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Bi-presymplectic chains of co-rank 1 and related Liouville integrable systems

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Abstract
Bi-presymplectic chains of 1-forms of co-rank 1 are considered. The conditions under which such chains represent some Liouville integrable systems and the conditions under which there exist related bi-Hamiltonian chains of vector fields are derived. To present the construction of bi-presymplectic chains, the notion of a dual Poisson-presymplectic pair is used, and the concept of *d*-compatibility of Poisson bivectors and *d*-compatibility of presymplectic forms is introduced. It is shown that bi-presymplectic representation of a related flow leads directly to the construction of separation coordinates in a purely algorithmic way. As an illustration, bi-presymplectic and bi-Hamiltonian chains in $\mathbb{R}^3$ are considered in detail.

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1. Introduction

Symplectic structures play an important role in the theory of Hamiltonian dynamical systems. In the case of a non-degenerate Poisson tensor, the dual symplectic formulation of the dynamic can always be introduced via the inverse of the Poisson tensor. On the other hand, many dynamical systems admit Hamiltonian representation with a degenerate Poisson tensor. For such tensors, the notion of dual presymplectic structures was developed [2, 3, 6, 11].

The presymplectic picture is especially interesting for Liouville integrable systems. There is a well-developed bi-Hamiltonian theory of such systems, starting from the early work of Gel'fand and Dorfman [7]. Particularly interesting are these systems whose construction is based on Poisson pencils of the Kronecker type [8, 9], with a polynomial in pencil parameter Casimir functions, together with related separability theory (see [4, 10] and references quoted therein). The important question is whether it is possible to formulate an independent, alternative bi-presymplectic (bi-inverse Hamiltonian, in particular) theory of such systems with related separability theory and what is the way to relate these two theories to each other.

This paper develops the bi-presymplectic theory of Liouville integrable systems and related separability theory in the case when the co-rank of presymplectic forms is 1. The whole formalism is based on the notion of *d-compatibility* of presymplectic forms and *d-compatibility* of Poisson bivectors.

Let us point out that although the case of co-rank 1 is very special, nevertheless it is of particular importance. Actually, the majority of physically interesting Liouville integrable systems from classical mechanics belong to that class of problems. In particular, it contains all systems with first integrals, quadratic in momenta, whose configuration space is flat or of constant curvature. So, it seems that the case of co-rank 1 is worth of separate investigation. On the other hand, it is clear that in order to complete the new theory a generalization to a higher co-rank is necessary. In fact the work is in progress, although it is a non-trivial task as the systems with higher co-ranks show specific properties not shown in the case of co-rank 1.

Another question the reader can ask is about the relevance of the formalism presented. As we know that it is a well-established bi-Hamiltonian separability theory, the question is what can we gain when applying its dual bi-presymplectic (bi-inverse- Hamiltonian, in particular) counterpart. The answer is as follows. In the bi-Hamiltonian approach, the existence of the bi-Hamiltonian representation of a given flow is a necessary condition of separability but not a sufficient one. In order to construct separation coordinates, a Poisson projection of the second Hamiltonian structure onto a symplectic leaf of the first one has to be done. Unfortunately, it is far from a trivial non-algorithmic procedure that should be considered separately from case to case. Moreover, there is no proof that it is always possible. In contrast, once we find a bi-presymplectic representation of a flow considered, the construction of separation coordinates is a fully algorithmic procedure (in a generic case obviously), as the restriction of both presymplectic structures to any leaf of a given foliation is a simple task. For this reason, we do hope that the new formalism presented in the paper will be relevant to the modern separability theory and hence interesting for the readers.

The paper is organized as follows. In section 2, we give some basic information on Poisson tensors, presymplectic 2-forms, Hamiltonian and inverse Hamiltonian vector fields and dual Poisson-presymplectic pairs. In sections 3 and 4, the concepts of *d*-compatibility of Poisson bivectors and *d*-compatibility of closed 2-forms are developed. Then, in section 5, the main properties of bi-presymplectic chains of co-rank 1 are investigated. We present the conditions under which the bi-presymplectic chain is related to some Liouville integrable system and the conditions under which the chain is bi-inverse Hamiltonian. The conditions under which Hamiltonian vector fields, constructed from a given bi-presymplectic chain, constitute a related bi-Hamiltonian chain are also found. We also illustrate a construction of separation coordinates once a bi-presymplectic chain is given. In sections 6–8, we investigate in details, with many explicit calculations and examples, a special case of bi-presymplectic and bi-Hamiltonian chains in $\mathbb{R}^3$.

Finally, let us remark that our treatment in this work is local. Thus, even if it is not explicitly mentioned, we always restrict our considerations to the domain Σ of manifold M where appropriate functions, vector fields and 1-forms never vanish and respective Poisson tensors and presymplectic forms are of a constant co-rank. In some examples, we perform calculations in a particular local chart from Σ.

2. Preliminaries

On a manifold M, a Poisson tensor is a bivector with a vanishing Schouten bracket. A function $c : M \to \mathbb{R}$ is called the *Casimir function* of the Poisson operator Π if $\Pi \mathrm{d}c = 0$. A linear combination $\Pi_\lambda = \Pi_1 - \lambda \Pi_0$ ($\lambda \in \mathbb{R}$) of two Poisson operators Π_0 and Π_1 is called a *Poisson pencil* if the operator Π_λ is Poisson for any value of the parameter λ. In this case, we say that Π_0 and Π_1 are *compatible*. Having a Poisson tensor, we can define a Hamiltonian vector field on M. A vector field X_F related to a function $F \in C^\infty(M)$ by the relation

$$X_F = \Pi \mathrm{d}F \tag{2.1}$$

is called the Hamiltonian vector field with respect to the Poisson operator Π.

Further, a *presymplectic* operator Ω on M defines a 2-form that is closed, i.e. $\mathrm{d}\Omega = 0$, degenerated in general. Moreover, the kernel of any presymplectic form is always an integrable distribution. A vector field X^F related to a function $F \in C^\infty(M)$ by the relation

$$\Omega X^F = \mathrm{d}F \tag{2.2}$$

is called the inverse Hamiltonian vector field with respect to the presymplectic operator Ω.

Definition 1. *A Poisson bivector* Π *and a presymplectic form* Ω *are called compatible if* $\Omega\Pi\Omega$ *is a closed 2-form.*

Any non-degenerate closed 2-form on M is called a *symplectic* form. The inverse of a symplectic form is an *implectic* operator, i.e. invertible Poisson tensor on M and vice versa.

Definition 2. *A pair* (Π, Ω) *is called a dual implectic–symplectic pair on M if* Π *is a non-degenerate Poisson tensor,* Ω *is a non-degenerate closed 2-form and* $\Omega\Pi = \Pi\Omega = I$.

So, in the non-degenerate case, the dual implectic–symplectic pair is a pair of mutually inverse operators on M. Moreover, the Hamiltonian and the inverse Hamiltonian representations are equivalent because for any implectic bivector Π there is a unique dual symplectic form $\Omega = \Pi^{-1}$, and hence a vector field Hamiltonian with respect to Π is an inverse Hamiltonian with respect to Ω.

Let us extend these considerations onto a degenerate case. In order to do it, let us generalize the concept of the dual pair from [3]. Consider a manifold M of an arbitrary dimension m.

