Anamorphic fractional Fourier transform: optical implementation and applications

David Mendlovic, Yigal Bitran, Rainer G. Dorsch, Carlos Ferreira, Javier Garcia, and Haldun M. Ozaktaz

An additional degree of freedom is introduced to fractional-Fourier-transform systems by use of anamorphic optics. A different fractional Fourier order along the orthogonal principal directions is performed. A laboratory experimental system shows preliminary results that demonstrate the proposed theory. Applications such as anamorphic fractional correlation and multiplexing in fractional domains are briefly suggested.

Key words: Anamorphic systems, fractional Fourier transform, optical signal processing.

1. Introduction

Anamorphic systems are well known and widely used for various special systems. The so-called astigmatic processor was exploited in optical data processing as a device for one-dimensional (1-D) Fourier transforming and imaging in two mutually perpendicular directions.[1,2] Later, the idea of obtaining a nonsymmetrical Fourier transform with crossed cylindrical lenses of different focal lengths working under plane-wave illumination was presented.[3] Based on this idea, an anamorphic two-dimensional (2-D) optical processor was designed.[4] In order to extend the performance of the nonsymmetrical Fourier transformer, Andres *et al.*[5] proposed the use of spherical-wave illumination. In this case, with a two-crossed-cylindrical-lenses setup it is possible to obtain an exact Fourier transform and a greater angular-magnification coefficient. Based on the nonsymmetrical Fourier transformer working under spherical-beam illumination, a matched-filter anamorphic correlator with improved angular discrimination was suggested[6] and was also implemented with multiple matched filters for removing possible ambiguities in the recognition process (see, for example, Ref. 7). In Ref. 8, the anamorphic processor was applied for obtaining fine pseudocoloring encoding.

We investigate the application of the anamorphic approach to fractional-Fourier-transform (FRT) systems. The FRT was recently applied to optical applications by two different definitions: one based on the Wigner-distribution transformation[9] and the other on the propagation in a graded-index medium.[10] It was shown that both definitions are equivalent.[11] The FRT operation offers a considerable number of new applications. Some of them are discussed in Ref. 12.

The combination of an FRT system and anamorphic optics could result in a system that performs an FRT operation with two different fractional orders along the two main axes. This capability, which considerably extends the number of applications of FRT systems, was cited in Ref. 9 but was not elaborated on. In Section 2 the basic details regarding the FRT definition according to the Wigner-distribution transformation are given. Section 3 shows the anamorphic optical setup for performing a nonsymmetrical FRT; the relevant mathematical expressions for designing the optical setup are given. In Section 4 we show computer simulations with laboratory experimental results that demonstrate the feasibility of the suggested system. In Section 5 a list of applications is suggested.

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2. Fractional Fourier Transform and its Relation to the Wigner Distribution

Based on the Wigner-distribution chart definition, the FRT with order P of an input function $u(x)$ was

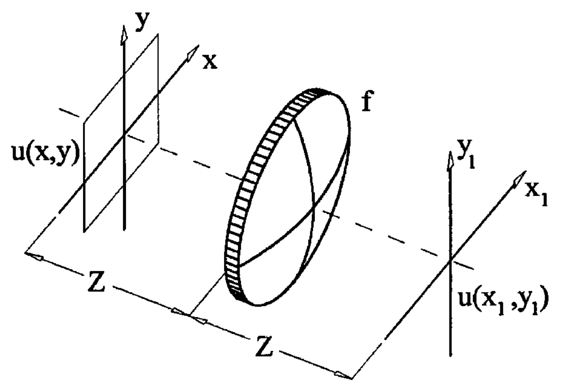

Fig. 1. Optical setup for performing a FRT.

defined as follows.[9] Performing the Pth FRT operation corresponds to rotating the Wigner distribution of $u(x)$ by an angle

$$\phi = P\frac{\pi}{2} \tag{1}$$

in the clockwise direction. On the basis of this definition, Lohmann showed that the bulk-optics system of Fig. 1 performs FRT's of order P provided that the following parameters are used:

$$f = \frac{f_1}{Q}, \tag{2}$$

$$Z = Rf_1, \tag{3}$$

where

$$R = \tan\left(\frac{\phi}{2}\right), \tag{4}$$

$$Q = \sin(\phi), \tag{5}$$

and f_1 is a constant, which connects the spatial frequency with the spatial coordinate in the FRT plane.

