Journal of Optimization Theory and Applications (2022) 194:1072–1080
https://doi.org/10.1007/s10957-022-02067-2

A Direct Proof of the Gale–Nikaido–Debreu Lemma Using Sperner's Lemma

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Received: 23 December 2021 / Accepted: 18 June 2022 / Published online: 14 July 2022

Abstract
The Gale–Nikaido–Debreu lemma plays an important role in establishing the existence of competitive equilibrium. In this paper, we use Sperner's lemma and basic elements of topology to prove the Gale–Nikaido–Debreu lemma.

Keywords Sperner lemma · Simplex · Subdivision · Fixed point theorem · Gale–Nikaido–Debreu lemma · General equilibrium

1 Introduction

The issue of existence of equilibrium is one of the fundamental questions in economics. [1] The Gale–Nikaido–Debreu lemma [11,20,32] plays an important role in establishing the existence of general equilibrium.[2] The proofs of the Gale–Nikaido–Debreu lemma require the use of the fixed point theorems. Indeed, Debreu [10,11] and Nikaido [32] used the Kakutani fixed point theorem while Gale [20] used the Knaster–Kuratowski–Mazurkiewicz lemma [26] to prove the Gale–Nikaido–Debreu lemma. Kuhn [27] provided another proof of the Gale–Nikaido–Debreu lemma, which is based on the Eilenberg–Montgomery fixed-point theorem.

[1] Looking back at history, Debreu [9] used the Eilenberg–Montgomery fixed point theorem to prove the existence of a social equilibrium. Then, by using this social equilibrium existence theorem, Arrow and Debreu [2] proved the existence of a general equilibrium for a competitive economy with productions. See Debreu [12] and Florenzano [16] for excellent treatments of the existence of equilibrium. See also Duppe and Weintraub [13], Khan [25] for discussions about the history of the general equilibrium theory.
[2] Another important lemma in the general equilibrium theory is Gale and Mas-Colell's lemma introduced and proved by Gale and Mas-Colell [21,22]. Their proofs are based on the Kakutani fixed point theorem [24] and Michael selection theorem [30]. See Florenzano [17] for the role of these two lemmas in the general equilibrium theory.

Communicated by Josh Taylor.

We would like to thank an Associate Editor and two Referees for their helpful and insightful comments which help us to substantially improve our paper.

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As mentioned by Duppe and Weintraub [13], Khan [25], Debreu wanted to discuss the question whether one could dispense with a fixed point theorem in proving the Gale–Nikaido–Debreu lemma. In the present paper, we address Debreu's question by providing a new proof of the Gale–Nikaido–Debreu lemma directly from Sperner's lemma which is a combinatorial result on colorings of triangulations.[3] More precisely, our proof relies on Sperner's lemma, Carathéodory's convexity theorem [7], and basic properties of topology such as the finite covering of a compact set. It should be noticed that the Sperner lemma and the mathematical tools that we use to prove the Gale–Nikaido–Debreu lemma date back to 1928.

The paper proceeds as follows. In Sect. 2, we review some basic concepts such as the notions of subsimplex, simplicial subdivision, and Sperner's lemma. In Sect. 3, we use Sperner's lemma to prove the Gale–Nikaido–Debreu lemma as well as its variants. Finally, Sect. 4 concludes the paper.

2 Preliminaries

In this section, we introduce basic terminologies and necessary background for our work.

2.1 On the Sperner Lemma

Consider the Euclidean space $\mathbb{R}^n$. Let $e^1 = (1, 0, 0, \ldots, 0)$, $e^2 = (0, 1, 0, \ldots, 0)$, ..., and $e^n = (0, 0, \ldots, 0, 1)$ denote the n unit vectors of $\mathbb{R}^n$. The unit-simplex Δ of $\mathbb{R}^n$ is the convex hull of $\{e^1, e^2, \ldots, e^n\}$. A $(m-1)$-dimensional simplex of $\mathbb{R}^n$, denoted by $[[x^1, x^2, \ldots, x^m]]$, is the convex hull of $\{x^1, x^2, \ldots, x^m\}$ where $x^i \in \mathbb{R}^n$ for any $i = 1, \ldots, m$, and the vectors $(x^1 - x^2, x^1 - x^3, \ldots, x^1 - x^m)$ are linearly independent, or equivalently, the vectors $(x^1, x^2, \ldots, x^m)$ are affinely independent (i.e., if $\sum_{i=1}^m \lambda_i x_i = 0$ and $\sum_{i=1}^m \lambda_i = 0$ imply that $\lambda_i = 0\ \forall i$).

