CONSTRUCTIVE APPROXIMATION OF FUNCTIONS
BANACH CENTER PUBLICATIONS, VOLUME 107
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2015

WIDOM FACTORS FOR THE HILBERT NORM

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Abstract. Given a probability measure μ with non-polar compact support K, we define the n-th Widom factor $W_n^2(\mu)$ as the ratio of the Hilbert norm of the monic n-th orthogonal polynomial and the n-th power of the logarithmic capacity of K. If μ is regular in the Stahl–Totik sense then the sequence $(W_n^2(\mu))_{n=0}^{\infty}$ has subexponential growth. For measures from the Szegő class on $[-1,1]$ this sequence converges to some proper value. We calculate the corresponding limit for the measure that generates the Jacobi polynomials, analyze the behavior of the corresponding limit as a function of the parameters and review some other examples of measures when Widom factors can be evaluated.

1. Introduction. Let μ be a positive Borel measure on $\mathbb{C}$ with compact support K containing infinitely many points. The Gram–Schmidt process in the space $L^2(\mu)$ defines the unique sequence of orthonormal polynomials $p_n(z) = \kappa_n z^n + \dots$ provided $\kappa_n > 0$. By q_n with $n \in \mathbb{Z}_+$ we denote the monic orthogonal polynomials, that is $q_n = \kappa_n^{-1} p_n$. It is known (see e.g. [15], p. 78) that $\|q_n\|_2 = \kappa_n^{-1}$ realizes $\inf_{Q \in \mathcal{M}_n} \|Q\|_2$ where $\mathcal{M}_n$ stands for the class of all monic polynomials of degree at most n. If $K \subset \mathbb{R}$ then (see e.g. [15], p. 79) a three-term recurrence relation

$$x q_n(x) = q_{n+1}(x) + b_n q_n(x) + a_{n-1}^2 q_{n-1}(x)$$

is valid with the Jacobi parameters $a_n = \kappa_n / \kappa_{n+1}$ and $b_n = \int x p_n^2(x)\, d\mu(x)$. If, in addition, $\mu(\mathbb{R}) = 1$ then $p_0 = q_0 \equiv 1$, so $\kappa_0 = 1$ and $a_0 a_1 \dots a_{n-1} = \kappa_n^{-1}$.

2010 *Mathematics Subject Classification*: 42C05, 33C45, 30C85, 47B36.
Key words and phrases: Orthogonal polynomials, Jacobi weight, Szegő class, Widom condition, Julia sets.
The research was partially supported by TÜBÍTAK (Scientific and Technological Research Council of Turkey), Project 115F199.
The paper is in final form and no version of it will be published elsewhere.

Suppose μ is a probability Borel measure on $\mathbb{C}$ and the logarithmic capacity $\mathrm{Cap}(K)$ is positive. Let us define *n-th Widom–Hilbert factor* as

$$W_n^2(\mu) := \frac{\|q_n\|_2}{\mathrm{Cap}^n(K)}\,.$$

Thus, for $K \subset \mathbb{R}$ we have $W_n^2(\mu) = (\kappa_n \cdot \mathrm{Cap}^n(K))^{-1}$ and, in particular, for $K = [-1, 1]$,

$$W_n^2(\mu) = \kappa_n^{-1} \cdot 2^n. \tag{1}$$

Example 1.1. The equilibrium measure $d\mu_e = \frac{dx}{\pi\sqrt{1-x^2}}$ generates the Chebyshev polynomials of the first kind $p_0 \equiv 1$, $p_n = \sqrt{2}\,T_n$ for $n \in \mathbb{N}$, where $T_n(x) = \cos(n \arccos x) = 2^{n-1}x^n + \ldots$ for $|x| \le 1$. Here, $\kappa_n = 2^{n-1/2}$ and $W_0^2(\mu_e) = 1$, $W_n^2(\mu_e) = \sqrt{2}$ for $n \in \mathbb{N}$. For the Chebyshev polynomials of the second kind (see e.g. [14], p. 3) we have to take $d\nu = \frac{2}{\pi}\sqrt{1-x^2}\,dx$. Then $p_n(x) = U_n(x) = 2^n x^n + \ldots$, so $\kappa_n = 2^n$ and $W_n^2(\nu) = 1$ for $n \in \mathbb{Z}_+$.