By analyzing the propagation of the input signal $u(x)$ through the optical system of Fig. 1, one could write explicitly the FRT operation of order P, $\mathscr{F}^P$, which is the amplitude distribution in the plane defined for the x_1 axis, as

$$u(x_1) = \mathscr{F}^P\{u(x)\} = \int u(x)\exp\left(i\pi\frac{x_1^2 + x^2}{T}\right)\exp\left(-i2\pi\frac{xx_1}{S}\right)\mathrm{d}x, \tag{6}$$

where

$$S = \lambda f_1 \sin(\phi), \tag{7}$$

$$T = \lambda f_1 \tan(\phi), \tag{8}$$

Note that although the above derivations are given for 1-D objects, the generalization to 2-D objects can be done as shown in Ref. 9.

3. Anamorphic Fractional-Fourier-Transform System

The bulk-optics optical system of Fig. 1 was extended for performing nonsymmetrical FRT operations by use of anamorphic optics. We assumed that the system should perform an FRT of order P_x along the x-axis direction and an FRT of order P_y along the y axis. For simplicity, it was also assumed that

$$P_y < P_x. \tag{9}$$

The suggested system for performing the nonsymmetrical FRT is shown in Fig. 2. It contains a symmetrical part for performing the FRT of order P_y, which is exactly the same as that in Fig. 1. Then an anamorphic part was added. In this part, along the x axis, a FRT of order $P_x - P_y$ is performed, while along the y axis, two imaging operations are done. Note

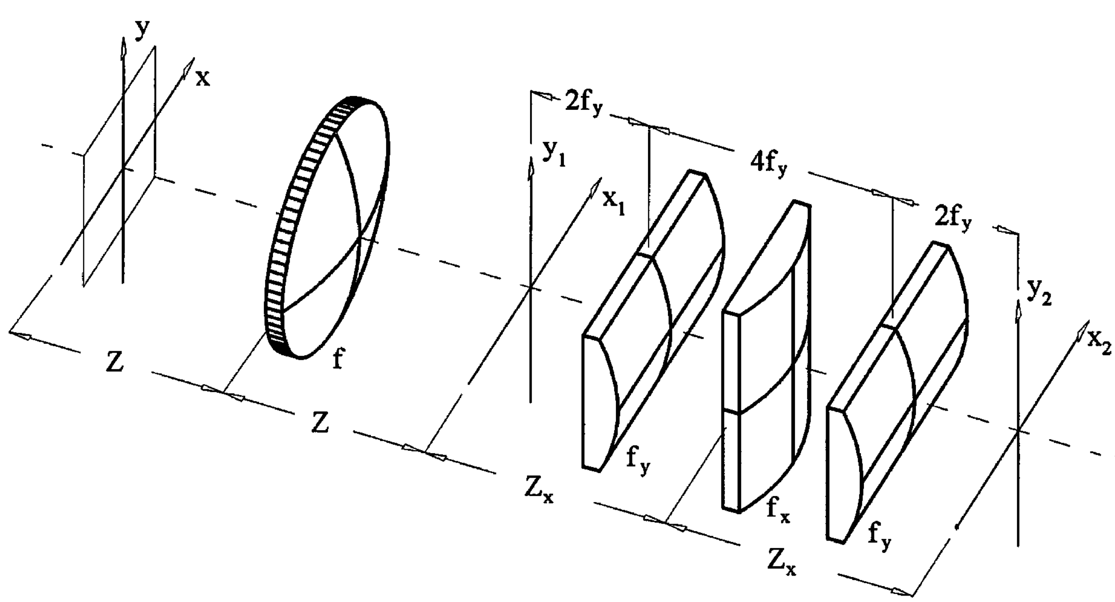

Fig. 2. Anamorphic optical setup for performing FRT's of different orders in the x and the y axes.

that an imaging operation means a FRT of order 2; thus two imaging operations gives a FRT of order 4, which is the self-Fourier order.[13]

Now we deal with the anamorphic part of the system. For performing the x-direction FRT, a cylindrical lens with focal length f_x is placed as shown in Fig. 2. This lens is active only along the x axis. In the y direction two imaging operations are obtained with two identical lenses with a focal length f_y. One can notice that imaging with lateral magnification equal to 1 is obtained only if the distance between the output of the symmetrical system and the output plane is $8f_y$. The system can be shortened to $4f_y$ if a lateral magnification of -1 is admitted.

Analogous to Eq. (1), we have

$$\phi' = (P_x - P_y)\pi/2. \tag{10}$$

Thus, owing to Eq. (3), one can write

$$4f_y = R_x f_1 = f_1 \tan\left(\frac{\phi'}{2}\right)\cdot \tag{11}$$

Equations (2) and (3) lead to

$$f_x = f_1/\sin(\phi'), \tag{12}$$

$$Z_x = R_x f_1. \tag{13}$$

Note that f_1 is the same constant that was used for calculating the symmetric part of the system. This provides the same spatial coordinates scale in both axes.

