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Decentralized stabilization: characterization of all solutions and genericity aspects

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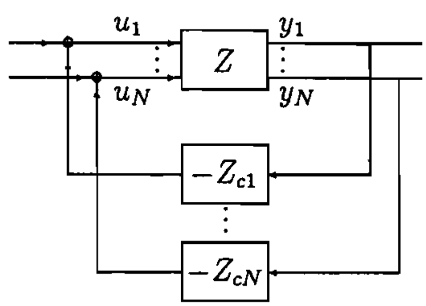
The decentralized stabilization problem of multivariable finite-dimensional systems is considered in a fractional set-up. A new synthesis procedure for decentralized stabilizing compensators is proposed. The class of all admissible local compensators that can be applied to a specified channel as an element of a decentralized compensator is identified. The conditions under which the class of admissible local compensators is generic are investigated. The problem of making a multi-channel system stabilizable and detectable from a single channel applying decentralized feedback around the other channels has been shown to be generically solvable for a given set of dynamic compensators if and only if the plant is strongly connected.

1. Introduction

In this paper we consider the decentralized stabilization problem of linear time-invariant, finite-dimensional systems. Referring to the Figure, let Z be a system having N input–output channels. The decentralized stabilization problem (DSP) is defined as determining N local feedback compensators $Z_{c1}, \ldots, Z_{cN}$, such that the overall closed-loop system is internally stable.



The decentralized feedback system.

In many feedback control problems, the controller is required to process constrained feedback information owing to some practical reasons, which make the centralized (full-feedback) control inefficient or impossible. With this motivation, many researchers have investigated the solvability conditions of DSP during the last two decades. As can be inferred from the use of a constrained feedback scheme, DSP has more restrictive solvability conditions in comparison with the full-feedback stabilization problem. It has been shown (Wang and Davison 1974) that DSP is solvable if and only if the open loop plant has no unstable decentralized fixed

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modes with respect to the proposed decentralized feedback constraint. The fixed modes of a plant are those open loop eigenvalues, which remain unchanged in the closed loop for all possible constant decentralized compensators. In Corfmat and Morse (1976), the solvability of DSP has been shown to be equivalent to the completeness of certain system matrices belonging to the interacting subsystems in case the open loop plant satisfies a connectivity condition called *strong connectedness*. The construction method of decentralized compensators proposed in Corfmat and Morse (1976) is obtained by making the closed loop system stabilizable and detectable from a single channel applying decentralized constant feedback around the other channels.

A direct proof of the equivalence of the completeness condition of Corfmat and Morse (1976) and the absence of decentralized fixed modes as defined by Wang and Davison (1973) has been given in Anderson and Clements (1981). It has later been shown by the fractional representation approach to DSP (Özgüler 1985, 1990, Vidyasagar and Viswanadham 1986, Gündeş and Desoer 1990, Ünyelioğlu and Özgüler 1990) that the strong connectedness assumption can also be removed by applying dynamic compensation to each of the channels instead of constant compensation.

The purpose of this paper is to discuss several synthesis issues concerning decentralized stabilizing compensators. Although the obtained results are technique-independent, we extensively use a stable proper fractional representation technique, since it provides a suitable algebraic and topological structure for handling dynamic feedback problems in lumped parameter systems. This structure is also suitable for considering similar constrained feedback stabilization problems in more general set-ups such as distributed parameter systems (Ünyelioğlu and Özgüler 1991 a) and linear time-varying systems (Poolla and Khargonekar 1987).

Below we summarize the main results of this paper.

(1) A *hierarchically internally stable synthesis procedure* for decentralized compensators is proposed in the constructive proof of Theorem 3.2.

(2) The set of all admissible local compensators that can be applied to a specified channel, as an element of some decentralized stabilizing compensator, is characterized in (I) of Theorem 4.1. The characterization is obtained in terms of only two parameters, independent of the number of channels. This yields the characterization of all decentralized stabilizing compensators of a plant.

(3) The conditions under which the class of admissible local compensators is generic have been determined in (II) of Theorem 4.1. These are purely structural conditions and correspond to certain connectivity relations among the subsystems. It has further been shown in (III) of Theorem 4.1 that, in case these conditions fail to hold, the set of admissible local compensators is precisely the set of internally stabilizing compensators of the corresponding channel. The proof of Theorem 3.2 also yields that the internally stabilizing compensators of a channel is generically admissible for that channel, independent of structural conditions.

(4) The problem of making a multi-channel system stabilizable and detectable from a single channel applying decentralized feedback around the other channels has been shown to be generically solvable for a given set of dynamic local compensators, if and only if the plant is strongly connected (Theorem 4.2).

The paper is organized as follows. In the next section we give some preliminary results which are needed for subsequent discussions. In § 3, the solution of DSP is stated with a new synthesis procedure. In § 4 we discuss several characterization results. These results are illustrated by an example in § 5.

2. Preliminaries

Throughout this paper $\mathscr{R}$ denotes the field of real numbers. $\mathbf{R}(s)$ denotes the field of transfer functions in the indeterminate s with real coefficients. $\mathbf{R}$ and $\mathbf{S}$ denote the rings of proper transfer functions and the stable proper transfer functions of s, respectively. We also denote by $\mathbf{R}[s]$, the ring of polynomials with real coefficients in s. Notice that $\mathbf{R}(s)$ is the field of fractions associated with $\mathbf{S}$. For further details the reader is referred to § 2.1 of Vidyasagar (1985).

By $\mathbf{S}^{k \times l}$, we denote the set of $k \times l$ matrices with the entries over $\mathbf{S}$. If $A \in \mathbf{S}^{k \times l}$, then $|A|$ denotes the determinant of A. Let A, B have the same number of rows. If $[A \;\; B]$ has a right inverse over $\mathbf{S}$, then $[A \;\; B]$ is said to be *left unimodular*. A similar statement can be given for the *right unimodular* matrices, whenever $[C' \;\; D']'$ has a left inverse over $\mathbf{S}$, where $'$ denotes transpose. Sometimes we say, alternatively, that (A, B) is *left coprime* if $[A \;\; B]$ is left unimodular, and (D, C) is *right coprime* if $[C' \;\; D']'$ is right unimodular. A square matrix V is said to be *unimodular* if it has an inverse over $\mathbf{S}$. A *greatest common left factor* L of the pair (A, B) is a square matrix such that $A = L\bar{A}$, $B = L\bar{B}$, for some $\bar{A}$, $\bar{B}$, and for some unimodular matrix V, $[A \;\; B]V = [L \;\; 0]$. In this case $(\bar{A}, \bar{B})$ is left coprime. Also if $[A \;\; B]$ is full row rank, then L is non-singular. Similar statements apply to pairs of matrices with the same number of columns. In this case the word 'row' is replaced by 'column', and 'left' is replaced by 'right'. Two matrices A and B of the same size are said to be equivalent over $\mathbf{S}$, if there are unimodular matrices U and V such that $A = UBV$.