Definition 3. *A pair of tensor fields* (Π, Ω) *on M of co-rank r, where* Π *is a Poisson tensor and* Ω *is a closed 2-form, is called a dual pair (Poisson-presymplectic pair) if there exist r 1-forms* α_i *and r linearly independent vector fields* Z_i*, such that the following conditions are satisfied.*

(i) $\alpha_i(Z_j) = \delta_{ij}, i = 1, 2, \ldots, r$.
(ii) $\ker \Pi = Sp\{\alpha_i : i = 1, \ldots, r\}$.
(iii) $\ker \Omega = Sp\{Z_i : i = 1, \ldots, r\}$.
(iv) The following partition of unity holds on TM*, respectively on* T^*M*,*

$$I = \Pi\Omega + \sum_{i=1}^{r} Z_i \otimes \alpha_i, \qquad I = \Omega\Pi + \sum_{i=1}^{r} \alpha_i \otimes Z_i. \tag{2.3}$$

In contrast to the non-degenerated case, for a given Poisson tensor Π the choice of its dual is not unique. Also for a given presymplectic form Ω, the choice of the dual Poisson tensor is not unique. The details are given in the following section. For the degenerate case, the Hamiltonian and the inverse Hamiltonian vector fields are defined in the same way as for the non-degenerate case. But for degenerate structures, the notions of the Hamiltonian and inverse Hamiltonian vector fields do not coincide. For a degenerate dual pair, it is possible to find a Hamiltonian vector field that is not inverse Hamiltonian and an inverse Hamiltonian vector field that is not Hamiltonian. Actually, assume that (Π, Ω) is a dual pair, $X_F = \Pi \mathrm{d}F$ is a Hamiltonian vector field and $\mathrm{d}F = \Omega X^F$ is an inverse Hamiltonian 1-form, where X^F is an inverse Hamiltonian vector field. Having applied Ω to both sides of the Hamiltonian vector field, Π to both sides of the inverse Hamiltonian 1-form and using decomposition (2.3), we get

$$\mathrm{d}F = \Omega(X_F) + \sum_{i=1}^{r} Z_i(F)\alpha_i, \qquad X_F = X^F - \sum_{i=1}^{r} \alpha_i(X^F) Z_i. \tag{2.4}$$

It means that an inverse Hamiltonian vector field X^F is simultaneously a Hamiltonian vector field X_F, i.e. $X^F = X_F$, if $\mathrm{d}F$ is annihilated by $\ker(\Omega)$ and X^F is annihilated by $\ker(\Pi)$.

Finally, for a dual pair (Π, Ω), the following important relations hold:

$$[Z_i, Z_j] = 0, \qquad L_{X_F}\Pi = 0, \qquad L_{Z_i}\Pi = 0, \qquad L_{X^F}\Omega = 0, \qquad L_{Z_i}\Omega = 0, \tag{2.5}$$

where L_X is the Lie-derivative operator in the direction of vector field X and $[.\,,.]$ is a commutator.

3. *D*-compatibility for a non-degenerate case

In this section we introduce a notion of *d*-compatibility when a dual pair is an implectic–symplectic one, i.e. when it is of co-rank 0. Let M be a manifold of even dimension $m = 2n$.

Definition 4. *We say that a closed 2-form Ω_1 is d-compatible with a symplectic form Ω_0 if $\Pi_0\Omega_1\Pi_0$ is a Poisson tensor and $\Pi_0 = \Omega_0^{-1}$ is dual to Ω_0.*

Definition 5. *We say that a Poisson tensor Π_1 is d-compatible with an implectic tensor Π_0 if $\Omega_0\Pi_1\Omega_0$ is closed and $\Omega_0 = \Pi_0^{-1}$ is dual to Π_0.*

Now, the following lemma relates *d*-compatible Poisson structures, of which one is implectic, and *d*-compatible closed 2-forms, of which one is symplectic.

Lemma 6.

(i) Let an implectic tensor Π_0 and a symplectic form Ω_0 be a dual pair. Let a Poisson tensor Π_1 be d-compatible with Π_0. Then Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0$ are d-compatible closed 2-forms.

(ii) Let an implectic tensor Π_0 and a symplectic form Ω_0 be a dual pair. Let a closed 2-form Ω_1 be d-compatible with Ω_0. Then Π_0 and $\Pi_1 = \Pi_0\Omega_1\Pi_0$ are d-compatible Poisson tensors.

Proof. We have $\Pi_0\Omega_0 = \Omega_0\Pi_0 = I$.

(i) The form $\Omega_0\Pi_1\Omega_0$ is closed since (Π_0, Π_1) are *d*-compatible. The forms (Ω_0, Ω_1) are *d*-compatible as the tensor

$$\Pi_0\Omega_1\Pi_0 = \Pi_0\Omega_0\Pi_1\Omega_0\Pi_0 = \Pi_1$$

is a Poisson tensor.

(ii) The tensor Π_1 is Poisson since (Ω_0, Ω_1) are *d*-compatible. The Poisson tensors (Π_0, Π_1) are *d*-compatible as the form

$$\Omega_0\Pi_1\Omega_0 = \Omega_0\Pi_0\Omega_1\Pi_0\Omega_0 = \Omega_1$$

is closed. □

What is important in the case considered is that the notions of *d*-compatibility and compatibility of Poisson tensors are equivalent. Actually, one can show (see for example [5]) that if $\Omega_0\Pi_1\Omega_0$ is closed (which means *d*-compatibility of $\Pi_0 = \Omega_0^{-1}$ and Π_1), then Π_0 and Π_1 are compatible and vice versa; if Π_0 and Π_1 are compatible, then $\Omega_0\Pi_1\Omega_0$ is closed and hence Π_0 and Π_1 are *d*-compatible [2].

4. *D*-compatibility for a degenerate case

Let us extend the notion of *d*-compatibility onto the degenerate case.

Definition 7. *A closed 2-form Ω_1 is d-compatible with a closed 2-form Ω_0 if there exists a Poisson tensor Π_0, dual to Ω_0, such that $\Pi_0\Omega_1\Pi_0$ is Poisson. Then we say that Ω_1 is d-compatible with Ω_0 with respect to Π_0.*

Definition 8. *A Poisson tensor Π_1 is d-compatible with a Poisson tensor Π_0 if there exists a presymplectic form Ω_0, dual to Π_0, such that $\Omega_0\Pi_1\Omega_0$ is closed. Then we say that Π_1 is d-compatible with Π_0 with respect to Ω_0.*

In the rest of this paper we restrict our considerations to the simplest case, when the dual pair considered is of co-rank 1 and our manifold $\mathcal{M}$ is of odd dimension $\dim \mathcal{M} = m = 2n+1$.

As was mentioned in the previous section, a presymplectic form dual to a given Poisson tensor is not unique. The set of all presymplectic forms dual to Π is parametrized by an arbitrary differentiable function on $\mathcal{M}$. Moreover, as Π is a Poisson tensor then an arbitrary element of its one-dimensional kernel has the form $\alpha = \mu \mathrm{d}H$, where μ is an arbitrary differentiable function on $\mathcal{M}$ and H is a Casimir function of Π.

Lemma 9. *Let Π be a fixed Poisson tensor and Ω be a dual presymplectic form. Assume that $\alpha = \mu\mathrm{d}H \in \ker \Pi$, $Z \in \ker \Omega$ and $\alpha(Z) = 1$. A presymplectic form Ω' is dual to Π if and only if*

$$\Omega' = \Omega + \mathrm{d}H \wedge \mathrm{d}F, \tag{4.1}$$

where F is an arbitrary differentiable function on $\mathcal{M}$.

Proof. First, observe that $Z' = Z + \frac{1}{\mu}\Pi\mathrm{d}F$ is an element of $\ker \Omega'$ and that $\mu Z'(H) = \mu Z(H) = 1$. Then,

$$\Pi\Omega' = \Pi\Omega - \Pi\mathrm{d}F \otimes \mathrm{d}H = I - \mu Z \otimes \mathrm{d}H - \Pi\mathrm{d}F \otimes \mathrm{d}H = I - \mu Z' \otimes \mathrm{d}H,$$

so Ω' is dual to Π.

Let Ω and Ω' be presymplectic forms dual to Π. Let $Z' \in \ker \Omega'$ and $\mu Z(H) = \mu Z'(H) = 1$. We have

$$\begin{aligned} \Pi\Omega &= I - \mu Z \otimes \mathrm{d}H. \\ \Pi\Omega' &= I - \mu Z' \otimes \mathrm{d}H. \end{aligned} \tag{4.2}$$

Multiplying (4.2) by Ω, we get

$$\Omega\Pi\Omega' = \Omega - \mu\Omega(Z') \otimes \mathrm{d}H.$$

Then, using the partition of unity, we find

$$(I - \mu \mathrm{d}H \otimes Z)\Omega' = \Omega - \mu\Omega(Z') \otimes \mathrm{d}H$$

and

$$\Omega' - \Omega = -\mu \mathrm{d}H \otimes \Omega'(Z) - \mu\Omega(Z') \otimes \mathrm{d}H.$$

Since $\Omega' - \Omega$ is a closed form, we have

$$\mu\Omega(Z') = -\mu\Omega'(Z) = \mathrm{d}F - Z(F)\alpha$$

and hence (4.1). □

We also have freedom in the choice of a Poisson tensor dual to a given 2-form. The set of all Poisson tensors dual to Ω is parametrized by an arbitrary vector field K which is both Hamiltonian and inverse Hamiltonian with respect to a dual pair.