In general, for $1 \le p \le \infty$, we can define $W_n^p(\mu)$ as $\frac{\inf_{\mathcal{M}_n} \|Q\|_p}{\mathrm{Cap}^n(K)}$ where $\|\cdot\|_p$ is the norm in the space $L^p(\mu)$. In the case $p = \infty$ we get the Widom–Chebyshev factors considered in [7]. Since $\mu(\mathbb{C}) = 1$, by Hölder's inequality, $W_n^p(\mu) \le W_n^r(\mu)$ for $1 \le p \le r \le \infty$.

As in the case $p = \infty$, the value W_n^p is invariant under dilation and translation. Indeed, the map $\varphi(z) = w = az + b$ with $a \ne 0$ transforms μ_0 into μ with $d\mu(w) = d\mu_0(\frac{w-b}{a})$. If $q_n(\mu_0, z) = z^n + \ldots$ realizes the infimum of norm in $L^p(\mu_0)$ then $q_n(\mu, w) = a^n q_n(\mu_0, \frac{w-b}{a})$ does so in the space $L^p(\mu)$. Therefore, $\|q_n(\mu,\cdot)\|_p = |a|^n \cdot \|q_n(\mu_0,\cdot)\|_p$. On the other hand, $\mathrm{Cap}(aK + b) = |a|\,\mathrm{Cap}(K)$. From here, $W_n^p(\mu) = W_n^p(\mu_0)$.

Example 1.2. The monic Chebyshev polynomials $(2^{1-n}T_n)_{n=1}^\infty$ have a remarkable property: They realize $\inf_{\mathcal{M}_n} \|\cdot\|_p$ in the space $L^p(\mu_e)$ for each $1 \le p \le \infty$ (see e.g. [11], p. 96). For proper p, it is easy to check that $\int_0^\pi |\cos nt|^p\,dt = \int_0^\pi \sin^p t\,dt$ which does not depend on $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$ we have

$$W_n^p(\mu_e) = 2 \cdot \Big(\frac{1}{\pi}\int_0^\pi \sin^p t\,dt\Big)^{1/p},$$

which increases to $W_n^\infty(\mu_e) = 2$ as $p \to \infty$.

The Hilbert case $p = 2$ is of interest since some important classes of measures in the theory of general orthogonal polynomials can be described in terms of behavior of Widom factors. For example, a measure μ is regular in the Stahl–Totik sense ($\mu \in \mathrm{Reg}$) if and only if the sequence of Widom factors has subexponential growth.

Recall that $\mu \in \mathrm{Reg}$ ([12], Def. 3.1.2) if $\kappa_n^{-1/n} \to \mathrm{Cap}(K)$ as $n \to \infty$ and a sequence $(a_n)_{n=1}^\infty$ with $a_n > 0$ has subexponential growth if $a_n = \exp(n \cdot \varepsilon_n)$ with $\varepsilon_n \to 0$ as $n \to \infty$. In the case of Chebyshev norm ($p = \infty$), by G. Szegő, the sequence of Widom factors has subexponential growth for each non-polar compact set K.

By the celebrated Szegő's result ([14], p. 297), for a wide class of measures on $[-1, 1]$ the sequence $(W_n^2(\mu))_{n=0}^\infty$ converges. In Section 2 we calculate the corresponding limit for the measure that generates the Jacobi polynomials. In Section 3 we discuss the Widom characterization of Szegő's class. In Section 4 we consider the behavior of the Widom factors for the Pollaczek polynomials—a typical example of polynomials that are generated by a regular measure beyond the Szegő class. Also, following methods from [12], we consider Widom factors for some irregular measures. Section 5 is devoted to the review of related results for orthogonal polynomials on Julia sets.

The motivation of our research is the problem to define the Szegő class for the general case, particularly for strictly singular measures. The Szegő type condition for the finite gap case is given in terms of the Radon–Nikodym derivative of the spectral measure with respect to the Lebesgue measure. Therefore it cannot be directly applied for the strictly singular case. But the Widom condition (Section 3), which characterizes the Szegő class in known cases, is given only in terms of properties of $(W_n^2(\mu))_{n=0}^{\infty}$.