Let Z be a $k \times l$ transfer matrix, i.e. $Z \in \mathbf{R}^{k \times l}$. It is known that there are matrices P, Q, R, W, P_r, Q_r, Q_l, R_l over $\mathbf{S}$ such that

$$Z = PQ^{-1}R + W = P_r Q_r^{-1} = Q_l^{-1} R_l \tag{1}$$

The first representation is called *bicoprime* if (Q, R) is left coprime and (P, Q) is right coprime. The second representation is called a *right coprime representation*, if (P_r, Q_r) is right coprime. Similarly, $Z = Q_l^{-1} R_l$ is a *left coprime representation* if (Q_l, R_l) is left coprime. Note that in obtaining a bicoprime fractional representation of Z one can always choose $W = 0$. A compensator Z_c internally stabilizes $Z = PQ^{-1}R + W$ if and only if

$$\begin{bmatrix} Q & RP_c \\ -P & Q_c + WP_c \end{bmatrix}$$

is unimodular over $\mathbf{S}$, where $Z_c = P_c Q_c^{-1}$ for a right coprime pair of matrices (P_c, Q_c) over $\mathbf{S}$.

Let S be a set. If $S_1 \subset S$ and $S - S_1$ is non-empty, we say that S_1 is a *proper subset of* S. Define $\mathbf{N} := \{1, 2, \ldots, N\}$. Let $\mathbf{N}(1)$ denote the collection of all proper subsets of $\mathbf{N}$ including 1. We shall use the following abbreviations:

$$R_1 = [R_1 \ldots R_l], \quad P_k = [P_1' \ldots P_k']', \quad \text{and} \quad Z_{k,1} := P_k Q^{-1} R_1.$$

It is possible to view **S** and the sets of matrices over **S** as normed algebras. A detailed study of the H_∞ norm on **S**, and the natural topology induced by that norm can be found in Vidyasagar (1985). Here, we give a few definitions concerning the subsets of the topological space $\mathbf{S}^{k\times l}$. A subset of $\mathbf{S}^{k\times l}$ is called *generic* in $\mathbf{S}^{k\times l}$, if it is open and dense in $\mathbf{S}^{k\times l}$. If a property holds true for the elements of a generic subset of $\mathbf{S}^{k\times l}$, then we say that the property holds true *for almost all elements* of $\mathbf{S}^{k\times l}$. Let $G = P_r Q_r^{-1} \in \mathbf{R}^{k\times l}(s)$ be a right coprime fractional representation. There exists a positive real number $\mu(P_r, Q_r)$ such that for all (P, Q) with

$$\left\| \begin{bmatrix} Q \\ P \end{bmatrix} - \begin{bmatrix} Q_r \\ P_r \end{bmatrix} \right\| < \mu(P_r, Q_r)$$

(P, Q) is right coprime and Q is non-singular. Then, a *basic neighbourhood* of G with respect to the chosen factorization $G = P_r Q_r^{-1}$, is defined as

$$B(P_r, Q_r, \varepsilon) = \left\{ \bar{Z} = \bar{P}_r \bar{Q}_r^{-1} \; \left\| \begin{bmatrix} Q_r - \bar{Q}_r \\ P_r - \bar{P}_r \end{bmatrix} \right\| < \varepsilon \right\}$$

for any $\varepsilon < \mu(P_r, Q_r)$. Clearly, a basic neighbourhood is not unique. It is shown in Vidyasagar (1985) that the union of basic neighbourhoods over all rational matrices $G \in \mathbf{R}^{k\times l}(s)$ is a base for a topology Γ over $\mathbf{R}^{k\times l}(s)$. We note that Γ induces a topology over $\mathbf{R}^{k\times l}$ in a natural way: let Γ' be the collection of all intersections with $\mathbf{R}^{k\times l}$ sets of Γ. Then Γ' is the subspace topology over $\mathbf{R}^{k\times l}$. A similar topology can be defined by using the left coprime factorizations. In this case, a property holds true for almost all elements of $\mathbf{S}^{k\times l}$ with respect to one of the topologies if and only if it holds true with respect to the other topology.

Let $Z = PQ^{-1}R + W$ be a given fractional representation. The quadruple (P, Q, R, W) (the triple (P, Q, R), in case $W = 0$) is *complete* (over **S**) if and only if the Smith normal form over **S** of the system matrix

$$\Pi := \begin{bmatrix} Q & R \\ -P & W \end{bmatrix}$$

has at least $r := \text{size}(Q)$ identity elements, i.e. if and only if at least r invariant factors of Π over **S** are identity. Alternatively, we sometimes refer to the system matrix Π as complete whenever (P, Q, R, W) is complete.

Equivalently, (P, Q, R, W) is complete if and only if there exist unimodular matrices U and V of suitable sizes such that

$$U \begin{bmatrix} Q & R \\ -P & W \end{bmatrix} V = \begin{bmatrix} I_r & 0 \\ 0 & \Omega \end{bmatrix} \tag{2}$$

for some matrix Ω, where I_r denotes the identity matrix of size r. A summary of several consequences of this definition can be found in § 2 of Özgüler (1990).

3. Main results

In this section a rigorous definition of DSP is given and a new synthesis method is proposed.

Consider an N-channel plant described by $y = Zu$, where $Z = [Z_{ij}]$, $Z_{ij} \in \mathbf{R}^{p_i \times m_j}$ for $i, j = 1, \ldots, N$, $y = [y'_1 \ldots y'_N]'$, and $u = [u'_1 \ldots u'_N]'$.

Decentralized stabilization problem (DSP)

Given the N-channel plant Z, determine N compensators $Z_{c1} \in \mathbf{R}^{m_1 \times p_1}, \ldots, Z_{cN} \in \mathbf{R}^{m_N \times p_N}$, such that the pair of plants (Z, Z_c) is internally stable, where $Z_c = \text{bdiag}\{Z_{c1}, \ldots, Z_{cN}\}$.

Now, let the plant have the following bicoprime fractional representation,

$$\begin{bmatrix} Z_{11} & \cdots & Z_{1N} \\ \vdots & & \vdots \\ Z_{N1} & \cdots & Z_{NN} \end{bmatrix} = \begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix} Q^{-1}[R_1 \ldots R_N] \tag{3}$$

where $P_i \in \mathbf{S}^{p_i \times r}$, $R_i \in \mathbf{S}^{r \times m_i}$, and $Q \in \mathbf{S}^{r \times r}$. It follows that DSP is solvable if and only if

$$\Sigma_N := \begin{bmatrix} Q & R_1 P_{c1} & \cdots & R_N P_{cN} \\ -P_1 & Q_{c1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ -P_N & 0 & \cdots & Q_{cN} \end{bmatrix} \tag{4}$$

is unimodular, where $Z_{ci} = P_{ci} Q_{ci}^{-1}$ for right coprime pair of matrices (P_{ci}, Q_{ci}), $i = 1, \ldots, N$.

A closely related problem to DSP is the single channel canonicity (more precisely, the stabilizability and detectability) problem which is defined as follows.

Single channel canonicity problem (SCCP)

Given the N-channel plant (3), determine $N-1$ compensators $Z_{c2}, \ldots, Z_{cN}$ such that the closed loop system that results from the application of feedback $u_i = -Z_{ci} y_i$, $i = 2, \ldots, N$ is stabilizable from u_1 and detectable at y_1, i.e. the fractional representation of the closed loop transfer matrix bicoprime: $[P_1 \ 0 \ \ldots \ 0]$ $(\Sigma_{N-1})^{-1}$ $[R_1' \ 0 \ \ldots \ 0]'$, where

$$\Sigma_{N-1} := \begin{bmatrix} Q & R_2 P_{c2} & \cdots & R_N P_{cN} \\ -P_2 & Q_{c2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ -P_N & 0 & \cdots & Q_{cN} \end{bmatrix} \tag{5}$$

Remark

It is immediate from above that if SCCP is solvable then DSP is solvable by applying an internally stabilizing compensator to the first channel. Conversely, if DSP is solvable then SCCP is solvable by the compensators applied to the channels $2, \ldots, N$. We thus obtain the conclusion that DSP is solvable if and only if SCCP is solvable. (See also Theorem 3.2 of Özgüler (1990).) The same result is also stated in Corfmat and Morse (1976) for *strongly connected* plants, where $Z_{c2}, \ldots, Z_{cN}$ are restricted to be constant compensators. □

We now proceed by giving a description of the set of all proper compensators and an explicit characterization of all internally stabilizing proper compensators of a given plant.