Lemma 10. *Let Ω be a fixed presymplectic form and Π be a dual Poisson tensor. Assume that $Z \in \ker\Omega$, $\alpha \in \ker\Pi$ and $\alpha(Z) = 1$. Let K be a vector field such that*

$$K = \Pi \mathrm{d}F, \qquad \mathrm{d}F = \Omega K \qquad \Rightarrow \qquad Z(F) = 0, \qquad K(\alpha) = 0 \tag{4.3}$$

for some function F. Then, a Poisson tensor Π' is dual to Ω if and only if it has a form

$$\Pi' = \Pi + Z \wedge K. \tag{4.4}$$

Proof. First, we show that Π' is Poisson. Indeed, consider a Schouten bracket

$$[\Pi', \Pi']_S = -Z \wedge L_K\Pi + K \wedge L_Z\Pi - 2K \wedge [Z, K] \wedge Z.$$

Since $L_K\Pi = 0$, $L_Z\Pi = 0$ and $[Z, K] = 0$, we have $[\Pi', \Pi']_S = 0$. Let $\alpha = \mu \mathrm{d}H$; then observe that $\alpha' \in \ker\Pi'$ takes the form $\alpha' = \mu \mathrm{d}H' = \mu \mathrm{d}H + \mathrm{d}F$. Moreover, $\mu Z(H) = \mu Z(H') = 1$ and

$$\Pi'\Omega = \Pi\Omega - Z \otimes \Omega K = I - \mu Z \otimes \mathrm{d}H - Z \otimes \mathrm{d}F = I - \mu Z \otimes \mathrm{d}H',$$

so Π' is dual to Ω.

Let Π and Π' be Poisson tensors dual to Ω. Let $\mu \mathrm{d}H \in \ker\Pi$, $\mu \mathrm{d}H' \in \ker\Pi'$ and $\mu Z(H) = \mu Z(H') = 1$. Using the partition of unity, we get

$$\Omega\Pi = I - \mu \mathrm{d}H \otimes Z$$

and

$$\Omega\Pi' = I - \mu \mathrm{d}H' \otimes Z. \tag{4.5}$$

Multiplying equation (4.5) by Π, we get

$$\Pi\Omega\Pi' = \Pi - \mu(\Pi \mathrm{d}H') \otimes Z$$

and

$$(I - \mu Z \otimes \mathrm{d}H)\Pi' = \Pi - \mu(\Pi \mathrm{d}H') \otimes Z.$$

Transforming the above equality, we find

$$\Pi' = \Pi - \mu Z \otimes \Pi' \mathrm{d}H - \mu(\Pi \mathrm{d}H') \otimes Z.$$

As Π' is skew-symmetric, we can put $-\mu\Pi'\mathrm{d}H = \mu\Pi \mathrm{d}H' = K$, so $K = \Pi \mathrm{d}F$, $\Omega K = \mathrm{d}F$ and hence (4.4). □

Theorem 11. *Let a Poisson tensor Π_0 and a closed 2-form Ω_0 form a dual pair. Let $Y_0 \in \ker\Omega_0$, $\mu \mathrm{d}H_0 \in \ker\Pi_0$ and $\mu Y_0(H_0) = 1$.*

(i) If Π_1 is a Poisson tensor d-compatible with Π_0 with respect to Ω_0, then forms Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0$ are d-compatible.

(ii) If Ω_1 is a closed 2-form d-compatible with Ω_0 with respect to Π_0, then Poisson tensors Π_0 and $\Pi_1 = \Pi_0\Omega_1\Pi_0$ are d-compatible, provided that

$$\mu\Pi_0\Omega_1 Y_0 = \Pi_0\,\mathrm{d}F \tag{4.6}$$

for some function F.

Proof.

(i) Ω_1 is closed as Π_1 is d-compatible with Π_0. Then, $\Pi_0\Omega_1\Pi_0 = \Pi_0\Omega_0\Pi_1\Omega_0\Pi_0$ is Poisson (as was shown in [2]).

(ii) From the d-compatibility of Ω_0 and Ω_1, it follows that Π_1 is Poisson. Then,

$$\begin{aligned}\Omega_0\Pi_1\Omega_0 &= \Omega_0\Pi_0\Omega_1\Pi_0\Omega_0 = (I - \mu\mathrm{d}H_0 \otimes Y_0)\Omega_1(I - \mu Y_0 \otimes \mathrm{d}H_0)\\ &= \Omega_1 + \mu\mathrm{d}H_0 \wedge \Omega_1(Y_0).\end{aligned}$$

From the assumption $\Pi_0\Omega_1\mu Y_0 = \Pi_0\,\mathrm{d}F$, it follows that either

$$\Omega_1(\mu Y_0) = \mathrm{d}F \quad \text{if} \quad Y_0(F) = 0$$

or

$$\Omega_1(\mu Y_0) = \mathrm{d}F - \mu Y_0(F)\,\mathrm{d}H_0 \quad \text{if} \quad Y_0(F) \neq 0.$$

In both cases, $\Omega_0\Pi_1\Omega_0 = \Omega_1 + \mathrm{d}H_0 \wedge \mathrm{d}F$ is closed. □

Theorem 12. *Let a Poisson tensor Π_0 and a closed 2-form Ω_0 form a dual pair. Let $Y_0 \in$* ker Ω_0, $\mu\mathrm{d}H_0 \in$ ker Π_0 *and $\mu Y_0(H_0) = 1$.*

(i) If Π_1 is a Poisson tensor d-compatible with Π_0 with respect to Ω_0 and

$$X = \Pi_1\,\mathrm{d}H_0 = \Pi_0\,\mathrm{d}H_1 \tag{4.7}$$

is a bi-Hamiltonian vector field, then Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0 + \mathrm{d}H_1 \wedge \mathrm{d}H_0$ are a d-compatible pair of presymplectic forms.

(ii) If Ω_1 is a presymplectic form d-compatible with Ω_0 with respect to Π_0 and

$$\beta = \mu\Omega_0 Y_1 = \mu\Omega_1 Y_0 \tag{4.8}$$

is a bi-presymplectic 1-form, then Π_0 and $\Pi_1 = \Pi_0\Omega_1\Pi_0 + X \wedge \mu Y_0$ are d-compatible Poisson tensors if there exist some functions F and G such that

$$\mu\Pi_0\Omega_0 Y_1 = \Pi_0\,\mathrm{d}F, \qquad \mu\Pi_0\Omega_1 Y_1 = \Pi_0\,\mathrm{d}G, \tag{4.9}$$

where $X = \Pi_0\beta = \Pi_0\,\mathrm{d}F$.

Proof. (i) Ω_1 is closed as Π_1 is d-compatible with Π_0. Then, $\Pi_0\Omega_1\Pi_0 = \Pi_0\Omega_0\Pi_1\Omega_0\Pi_0$ is Poisson (as was shown in [2]).

(ii) From (4.9), it follows that either $Y_0(F) \neq 0$, $Y_0(G) \neq 0$ and

$$\begin{aligned}&\mu\Omega_0 Y_1 = \mathrm{d}F - \mu Y_0(F)\,\mathrm{d}H_0, \qquad \mu\Omega_1 Y_1 = \mathrm{d}G - \mu Y_0(G)\,\mathrm{d}H_0,\\ &\mu Y_1 = X + \mu^2 Y_0(F) Y_0,\end{aligned}$$

or $Y_0(F) = Y_0(G) = 0$ and

$$\mu Y_1 = X, \qquad \mu\Omega_0 Y_1 = \Omega_0 X = \mathrm{d}F, \qquad \mu\Omega_1 Y_1 = \Omega_1 X = \mathrm{d}G.$$

By part (ii) of the previous theorem, the form $\Omega_0\Pi_1\Omega_0 = \Omega_0\Pi_0\Omega_1\Pi_0\Omega_0$ is closed. Let us prove that Π_1 is a Poisson tensor. We show that the Schouten bracket of Π_1 is zero. First, observe that

$$[\Pi_1, \Pi_1]_S = 2[\Pi_0\Omega_1\Pi_0, X \wedge \mu Y_0]_S + [X \wedge \mu Y_0, X \wedge \mu Y_0]_S,$$

as by previous theorem $[\Pi_0\Omega_1\Pi_0, \Pi_0\Omega_1\Pi_0]_S = 0$. Next,

$$[\Pi_0\Omega_1\Pi_0, X \wedge \mu Y_0]_S = \mu Y_0 \wedge \Pi_0\, \mathrm{d}(\Omega_1 X)\Pi_0 - X \wedge \Pi_0\, \mathrm{d}(\Omega_1\mu Y_0)\Pi_0$$

and

$$[X \wedge \mu Y_0, X \wedge \mu Y_0]_S = 2X \wedge \mu Y_0 \wedge [\mu Y_0, X].$$