We suggest the name *Widom factor* for $W_n^2(\mu)$ because of the fundamental paper [17], where H. Widom considered the corresponding values for $K \subset \mathbb{C}$ which is a finite union of smooth Jordan curves.

For basic notions of logarithmic potential theory we refer the reader to [10], log denotes natural logarithm. The symbol $\sim$ denotes the strong equivalence: $a_n \sim b_n$ means that $a_n = b_n(1 + o(1))$ for $n \to \infty$.

2. Jacobi weight. Let us find the limit of the sequence $(W_n^2(\mu))_{n=0}^{\infty}$ where $d\mu/dx$ is the density of a beta-distribution on $[-1, 1]$. Here, μ generates the classical (Jacobi) orthogonal polynomials on $[-1, 1]$. The Jacobi polynomials are orthogonal with respect to the weight $h_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ with $-1 < \alpha, \beta < \infty$. Let $C_{\alpha,\beta} = \int_{-1}^{1} h_{\alpha,\beta}(x)\, dx$. Then the measure $d\mu_{\alpha,\beta} = C_{\alpha,\beta}^{-1} h_{\alpha,\beta}(x)\, dx$ has unit mass and we will consider $W_{n,\alpha,\beta} := W_n^2(\mu_{\alpha,\beta})$.

LEMMA 2.1. *We have* $\int_0^{\pi/2} (2\sin t)^{\alpha}(2\cos t)^{\beta}\, dt \ge \pi/2$ *for each* $-1 < \alpha, \beta < \infty$. *If* $\alpha^2 + \beta^2 > 0$ *then the inequality is strict.*

Proof. For each $x \in \mathbb{R}$ we have the inequality $e^x \ge 1 + x + x^2/2 \cdot \chi_{(0,\infty)}$, which is strict if $x \neq 0$. Let us take $x = \log[(2\sin t)^{\alpha}(2\cos t)^{\beta}]$. Then

$$(2\sin t)^{\alpha}(2\cos t)^{\beta} \ge 1 + \alpha \log(2\sin t) + \beta \log(2\cos t) + x^2/2 \cdot \chi_{(0,\infty)}.$$

Since

$$\int_0^{\pi/2} \log(2\sin t)\, dt = \int_0^{\pi/2} \log(2\cos t)\, dt = 0,$$

(see e.g. [18], p. 402, form. 688), we get the desired inequality. Let us check its strictness if at least one of the parameters is not zero. It is enough to find $t \in (0, \pi/2)$ such that $x(t) > 0$. Then, by continuity, x is positive in some neighborhood of t and $\int_0^{\pi/2} x^2(t) \cdot \chi_E\, dt > 0$. Here, $E = \{t \in (0, \pi/2) : x(t) > 0\}$.

Suppose $\alpha + \beta > 0$. Then $x(\pi/4) = (\alpha + \beta)/2 \cdot \log 2 > 0$.

If $\alpha + \beta < 0$ and $\beta < 0$ with $\alpha \ge \beta$, then for $t = \pi/2 - \varepsilon$ with small enough ε we get $x(t) = \log[(2\sin 2\varepsilon)^{\beta}(2\cos\varepsilon)^{\alpha-\beta}] > (\alpha - \beta)\log(2\cos\varepsilon) \ge 0$.

Similarly, if $\alpha + \beta < 0$ and $\alpha < 0$ with $\alpha < \beta$, then one can take $t = \varepsilon$.

Finally, let $\alpha + \beta = 0$ and, without loss of generality, $\alpha > 0$. Then, for $t > \pi/4$, $x(t) = \log(\tan t)^{\alpha} > 0$. ■

LEMMA 2.2. *For* $-1 < \alpha, \beta < \infty$, *let* $C_{\alpha,\beta}$ *be defined as above. Then* $2^{\alpha+\beta} C_{\alpha,\beta} \ge \pi/2$. *The inequality is strict if* $(\alpha, \beta) \neq (-1/2, -1/2)$. *If* (α, β) *approaches the boundary of the domain* $(-1, \infty)^2$ *then* $2^{\alpha+\beta} C_{\alpha,\beta} \to \infty$.