Let

$$\bar{Z}_{11} = D_l^{-1} N_l = N_r D_r^{-1} \tag{6}$$

be some left and right coprime fractional representations of a plant transfer matrix $\bar{Z}_{11} \in \mathbf{R}^{p \times m}$. Then, there exist matrices T_l, S_l, S_r, T_r over $\mathbf{S}$ such that

$$\begin{bmatrix} T_l & S_l \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -S_r \\ N_r & T_r \end{bmatrix} = I \tag{7}$$

Using this result, it is seen that given any transfer matrix $Z_c = P_c Q_c^{-1}$ in right fractional representation, there exists a transfer matrix $X = X_2 X_1^{-1}$ such that

$$\begin{bmatrix} Q_c \\ P_c \end{bmatrix} = \begin{bmatrix} T_r & -N_r \\ S_r & D_r \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{8}$$

Conversely, as P_c, Q_c defined as above, any X_1, X_2 with X_1 is non-singular, $X_2 X_1^{-1}$ is proper, yield a transfer matrix $P_c Q_c^{-1}$. Moreover (P_c, Q_c) is right coprime iff (X_2, X_1) is. Similarly, given any transfer matrix $Z_c = \bar{Q}_c^{-1} \bar{R}_c$ in left coprime fractional representation, there exists $Y = Y_1^{-1} Y_2$ such that

$$[\bar{Q}_c \quad \bar{R}_c] = [Y_1 \quad Y_2] \begin{bmatrix} T_l & S_l \\ -N_l & D_l \end{bmatrix} \tag{9}$$

and conversely. The pair $(\bar{Q}_c, \bar{R}_c)$ is left coprime if and only if (Y_1, Y_2) is left coprime. Note that the set of pairs (X_1, X_2) and (Y_1, Y_2) serve as alternative descriptions of all proper compensators.

It follows from the standard Youla–Bongiorno–Jabr–Kučera parametrization that a transfer matrix $Z_c \in \mathbf{R}^{p \times m}$ is an internally stabilizing compensator for $\bar{Z}_{11}$, i.e. the pair $(\bar{Z}_{11}, Z_c)$ is internally stable if and only if

$$\begin{aligned} Z_c &= (S_r + D_r X)(T_r - N_r X)^{-1} \\ &= (T_l - X N_l)^{-1}(S_l + X D_l) \end{aligned} \tag{10}$$

for some $X \in \mathbf{S}^{m \times p}$, provided that $(T_r - N_r X)$ is biproper. This result is now utilized to define a topology over $\mathscr{Z}_c(\bar{Z}_{11})$, the set of all proper internally stabilizing compensators of $\bar{Z}_{11}$. Let $P_c(X) := S_r + D_r X$ and $Q_c(X) := T_r - N_r X$. If $Z_c \in \mathscr{Z}_c(\bar{Z}_{11})$, then for some X, $Z_c = P_c(X) Q_c^{-1}(X)$. Let a basic neighbourhood around $Z_{c0} \in \mathscr{Z}_c(\bar{Z}_{11})$, where $Z_{c0} = P_c(X_0) Q_c^{-1}(X_0)$ be defined as

$$\{P_c(X) Q_c^{-1}(X) \in \mathbf{R}^{p \times m} \mid \|X - X_0\| < \varepsilon\}, \quad \varepsilon > 0$$

Then, using arguments similar to those in § 7.2 of Vidyasagar (1985), it can be shown that the collection of the basic neighbourhoods is a base for a topology on $\mathscr{Z}_c(\bar{Z}_{11})$.

The constructive proof of the 'If' part of the following theorem is the main result of Özgüler (1990), and states the solution of DSP when $N = 2$.

Theorem 3.1

Given the plant (3) with $N = 2$, DSP (and equivalently SCCP) is solvable if and only if (P_1, Q, R_1) and (P_1, Q, R_2) are complete. □

The synthesis procedure of Özgüler (1990) consists of solving SCCP through the application of a compensator at the second channel. As the closed loop system obtained is stabilizable and detectable, any internally stabilizing compensator at the first channel solves DSP. We follow the same approach in the constructive proof of Theorem 3.2 for N-channel systems. It must be noted that for *strongly connected* systems, a similar procedure of solving DSP via obtaining a solution of SCCP is proposed in Corfmat and Morse (1976).

To obtain the solution of N-channel DSP, we use the following lemma, which gives conditions for a closed loop system matrix to be complete. A proof of Lemma 3.1 is given in the Appendix.

Lemma 3.1

Consider the triple

$$\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, Q_{11}, [S_1 \quad S_2]\right)$$

Define $\bar{Z}_{11} := T_1 Q_{11}^{-1} S_1 \in \mathbf{R}^{p \times m}$.

Let $(T_2, Q_{11}, [S_1 \quad S_2])$ and $\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, Q_{11}, S_2\right)$ be complete. Then the following statements hold.

(1) For almost all $Z_c \in \mathscr{Z}_c(\bar{Z}_{11})$

$$\left([T_2 \quad 0], \begin{bmatrix} Q_{11} & S_1 P_c \\ -T_1 & Q_c \end{bmatrix}, \begin{bmatrix} S_2 \\ 0 \end{bmatrix}\right) \tag{11}$$

is complete, where $P_c Q_c^{-1}$ is a right coprime fractional representation of Z_c.

(2) For almost all $Z_c \in \mathbf{R}^{m \times p}$ the triple in (11) is complete if and only if at least one of $\bar{Z}_{12} := T_1 Q_{11}^{-1} S_2$, $Z_{21} := T_2 Q_{11}^{-1} S_1$, and $\bar{Z}_{22} := T_2 Q_{11}^{-1} S_2$ is non-zero, where $Z_c = P_c Q_c^{-1}$ is a right coprime fractional representation of Z_c. □

The constructive proof of the following theorem is one of the main contributions of this paper.

Theorem 3.2

DSP (and equivalently SCCP) is solvable if and only if $(P_{\mathbf{N}-\mathbf{r}}, Q, R_{\mathbf{r}})$ and $(P_{\mathbf{r}}, Q, R_{\mathbf{N}-\mathbf{r}})$ are complete for all $\mathbf{r} \in \mathbf{N}(1)$. □

Remark

For any $\mathbf{r} \in \mathbf{N}(1)$, $(P_{\mathbf{N}-\mathbf{r}}, Q, R_{\mathbf{r}})$, and $(P_{\mathbf{r}}, Q, R_{\mathbf{N}-\mathbf{r}})$ are called as the complementary subsystems including channel 1 (Corfmat and Morse 1976). Thus, DSP is solvable if and only if all complementary subsystems including channel 1 are complete. Since the role of each channel is symmetric, once this condition holds true for channel 1, it holds true for all other channels in the system as well. So, it is enough to check the condition for any fixed but otherwise arbitrary channel. □

Proof of Theorem 3.2

If. The proof of the If part is established by induction. Let $N = 2$. The statement reduces to two-channel DSP. Indeed, in this case $\mathbf{N}(1) = \{1\}$ and the hypothesis implies, (P_2, Q, R_1) and (P_1, Q, R_2) are complete. So, using theorem 3.1 the solution is obtained.