Given a simplex $S = [[x^1, x^2, \ldots, x^m]]$, a face of S is the convex hull $[[x^{i_1}, x^{i_2}, \ldots, x^{i_h}]]$ with $h < m$, and $\{i_1, i_2, \ldots, i_h\} \subset \{1, 2, \ldots, m\}$. In particular, a vertex of S is x^i with $i \in \{1, \ldots, m\}$.

We now define the notions of simplicial subdivision (or triangulation) and labeling (see Border [6], Su [38] or Chapter 23 in Maschler, Solan, and Zamir [31] for a general treatment) before stating Sperner's lemma.

Definition 1 The set $T = \{\Delta_i : i = 1, \ldots, p\}$, of simplices, is a simplicial subdivision of Δ if

1. $\Delta = \cup_{i=1}^p \Delta_i$,
2. For any $i, j \in \{1, \ldots, p\}$, the intersection $\Delta_i \cap \Delta_j$ is either empty or a face of both Δ_i and Δ_j.

[3] Sperner's lemma [37] can be viewed as a combinatorial variant of the Brouwer fixed point theorem [5,23] and actually equivalent to it. For instance, Knaster, Kuratowski and Mazurkiewicz [25] used the Sperner lemma to prove the Knaster–Kuratowski–Mazurkiewicz theorem which implies the Brouwer theorem. Meanwhile, Yoseloff [40], Park and Jeong [34] proved the Sperner lemma by using the Brouwer theorem. The reader is referred to Park [33] for a more complete survey of fixed point theorems and Ben-El-Mechaiekh, Bich, and Florenzano [3] for a survey of general equilibrium and fixed point theory.

3. For any $i \in \{1, \ldots, p\}$, all of the faces of Δ_i are in T.

Remark 1 For any positive integer K, there is a simplicial subdivision $T^K = \{\Delta_1^K, \ldots, \Delta_{p(K)}^K\}$ of Δ such that $Mesh(T^K) \equiv \max_{i\in\{1,\ldots,p(K)\}} \sup_{x,y}\{\|x - y\| : x, y \in \Delta_i^K\} < 1/K$. For example, we can take equilateral subdivisions or barycentric subdivisions.

Definition 2 Consider a simplicial subdivision of Δ. Let V denote the set of vertices of all the subsimplices of Δ. A labeling R is a function from V into $\{1, 2, \ldots, n\}$. A labeling R is said to be proper if it satisfies the **Sperner condition**:

$$\text{For any } m \le n, \; if x \in \text{ri}[[e^{i_1}, e^{i_2}, \ldots, e^{i_m}]] \text{ then } R(x) \in \{i_1, i_2, \ldots, i_m\}.^4$$

In particular, $R(e^i) = i, \forall i$.

Note that the Sperner condition implies that all vertices of the simplex are labeled distinctly. Moreover, the label of any vertex on the edge between the vertices of the original simplex matches with another label of these vertices. With these in mind, we can now state Sperner's lemma.

Lemma 1 (Sperner) *Let $T = \{\Delta_1, \ldots, \Delta_p\}$ be a simplicial subdivision of Δ. Let R be a labeling which satisfies the Sperner condition. Then there exists a subsimplex $\Delta_i \in T$ which is completely labeled, i.e., $\Delta_i = [[x^1(i), \ldots, x^n(i)]]$ with $R(x^l(i)) = l, \forall l = 1, \ldots, n$.*

Sperner's lemma guarantees the existence of a completely labeled subsimplex for any simplicially subdivided simplex in accordance with the Sperner condition. A proof of this lemma can be found in several textbooks [4,6,31,35] or papers [29, 37]. In particular, the original proof uses an inductive argument based on a complete enumeration of all completely labeled simplices for a series of lower dimensional problems. Meanwhile, proofs using constructive arguments date back to Cohen [8] and Kuhn [28] (see Scarf [36] for a demonstration of the constructive proof).