Proof. By substitution $x = \cos 2t$, we have

$$C_{\alpha,\beta} = \int_{-1}^{1} (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2} \frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} (2\sin^2 t)^{\alpha+1/2}(2\cos^2 t)^{\beta+1/2}\, 2\, dt.$$

From here, $2^{\alpha+\beta} C_{\alpha,\beta} = \int_0^{\pi/2} (2\sin t)^A (2\cos t)^B dt$ with $A = 2\alpha+1$, $B = 2\beta+1$. Since $-1 < A, B < \infty$, Lemma 2.1 can be applied. The equality $2^{\alpha+\beta} C_{\alpha,\beta} = \pi/2$ occurs only if $A = B = 0$, that is $\alpha = \beta = -1/2$.

Let us analyze the boundary behavior of the function $f(\alpha,\beta) := 2^{\alpha+\beta} C_{\alpha,\beta}$. First we consider the symmetric case. For large $m \in \mathbb{N}$ we have

$$f(m,m) = 4^m \int_{-1}^{1} (1-x^2)^m\, dx = 4^m \int_0^{\pi} \sin^{2m+1} t\, dt = 4^m\, \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{2}{3} \cdot 2$$

and $f(m,m) \sim 4^m \sqrt{\pi/m}$.

For the opposite case, let ε be small and positive. Then

$$f(-1+\varepsilon, -1+\varepsilon) = 4^{-1+\varepsilon} \cdot 2 \int_0^1 (1-x^2)^{-1+\varepsilon}\, dx > \frac{1}{4} \int_0^1 (1-x)^{-1+\varepsilon}\, dx = \frac{1}{4\varepsilon}.$$

In general, let us estimate from below $f(\alpha, m) = 2^{\alpha+m} \int_{-1}^{1} (1-x)^\alpha (1+x)^m\, dx$ for large m. If $\alpha < 0$ then $f(\alpha,m) > 2^{m-1} \int_0^1 (1+x)^m\, dx \sim 4^m/m$.

If $\alpha \geq 0$ then $f(\alpha,m) > 2^{m+\alpha} \int_0^{1/2} (1-x)^\alpha (1+x)^m\, dx > 2^m \int_0^{1/2} (1+x)^m\, dx \sim 3^m/m$.

Similarly, $f(-1+\varepsilon, \beta) > 2^{-1+\beta} \int_0^1 \frac{(1+x)^\beta}{(1-x)^{1-\varepsilon}}\, dx$. If $\beta < 0$ then $(1+x)^\beta > 2^{-\beta}$ and $f(-1+\varepsilon,\beta) > \frac{1}{2\varepsilon}$. If $\beta \geq 0$ then $2^\beta(1+x)^\beta \geq 1$, which gives the same lower bound $f(-1+\varepsilon,\beta)$ as above. Clearly, $f(\alpha, -1+\varepsilon)$ and $f(m,\beta)$ can be estimated in the same way. ■

The leading coefficient for Jacobi polynomials is given in terms of the gamma function $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$ with $p > 0$. It is known (see e.g. [13], Lemma 4.3) that

$$\Gamma(n+1) = n \cdot \Gamma(n) = n!, \qquad \Gamma(n+p) \sim n^p\, \Gamma(n) \qquad \text{for} \quad n \in \mathbb{N},\ p > 0. \tag{2}$$

From here and by Stirling's formula,

$$\frac{\Gamma(2n)}{\Gamma^2(n)} \sim \frac{1}{2} \sqrt{\frac{n}{\pi}}\, 4^n. \tag{3}$$

Let

$$W_{\alpha,\beta} := \sqrt{\frac{\pi}{2^{\alpha+\beta} C_{\alpha,\beta}}}\,.$$

THEOREM 2.3. *We have*

(i) *for each* $-1 < \alpha, \beta < \infty$, $\quad W_{n,\alpha,\beta} \to W_{\alpha,\beta}$ *as* $n \to \infty$,
(ii) $\sup_{-1<\alpha,\beta<\infty} W_{\alpha,\beta} = W_{-1/2,-1/2} = \sqrt{2}$, *which is the only maximum.*
(iii) $W_{\alpha,\beta} \to 0$ *as* (α,β) *approaches the boundary of the domain* $(-1,\infty)^2$.