In the case when $\Omega_0 X = \mathrm{d}F$ and $\Omega_1 X = \mathrm{d}G$, we have $[\mu Y_0, X] = -X(\mu)Y_0$ and the proof is completed. In the second case,

$$\begin{aligned}[\mu Y_0, X] &= [\mu Y_0, \Pi_0\Omega_1\mu Y_0] = L_{\mu Y_0}(\Pi_0\Omega_1)\mu Y_0 = \Pi_0(L_{\mu Y_0}\Omega_1)\mu Y_0 - (\Pi_0\, \mathrm{d}\mu \wedge Y_0)\beta \\ &= \Pi_0\, \mathrm{d}(\Omega_1\mu Y_0)\mu Y_0 + \beta(\Pi_0\, \mathrm{d}\mu)Y_0 = \Pi_0(\mathrm{d}\beta)\mu Y_0 + \beta(\Pi_0\, \mathrm{d}\mu)Y_0 \\ &= -\Pi_0\, \mathrm{d}(\mu Y_0(F)) + \beta(\Pi_0\, \mathrm{d}\mu)Y_0.\end{aligned}$$

Also,

$$\mu\Omega_1 Y_1 = \Omega_1 X + \mu Y_0(F)\beta;$$

hence

$$\Omega_1 X = \mathrm{d}G - \mu Y_0(F)\, \mathrm{d}F + [\mu Y_0(F)]^2\, \mathrm{d}H_0 - \mu Y_0(G)\, \mathrm{d}H_0.$$

So,

$$\Pi_0\, \mathrm{d}(\Omega_1 X)\Pi_0 = -\Pi_0\, \mathrm{d}(\mu Y_0(F)) \wedge X.$$

Finally,

$$\Pi_0\, \mathrm{d}(\Omega_1\mu Y_0)\Pi_0 = \Pi_0\, \mathrm{d}\beta\Pi_0 = 0$$

and the proof is completed. □

5. Bi-presymplectic chains

Now we are ready to present the main result of the paper.

Theorem 13. *Assume that on $\mathcal{M}$, we have a bi-presymplectic chain of 1-forms:*

$$\beta_i = \mu\Omega_0 Y_i = \mu\Omega_1 Y_{i-1}, \qquad i = 1, 2, \ldots, n, \tag{5.1}$$

with a d-compatible pair (Ω_0, Ω_1) with respect to some Π_0, which starts with a kernel vector field Y_0 of Ω_0 and terminates with a kernel vector field Y_n of Ω_1, where μ is an arbitrary function. Then,

(i)

$$\Omega_0(Y_i, Y_j) = \Omega_1(Y_i, Y_j) = 0, \qquad i = 1, 2, \ldots, n. \tag{5.2}$$

Moreover, let us assume that

$$\Pi_0\beta_i = X_i = \Pi_0\, \mathrm{d}H_i, \qquad i = 1, 2, \ldots, n, \tag{5.3}$$

which implies

$$\beta_i = \mathrm{d}H_i - \mu Y_0(H_i)\, \mathrm{d}H_0, \qquad \mu Y_i = X_i + \mu^2 Y_i(H_0)Y_0, \tag{5.4}$$

where $\Pi_0\, \mathrm{d}H_0 = 0$. Then,

(ii)

$$\Pi_0(\mathrm{d}H_i, \mathrm{d}H_j) = 0 \qquad [X_i, X_j] = 0 \tag{5.5}$$

and equation (5.1) defines a Liouville integrable system.

Additionally, if $Y_i(H_0) = Y_0(H_i)$, then

(iii) Hamiltonian vector fields X_i (5.3) form a bi-Hamiltonian chain:

$$X_i = \Pi_0\,\mathrm{d}H_i = \Pi_1\,\mathrm{d}H_{i-1}, \qquad i = 1, 2, \ldots, n, \tag{5.6}$$

where $\Pi_1 = \Pi_0\Omega_1\Pi_0 + X_1 \wedge \mu Y_0$. The chain starts with H_0, a Casimir of Π_0, and terminates with H_n, a Casimir of Π_1.

Proof.

(i) From (5.1), we have

$$\Omega_0(Y_i, Y_j) = \Omega_0(Y_{i-1}, Y_{j+1})$$
$$\Omega_1(Y_i, Y_j) = \Omega_0(Y_{i+1}, Y_{j-1}).$$

Then (5.2) follows from

$$\Omega_0(Y_i, Y_0) = 0 \qquad \Omega_1(Y_i, Y_n) = 0.$$

(ii) From the properties of the dual pair (Π_0, Ω_0), if $X_i = \Pi_0\,\mathrm{d}H_i$, then

$$\Pi_0(\mathrm{d}H_i, \mathrm{d}H_j) = \Omega_0(X_i, X_j).$$

On the other hand, as $X_i = \mu Y_i - \alpha_i Y_0$ it follows that

$$\Omega_0(X_i, X_j) = \Omega_0(Y_i, Y_j).$$

(iii) We have

$$\begin{aligned} X_i = \Pi_0\,\mathrm{d}H_i = \mu\Pi_0\Omega_1 Y_{i-1} &= \Pi_0\Omega_1(X_{i-1} + \mu^2 Y_0(H_{i-1})Y_0) \\ &= \Pi_0\Omega_1\Pi_0\,\mathrm{d}H_{i-1} + \mu Y_0(H_{i-1})X_1 \\ &= (\Pi_0\Omega_1\Pi_0 + X_1 \wedge \mu Y_0)\,\mathrm{d}H_{i-1} = \Pi_1\,\mathrm{d}H_{i-1}. \end{aligned}$$

From theorem 12, we know that Π_1 is a Poisson tensor d-compatible with Π_0. We have

$$\begin{aligned} \Pi_1\,\mathrm{d}H_n &= (\Pi_0\Omega_1\Pi_0 + X_1 \wedge \mu Y_0)\,\mathrm{d}H_n = \Pi_0\Omega_1 X_n + \mu Y_0(H_n)X_1 \\ &= \mu\Pi_0\Omega_1(Y_n - \mu Y_0(H_n)Y_0) + \mu Y_0(H_n)X_1 = -\mu Y_0(H_n)X_1 + \mu Y_0(H_n)X_1 = 0. \end{aligned}$$

□

A simple example of a bi-presymplectic chain and its equivalent bi-Hamiltonian representation was given in [2] where the extended Henon–Heiles system on $\mathbb{R}^5$ was considered. Actually, it is the system with Hamiltonians

$$\begin{aligned} H_1 &= \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + q_1^3 + \tfrac{1}{2}q_1 q_2^2 - cq_1, \\ H_2 &= \tfrac{1}{2}q_2 p_1 p_2 - \tfrac{1}{2}q_1 p_2^2 + \tfrac{1}{16}q_2^4 + \tfrac{1}{4}q_1^2 q_2^2 - \tfrac{1}{4}cq_2^2, \end{aligned} \tag{5.7}$$

where (q, p) are canonical coordinates and c is a Casimir coordinate. We will come back to this example in the end of this section.