2.2 On Correspondences

Let $X \subset \mathbb{R}^l$, $Y \subset \mathbb{R}^m$. A correspondence Γ from X into Y is a mapping from X into the set of subsets of Y. The graph of Γ is the set graph$\Gamma = \{(x, y) \in X \times Y : y \in \Gamma(x)\}$. A correspondence $\Gamma : X \to Y$ is *closed* if its graph is closed.

Definition 3 A correspondence $\Gamma : X \to Y$ is *upper semicontinuous* at point x if for every open set V of Y for which $\Gamma(x) \subset V$, there exists a neighborhood U of x such that $\Gamma(x') \in V \; \forall x' \in U$. Γ is said to be upper semicontinuous on X if it is upper semicontinuous at every point of X.

Notice that if Y is compact then Γ is upper semicontinuous if and only if Γ is closed. It is also clear that if Γ is upper semicontinuous and $K \subset X$ is compact, then (i) $\Gamma(K)$ is compact if Y is compact and Γ has closed values, and (ii) $\Gamma(K)$ is compact if Γ is compact valued.

[4] Recall that if $\Delta_i = [[x^{i_1}, x^{i_2}, \ldots, x^{i_m}]]$, then $\text{ri}(\Delta_i) \equiv \{x | x = \sum_{k=1}^m \alpha_k x^k(i); \sum_k \alpha_k = 1$; and $\forall k : \alpha(k) > 0\}$.

3 Using Sperner's Lemma to Prove the Gale–Nikaido–Debreu Lemma

In the general equilibrium theory, when we consider preference orders for the consumers and production sets for the firms, the demands of the consumers or of the firms are correspondences (not necessarily single-valued). In this case, the customary proofs of the equilibrium existence make use of either the Gale–Nikaido–Debreu lemma [10,11,20,32] or the Gale and Mas-Colell lemma [21,22] whose proofs, in turn, require the Kakutani fixed point theorem or the Knaster–Kuratowski–Mazurkiewicz lemma. In what follows, we use the Sperner lemma and well-known mathematical results to prove several versions of the Gale–Nikaido–Debreu lemma.

Let us start with the following version (Theorem 1 in Debreu [11], page 82).

Lemma 2 (Gale–Nikaido–Debreu lemma) *Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be an upper semi-continuous correspondence with non-empty, compact, convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the following condition:*

$$\forall p \in \Delta,\ \forall z \in \zeta(p),\ p \cdot z \leq 0. \tag{1}$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$.

Proof Let $A = \max\{\|z\| \equiv (\sum_{i=1}^N z_i^2)^{1/2} : z \in \zeta(\Delta)\}$.

Step 0. Let $\epsilon \in (0, 1)$. Since Δ is compact, there exists a finite covering of Δ with a finite family of open balls $\left(B\left(x^i(\epsilon), \epsilon\right)\right)_{i=1,\ldots,I(\epsilon)}$. Let d denote the distance function in $\mathbb{R}^N$. Let $B^c\left(x^i(\epsilon), \epsilon\right)$ denote the complementary set of $B\left(x^i(\epsilon), \epsilon\right)$. For $i = 1, \ldots, I(\epsilon)$, define the non-negative function $\alpha_i(x) = \frac{d(x, B^c(x^i(\epsilon), \epsilon))}{\sum_{i=1}^{I(\epsilon)} d(x, B^c(x^i(\epsilon), \epsilon))}$. One can easily check that, for any i, α_i is continuous, Supp $\alpha_i \subset B\left(x^i(\epsilon), \epsilon\right)$ and $\sum_{i=1}^{I(\epsilon)} \alpha_i(x) = 1, \forall x \in \Delta$.[5]

Step 1 Take $y^i(\epsilon) \in \zeta(x^i(\epsilon))\ \forall i$. We define the function $f^\epsilon : \Delta \to \mathbb{R}^N$ by $f^\epsilon(x) = \sum_{i=1}^{I(\epsilon)} \alpha_i(x) y^i(\epsilon)$. This function is continuous.