Based on the knowledge of f_1 and ϕ', and given the lens parameters f_y and f_x, the output of the anamorphic system when the object $u(x, y)$ is placed at the input plane is

$$\begin{aligned} u(x_2, y_2) &= \mathcal{F}_x^{P_x}\mathcal{F}_y^{P_y}\{u(x, y)\} \\ &= \iint u(x, y)\exp\left[\left(i\pi\frac{x_2^2 + x^2}{T_x}\right) + \left(i\pi\frac{y_2^2 + y^2}{T_y}\right)\right] \\ &\quad \times \exp\left(-i2\pi\frac{xx_2}{S_x} - i2\pi\frac{yy_2}{S_y}\right)\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{14}$$

and T_x, T_y, S_x, and S_y are calculated according to Eqs. (7) and (8) with

$$\phi_x = P_x\pi/2, \tag{15}$$

$$\phi_y = P_y\pi/2. \tag{16}$$

4. Simulation and Experimental Results

The suggested anamorphic processor was simulated by computer and experimentally demonstrated. For the computer simulations, MATLAB software for simulating the integral of Eq. (14) was written. Note that it involves a 2-D integral that contains a chirp factor, which requires a huge number of calculations. Thus the estimation of this integral was done by use of a more efficient procedure that is based on the gradient-index medium definition of the FRT.[10] Figure 3(a) shows the input pattern that was used for the simulation. The order $P_x = 0.667$ and $P_y = 0$ was used. Figure 3(b) depicts the corresponding simulation result.

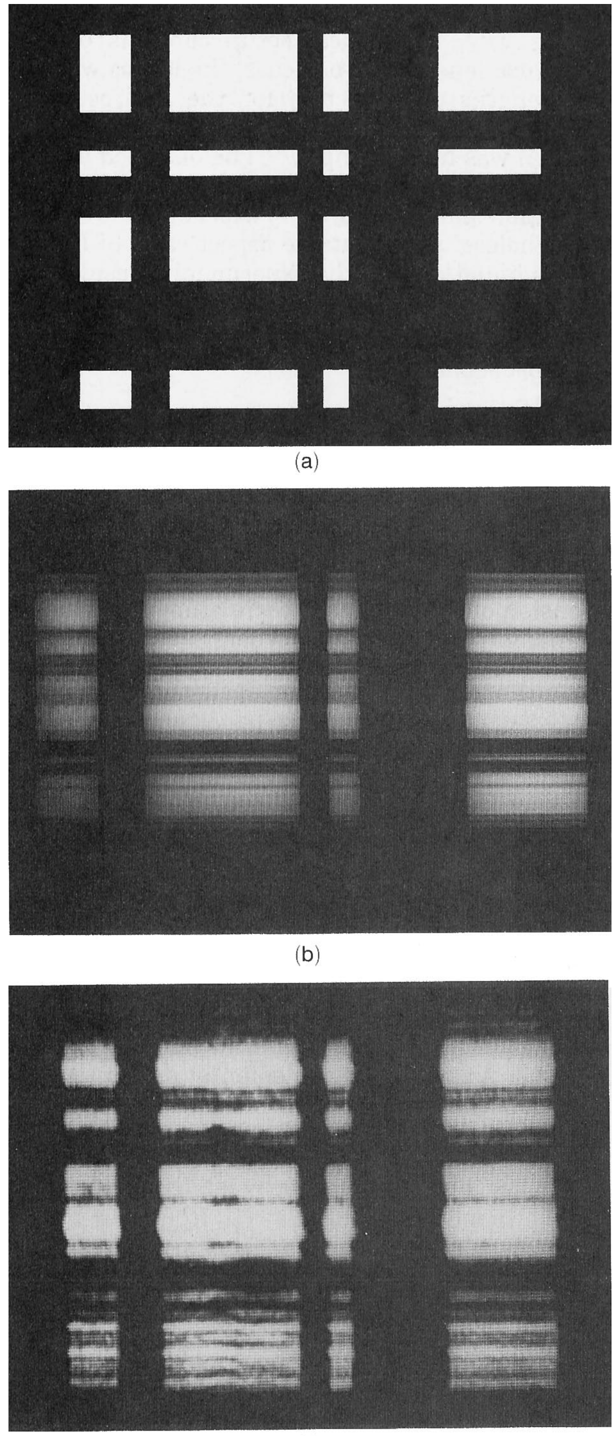

Fig. 3. (a) Input pattern used in experiments, (b) anamorphic FRT of orders $P_x = 0.667$ and $P_y = 0$ calculated digitally, and (c) experimental optical result.