Assume that the theorem is true for $N = H \geqslant 2$. Define $L := H + 1$. Note that, if $\mathbf{r} \in \mathbf{H}(1)$, then $\mathbf{r} \in \mathbf{L}(1)$, $L \cup \{\mathbf{H} - \mathbf{r}\} = \mathbf{L} - \mathbf{r}$, $\mathbf{r} \cup L \in \mathbf{L}(1)$ and $\mathbf{H} - \mathbf{r} = \mathbf{L} - (\mathbf{r} \cup L)$. Also $L = \mathbf{L} - \mathbf{H}$. Moreover, by the hypothesis, $(P_{\mathbf{L}-\mathbf{r}}, Q, R_{\mathbf{r}})$ are complete for all $\mathbf{r} \in \mathbf{L}(1)$. These statements imply that,

$$\left(\begin{bmatrix} P_L \\ P_{\mathbf{H}-\mathbf{r}} \end{bmatrix}, Q, R_{\mathbf{r}}\right) \quad \text{and} \quad \left(P_{\mathbf{H}-\mathbf{r}}, Q, [R_L \quad R_{\mathbf{r}}]\right)$$

are complete for all $\mathbf{r} \in \mathbf{H}(1)$, and $(P_L, Q, R_{\mathbf{H}})$ is complete.

Now fix any $\mathbf{r} \in \mathbf{H}(1)$ and let $Q_{11} := Q$, $T_1 := P_L$, $T_2 := P_{\mathbf{H}-\mathbf{r}}$, $S_1 := R_L$, and $S_2 := R_{\mathbf{r}}$. Applying Lemma 3.1 we have that

$$\left([P_{\mathbf{H}-\mathbf{r}} \quad 0], \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathbf{r}} \\ 0 \end{bmatrix}\right)$$

is complete for almost all $Z_c \in \mathscr{Z}_c(Z_{LL})$. Let $\mathscr{Z}_c^{\mathbf{r}}$ denote the set of these compensators. Clearly $\mathscr{Z}_c^{\mathbf{r}}$ is generic in $\mathscr{Z}_c(Z_{LL})$. Since $\mathbf{r}$ is fixed but otherwise arbitrary, this result is true for all $\mathbf{r} \in \mathbf{H}(1)$. Moreover, $\bigcup_{\mathbf{r} \in \mathbf{H}(1)} \mathscr{Z}_c^{\mathbf{r}}$ is generic in $\mathscr{Z}_c(Z_{LL})$. Now let $Q_{11} := Q$, $T_1 := P_L$, $T_2 := R_L$, and $S_2 := R_{\mathbf{H}}$. Lemma 3.1 and the fact that the system we consider is bicoprime, give us that

$$\left([0 \quad 0], \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathbf{H}} \\ 0 \end{bmatrix}\right)$$

is complete for all Z_c included in a generic subset of $\mathscr{Z}_c(Z_{LL})$. In other words

$$\left(\begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathbf{H}} \\ 0 \end{bmatrix}\right)$$

is left coprime for almost all $Z_c \in \mathscr{Z}_c(Z_{LL})$. Now, consider the following conditions.

(i) $\left([P_{\mathbf{H}-\mathbf{r}} \quad 0], \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathbf{r}} \\ 0 \end{bmatrix}\right)$ are complete for all $\mathbf{r} \in \mathbf{H}(1)$.

(ii) $\left(\begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathbf{H}} \\ 0 \end{bmatrix}\right)$ is left coprime.

(i′) $\left([P_{\mathbf{r}}, 0], \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathbf{H}-\mathbf{r}} \\ 0 \end{bmatrix}\right)$ are complete for all $\mathbf{r} \in \mathbf{H}(1)$.

(ii′) $\left([P_{\mathbf{H}} \quad 0], \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}\right)$ is right coprime.

By using the above discussion, we obtain the conclusion that for almost all $Z_c \in \mathscr{Z}_c(Z_{LL})$, the conditions (i) and (ii) hold simultaneously. The dual of this result says that for almost all $Z_c \in \mathscr{Z}_c(Z_{LL})$, (i′) and (ii′) hold simultaneously. Since the intersection of open and dense subsets is open and dense, the set of compensators satisfying (i), (ii), (i′) and (ii′) simultaneously, is also open and dense in $\mathscr{Z}_c(Z_{LL})$. Now fix one such compensator, and consider the closed loop system represented by

$$\left([P_{\mathbf{H}} \quad 0], \begin{bmatrix} Q & R_L P_c \\ -P_L & Q_c \end{bmatrix}, \begin{bmatrix} R_{\mathbf{H}} \\ 0 \end{bmatrix}\right)$$

Using (i), (ii), (i′), (ii′), and by the inductive hypothesis for $N = H(= (L-1))$, DSP is solvable for the L-channel system.

Only if. The proof closely follows the proof of the 'Only if' part of Theorem 3.1 of Özgüler (1990), with obvious extensions of the arguments to the N-channel case. □

Now assume that the completeness conditions of Theorem 3.2 hold. The design methodology in the theorem is to apply a compensator to Channel N such that the closed loop system (with the remaining $N-1$ channels) satisfies the following two conditions:

(A) The $N-1$-channel system is jointly stabilizable and detectable.

(B) All complementary subsystems including Channel 1 of the $(N-1)$-channel system are complete.

The synthesis procedure continues inductively, and ends up with the first channel, from which the closed-loop system is now stabilizable and detectable. By applying to the first channel an internally stabilizing compensator for the closed-loop system, the synthesis procedure is terminated. This also leads to a *hierarchically internally stable synthesis procedure*, since at each step the local compensator can be chosen as an internally stabilizing compensator of the respective channel in the closed-loop. It has been shown (Ünyelioğlu and Özgüler 1991 b) that this procedure improves the reliability of the closed-loop system due to certain types of subsystem interconnection breakdowns which result in discrete (on–off) changes in the plant parameters.

4. Characterization results

In this section we utilize the synthesis procedure of Theorem 3.2 in order to characterize the class of all local feedback compensators that can be applied to a specified channel, as an element of some decentralized stabilizing compensator. More explicitly we consider the following problem. We say that Z_{cN} is an admissible local compensator for Channel N, if there exist compensators $Z_{c1}, \ldots, Z_{cN-1}$, such that the decentralized compensator diag $\{Z_{c1}, \ldots, Z_{cN-1}, Z_{cN}\}$ internally stabilizes Z. The characterization of the set of all admissible compensators for each channel also yields a characterization of all decentralized stabilizing compensators of the plant in the following way.

For simplicity let $N = 2$. One can obtain the characterization of admissible local compensators for channel 2. (This also yields the characterization of all compensators solving SCCP.) After a fixed compensator is applied around the second channel, the class of all stabilizing compensators for the single channel system can be obtained by known methods (see also Özgüler 1990). This procedure can be repeated for all admissible compensators of the second channel, and hence all decentralized stabilizing compensators can be obtained by repeating the process. Although this is a tedious work (especially when N is large), we believe that there is no alternative way of giving a simpler characterization of all decentralized stabilizing compensators because of the complex nature of the problem. The alternative parametrization of all decentralized stabilizing compensators in Gündeş and Desoer (1990), for example, is given in terms of a solution of a multi-parameter (depending on N) unimodularity equation. This not only makes the characterization of stabilizing decentralized compensators, but also that of admissible local compensators for a specified channel, quite difficult to obtain. As can be seen from (I) of Theorem 4.1, our characterization of admissible local compensators is given in terms only of two parameters (independent of N) which satisfy certain coprimeness and completeness relations. In (II) of Theorem 4.1, we give certain connectivity conditions under which the class of admissible local compensators is generic among all compensators. From statement (III) of Theorem 4.1, if these conditions fail to hold, then the class of admissible local compensators is precisely the set of internally stabilizing compensators of the corresponding channel. We remind that, from the proof of Theorem 3.2, any internally stabilizing compensator of a channel, independent of connectivity conditions, is generically an admissible compensator.

