^2 > l_2^3 + r_1^2 + r_1^2 > l_2^3 + r_1^2$ (Proposition 2).
- $r_2^3 > r_3^2 > r_1^2 > r_1^2 > r_1^2 > r_1^2$ (Proposition 3).
- $l_2^3 > l_2^3 > r_1^2 > r_1^2 > r_1^2$ and $r_2^3 + r_1^2 > r_2^3 + r_1^2 > r_2^3 + r_1^2$ (Proposition 4).
- All effort levels are increasing in $W$ and $L$ (Proposition 4).
- Sum of efforts are all increasing in $W$ and $L$ (Corollary 1).
- $l_2^3 + r_2^3 > l_2^3 + r_2^3$ and $l_2^3 + r_2^3 > l_2^3 + r_2^3$ (Corollary 1).
- $r_2^3 > r_2^3 > r_2^3 > r_2^3 > r_2^3$ (Corollary 2).
Finally, as can also be seen from Table 1, the likelihood of rent over-dissipation decreases (increases) when $W$ and $L$ are both multiplied with a scalar $\alpha > 1$ ($0 < \alpha < 1$). Unfortunately, a full-fledged analysis of rent-dissipation, similar to the one conducted by Konrad and Kovenock (2009) for multi-stage race in which players compete in a sequence of simultaneous move component contests and the first player to obtain a certain number of victories is the winner, cannot be performed here. The reason is that, each node can be visited arbitrarily many times in the tug-of-war game we study, which makes the calculation of the expected total effort cost non-trivial.

4. Concluding remarks

We study a tug-of-war game where the battle outcome is determined by a lottery CSF and the cost of effort is quadratic. We provide a characterization of the pure strategy symmetric Markov perfect equilibrium. Both players exert positive (equilibrium) efforts until the very last battle. The player closer to winning exerts more effort and the asymmetry between the two players’ efforts increase as the advantageous player approaches winning the contest. Furthermore, the total effort decreases as either player gets closer to winning. Finally, the equilibrium effort levels at all interior nodes increase with the difference between the values of winning and losing prizes, and decrease with the number of nodes.

McAfee (2000) and Agastya and McAfee (2006) also reported pervasiveness (i.e., all states are reached with positive probabilities) in a game of tug-of-war, despite using an all-pay CSF. However, these papers added a negative losing prize and changed the tie-breaking assumption. In our model we do not need negative losing prizes or alternative tie-breaking assumptions: the lottery CSF provides sufficient incentives to players. Konrad and Kovenock (2009) also reported pervasiveness in a model of multi-stage race. In that paper the presence of intermediate prizes was an important factor leading to this result. As suggested by an anonymous reviewer, it would be interesting to see the influence of intermediate prizes on effort dynamics. However, incorporating them to our model is far from trivial and as such left for future research.

Deck and Sheremata (2019) conducted the first laboratory experiment on tug-of-war contests. Their experimental design was built on Konrad and Kovenock (2005). They offered behavioral explanations to bridge the gap between subjects’ behavior and the predictions of the tug-of-war model with the all-pay auction CSF. Our model relies on standard preferences yet delivers empirically more appealing predictions. We believe that our model will be of practical value to experimental researchers who study tug-of-war contests in the lab.

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