Proof. The leading coefficient of the Jacobi polynomial $P^{(\alpha,\beta)}$ for the measure $d\mu = h_{\alpha,\beta}(x)\, dx$ is given by the formula (25) in [13], 7.1. Therefore, for the normalized case, we have

$$\kappa_n(\alpha,\beta) = \frac{\sqrt{C_{\alpha,\beta}}}{2^n} \sqrt{\frac{\alpha+\beta+2n+1}{n!\, 2^{\alpha+\beta+1}}}\, \frac{\Gamma(\alpha+\beta+2n+1)}{\sqrt{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}}\,.$$

By (2), the last fraction is equivalent to $\frac{\Gamma(2n)}{\Gamma^{3/2}(n)} \frac{2^{\alpha+\beta+1}}{\sqrt{n}}$. Therefore,

$$\kappa_n(\alpha,\beta)\cdot 2^n \sim \sqrt{2^{\alpha+\beta+1}C_{\alpha,\beta}}\sqrt{\frac{\alpha+\beta+2n+1}{n}}\,\frac{\Gamma(2n)}{\sqrt{n!}\,\Gamma^{3/2}(n)}.$$

By (2) and (3), the last fraction here is equivalent to $\frac{1}{2}\frac{1}{\sqrt{\pi}}4^n$. Thus, $\kappa_n(\alpha,\beta)\cdot 2^{-n} \sim \sqrt{2^{\alpha+\beta}C_{\alpha,\beta}/\pi}$, which is, by (1), the desired result.

The statements (ii) and (iii) follow from Lemma 2.2. ∎

For example, for the Legendre polynomials we have $C_{0,0}=2$ and $W_{0,0}=\sqrt{\pi/2}$.

3. The Szegő class. The measures $\mu_{\alpha,\beta}$ from the previous section satisfy the Szegő condition. Recall that a probability measure $d\mu(x)=\omega(x)\,dx$ with support $[-1,1]$ belongs to the Szegő class ($\mu\in(S)$) if $I(\omega):=\int_{-1}^{1}\frac{\log\omega(x)}{\pi\sqrt{1-x^2}}\,dx>-\infty$, which means that this integral converges for it cannot be $+\infty$. Orthogonal polynomials generated by $\mu\in(S)$ enjoy several nice asymptotics. The basic of them is the asymptotics of $p_n(z)(z+\sqrt{z^2-1})^{-n}$ outside $[-1,1]$ as $n\to\infty$ (see [14], p. 297, or e.g. Theorem 1.7 in [16]). Here, we take the branch of $\sqrt{z^2-1}$ that behaves like z near infinity, so the modulus of the second term above is $\exp(-n\cdot g(z))$, where g is the Green function of $\mathbb{C}\setminus[-1,1]$ with pole at infinity. By setting $z=\infty$, we get (see (12.7.2) in [14] or [16], p. 26)

$$\lim_n W_n^2(\mu)=\sqrt{\pi}\exp(I(\omega)/2), \tag{4}$$

which gives another way to calculate $W_{\alpha,\beta}$.

Thus, for any measure from the Szegő class, the sequence of Widom factors converges to some positive value. The inverse implication is also valid: if $\lim_n W_n^2(\mu)$ exists in $(0,\infty)$ then $\mu\in(S)$ (see e.g. Theorem 2.4 in [6]).

We see that $I(\omega)=\int\log\omega\,d\mu_e$. Let us calculate this value for the equilibrium density $\omega_e=d\mu_e/dx=\frac{1}{\pi\sqrt{1-x^2}}$. Here, $I(\omega_e)=-\log\pi-\int_0^\pi\log\sin t\,dt/\pi=\log(2/\pi)$. As a generalization of Theorem 2.3, let us show that $I(\omega_e)$ realizes maximum of $I(\omega)$ among all densities from the Szegő class (compare with (4.7) in [6]).