We proceed by recalling the definition of a strongly connected system (Corfmat and Morse 1976).

Definition

The plant in (3) is said to be *strongly connected* if the transfer matrix of each complementary subsystem including channel 1 is non-zero. That is, the plant is strongly connected if $Z_{\mathbf{N}-\mathbf{r},\mathbf{r}} \neq 0$ and $Z_{\mathbf{r},\mathbf{N}-\mathbf{r}} \neq 0$, for all $\mathbf{r} \in \mathbf{N}(1)$. □

We now obtain a useful characterization for the set

$$\mathscr{Z}_{cN} := \{ Z \in \mathbf{R}^{m_N \times p_N} \mid \text{There exists } \{Z_1, \ldots, Z_{N-1}\} \in \mathbf{R}^{m_1 \times p_1} \times \ldots \times \mathbf{R}^{m_{N-1} \times p_{N-1}}, \text{ such that } \{Z_1, \ldots, Z_{N-1}, Z\} \text{ solves DSP}\}$$

which is the set of all admissible local compensators of channel N. Thus, $\mathscr{Z}_{cN}$ is the set of compensators $Z_{cN} = P_c Q_c^{-1}$ such that (i), (ii), (i′) and (ii′) in the proof of Theorem 3.2 are satisfied with $H = N - 1$. The characterization of $\mathscr{Z}_{cN}$ depends heavily on various quantities defined in the proof of Lemma 3.1. Let $H := N - 1$ and consider the conditions (i), (ii), (i′) and (ii′) of § 3.

Let $Z_{cN} = P_c Q_c^{-1} \in \mathscr{Z}_{cN}$, where P_c, Q_c are parametrized as in (8), in terms of X_1, X_2, such that X_2, X_1^{-1} is proper.

Now fix any $r \in \mathbf{H}(1)$. Letting $Q_{11} := Q$, $T_1 := P_N$, $T_2 := P_{\mathbf{H}-\mathbf{r}}$, $S_1 := R_N$, $S_2 := R_{\mathbf{r}}$, and following the arguments in the proofs of Theorem 3.2 and Lemma 3.1, it is seen that there exist $A_{\mathbf{r}}$, $B_{\mathbf{r}}$, given by (26), $\Psi_{\mathbf{r}}$, given by (19), $\Theta_{\mathbf{r}}$, given by (22), and $\Omega_{\mathbf{r}}$, $\Gamma_{\mathbf{r}}$, given by (28) such that (i) holds for $\mathbf{r}$ if and only if

$$(-\Omega_{\mathbf{r}}(S_r X_1 + D_r X_2), A_{\mathbf{r}} X_1 + B_{\mathbf{r}} X_2, \Theta_{\mathbf{r}}, \Gamma_{\mathbf{r}}, \Psi_{\mathbf{r}})$$

is complete. Moreover, by letting $Q_{11} := Q$, $T_1 := P_N$, $T_2 := P_{\mathbf{r}}$, $S_1 := R_N$, $S_2 := R_{\mathbf{H}-\mathbf{r}}$, and following the same arguments, there exist $A_{\mathbf{H}-\mathbf{r}}$, $B_{\mathbf{H}-\mathbf{r}}$, given by (26), $\Psi_{\mathbf{H}-\mathbf{r}}$, given by (19), $\Theta_{\mathbf{H}-\mathbf{r}}$, given by (22), and $\Omega_{\mathbf{H}-\mathbf{r}}$, $\Gamma_{\mathbf{H}-\mathbf{r}}$, given by (28), such that (i′) holds for $\mathbf{r}$ if and only if

$$(-\Omega_{\mathbf{H}-\mathbf{r}}(S_r X_1 + D_r X_2), A_{\mathbf{H}-\mathbf{r}} X_1 + B_{\mathbf{H}-\mathbf{r}} X_2, \Theta_{\mathbf{H}-\mathbf{r}} \Gamma_{\mathbf{H}-\mathbf{r}}, \Psi_{\mathbf{H}-\mathbf{r}})$$

is complete.

In the special case $\mathbf{r} = \mathbf{H}$, letting $Q_{11} := Q$, $T_1 := P_N$, $T_2 := 0$, $S_1 := R_N$, $S_2 := R_{\mathbf{H}}$, and following Theorem 3.2 and Lemma 3.1, there exist $A_{\mathbf{H}}$, $B_{\mathbf{H}}$, $\Theta_{\mathbf{H}}$, and $\Gamma_{\mathbf{H}}$ such that (ii) holds if and only if

$$(A_{\mathbf{H}} X_1 + B_{\mathbf{H}} X_2, \Theta_{\mathbf{H}} \Gamma_{\mathbf{H}})$$

is left coprime. Similarly, in the special case $\mathbf{r} = \emptyset$, letting $Q_{11} := Q$, $T_1 := P_N$, $T_2 := P_{\mathbf{H}}$, $S_1 := R_N$, $S_2 := 0$, and following Theorem 3.2 and Lemma 3.1, there exist $A_\emptyset$, $B_\emptyset$, $\Phi_\emptyset$, and $\Omega_\emptyset$, such that (ii′) holds if and only if

$$(-\Omega_\emptyset \Phi_\emptyset, A_\emptyset X_1 + B_\emptyset X_2)$$

is right coprime.

We summarize these results in Theorem 4.1 below.

Theorem 4.1

(I) $\mathscr{Z}_{cN}$ consists of $Z_{cN} = P_c Q_c^{-1}$, where P_c, Q_c are parametrized as in (8), in terms of X_1, X_2 such that $X_2 X_1^{-1}$ is proper, and (*a*), (*b*) and (*c*) below simultaneously hold:

(*a*) $$(-\Omega_{\mathbf{r}}(S_r X_1 + D_r X_2), A_r X_1 + B_{\mathbf{r}} X_2, \Theta_r \Gamma_{\mathbf{r}}, \Psi_{\mathbf{r}})$$

and

$$(-\Omega_{\mathbf{H}-\mathbf{r}}(S_r X_1 + S_r X_2), A_{\mathbf{H}-\mathbf{r}} X_1 + B_{\mathbf{H}-\mathbf{r}} X_2, \Theta_{\mathbf{H}-\mathbf{r}} \Gamma_{\mathbf{H}-\mathbf{r}}, \Psi_{\mathbf{H}-\mathbf{r}})$$

are complete for all $\mathbf{r} \in \mathbf{H}(1)$,

(*b*) $$(A_{\mathbf{H}} X_1 + B_{\mathbf{H}} X_2, \Theta_{\mathbf{H}} \Gamma_{\mathbf{H}})$$

is left coprime,

(*c*) $$(-\Omega_\emptyset \Phi_\emptyset, A_\emptyset X_1 + B_\emptyset X_2)$$

is right coprime.