Step 2 We claim that: $x \cdot f^\epsilon(x) \leq \epsilon A, \forall x \in \Delta$. Let $x \in \Delta$, there exists a set $J(x) \subset \{1, \ldots, I(\epsilon)\}$ such that $x \in \cap_{i \in J(x)} B\left(x^i(\epsilon), \epsilon\right)$. Recall $f^\epsilon(x) = \sum_{i \in J(x)} \alpha_i(x) y^i(\epsilon)$ with $\sum_{i \in J(x)} \alpha_i(x) = 1$. Observe that $\forall i \in J(x), x = x^i(\epsilon) + \epsilon u^i(x)$, with some $u^i(x) \in B(0, 1)$. Then, we have

$$\begin{aligned} x \cdot f^\epsilon(x) &= \sum_{i=1}^{J(x)} \alpha_i(x)\big(x^i(\epsilon) + \epsilon u^i(x)\big) \cdot y^i(\epsilon) \\ &= \left[\sum_{i=1}^{J(x)} \alpha_i(x) x^i(\epsilon) \cdot y^i(\epsilon)\right] + \epsilon \left[\sum_{i=1}^{J(x)} \alpha_i(x) u^i(x) \cdot y^i(\epsilon)\right] \end{aligned}$$

[5] These functions α_i constitute a partition of unity subordinate to the covering $\left(B\left(x^i(\epsilon), \epsilon\right)\right)_{i=1,\ldots,I(\epsilon)}$. See, for instance, Section 2.19 in Aliprantis and Border [1].

$$\leq \epsilon \left[\sum_{i=1}^{J(x)} \alpha_i(x) u^i(x)) \cdot y^i(\epsilon) \right] \text{ (since } x^i(\epsilon) \cdot y^i(\epsilon) \leq 0, \text{ from (1))}$$

$$\leq \epsilon \sum_{i=1}^{J(x)} \alpha_i(x) \|u^i(x))\| \cdot \|y^i(\epsilon)\| \leq \epsilon A,$$

where the last inequalities follow $\|u^i(x))\| \leq 1$ and $\|y^i(\epsilon)\| \leq A$.

Step 3 We prove that:

$$\forall x \in \Delta, \text{there exists i satisfying:} f_i^\epsilon(x) \leq \epsilon A \text{and} x_i \neq 0. \tag{2}$$

Indeed, let $x \in \Delta$, we have $x \cdot f^\epsilon(x) = \sum_{i=1}^N x_i f_i^\epsilon(x) = \sum_{i:x_i \neq 0} x_i f_i^\epsilon(x)$. As proved above, we have $x \cdot f^\epsilon(x) \leq \epsilon A$. Therefore, we get that $\sum_{i:x_i \neq 0} x_i f_i^\epsilon(x) \leq \epsilon A$. By combining this with $\sum_{i:x_i \neq 0} x_i = \sum_{i=1}^N x_i = 1$, there exists i such that $x_i \neq 0$ and $f_i^\epsilon(x) \leq \epsilon A$.

Step 4 (using the Sperner lemma). Let $K > 0$ be an integer and consider a simplicial subdivision T^K of the unit-simplex Δ of $\mathbb{R}^N$ such that $Mesh(T^K) < 1/K$

and define the labeling R as follows:

$$\forall x \in \Delta, \ R(x) = i, \text{ where i is one of the indices satisfying } f_i^\epsilon(x) \leq \epsilon A \text{ and } x_i \neq 0.$$

According to (2), this labeling is well-defined. It also satisfies the Sperner condition. Indeed, let $x \in$ ri$[[e^{i_1}, e^{i_2}, \ldots, e^{i_m}]]$ where $m \leq N$. Then $\{i : x_i \neq 0\} = \{i_1, \ldots, i_m\}$. By definition of $R(x)$, we have $R(x) \in \{i : x_i \neq 0 \text{ and } f_i^\epsilon(x) \leq \epsilon A\} \subset \{i : x_i \neq 0\}$. So, $R(x) \in \{i_1, \ldots, i_m\}$.

The Sperner lemma implies that there exists a completely labeled subsimplex $[[x^{K,1}, \ldots, x^{K,N}]]$ with $R(x^{K,l}) = l, \forall l = 1, \ldots, N$. Hence, we have $f_l^\epsilon(x^{K,l}) \leq \epsilon A, \forall l = 1, \ldots, N$.