Proposition 3.1. *Suppose* $\omega\overset{\text{a.e.}}{>}0$ *with* $\int_{-1}^{1}\omega(x)\,dx=1$ *and* $I(\omega)>-\infty$*. Then* $I(\omega)\le\log(2/\pi)$ *with equality if and only if* $\omega\overset{\text{a.e.}}{=}\omega_e$*.*

Proof. We have $I(\omega)=\int\log\omega_e\,d\mu_e+\int\log(\omega/\omega_e)\,d\mu_e$. The first term here is $\log(2/\pi)$, for the latter we use Jensen's inequality (see e.g. [5], p. 141):

$$\int\log(\omega/\omega_e)\,d\mu_e\le\log\int\omega/\omega_e\,d\mu_e=\log\int_{-1}^{1}\omega(x)\,dx=0.$$

Since $\log(\cdot)$ is strictly concave, the equality above is possible if and only if $\omega/\omega_e\overset{\mu\text{-a.e.}}{=}1$, that is $\omega\overset{\text{a.e.}}{=}\omega_e$. ∎

Corollary 3.2. *Let* $\mu\in(S)$ *and* $W(\mu):=\lim_n W_n^2(\mu)$*. Then* $W(\mu)\le W(\mu_e)$ *with equality if and only if* $\mu=\mu_e$*.*

During the last two decades significant progress was achieved in the generalization of Szegő's theory to the case of finite gap Jacobi matrices J (see e.g. the review [6]). For such matrices, the essential spectrum $K=\sigma_{\text{ess}}(J)$ is a finite union of closed intervals. If the spectral measure μ is absolutely continuous, that is $d\mu(x) = \omega(x)\,dx$, then the Szegő class can be defined as (4.6) in [6]: $\mu \in (S)$ if

$$\int_K \frac{\log \omega(x)}{\sqrt{\operatorname{dist}(x, \mathbb{R} \setminus K)}}\,dx > -\infty.$$

Here, we have the Widom characterization ((4.11) in [6])

$$\mu \in (S) \quad \Longleftrightarrow \quad \limsup_{n\to\infty} W_n^2(\mu) > 0. \tag{5}$$

As in the case W_n^∞ (see [1], [2] or e.g. [7]), the behavior of $(W_n^2(\mu))$ for such measures is rather irregular. This sequence may have a finite number of accumulation points or the set of its accumulation points may fill a whole interval. For asymptotics of $W_n^2(\mu)$ we refer the reader to [15], p. 101.

As an example, let us consider the Jacobi matrix with periodic coefficients (a_n) and zero (or slowly oscillating) main diagonal. Recall that periodic coefficients gives a Jacobi matrix in the Szegő class. We follow [8] here.

EXAMPLE 3.3. Let $a_{2n-1} = a$, $a_{2n} = b$ for $n \in \mathbb{N}$ with $b > 0$ and $a = b + 2$. These values with $b_n = 0$ define a Jacobi matrix B_0 with spectrum $\sigma(B_0) = [-b-a, b-a] \cup [a-b, a+b]$ ([8], Lemma 2.1). The same values $(a_n)_{n=1}^\infty$ with $b_n = \sin n^\gamma$ for $0 < \gamma < 1$ give a matrix B with $\sigma(B) = [-b-a-1, b-a+1] \cup [a-b-1, a+b+1]$ ([8], Theorem 2.6). Let μ_0 and μ be spectral measures for B_0 and B correspondingly. We know (see e.g. [10], Corollary 5.2.6) that the capacity of $[-B, -A] \cup [A, B]$ for $0 < A < B$ is $\frac{1}{2}\sqrt{B^2 - A^2}$. Therefore, $\operatorname{Cap}(\sigma(B_0)) = \sqrt{ab}$, $\operatorname{Cap}(\sigma(B)) = \sqrt{a(b+1)}$. From here, $W_{2n}^2(\mu_0) = 1$ and $W_{2n-1}^2(\mu_0) = \sqrt{a/b}$ for $n \in \mathbb{N}$. The measure μ_0 is absolutely continuous with respect to the Lebesgue measure (see e.g. [16], Lemma 2.15). Here, $\mu_0 \in (S)$, as we expected.

On the other hand, $W_{2n}^2(\mu) = (\frac{b}{b+1})^n$ and $W_{2n+1}^2(\mu) = (\frac{b}{b+1})^n\sqrt{\frac{a}{b+1}}$. Thus, $W_n^2(\mu) \to 0$ as $n \to \infty$, $\mu \notin (S)$ and $\mu \notin \operatorname{Reg}$.