An optical setup similar to the one in Fig. 2 was constructed. In order to eliminate a cylindrical lens, we used a setup that performs $P_y = 2$. That is, along the y direction, we perform an imaging with inversion. In this way, the optical setup contains only two cylindrical lenses. In our case, the lenses were with focal lengths of 304.8 mm for the x direction and 76.2 mm for the y direction. Again, the pattern of Fig. 3(a) was used as input. The obtained output is presented in Fig. 3(c). A fair agreement between the simulation and the laboratory experiment is obtained. Nevertheless, note that the aspect ratio of the computer simulation and the experimental results is not the same.

5. Applications

A. Anamorphic Fractional Correlation

Based on the symmetric FRT, various definitions for the fractional correlation operation were suggested in Ref. 14. It was shown that the fractional correlation is a useful tool for shift-variant pattern recognition and for object localization. Below, we suggest the extension of the fractional correlation definition also to anamorphic fractional orders. If we take into account the conclusion of Ref. 14 that the amount of the shift-variance property could be controlled by the fractional order, the motivation of this anamorphic definition might be a system that has different shift-invariant properties along the two main axes. For instance, when letters written in an x direction line should be recognized, the shift-invariant property along the y direction is not necessary, and we can save it for improving the noise performance of the system.

Figure 4 shows a block diagram of the algorithm for performing an anamorphic fractional correlation. It contains the filter generation procedure and the fractional correlation itself. Note that this procedure is based on the first definition that was given in Ref. 14. Anamorphic implementations of the other definitions can be done in a similar way.

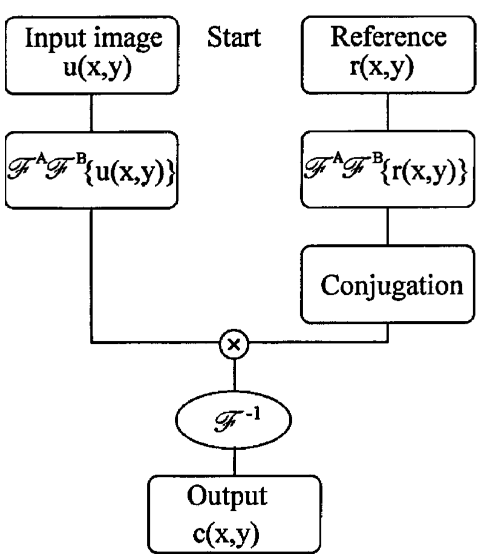

Fig. 4. Block diagram showing the fractional correlation procedure.

B. Efficient Multiplexing

In many practical cases a single transmission line is used to transmit several signals packed together. Multiplexing techniques permit the signal compression for the transmission. An analogous situation arises when several signals are to be stored in a recording medium. In these cases it is important to use in the most efficient way the space–bandwidth product of the recording medium or the transmission line, which permits the transmission of the maximum possible amount of information. Two typical ways for multiplexing are in the space and in the frequency domains. In the first case the signals are split into spatial (or temporal) pieces, which are interleaved for several signals in space (time) sequence. For frequency multiplexing the process is based on adding a bias frequency that is different for every signal, which ensures that the spectra do not overlap. In this case several signals can share the same space without cross talk. The procedure may be easily extended to both space- and frequency-domain multiplexing. The useful area of the transmission line, defined by the aperture of the system and the bandwidth, may be covered by signals of smaller extent.

The multiplexing can be easily illustrated for 1-D signals in the Wigner-distribution space. This joint representation involves simultaneously the space and the frequency domains. The space and the frequency cutoff of the system or transmission line defines a rectangular area $(\Delta X_{\text{Total}}, \Delta\nu_{\text{Total}})$, which limits the signals that are input into the system. Analogously, every signal may be circumscribed in a rectangle of dimensions $(\Delta x, \Delta\nu)$. The packing of signals in the Wigner space consists of shifting every signal, creating an array inside the bandwidth of the system. An example is shown in Fig. 5 for the 1-D case.