Let $K \to +\infty$, there is a subsequence (K_t) such that

$$\forall l, x^{K_t,l} \to x^\epsilon \in \Delta, \quad f^\epsilon(x^{K_t,l}) \to f^\epsilon(x^\epsilon)$$
$$\text{and, therefore,} \quad f_l^\epsilon(x^\epsilon) \leq \epsilon A, \forall l = 1, \ldots, N.$$

Step 5 Since $\left(B\left(x^i(\epsilon), \epsilon\right)\right)_{i=1,\ldots,I(\epsilon)}$ is a covering of Δ, there exists a set $J(x^\epsilon) \subset \{1, \ldots, I(\epsilon)\}$ such that $x \in \cap_{i \in J(x^\epsilon)} B\left(x^i(\epsilon), \epsilon\right)$. We have $f^\epsilon(x^\epsilon) = \sum_{i \in J(x^\epsilon)} \alpha_i(x^\epsilon) y^i(x^\epsilon)$ with $\sum_{i \in J(x^\epsilon)} \alpha_i(x^\epsilon) = 1$. By using Carathéodory's convexity theorem,[6] we get a decomposition

[6] Carathéodory's convexity theorem states that: In an n-dimensional vector space, every vector in the convex hull of a non-empty set can be written as a convex combination using no more than $n + 1$ vectors from the set. For a simple proof, see Proposition 1.1.2 in Florenzano and Le Van [19] or Theorem 5.32 in Aliprantis and Border [1].

$$f^{\epsilon}(x^{\epsilon}) = \sum_{i=1}^{N+1} \beta_i(x^{\epsilon})\tilde{y}^i(x^{\epsilon})$$

with $\tilde{y}^i(x^{\epsilon}) \in \zeta\big(B(x^{\epsilon}, \epsilon)\big)$, $\beta_i(x^{\epsilon}) \geq 0$, $\sum_{i=1}^{N+1} \beta_i(x^{\epsilon}) = 1$.
Step 6 Let $\epsilon \to 0$, without loss of generality, we can assume that

$$x^{\epsilon} \to \bar{x} \in \Delta, \quad \beta_i(x^{\epsilon}) \to \bar{\beta}_i \geq 0, \sum_{i=1}^{N+1} \bar{\beta}_i = 1, \quad \tilde{y}^i(x^{\epsilon}) \to \bar{y}^i, \forall i = 1, \ldots, N+1.$$

We also have $\bar{y}^i \in \zeta(\bar{x})$ because ζ has a closed graph. Therefore, we get

$$f^{\epsilon}(x^{\epsilon}) \overset{\epsilon \to 0}{\longrightarrow} \bar{z} = \sum_{i=1}^{N+1} \bar{\beta}_i \bar{y}^i \in \zeta(\bar{x}) \text{ (because } \zeta(\bar{x}) \text{ is convex).}$$

Moreover, the condition $f_l^{\epsilon}(x^{\epsilon}) \leq \epsilon A$, $\forall l = 1, \ldots, N$ implies that $\bar{z}_l \leq 0, \forall l = 1, \ldots, N$. Define $\bar{p} \equiv \bar{x}$, we have $\zeta(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$ because $\bar{z} \in \zeta(\bar{p}) \cap \mathbb{R}^N_-$. The proof is over. □

From Lemma 2, we can additionally derive two stronger versions of the Gale–Nikaido–Debreu lemma. Each of them is stated and proved below.

Lemma 3 *Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be an upper semicontinuous correspondence with nonempty, compact, convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the condition*

$$\forall p \in \Delta, \ \exists z \in \zeta(p) \text{ which satisfies } p \cdot z \leq 0.$$

Then there exists $\bar{p} \in \Delta$ such that $\zeta(\bar{p}) \cap \mathbb{R}^N_- \neq \emptyset$.

Proof For $p \in \Delta$, let $\tilde{\zeta}(p) = \{z \in \zeta(p) : z \cdot p \leq 0\}$. The correspondence $\tilde{\zeta}$ is upper semicontinuous, convex, and compact valued from Δ into $\mathbb{R}^N$. It satisfies the assumptions of Lemma 2. Hence there exist $\bar{p}$ and $\bar{z} \in \tilde{\zeta}(\bar{p}) \subset \zeta(\bar{p})$, such that $\bar{z} \leq 0$.□

Lemma 4 *Let Δ be the unit-simplex of $\mathbb{R}^N$. Let ζ be an upper semicontinuous correspondence with nonempty, compact, convex values from Δ into $\mathbb{R}^N$. Suppose ζ satisfies the condition*

$$\forall p \in \Delta, \ \forall z \in \zeta(p), \text{ we have } p \cdot z = 0.$$

Then there exist $\bar{p}$, $\bar{z} \in \zeta(\bar{p})$ such that (1) $\bar{z} \leq 0$, and (2) $\forall i = 1, \ldots, N$, $\bar{p}_i \neq 0 \Rightarrow \bar{z}_i = 0$.