4. Outside the Szegő class. The measure μ that generates the Pollaczek polynomials (see [14], Appendix, [9], p. 80, [16], p. 6) presents a typical example of a regular absolutely continuous measure beyond the Szegő class.

EXAMPLE 4.1. For real parameters a and b with $a \geq |b|$, in the simplest case ($\lambda = 1/2$), the weight function for the Pollaczek polynomials is ([14], p. 394, [16], p. 6)

$$\omega(x) = \frac{1+a}{2\pi}\,\exp(-2t \cdot \arcsin x) \cdot |\Gamma(1/2 + it)|^2$$

with $t = \frac{ax+b}{2\sqrt{1-x^2}}$ for $|x| \leq 1$.

By the Erdős–Turán criterion (see e.g [12], p. 101), the measure $d\mu(x) = \omega(x)\,dx$ is regular. On the other hand, ω goes to zero exponentially fast near the endpoints of the interval $[-1, 1]$, thus the integral $I(\omega)$ diverges and $\mu \notin (S)$. Substituting $x = \infty$ in [16], (1.3.19), we get

$$\lim_n W_n^2(\mu) \cdot n^{a/2} = \Gamma\Big(\frac{a+1}{2}\Big).$$

Here, the sequence $(W_n^2(\mu))_{n=0}^\infty$ converges to zero, but slowly, which corresponds with the regularity of μ.

In Example 3.3 the sequence $(W_n^2(\mu))_{n=0}^\infty$ converges to zero with an exponential rate. Using techniques from [12], let us show that any rate of decrease, which is faster than exponential, can be achieved.

EXAMPLE 4.2. Let Z_n be the set of zeros of the Chebyshev polynomial T_{3^n} for $n \in \mathbb{Z}_+$. Since $T_{3^{n+1}}(x) = T_3(T_{3^n}(x))$, we have $Z_n \subset Z_{n+1}$. Let $\mu_n = 3^{-n} \sum_{x \in Z_n} \delta_x$ for $n \in \mathbb{N}$, where δ_x is the Dirac measure at x. Given a sequence $(a_n)_{n=1}^\infty$ with $a_n > 0$, $\sum_{n=1}^\infty a_n = 1$, we consider the measures $\mu = \sum_{n=1}^\infty a_n \mu_n$ and $\nu_n = \sum_{j=n}^\infty a_j \mu_j$, $n \in \mathbb{N}$. Clearly, μ is a probability measure with support $[-1, 1]$. Let us take $\varepsilon_n \searrow 0$ with $\varepsilon_{n+1} \le \varepsilon_n/2$ for all n. Set $A := (\sum_{n=1}^\infty \varepsilon_n)^{-1}$ and $a_n = A \cdot \varepsilon_n$ for $n \in \mathbb{N}$. Then $a_n < \|\nu_n\| = \sum_{j=n}^\infty a_j \le 2a_n$. Let t_m for $m \in \mathbb{N}$ be the monic Chebyshev polynomial, so $\|t_m\|_\infty = 2^{1-m}$ for $m \in \mathbb{N}$. As in Example 3.5.2 in [12], for $3^{n-1} \le m < 3^n$, let us take the polynomial $Q_m = t_{m-3^{n-1}} \cdot t_{3^{n-1}}$. We see that $\int |Q_m|^2 \, d\mu_k = 0$ for $1 \le k \le n-1$ and $\|Q_m\|_\infty \le 2^{2-m}$. By the minimality property of the monic orthogonal polynomials q_n, we get

$$\kappa_m^{-2}(\mu) = \|q_m\|_2^2 \le \|Q_m\|_2^2 = \int |Q_m|^2 \, d\mu = \int |Q_m|^2 \, d\nu_n \le 2^{4-2m} \cdot 2a_n.$$

Therefore, by (1), $W_m^2(\mu) \le 4\sqrt{2A} \cdot \sqrt{\varepsilon_n}$ for $3^{n-1} \le m < 3^n$. Here, $W_m^2(\mu) \searrow 0$ as fast as we wish for a suitable choice of $(\varepsilon_n)_{n=1}^\infty$.