(II) $\mathscr{Z}_{cN}$ is an open dense subset of $\mathbf{R}^{m_N \times p_N}$ if and only if (*a*) and (*b*) below simultaneously hold

(*a*) $\mathscr{Z}_{N,\mathbf{H}} = P_N Q^{-1} R_{\mathbf{H}} \neq 0$ and $Z_{\mathbf{H},N} = P_{\mathbf{H}} Q^{-1} R_N \neq 0$

(*b*) For each $\mathbf{r} \in \mathbf{H}(1)$,

$$Z_{(N \cup \mathbf{H}) - \mathbf{r},\mathbf{r}} \neq 0, \quad \text{or} \quad Z_{\mathbf{H}-\mathbf{r}, N \cup \mathbf{r}} \neq 0,$$

and

$$Z_{\mathbf{r},(N \cup \mathbf{H}) - \mathbf{r}} \neq 0 \quad \text{or} \quad Z_{N \cup \mathbf{r}, \mathbf{H}-\mathbf{r}} \neq 0$$

(II) If one of (*a*) or (*b*) of (II) is violated, then $\mathscr{Z}_{cN} = \mathscr{Z}_c(Z_{NN})$. □

For the proofs of statements (II) and (III) in Theorem 4.1, we need the technical lemma below, whose simple proof is omitted.

Lemma 4.1

Consider the triple

$$\left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, Q_{11}, [S_1 \quad S_2]\right)$$

where $(Q_{11}, [S_1 \; S_2])$ is left and $(Q_{11}, [T_1' \; T_2']')$ is right coprime pairs. Also let

(T_1, Q_{11}, S_2) and (T_2, Q_{11}, S_1) be complete triples. Consider

$$\begin{bmatrix} Q_{11} & S_1 P_{c1} & S_2 P_{c2} \\ -T_1 & Q_{c1} & 0 \\ -T_2 & 0 & Q_{c2} \end{bmatrix} \tag{12}$$

where (P_{c1}, Q_{c1}) and (P_{c2}, Q_{c2}) are coprime pairs.

In the case of one $\bar{Z}_{12} := T_1 Q_{11}^{-1} R_2$ or $\bar{Z}_{21} := T_2 Q_{11}^{-1} R_1$ is zero, the matrix in (12) is unimodular if and only if $(\bar{Z}_{11}, P_{c1} Q_{c1}^{-1})$ and $(\bar{Z}_{22}, P_{c2} Q_{c2}^{-1})$ are internally stable, where $\bar{Z}_{11} := T_1 Q_{11}^{-1} R_1$ and $\bar{Z}_{22} := T_2 Q_{11}^{-1} R_2$. □

Lemma 4.1 states that the decentralized compensator diag $\{Z_{c1}, Z_{c2}\}$ solves the decentralized stabilization problem for a two-channel not-strongly connected plant with no unstable decentralized fixed modes if and only if Z_{c1} and Z_{c2} internally stabilize channels 1 and 2, respectively.

Proof of Theorem 4.1

Proof of (I) follows from the discussion preceding the theorem. We will now prove the 'If' part of (II). Assume that for all $\mathbf{r} \in \mathbf{H}(1)$, at least one of $\Gamma_{\mathbf{r}}$, $\Omega_{\mathbf{r}}$ and $\Psi_{\mathbf{r}}$ is non-zero and $\Gamma_{\mathbf{H}}$ is non-zero. In this case Lemma 3.6, (2) of Lemma 3.1, and the fact that the union of generic sets is generic, reveal that for almost all compensators $Z_c \in \mathbf{R}^{r_N \times p_N}$, (i) and (ii) in the proof of Theorem 3.2 hold. Similarly, assume that for all $\mathbf{r} \in \mathbf{H}(1)$ at least one of $\Gamma_{\mathbf{H}-\mathbf{r}}$, $\Omega_{\mathbf{H}-\mathbf{r}}$ and $\Psi_{\mathbf{H}-\mathbf{r}}$ is non-zero and $\Omega_\emptyset$ is non-zero. Then, for almost all compensators $Z_c \in \mathbf{R}^{r_N \times p_N}$, (i') and (ii') in the proof of Theorem 3.2 hold. On the other hand, a closer inspection of the proof of Lemma 3.1 reveals that for some $\mathbf{r} \in \mathbf{H}(1)$, $\Gamma_{\mathbf{r}}$, $\Omega_{\mathbf{r}}$ and $\Psi_{\mathbf{r}}$ are all zero if and only if

$$Z_{\mathbf{H}-\mathbf{r},\mathbf{r}} = 0, \quad Z_{N,\mathbf{r}} = 0, \quad Z_{\mathbf{H}-\mathbf{r},N} = 0$$

or, equivalently

$$Z_{(N \cup \mathbf{H})-\mathbf{r},\mathbf{r}} = 0, \quad Z_{\mathbf{H}-\mathbf{r},N \cup \mathbf{r}} = 0$$

and $\Gamma_{\mathbf{H}} = 0$ if and only if $Z_{N,\mathbf{H}} = 0$.

Similarly, for some $\mathbf{r} \in \mathbf{H}(1)$, $\Gamma_{\mathbf{H}-\mathbf{r}}$, $\Omega_{\mathbf{H}-\mathbf{r}}$, and $\Psi_{\mathbf{H}-\mathbf{r}}$ are all zero if and only if

$$Z_{\mathbf{r},(N \cup \mathbf{H})-\mathbf{r}} = 0, \quad Z_{N \cup \mathbf{r},\mathbf{H}-\mathbf{r}} = 0,$$

and $\Omega_\emptyset = 0$ if and only if $Z_{N,\mathbf{H}} = 0$. This completes the 'If' part of the proof.

Now, we will prove (III) and the 'Only if' part of (II). Assume, $Z_{(N \cup \mathbf{H})-\mathbf{r},\mathbf{r}} = 0$ and $Z_{\mathbf{H}-\mathbf{r},N \cup \mathbf{r}} = 0$ for some $\mathbf{r} \in \mathbf{H}(1)$. Then, by a suitable permutation at the inputs and outputs, the transfer matrix structure of Z takes the following form.

	$\mathbf{H}-\mathbf{r}$	N	$\mathbf{r}$
$\mathbf{H}-\mathbf{r}$	×	0	0
N	×	×	0
$\mathbf{r}$	×	×	×

where the × subblocks are not important for our discussion. In this case applying Lemma 4.1 repeatedly, first by letting

$$\begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} \end{bmatrix} := \begin{bmatrix} Z_{(N \cup \mathbf{H})-\mathbf{r},(N \cup \mathbf{H})-\mathbf{r}} & Z_{(N \cup \mathbf{H})-\mathbf{r},\mathbf{r}} \\ Z_{\mathbf{r},(N \cup \mathbf{H})-\mathbf{r}} & Z_{\mathbf{r},\mathbf{r}} \end{bmatrix}$$

and then letting

$$\begin{bmatrix}\bar{Z}_{11} & \bar{Z}_{12}\\ \bar{Z}_{21} & \bar{Z}_{22}\end{bmatrix} := \begin{bmatrix}Z_{\mathbf{H}-\mathbf{r},\mathbf{H}-\mathbf{r}} & Z_{\mathbf{H}-\mathbf{r},N}\\ Z_{N,\mathbf{H}-\mathbf{r}} & Z_{N,N}\end{bmatrix}$$

we conclude that $\mathscr{Z}_{cN} = \mathscr{Z}_c(Z_{NN})$. In the case $Z_{N,\mathbf{H}} = 0$ applying Lemma 4.1 by letting