Note that theorem 13 holds in an important special case when (5.1) is *bi-inverse Hamiltonian*, i.e. $\beta_i = \mathrm{d}H_i$, $Y_0(H_i) = 0$, $i = 1, \ldots, n$. Obviously, it does not have a bi-Hamiltonian counterpart until $\gamma_i \equiv Y_i(H_0) \neq 0$, but has equivalent quasi-bi-Hamiltonian representation on $2n$ dimensional manifold M. Indeed, as $\beta_i = \mathrm{d}H_i$,

$$\Pi_0\,\mathrm{d}H_i = \Pi_0\Omega_1\mu Y_{i-1} = \Pi_0\Omega_1(X_{i-1} + \gamma_i\mu^2 Y_0) = \Pi_0\Omega_1\Pi_0\,\mathrm{d}H_{i-1} + \gamma_i\Pi_0\,\mathrm{d}H_1.$$

Note that both Poisson structures Π_0 and $\Pi_0\Omega_1\Pi_0$ share the same Casimir H_0 and all Hamiltonians H_i are independent of the Casimir coordinate $H_0 = c$, so the quasi-bi-Hamiltonian dynamics can be restricted immediately to any common leaf M of dimension $2n$:

$$\pi_0\,\mathrm{d}H_i = \pi_1\,\mathrm{d}H_{i-1} + \gamma_i\pi_0\,\mathrm{d}H_1, \qquad i = 1,\ldots,n, \tag{5.8}$$

where

$$\pi_0 = \Pi_0|_M, \qquad \pi_1 = (\Pi_0\Omega_1\Pi_0)|_M$$

are restrictions of respective Poisson structures to M. Hence, we deal with a Stäckel system whose separation coordinates are eigenvalues of the recursion operator $N = \pi_1\pi_0^{-1}$ [12], provided that N has n distinct and functionally independent eigenvalues at any point of M, i.e. we are in a generic case.

The advantage of bi-inverse-Hamiltonian representation when compared to bi-Hamiltonian ones is that the existence of the first guarantees that the related Liouville integrable system is separable and the construction of separation coordinates is purely algorithmic (in a generic case), while the bi-Hamiltonian representation does not guarantee the existence of quasi-bi-Hamiltonian representation and hence separability of the related system. Moreover, the projection of the second Poisson structure onto the symplectic foliation of the first one, in order to construct a quasi-bi-Hamiltonian representation, is far from being a trivial non-algorithmic procedure.

Let us illustrate the case on the example of the Henon–Heiles system on $\mathbb{R}^4$ given by two constants of motion:

$$H_1 = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + q_1^3 + \tfrac{1}{2}q_1q_2^2, \qquad H_2 = \tfrac{1}{2}q_2p_1p_2 - \tfrac{1}{2}q_1p_2^2 + \tfrac{1}{16}q_2^4 + \tfrac{1}{4}q_1^2q_2^2. \tag{5.9}$$

On $\mathbb{R}^5$, differentials $\mathrm{d}H_1$ and $\mathrm{d}H_2$ have bi-inverse-Hamiltonian representation of the form

$$\begin{aligned}
&\Omega_0Y_0 = 0\\
&\Omega_0Y_1 = \mathrm{d}H_1 = \Omega_1Y_0\\
&\Omega_0Y_2 = \mathrm{d}H_2 = \Omega_1Y_1\\
&0 = \Omega_1Y_2,
\end{aligned}$$

where $\mu = 1$, vector fields Y_i are

$$\begin{aligned}
Y_0 &= (0,0,0,0,1)^T\\
Y_1 &= X_1 + Y_1(H_0)Y_0 = \left(p_1, p_2, -3q_1^2 - \tfrac{1}{2}q_2^2, -q_1q_2, -q_1\right)^T\\
Y_2 &= X_2 + Y_2(H_0)Y_0 = \left(\tfrac{1}{2}q_2p_2, \tfrac{1}{2}q_2p_1 - q_1p_1, \tfrac{1}{2}p_2^2 - \tfrac{1}{2}q_1q_2^2,\right.\\
&\qquad \left.-\tfrac{1}{2}p_1p_2 - \tfrac{1}{4}q_2^3 - \tfrac{1}{2}q_1^2q_2, -\tfrac{1}{4}q_2^2\right)^T
\end{aligned}$$

and presymplectic forms

$$\Omega_0 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega_1 = \begin{pmatrix} 0 & -\tfrac{1}{2}p_2 & -q_1 & -\tfrac{1}{2}q_2 & 3q_1^2 + \tfrac{1}{2}q_2^2\\ \tfrac{1}{2}p_2 & 0 & -\tfrac{1}{2}q_2 & 0 & q_1q_2\\ q_1 & \tfrac{1}{2}q_2 & 0 & 0 & p_1\\ \tfrac{1}{2}q_2 & 0 & 0 & 0 & p_2\\ -3q_1^2 - \tfrac{1}{2}q_2^2 & -q_1q_2 & -p_1 & -p_2 & 0 \end{pmatrix}$$

are d-compatible with respect to the canonical Poisson tensor dual to the Ω_0 one. The chain starts with a kernel vector field Y_0 of Ω_0 and terminates with a kernel vector field Y_2 of Ω_1. On $\mathbb{R}^4$, we have

$$\omega_0 = \Omega_0|_{\mathbb{R}^4} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \omega_1 = \Omega_1|_{\mathbb{R}^4} \begin{pmatrix} 0 & -\frac{1}{2}p_2 & -q_1 & -\frac{1}{2}q_2 \\ \frac{1}{2}p_2 & 0 & -\frac{1}{2}q_2 & 0 \\ q_1 & \frac{1}{2}q_2 & 0 & 0 \\ \frac{1}{2}q_2 & 0 & 0 & 0 \end{pmatrix}$$

and the quasi-bi-Hamiltonian representation takes form (5.8), where

$$\pi_0 = \Pi_0|_{\mathbb{R}^4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \omega_0^{-1},$$

$$\pi_1 = \Pi_0\Omega_1\Pi_0|_{\mathbb{R}^4} = \begin{pmatrix} 0 & 0 & q_1 & \frac{1}{2}q_2 \\ 0 & 0 & \frac{1}{2}q_2 & 0 \\ -q_1 & -\frac{1}{2}q_2 & 0 & \frac{1}{2}p_2 \\ -\frac{1}{2}q_2 & 0 & -\frac{1}{2}p_2 & 0 \end{pmatrix} = \pi_0\omega_1\pi_0,$$

$\gamma_1 = -q_1$ and $\gamma_2 = -\frac{1}{4}q_2^2$. Separation coordinates (λ_1, λ_2), which are eigenvalues of the recursion operator $N = \pi_1\pi_0^{-1} = \omega_0^{-1}\omega_1$, are related to (q_1, q_2) coordinates by the following point transformation:

$$q_1 = \lambda_1 + \lambda_2, \qquad \frac{1}{4}q_2^2 = -\lambda_1\lambda_2.$$

Obviously, Hamiltonians (5.9) do not form a related bi-Hamiltonian chain in contrast to Hamiltonians (5.7).

6. Poisson and presymplectic structures in $\mathbb{R}^3$

In this section, we consider the Poisson and presymplectic structures in $\mathbb{R}^3$. In this case, we have a convenient description of the Poisson tensors and presymplectic forms and can obtain simple conditions for compatibility. In $\mathbb{R}^3$, all Poisson tensors are described by the following theorem [1].

Theorem 14. *Any Poisson tensor Π in $\mathbb{R}^3$, except at some irregular points, has the form*

$$\Pi^{ij} = \mu\epsilon^{ijk}\partial_k H. \tag{6.1}$$

Here μ and H are some differentiable functions in $\mathbb{R}^3$ and ϵ^{ijk} is a Levi-Civita symbol.

Note that for the above Poisson tensor, we have $\Pi\mathrm{d}H = 0$, that is, the kernel of Π is spanned by the form $\mathrm{d}H$. To have consistency, we chose the function μ in (6.1) the same as that used in (5.1). The compatible Poisson tensors in $\mathbb{R}^3$ are characterized by the following theorem [1].

Theorem 15. *Let Poisson tensors Π_0 and Π_1 be given by $(\Pi_0)^{ij} = \mu_0\epsilon^{ijk}\partial_k H_0$ and $(\Pi_1)^{ij} = \mu_1\epsilon^{ijk}\partial_k H_1$, respectively, where μ_0, μ_1 and H_0, H_1 are some differentiable functions. Then Π_0 and Π_1 are compatible if and only if there exists a differentiable function $\Phi(H_0, H_1)$ such that*

$$\mu_1 = \mu_0 \frac{\partial_{H_1}\Phi}{\partial_{H_0}\Phi} \tag{6.2}$$

provided that $\partial_{H_1}\Phi = \partial\Phi/\partial H_1 \neq 0$ and $\partial_{H_0}\Phi = \partial\Phi/\partial H_0 \neq 0$.

For example, from the above theorem it follows that a Poisson tensor Π_0, given by μ and a function H_0, and a Poisson tensor Π_1, given by $-\mu$ and a function H_1, are compatible. One should take $\Phi = H_0 - H_1$. The presymplectic forms in $\mathbb{R}^3$ are described by the following lemma.