Proof Since "$\forall p \in \Delta, \ \forall z \in \zeta(p), p \cdot z = 0$" $\Rightarrow$ "$\forall p \in \Delta, \ \forall z \in \zeta(p), p \cdot z \leq 0$", from Lemma 2, there exist $\bar{p}$ and $\bar{z} \in \zeta(\bar{p})$ such that $\bar{z} \leq 0$. Since $\bar{p} \cdot \bar{z} = 0$, the conclusion is immediate. □

Remark 2 Florenzano [16] (Lemma 2.1.1) provides another version of the Gale–Nikaido–Debreu lemma (her proof of this result makes use of the separation and the Brouwer fixed point theorems). However, the point $\bar{p}$ in her Lemma 2.1.1 is not proved to be different from zero. In Lemmas 2 and 3, the price $\bar{p}$ is in the unit-simplex and hence not equal to zero (see Florenzano [15], and Florenzano and Le Van [18] for more detailed discussions).

Remark 3 [the Kakutani fixed point theorem and the Gale–Nikaido–Debreu lemma] We emphasize that the Kakutani fixed point theorem can be obtained as a corollary of the Gale–Nikaido–Debreu lemma. We prove this by adapting the argument of Uzawa [39] for continuous mapping.

Let ζ be an upper semicontinuous correspondence, with non-empty convex compact values from Δ into itself. Define, for $p \in \Delta$,

$$\psi(p) = \left\{ y : y = z - \frac{p \cdot z}{\sum_{i=1}^{N} p_i^2} p, \text{ with } z \in \zeta(p) \right\}.$$

One can check that ψ is upper semicontinuous and convex valued. Moreover, for any $p \in \Delta$, any $y \in \psi(p)$, we have $p \cdot y = 0$. Hence, from Lemma 4, there exist $\bar{p} \in \Delta$ and $\bar{y} \in \psi(\bar{p})$ which satisfy $\bar{y} \leq 0$, and $\forall i = 1, \ldots, N$, $\bar{p}_i \neq 0 \Rightarrow \bar{y}_i = 0$. In other words, there exist $\bar{p} \in \Delta$ and $\bar{z} \in \zeta(\bar{p})$ satisfying two conditions:

1. $\forall i = 1, \ldots, N,\ \bar{z}_i \leq \frac{\bar{p} \cdot \bar{z}}{\sum_{i=1}^{N} \bar{p}_i^2} \bar{p}_i$.
2. $\forall i = 1, \ldots, N,\ \bar{p}_i \neq 0 \Rightarrow \bar{z}_i = \frac{\bar{p} \cdot \bar{z}}{\sum_{i=1}^{N} \bar{p}_i^2} \bar{p}_i$.

Hence, if $\bar{p}_i = 0$, we have $0 \leq \bar{z}_i \leq 0$ which in turn implies that $\bar{z}_i = 0$. Let $\mu = \frac{\bar{p} \cdot \bar{z}}{\sum_{i=1}^{N} \bar{p}_i^2}$. We obtain that $\bar{z}_i = \mu \bar{p}_i$ for any $i = 1, \ldots, N$. Since $\bar{z} \in \Delta$, $\bar{p} \in \Delta$, we have $\mu = 1$. Hence, $\bar{p} = \bar{z} \in \zeta(\bar{p})$.

Notice that Florenzano (see Corollary 3 in [14] or Proposition 2 in [15]) also proved the Kakutani fixed point theorem from the Gale–Nikaido–Debreu lemma but she considers for the unit ball instead of the simplex Δ and she makes use of the separation theorem.

4 Conclusion

We have made use of the Sperner lemma to provide a new proof of the Gale–Nikaido–Debreu lemma. It is interesting to notice that, by using the Sperner lemma and algorithms of a combinatorial nature, we can approximate the equilibrium price (see Scarf and Hansen [35], Scarf [36] for more details). By consequence, we hope that our paper provides a fresh alternative way in studying the equilibrium existence, and, potentially, in computing economic equilibria.

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