$$\begin{bmatrix}\bar{Z}_{11} & \bar{Z}_{12}\\ \bar{Z}_{21} & \bar{Z}_{22}\end{bmatrix} := \begin{bmatrix}Z_{\mathbf{H},\mathbf{H}} & Z_{\mathbf{H},N}\\ Z_{N,\mathbf{H}} & Z_{N,N}\end{bmatrix}$$

we conclude that $\mathscr{Z}_{cN} = \mathscr{Z}_c(Z_{NN})$. Dual arguments follow for the case when $Z_{\mathbf{H},N}$ is zero, or $Z_{\mathbf{r},(N\cup\mathbf{H})-\mathbf{r}}$ and $Z_{N\cup\mathbf{r},\mathbf{H}-\mathbf{r}}$ is zero. This completes the proof of (III). Now note that $\mathscr{Z}_c(Z_{NN})$ is not dense in $\mathbf{R}^{m_N \times p_N}$. To see this let $Z_{c_0} \in \mathbf{R}^{m_N \times p_N}$ be such that the closed loop characteristic polynomial of (Z_{NN}, Z_{c_0}) has unstable zeros other than zero. Then, for all Z_c belonging to a sufficiently small open ball around Z_{c_0}, the closed loop characteristic polynomial of (Z_{NN}, Z_c) still contains unstable zeros, which implies that $\mathscr{Z}_c(Z_{NN})$ is not dense in $\mathbf{R}^{m_N \times p_N}$. This completes the proof of the 'Only if' part of (II). □

We now consider the class of compensators solving SCCP. Theorem 4.2 below states that once the solvability conditions are satisfied, then the class of compensators solving SCCP is generic, if and only if the plant is strongly connected.

Theorem 4.2

Let SCCP be solvable. The set of compensators $\{Z_{c2}, \ldots, Z_{cN}\}$, where $Z_{ci} = P_{ci}Q_{ci}^{-1}$, (P_{ci}, Q_{ci}) is right coprime $i = 2, \ldots, N$, such that

$$[P_1 \quad 0 \quad \ldots \quad 0](\Sigma_{N-1})^{-1}[R_1' \quad 0 \ \vdots \ 0]' \tag{13}$$

is bicoprime, where Σ_{N-1} is given by (5), is generic in $\mathbf{R}^{m_2 \times p_2} \times \ldots \times \mathbf{R}^{m_N \times p_N}$ (with respect to the product topology induced by $\mathbf{R}^{m_i \times p_i}$, $i = 1, \ldots, N$) if and only if the plant is strongly connected. □

The proof of Theorem 4.2 requires the following lemma which gives necessary and sufficient conditions for a closed loop transfer matrix to be non-zero. A proof of Lemma 4.2 is given in the Appendix.

Lemma 4.2

Consider the triple $([T_1' \ \ T_2']', Q_{11}, [S_1 \ \ S_2])$. Then,

$$[T_2 \quad 0]\begin{bmatrix}Q_{11} & S_1P_c\\ -T_1 & Q_c\end{bmatrix}^{-1}\begin{bmatrix}S_2\\ 0\end{bmatrix} \neq 0 \tag{14}$$

for some $Z_c = P_cQ_c^{-1} \in \mathbf{R}^{m \times p}$ where (P_c, Q_c) is right coprime, if and only if

$$\bar{Z}_{2,\{1,2\}} \neq 0, \quad \text{and} \quad \bar{Z}_{\{1,2\},2} \neq 0, \tag{15}$$

where $\bar{Z}_{2,\{1,2\}} := T_2Q_{11}^{-1}[S_1 \ \ S_2]$, and $\bar{Z}_{\{1,2\},2}$ is defined similarly.

Moreover, if (15) holds, then the set of $Z = P_cQ_c^{-1}$ for which (14) holds is an open and dense subset of $\mathbf{R}^{m \times p}$. □

Proof of Theorem 4.2

Only if. Assume that for some $\mathbf{r} \in \mathbf{N}(1)$, $Z_{N-\mathbf{r},\mathbf{r}} = 0$. If $\mathbf{r} = \mathbf{H}$, with $H := N-1$, then Theorem 4.1 states that $\mathscr{Z}_{cN}$ is only an open and dense subset of $\mathscr{Z}_c(Z_{NN})$. Otherwise Lemma 4.2 reveals that

$$[P_{\mathbf{H}-\mathbf{r}'} \quad 0]\begin{bmatrix} Q & R_N P_c \\ -P_N & Q_c \end{bmatrix}^{-1}\begin{bmatrix} R_{\mathbf{r}'} \\ 0 \end{bmatrix} = 0$$

for some $\mathbf{r}' \in \mathbf{H}(1)$. Repeating this inductively until $N = 1$, it is observed that at some step $\tilde{Z}_{N,\mathbf{H}} = 0$, where $\tilde{}$ denotes the closed loop transfer matrix. In this case $\mathscr{Z}_{cN}$ is an open and dense subset of $\mathscr{Z}_c(\tilde{Z}_{NN})$, because of Theorem 4.1. Dual arguments follow, if, for some $\mathbf{r} \in \mathbf{N}(1)$, $Z_{\mathbf{N}-\mathbf{r},\mathbf{N}} = 0$. On the other hand, it can be shown that $\mathscr{Z}_c(Z_{NN})$ is not dense in $\mathbf{R}^{m_N \times p_N}$. (See the proof of Theorem 4.1.) This completes the proof of the necessity part.

If. If the hypothesis is true, (*a*) and (*b*) in (II) of Theorem 4.1 hold. Hence, $\mathscr{Z}_{cN}$ is open and dense in $\mathbf{R}^{m_N \times p_N}$. Also, applying Lemma 4.2 it is seen that $\tilde{Z}_{\mathbf{H}-\mathbf{r},\mathbf{r}} \neq 0$ and $\tilde{Z}_{\mathbf{r},\mathbf{H}-\mathbf{r}} \neq 0$ for all $\mathbf{r} \in \mathbf{H}(1)$, for almost all compensators applied to the Nth channel, where $\tilde{}$ denotes the resulting closed loop transfer matrix. Since the union of generic sets is generic, $\tilde{Z}_{\mathbf{H}-\mathbf{r},\mathbf{r}} \neq 0$ and $\tilde{Z}_{\mathbf{r},\mathbf{H}-\mathbf{r}} \neq 0$ for all $\mathbf{r} \in \mathbf{H}(1)$, for almost all $Z_c \in \mathscr{Z}_{cN}$. Repeating these arguments inductively until $N = 1$, at each step the set $\mathscr{Z}_{cN}$ appears to be generic in $\mathbf{R}^{m_N \times p_N}$. It can be shown that $\{Z_{c2}, \ldots, Z_{cN} \mid Z_{ci}$ is generic $\in \mathbf{R}^{m_i \times p_i}\ i = 2, \ldots, N\}$ is generic in the product topology of $\mathbf{R}^{m_2 \times p_2} \times \ldots \times \mathbf{R}^{m_N \times p_N}$. This completes the proof. □

For those plants which are not strongly connected we can use Lemma 4.1 to classify the class of compensators solving SCCP. It is easy to see that in this case the plant can be decomposed into its strongly connected components, where the class of compensators solving DSP can be considered for each of the subsystems independently. Also note that the 'If' part of Theorem 4.2 is implicit in Theorem 1 of Corfmat and Morse (1976).