Lemma 16. *Any closed 2-form Ω in $\mathbb{R}^3$ has the form*

$$\Omega_{ij} = \epsilon_{ijk} Y^k, \tag{6.3}$$

where $Y = (Y^1, Y^2, Y^3)^T$ is a divergence free vector:

$$\nabla \cdot Y = \partial_i Y^i = 0. \tag{6.4}$$

Note that for the above presymplectic form, we have $\Omega Y = 0$, that is, the kernel of Ω is spanned by the vector Y. Next, let us consider a dual pair.

Lemma 17. *Consider a Poisson tensor Π, $\Pi^{ij} = \mu\epsilon^{ijk}\partial_k H$, and a presymplectic form Ω, $\Omega_{ij} = \epsilon_{ijk} Y^k$. Then (Π, Ω) is a dual pair if and only if*

$$\mu Y(H) = \mu Y^i \partial_i H = 1. \tag{6.5}$$

Proof. The form Ω is dual to the Poisson tensor Π if the following partition of the unit operator holds:

$$I = \Pi\Omega + \mu Y \otimes \mathrm{d}H.$$

The above equality is equivalent to (6.5). □

We have a simple condition for compatibility of a Poisson tensor and a presymplectic form.

Lemma 18. *The Poisson tensors Π, given by $(\Pi)^{ij} = \mu\epsilon^{ijk}\partial_k H$, and the presymplectic form Ω, given by $(\Omega)_{ij} = \epsilon_{ijk} Y^k$, are compatible if*

$$Y(\mu[Y(H)]) = Y^i \partial_i (\mu Y(H)) = 0. \tag{6.6}$$

Proof. We have

$$\Omega\Pi\Omega = \mu Y(H)\Omega.$$

The above form is given in terms of a vector $Y(H)Y$. It is closed if

$$\nabla \cdot (\mu Y(H)Y) \equiv Y(\mu Y(H)) = 0.$$

Since $\nabla \cdot Y = 0$, the above equation is equivalent to (6.6). □

As a corollary of the previous lemma, we have the condition for the *d*-compatibility of two Poisson tensors.

Lemma 19. *Consider a dual pair (Π_0, Ω_0) where the Poisson tensor Π_0 is given by $(\Pi_0)^{ij} = \mu\epsilon^{ijk}\partial_k H_0$ and the presymplectic form Ω_0 is given by $(\Omega_0)_{ij} = \epsilon_{ijk} Y_0^k$. Then the Poisson tensor Π_1, $(\Pi_1)^{ij} = -\mu\epsilon^{ijk}\partial_k H_1$, is d-compatible with the Poisson tensor Π_0 if*

$$Y_0(\mu Y_0(H_1)) = 0. \tag{6.7}$$

The condition for *d*-compatibility of two presymplectic forms in $\mathbb{R}^3$ is given in the following lemma.

Lemma 20. *Consider a dual pair* (Π_0, Ω_0) *where the Poisson tensor* Π_0 *is given by* $(\Pi_0)^{ij} = \mu\epsilon^{ijk}\partial_k H_0$ *and the presymplectic form* Ω_0 *is given by* $(\Omega_0)_{ij} = \epsilon_{ijk}Y_0^k$*. Then the presymplectic form* Ω_1, $(\Omega_1)_{ij} = \epsilon_{ijk}Y_1^k$*, is d-compatible with the presymplectic form* Ω_0 *if*

$$Y_1(H_0) \neq 0. \tag{6.8}$$

Proof. We have

$$\Pi_0\Omega_1\Pi_0 = \mu Y_1(H_0)\Pi_0.$$

Since Π_0 is a Poisson tensor, the above tensor is a Poisson tensor if $Y_1(H_0) \neq 0$. □

It turns out that in $\mathbb{R}^3$, any two forms and any two Poisson tensors are *d*-compatible.

Lemma 21. *Let* Ω_0, Ω_1 *be two presymplectic forms in* $\mathbb{R}^3$*, given by* $(\Omega_0)_{ij} = \epsilon_{ijk}Y_0^k$ *and* $(\Omega_1)_{ij} = \epsilon_{ijk}Y_1^k$*. Then* Ω_0 *and* Ω_1 *are d-compatible presymplectic forms.*

Proof. Take a function H_0 such that $Y_0(H_0) \neq 0$ and $Y_1(H_0) \neq 0$. Define a Poisson tensor Π_0 by $\Pi_0^{ij} = [Y_0(H_0)]^{-1}\epsilon^{ijk}\partial_k H_0$. Then by lemma 17, Π_0 and Ω_0 are dual and by lemma 20, the forms Ω_0 and Ω_1 are *d*-compatible. □

Lemma 22. *Let* Π_0, Π_1 *be two Poisson tensors in* $\mathbb{R}^3$*, given by* $(\Pi_0)^{ij} = \mu\epsilon^{ijk}\partial_k H_0$ *and* $(\Pi_1)^{ij} = -\mu\epsilon^{ijk}\partial_k H_1$*. Then* Π_0 *and* Π_1 *are d-compatible Poisson tensors.*

Proof. By the Darboux theorem, we can find the coordinates (t_1, t_2, t_3) such that Π_1 is given by $\mu_1 = 1$ and $H_1 = t_1$. We can construct a closed form Ω_0, $(\Omega_0)_{ij} = \epsilon_{ijk}Y_0^k$, dual to Π_0 and such that $\partial_1 Y_0^1 = 0$. Then

$$Y_0(\mu_1 Y_0(H_1)) = Y_0\left(Y_0^1\right) = 0,$$

so Ω_0 and Π_1 are compatible. That is, Π_0 and Π_1 are *d*-compatible. Such a form Ω_0 can be constructed as follows. Consider the coordinate change

$$u_1 = t_1, u_2 = t_2, u_3 = H_0(t_1, t_2, t_3).$$

In these coordinates, Π_0 is given by some $\tilde{\mu}_0$ and $\tilde{H}_0 = u_3$. Note that if a form is given by vector $\tilde{Y} = (A, B, C)^t$ in the (u_1, u_2, u_3) coordinates, then it is given by a vector $Y = (A\partial_3 H_0, B\partial_3 H_0, C - A\partial_1 H_0 - B\partial_2 H_0)$ in the (t_1, t_2, t_3) coordinates. We construct Ω_0 in the (u_1, u_2, u_3) coordinates in terms of the vector $\tilde{Y}_0 = (A, B, C)^t$. First, we choose $C = (\tilde{\mu})^{-1}$, so $\tilde{\mu}Y_0(\tilde{H}_0) = 1$. Hence, Π_0 and Ω_0 are dual. Then we choose A such that $A\partial_3 H_0$ does not depend on t_1 in the (t_1, t_2, t_3) coordinates, so Π_1 and Ω_0 are compatible. Then we choose B such that $\partial_1 A + \partial_2 B + \partial_3 C = 0$, so Ω_0 is closed. □

7. Bi-presymplectic chains in $\mathbb{R}^3$

Consider closed 2-forms Ω_0 and Ω_1 in some open domain of $\mathbb{R}^3$, given in terms of vectors Y_0 and Y_1 by

$$\Omega_{0,ij} = \epsilon_{ijk}Y_0^k \qquad \text{where} \qquad \partial_k Y_0^k = 0, \qquad i, j = 1, 2, 3,$$

and

$$\Omega_{1,ij} = \epsilon_{ijk}Y_1^k \qquad \text{where} \qquad \partial_k Y_1^k = 0, \qquad i, j = 1, 2, 3.$$

By lemma 21, there exists a Poisson tensor Π_0 such that Π_0 and Ω_0 are dual and Ω_0 and Ω_1 are *d*-compatible with respect to Π_0. We can choose a function H_0 such that $\mu Y_0(H_0) = 1$

and $Y_1(H_0) \neq 0$, so $\Pi_0^{ij} = \mu\epsilon^{ijk}\partial_k H_0$. It is easy to see that in $\mathbb{R}^3$, any two presymplectic forms Ω_0 and Ω_1 give a bi-presymplectic chain:

$$\begin{aligned} &\Omega_0 Y_0 = 0 \\ &\mu\Omega_0 Y_1 = \beta = \mu\Omega_1 Y_0 \\ &0 = \Omega_1 Y_1. \end{aligned} \tag{7.1}$$

Then, we can consider a vector field X:

$$X = \Pi_0\beta. \tag{7.2}$$

To construct bi-Hamiltonian representation of the above chain, we use theorem 13. Let chain (7.1) be such that

$$\Pi_0\beta = X = \Pi_0\,\mathrm{d}H_1 \tag{7.3}$$

and hence

$$\beta = \mathrm{d}H_1 - \mu Y_0(H_1)\,\mathrm{d}H_0. \tag{7.4}$$

Then, by theorem 13 (ii), the vector field X defines a Liouville integrable system.