5. Julia sets. Let us analyze the behavior of Widom factors for the equilibrium measure on Julia sets. Suppose a monic polynomial T of degree $k \ge 2$ is given. Let $T_0(z) = z$ and $T_n(z) = T_{n-1}(T(z))$ be the n-th iteration of T for $n \in \mathbb{N}$. The Julia set B_T for the polynomial T can be defined as the boundary of the domain of attraction of infinity $A(\infty) = \{z \in \overline{\mathbb{C}} : T_n(z) \to \infty \quad \text{as} \quad n \to \infty\}$. Due to H. Brolin [4], $\mathrm{Cap}(B_T) = 1$ and $\mathrm{supp}(\mu_e) = B_T$ for the equilibrium measure μ_e on B_T.

Following [3], we consider the Julia set corresponding to the polynomial $T(z) = z^3 - \lambda z$ with $\lambda > 3$. Here, $\deg T_n = 3^n$. Remark that, in the case $\lambda = 3$, we get the Chebyshev polynomials of degrees 3^n for $[-2, 2]$ and B_T coincides with this interval. For $\lambda > 3$, by [4], B_T is a Cantor type set on the real line. By [3], the Jacobi parameters satisfy the following conditions: $a_1 = 1$, $b_n = 0$ for all n and

$$a_{3n+1}^2 = 2\lambda/3 - a_{3n}^2, \qquad a_{3n+2}^2 = \lambda/3, \tag{6}$$

$$a_{3n} a_{3n-1} a_{3n-2} = a_n. \tag{7}$$

In addition, by Lemma 3 and Theorem 2 in [3], we have $\lim_{n\to\infty} a_{3^n} = 0$ and $a_{3n} < 1$. Therefore, by (6),

$$a_{3n+1}, a_{3n+2} > 1 \quad \text{for } n \in \mathbb{N}. \tag{8}$$

To shorten notation, we write W_k instead of $W_k^2(\mu_e)$. Since $\mathrm{Cap}(B_T) = 1$, we have $W_k = \kappa_n^{-1} = a_1 \, a_2 \cdots a_k$. Hence, by (7),

$$W_{3^n} = W_{3^{n-1}} = \ldots = a_1 \, a_2 \, a_3 = 1 \quad \text{and} \quad W_{3^n - 1} = W_{3^n}/a_{3^n} \to \infty \quad \text{as} \quad n \to \infty.$$

Thus, $\limsup_{k\to\infty} W_k = \infty$.

Let us show that $W_k \ge 1$ for all k, so $\liminf_{k\to\infty} W_k = 1$. Clearly we have the desired inequality for $k = 2$ and $k = 3$. Suppose $3^n < k < 3^{n+1}$ for some $n \in \mathbb{N}$. Then $k = k_n \cdot 3^n + \ldots + k_1 \cdot 3 + k_0$ with $k_n \in \{0, 1\}$ and $k_j \in \{0, 1, 2\}$ for $0 \le j \le n-1$, that is, k has the representation $(k_n\, k_{n-1} \cdots k_1\, k_0)$ in base 3. By (8), $W_k \geq W_{k'}$ for $k' = (k_n\, k_{n-1} \cdots k_1\, 0)$. By (7), $W_{k'} = W_m$, where $m = k'/3$ has the representation $(m_{n-1} \cdots m_1\, m_0)$ with $m_j = k_{j+1}$. Using (8) again, we get $W_m \geq W_{m'}$ with $m' = (m_{n-1} \cdots m_1\, 0)$. Proceeding this way, we deduce that $W_k > 1$ for such k.

The asymptotic self-reproducing property of the coefficients allow us to calculate accumulation points of the sequence $(W_k)_{k=1}^{\infty}$. For example, if $n \to \infty$ then $W_{3^n+1} = a_{3^n+1} = \sqrt{2\lambda/3 - a_{3^n}^2} \to \sqrt{2\lambda/3}\,, W_{3^n+2} = a_{3^n+1}a_{3^n+2} \to \sqrt{2\lambda/3}\sqrt{\lambda/3} = \sqrt{2}\,\lambda/3$, etc.

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