Nevertheless, this packing procedure, which uses the space and the frequency bandwidths of the signal, is not always the most efficient one. In many cases,

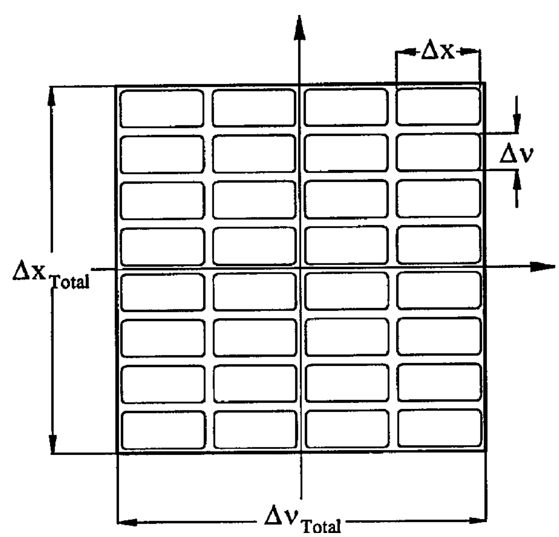

Fig. 5. Space and frequency multiplexing of the signals when the main axes of their domains are aligned with the (x, ν) axes.

the domain of the signal does not fully cover the rectangle defined by $(\Delta x, \Delta \nu)$. The previously described procedure for packing will result in a wasting of the transmission capacity of the system. An example is depicted in Fig. 6. In this situation a rectangle oblique to the (x, ν) axes would include the signal domain more accurately. This can be accomplished by rotation of the Wigner distribution of each signal before multiplexing. Then, adding a bias frequency and shifting every signal, one would obtain an arrangement analogous to that of Fig. 5. As mentioned in Section 1, a FRT of order P produces a rotation of the Wigner distribution by an angle $\phi = P(\pi/2)$. The procedure for demultiplexing would involve a FRT of order $-P$ for every signal. The generalization for 2-D signals is straightforward.

For the case of 2-D signals, if a symmetric optical system is used for performing the FRT, the order of the transformation is the same for both axes. A much greater flexibility is provided if an anamorphic system is used. In this case, it is possible to render a different-order FRT for each variable. The consequent four-dimensional rotation in the Wigner space may adjust the domain of the signal to the axis of the transmission line in a more accurate way. It is worth noting that the rotation can be distinct for every signal. Extension to nonequally spaced distributions of signals in the Wigner space is also possible.

C. Anamorphic Chirp Filtering

A common use of the Fourier transform is its use in spatial filtering. A simple case is that with an image corrupted with an additive single-frequency noise. The filtering of the Fourier transform with two small stops removes this noise without significantly altering the image. More complicated is the case in which the noise does not consist of a single frequency. A particular case that arises in some practical situations is the chirp noise. It consists of a signal with a spatially variable linear frequency, represented, for instance, by $u(x) = \sin(kx^2)$. It may appear as a

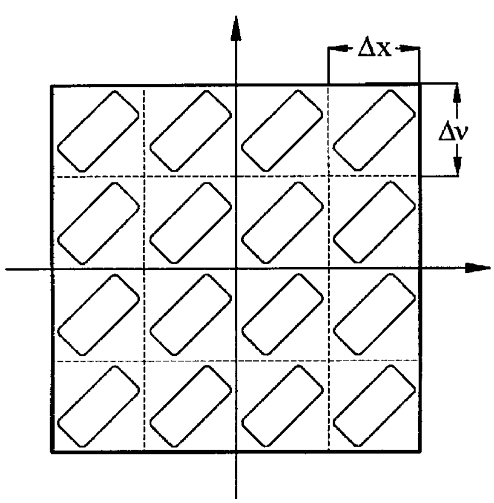

Fig. 6. Inefficient multiplexing of signals with the domains not aligned with the (x, ν) axes.

result of the interference of two spherical beams with different curvatures. This signal is not well located in either space or frequency domains, being, in practice, impossible to remove in any of these spaces by spatial filtering without the image being severely altered. A FRT can solve this problem. The reason is that a chirp signal has a Wigner distribution that is concentrated in a line in the (x, ν) space. Rotating it so that the line is perpendicular to the x axis, we can again obtain a spatial filtering by using a small stop. Recovery of the filtered image would involve an inverse FRT of the same order.

In the 2-D case, chirp noise with distinct characteristics in both spatial axes may appear. If the parameters of the chirps are not the same, a different rotation of the Wigner distribution may be needed to filter out each chirp noise. In a symmetric system, this would involve filtering in two different planes corresponding to different fractional orders. The use of anamorphic optics introduces the possibility of performing a different FRT for two orthogonal axes. The system can simultaneously filter two chirp noises with different characteristics.

6. Conclusions

In this paper the anamorphic fractional Fourier transform was optically implemented by means of a setup that contains cylindrical lenses. Such an anamorphic transform can increase the flexibility of the optical system. The degree of fractional Fourier transform can be set independently for two perpendicular directions. This provides the possibility of acting in different fractional domains in these axes. Experimental results match well with computer simulations. Finally, some applications of anamorphic fractional systems such as anamorphic fractional correlation and multiplexing in the fractional domain are suggested.

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