Let us obtain some relations that we will need later. Combining (7.1) and (7.4), we have

$$\mu\epsilon_{ijk}Y_0^k Y_1^j = H_{1,i} - \mu Y_0(H_1)H_{0,i}, \qquad i = 1, 2, 3,$$

that gives

$$Y_0(H_1) - \mu Y_0(H_1)Y_0(H_0) = 0$$

and

$$Y_1(H_1) = \mu Y_0(H_1)Y_1(H_0).$$

Using duality of Ω_0 and Π_0, we have

$$\mu Y_1^n = \mu^2 Y_1(H_0)Y_0^n + X^n, \qquad n = 1, 2, 3. \tag{7.5}$$

Note that if $Y_0(H_1) = 0$, then β is closed and $Y_1(H_1) = 0$. So,

$$Y_0(H_1) = Y_1(H_1) = 0. \tag{7.6}$$

Following [1], every Hamiltonian system in $\mathbb{R}^3$ has a bi-Hamiltonian representation. Thus the vector field $X = \Pi_0\,\mathrm{d}H_1$ can also be written as $X = \bar{\Pi}_1\,\mathrm{d}H_0$, where $(\bar{\Pi}_1)^{ij} = -\mu\epsilon^{ijk}\partial_k H_1$ for $i, j = 1, 2, 3$.

Theorem 13 also gives the bi-Hamiltonian representation of the vector field X. Let us show that these two representations coincide. Let $Y_0(H_1) = Y_1(H_0)$; then by theorem 13 (iii), we can define

$$\Pi_1 = \Pi_0\Omega_1\Pi_0 + \mu X \wedge Y_0, \tag{7.7}$$

that is,

$$\Pi_1^{ij} = -\mu^2 Y_1(H_0)\epsilon^{ijk}\partial_k H_0 + \mu\left(X^i Y_0^j - X^j Y_0^i\right), \quad i,\ j = 1, 2, 3.$$

Since $X^i = \epsilon^{ijk}\Pi_0^k H_{1,k}$, we can put

$$X^i Y_0^j - X^j Y_0^i = \epsilon^{ijk} W_k, \qquad i, j = 1, 2, 3.$$

So,

$$\Pi_1^{ij} = -\mu^2 Y_1(H_0)\epsilon^{ijk}\partial_k H_0 + \mu\epsilon^{ijk}W_k = \epsilon^{ijk}(-\mu^2 Y_1(H_0)\partial_k H_0 + \mu W_k),$$

for all $i, j = 1, 2, 3$. Since Π_1 is a Poisson tensor and $\mathrm{d}H_1$ belongs to the kernel of Π_1 we have

$$-\mu^2 Y_1(H_0)\partial_k H_0 + \mu W_k = -\mu\partial_k H_1, \tag{7.8}$$

where μ is an arbitrary function. For W_k, we have

$$\begin{aligned} W_k = \epsilon^{ijk} X^i Y_0^k = \mu\epsilon^{ijk}\epsilon^{imn} H_{0,n} H_{1,m} Y_0^j &= \mu\left(\delta_i^n\delta_n^k - \delta_m^k\delta_j^n\right) H_{0,n} H_{1,m} Y_0^j \\ = \mu Y_0(H_1) H_{0,k} - \mu Y_0(H_0) H_{1,k}, &\quad k = 1, 2, 3, \end{aligned}$$

where $H_{0,k} = \partial_k H_0$ and $H_{1,k} = \partial_k H_1$. Using the above equality for W_k in (7.8), we get

$$-\mu^2 Y_1(H_0)\partial_k H_0 + \mu^2 Y_0(H_1) H_{0,k} - \mu H_{1,k} = -\mu H_{1,k}, \quad k = 1, 2, 3,$$

which gives

$$Y_1(H_0) = Y_0(H_1). \tag{7.9}$$

Equations (7.9) and (7.5) are the only constraints on Y_0 and Y_1 respectively. We conclude that any presymplectic chain which fulfills condition (7.3) leads to a bi-Hamiltonian chain.

As the next example shows, there exist presymplectic chains that do not admit a dual bi-Hamiltonian representation.

Example 23. Consider closed 2-forms Ω_0 and Ω_1 in $\mathbb{R}^3$, given by

$$\Omega_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Omega_1 = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix},$$

where a, b and c are the functions of x_1, x_2 and x_3 respectively. Their kernels are spanned by vectors $Y_0 = (0, 0, 1)^t$ and $Y_1 = (a, b, c)^t$ respectively. Since $\nabla \cdot Y_1 = 0$, we then have

$$\partial_1 a + \partial_2 b + \partial_3 c = 0.$$

We take a Poisson tensor Π_0 in the form

$$\Pi_0 = \mu \begin{pmatrix} 0 & H_{0,3} & -H_{0,2} \\ -H_{0,3} & 0 & H_{0,1} \\ H_{0,2} & -H_{0,1} & 0 \end{pmatrix},$$

where μ and H_0 are arbitrary functions of x^1, x^2 and x^3. If $\mu H_{0,3} = 1$, then one can easily show that Π_0 and Ω_0 are dual and Ω_0 and Ω_1 are d-compatible with respect to Π_0. The forms Ω_0 and Ω_1 make a presymplectic chain:

$$\begin{aligned} &\Omega_0 Y_0 = 0 \\ &\mu\Omega_0 Y_1 = \beta = \mu\Omega_1 Y_0 \\ &0 = \Omega_1 Y_1, \end{aligned} \tag{7.10}$$

where $\beta = \mu(b, -a, 0)^t$. Consider a vector field X:

$$X = \Pi_0\beta = \mu(a, b, 0)^t.$$

We find that an additional condition

$$X = \Pi_0\,\mathrm{d}H_1$$

gives

$$a = H_{0,3}H_{1,2} - H_{0,2}H_{1,3}, \tag{7.11}$$

$$b = -H_{0,3}H_{1,1} + H_{0,1}H_{1,3}, \tag{7.12}$$

$$\mu(aH_{0,1} + bH_{0,2}) = H_{0,1}H_{1,2} - H_{0,2}H_{1,1}, \tag{7.13}$$

and from constraint (7.9) we get

$$H_{1,3} = aH_{0,1} + bH_{0,2} + cH_{0,3}. \tag{7.14}$$

Using a and b from equations (7.11) and (7.12) respectively, we show that (7.13) is identically satisfied. Using $\mu H_{0,3} = 1$ and identity (7.13) in (7.14), we get

$$c = \mu H_{1,3} - H_{0,1}H_{1,2} + H_{0,2}H_{1,1}. \tag{7.15}$$

As a summary, we are left with equations (7.11), (7.12), (7.15) for a, b and c and the duality condition $\mu H_{0,3} = 1$. When we use a, b and c in (7.10), we obtain that

$$(\mu H_{1,3})_{,3} = 0. \tag{7.16}$$

This is nothing else but the d-compatibility condition (6.7), i.e. $Y_0(\mu Y_0(H_1)) = 0$, of the Poisson tensors Π_0 and Π_1. Equation (7.16) means that

$$H_1 = h_1(x^1, x^2)H_0 + h_2(x^1, x^2), \tag{7.17}$$

where h_1 and h_2 are arbitrary functions of x^1 and x^2 respectively. Using (7.17), we get

$$a = (h_{1,2}H_0 + h_{2,2})H_{0,3}, \tag{7.18}$$

$$b = -(h_{1,1}H_0 + h_{2,1})H_{0,3}, \tag{7.19}$$

$$c = h_1 - (h_{1,2}H_0 + h_{2,2})H_{0,1} + (h_{1,1}H_0 + h_{2,1})H_{0,2}. \tag{7.20}$$

The above equations might be considered as differential equations to determine H_0, h_1 and h_2 with no conditions on a, b and c. When we use (7.18) and (7.19), we find that

$$H_0 = -\frac{ah_{2,1} + bh_{2,2}}{ah_{1,1} + bh_{1,2}}, \qquad H_{0,3} = \frac{ah_{1,1} + bh_{1,2}}{h_{1,1}h_{2,2} - h_{1,2}h_{2,1}}. \tag{7.21}$$

These equations put a constraint on the x^3 dependence on the given functions a, b and c. Hence, we may have a presymplectic structure with conditions (7.21) that are not satisfied and thus obtain a presymplectic chain with no dual bi-Hamiltonian chain.

8. Bi-Hamiltonian chains in $\mathbb{R}^3$

Suppose we have two compatible Poisson structures Π_0 and Π_1 in $\mathbb{R}^3$, given by $(\Pi_0)_{ij} = \mu\epsilon^{ijk}\partial_k H_0$ and $(\Pi_1)_{ij} = -\mu\epsilon^{ijk}\partial_k H_1$ $(i, j = 1, 2, 3)$. The Casimirs of Π_0 and Π_1 are $\mathrm{d}H_0$ and $\mathrm{d}H_1$ respectively. Then we can consider a bi-Hamiltonian chain

$$\begin{aligned} &\Pi_0\,\mathrm{d}H_0 = 0 \\ &\Pi_0\,\mathrm{d}H_1 = X = \Pi_1\,\mathrm{d}H_0 \\ &0 = \Pi_1\,\mathrm{d}H_1. \end{aligned} \tag{8.1}$$

Using theorem 11, we can construct a corresponding bi-presymplectic chain. To construct the bi-presymplectic chain, we have to find a closed form Ω_0 dual to the Poisson structure Π_0 and compatible with the Poisson structure Π_1. By lemma 22, such a form always exists. Having such a form Ω_0, the construction of the bi-presymplectic chain is straightforward. We start with $(\Omega_0)_{ij} = -\epsilon_{ijk}Y_0^k$, $i, j = 1, 2, 3$, where

$$\nabla\cdot Y_0 = 0, \qquad \mu Y_0(H_0) = 1, \tag{8.2}$$

$$Y_0(\mu Y_0(H_1)) = 0, \tag{8.3}$$

and Ω_1 is found from $Y_1 = \mu Y_1(H_0)Y_0 + \frac{1}{\mu}X$. Equation (8.3) is obtained from the divergence free condition of $Y_1 = \mu Y_1(H_0)Y_0 + \frac{1}{\mu}X$.

Example 24. Consider the Lorentz system [1]

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1 = \frac{1}{2}x_2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2 = -x_1x_3$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_3 = x_1x_2.$$

It admits a bi-Hamiltonian representation (8.1) with $H_0 = \frac{1}{4}(x_3 - x_1^2)$, $\mu = 1$ and $H_1 = x_2^2 + x_3^2$. The form Ω_0 dual to Π_0 and compatible with Π_1 is given by

$$\Omega_0 = -\begin{pmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ b & -\alpha & 0 \end{pmatrix}, \qquad \Pi_0 = \begin{pmatrix} 0 & 1/4 & 0 \\ -1/4 & 0 & -x_1/2 \\ 0 & x_1/2 & 0 \end{pmatrix},$$

where the vector $Y_0 = (\alpha, \beta, \gamma)^t$. The conditions on α, β and γ are

$$\nabla \cdot Y_0 = \partial_1\alpha + \partial_2\beta + \partial_3\gamma = 0, \qquad Y_0(H_0) = \tfrac{1}{4}\gamma - \tfrac{1}{2}x_1\alpha = 1.$$

One can find Ω_1 having determined Y_1 from (7.5):

$$Y_1 = \left(\tfrac{1}{2}x_2 + 2\alpha\eta, -x_1x_3 + 2\beta\eta, x_1x_2 + 2\gamma\eta\right),$$

where $\eta = \frac{1}{2}Y_0(H_1) = \beta x_2 + \gamma x_3$. We have an additional constraint on α, β and γ coming from $\nabla \cdot Y_1 = 0$, which reads as

$$Y_0(\eta) = \alpha\partial_1\eta + \beta\partial_2\eta + \gamma\partial_3\eta = 0.$$

A simple solution for the above presymplectic structures is given as $\alpha = -2/x_1, \beta = -2x_2\big/x_1^2, \gamma = 0$.

It is also possible to start with a dual pair and construct a second d-compatible Poisson structure with given properties. The following example gives hints on how to solve equations arising from d-compatible Poisson structures.

Example 25. We take a dual pair (Π_0, Ω_0) and construct a Poisson tensor Π_1, compatible with a given pair, such that Π_1 is nonlinear in x_3.

Let Π_0 be given in canonical coordinates. We take the form Ω_0 as follows:

$$\Omega_0 = \begin{pmatrix} 0 & -1 & f_1 \\ 1 & 0 & f_2 \\ -f_1 & -f_2 & 0 \end{pmatrix}, \qquad \Pi_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $f_1 = \partial_1 f$ and $f_2 = \partial_2 f$ for some function $f(x_1, x_2)$. Note that $(\Omega_0)_{ij} = -\epsilon_{ijk}Y_0^i$, where $Y_0 = (-f_2, f_1, 1)$ and $H_0 = x_3$. It is seen that $\nabla \cdot Y_0 = 0$, so by lemma 16 Ω_0 is closed and equality (6.5) holds; by lemma 17 it is dual to Π_0. We construct a Poisson tensor Π_1 compatible with Ω_0. Let Π_1 be given by $(\Pi_1)_{ij} = \epsilon_{ijk}\partial_k\chi$. Note that Π_1 is compatible with Π_0. By lemma 19, Ω_0 and Π_1 are compatible if equality (6.7) holds. Consider

$$Y_0\nabla\chi = -f_2\partial_1\chi + f_2\partial_2\chi + \partial_3\chi.$$

Let us perform the coordinate transformation

$$\xi = \alpha(x_1, x_2, x_3)$$
$$\eta = \beta(x_1, x_2, x_3)$$
$$\zeta = \gamma(x_1, x_2, x_3).$$

Then

$$\partial_1\chi = \partial_\xi\chi\,\partial_1\alpha + \partial_\eta\chi\,\partial_1\beta + \partial_\zeta\chi\,\partial_1\gamma$$
$$\partial_2\chi = \partial_\chi\,\partial_2\alpha + \partial_\eta\chi\,\partial_2\beta + \partial_\zeta\chi\,\partial_2\gamma$$
$$\partial_3\chi = \partial_\xi\chi\,\partial_3\alpha + \partial_\eta\chi\,\partial_3\beta + \partial_\zeta\chi\,\partial_3\gamma,$$

so

$$Y_0\cdot\nabla\chi = (-f_2\partial_1\alpha + f_1\partial_2\alpha + \partial_3\alpha)\partial_\xi\chi + (-f_2\partial_1\beta + f_1\partial_2\beta + \partial_3\beta)\partial_\eta\chi + (-f_2\partial_1\gamma + f_1\partial_2\gamma + \partial_3\gamma)\partial_\zeta\chi.$$

To simplify the above expression, we choose β, γ, α such that

$$-f_2\partial_1\beta + f_1\partial_2\beta + \partial_3\beta = 0$$
$$-f_2\partial_1\gamma + f_1\partial_2\gamma + \partial_3\gamma = 0$$
$$(-f_2\partial_1\alpha + f_1\partial_2\alpha + \partial_3\alpha) = 1;$$

hence,

$$Y_0\cdot\nabla\chi = \partial_\xi\chi.$$

Using the above technique, we can solve $Y_0(H_0) = 1$ and in particular $Y_0(Y_0(H_1)) = 0$ very easily. Equality (6.5) holds if $H_0 = \xi$. Then, $Y_0(Y_0(H_1)) = H_{1,\xi\xi} = 0$ and

$$H_1 = A_1(\zeta,\eta)\xi + A_2(\zeta,\eta),$$

where A_1 and A_2 are some arbitrary functions of ζ and η respectively. As an application, let

$$\eta = x_1x_2, \qquad \zeta = x_3 - \ln x_2, \qquad \xi = x_3$$

and $f = x_1x_2 = \eta$. Then, $H_0 = x_3$ and

$$H_1 = A_1(x_3 - \ln x_2, x_1x_2)x_3 + A_2(x_3 - \ln x_2, x_1x_2),$$

where A and B are functions of $(x_3 - \ln x_2)$ and x_1x_2.

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