5. Example

Consider the three-channel system below:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{(s+1)^2} & \dfrac{(s-1)}{(s+1)^3} & \dfrac{1}{(s+1)^2} \\ \dfrac{(2s-5)}{(s+1)(s-2)(s-3)} & \dfrac{1}{(s-2)(s+1)} & \dfrac{1}{(s-2)(s+1)} \\ \dfrac{(2s-3)}{(s-1)(s+1)(s-2)} & \dfrac{(2s-1)}{(s+1)^2(s-2)} & \dfrac{(2s-3)}{(s+1)(s-1)(s-2)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= Zu$$

Obtaining a bicoprime representation of Z over $\mathbf{S}$ we have $y = [P_1' \; P_2' \; P_3']' Q^{-1} [R_1 \; R_2 \; R_3] u$, where $P_1 = [(s-1)/(s+1)^2 \;\; 0 \;\; 0]$, $P_2 = [0 \;\; 1/(s+1) \;\; 1/(s+1)]$, $P_3 = [1/(s+1) \;\; 1/(s+1) \;\; 0]$, $R_1' = [1/(s+1) \;\; 1/(s+1) \;\; 1/(s+1)]'$, $R_2' = [(s-1)/(s+1)^2 \;\; 1/(s+1) \;\; 0]'$, $R_3' = [1/(s+1) \;\; 1/(s+1) \;\; 0]'$, and $Q = \operatorname{diag}\{(s-1)/(s+1), (s-2)/(s+1), (s-3)/(s+1)\}$.

Let $H = 2$, $\mathbf{H}(\mathbf{1}) = \{1\}$, and $\mathbf{r} = \{1\}$. We now determine $Z_{c3} = P_{c3}Q_{c3}^{-1} \in \mathbf{R}$, for coprime (P_{c3}, Q_{c3}) such that the closed loop system under feedback law $u_3 = -Z_{c3}y_3$ satisfies

(i) $\left([P_2 \quad 0], \begin{bmatrix} Q & R_3P_{c3} \\ -P_3 & Q_{c3} \end{bmatrix}, \begin{bmatrix} R_1 \\ 0 \end{bmatrix}\right)$ is complete

(ii) $\left(\begin{bmatrix} Q & R_3P_{c3} \\ -P_3 & Q_{c3} \end{bmatrix}, \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}\right)$ is left coprime

(i′) $\left([P_1 \quad 0], \begin{bmatrix} Q & R_3P_{c3} \\ -P_3 & Q_{c3} \end{bmatrix}, \begin{bmatrix} R_2 \\ 0 \end{bmatrix}\right)$ is complete

(ii′) $\left(\begin{bmatrix} P_1 & 0 \\ P_2 & 0 \end{bmatrix}, \begin{bmatrix} Q & R_3P_{c3} \\ -P_3 & Q_{c3} \end{bmatrix}\right)$ is right coprime.

Following Theorem 4.1 and the preceeding statements one can verify that (i) and (ii) hold for all $Z_{c3} \in \mathbf{R}$, whereas (i′) holds if and only if $Z_{c3}(1) \neq 0$ and $[Q_{c3} \ P_{c3}]_{s=3}[1 \ \ -\tfrac{1}{4}]'_{s=3} \neq 0$, and (ii′) holds if and only if $Z_{c3}(1) \neq 0$. So, by combining these results we conclude the following: $Z_{c3} = P_{c3}Q_{c3}^{-1} \in \mathbf{R}$, for coprime (P_{c3}, Q_{c3}) such that (i), (ii), (i′) and (ii′) hold, if and only if $P_{c3}(1) \neq 0$ and $[Q_{c3} \ P_{c3}]_{s=3}[1 \ \ -\tfrac{1}{4}]'_{s=3} \neq 0$.

In order to achieve an internally hierarchically stable design, we choose $P_{c3} = (97s - 113)/(s+1)$ and $Q_{c3} = (s^2 + 7s - 169)/(s+1)^2$. In this case $Z_{c3} = P_{c3}Q_{c3}^{-1}$ is a minimal order internally stabilizing compensator for Z_{33}. With this choice of Z_{c3} it can also be verified that (i), (ii), (i′) and (ii′) hold.

Repeating similar arguments for the resulting two-channel system $\tilde{Z}$, we obtain $Z_{c2} = 65$, which internally stabilizes the second channel of $\tilde{Z}$. We finally get $Z_{c1} = P_{c1}Q_{c1}^{-1}$, where

$$P_{c1} = \frac{65536(65s^6 + 390s^5 + 976s^4 + 1307s^3 + 805s^2 + 577s + 8)}{317(s+1)^6}$$

and

$$Q_{c1} = (317s^8 + 3804s^7 - 4237016s^6 - 25463940s^5 + 762902138s^4 - 633438348s^3 \\ - 2207193504s^2 + 692117428s + 1415227969)/317(s+1)^8$$

The resulting decentralized compensator has total order 10. It can be shown by following the approach in Corfmat and Morse (1976) that by using constant feedback compensators around the third and second channels, and a seventh order compensator around the third channel, a decentralized compensator of total order 7 could also be utilized to solve DSP. This, however, would not lead to an hierarchically internally stable design. Hence, the hierarchically internally stable design is achieved at the expense of increased compensator order. Also note that the design procedure yields a spread controller complexity (Anderson and Linnemann 1984, 1987) in this example.

Appendix

The appendix includes the proofs of Lemmata 3.1 and 4.2. For this we need the technical lemmata given below.

Lemma A.1

Let $A \in \mathbf{S}^{k \times k}$ and $B \in \mathbf{S}^{k \times c}$ be such that (A, B) is left coprime. Assume that $E \in \mathbf{S}^{k \times d}$ is non-zero. The set of X such that $(A + BX, E)$ is left coprime, is generic in $\mathbf{S}^{c \times k}$. □

Lemma A.2

Let $E \in \mathbf{S}^{k \times d}$ be non-zero. Then, the set of X such that (X, E) is left coprime is generic in $\mathbf{S}^{k \times k}$. □

Lemma A.3

Let $A \in \mathbf{S}^{k \times k}$ and $B \in \mathbf{S}^{k \times c}$ be such that the pair (A, B) is left coprime. Assume that $E \in \mathbf{S}^{k \times d}$ is non-zero. The set of $[X_1' \; X_2']'$ such that $(AX_1 + BX_2, E)$ is left coprime is generic in $\mathbf{S}^{k+c \times k}$. □

Lemma A.4

The set of biproper matrices is dense in $\mathbf{S}^{k \times k}$. □

Lemma A.5

Let $A \in \mathbf{S}^{k \times k}$ and $B \in \mathbf{S}^{k \times c}$ be such that the pair (A, B) is left coprime. Assume that $E \in \mathbf{S}^{k \times d}$ is non-zero. Express $Z \in \mathbf{R}^{c \times k}$ as $Z = ND^{-1}$, where (N, D) is right coprime. The set of $Z = ND^{-1}$ for which $(AD + BN, E)$ is left coprime is open and dense in $\mathbf{R}^{c \times k}$. □

Lemma A.1 states in system theoretic words that, 'A stabilizable and detectable system can be made stabilizable from a single output connected to any of its states under almost all stable dynamic compensators'. Lemma A.5 is a revised version of Lemma A.1 and says that, 'For almost all dynamic proper feedback compensators, a canonical system can be made stabilizable from a single input, connected to any of its states'. Lemmata A.2, A.3 and A.4 are used as intermediate steps in the proof of Lemma A.5.

Proof of Lemma A.1

We prove the lemma for the case A is non-singular. The extension of the proof to the general case is straightforward by using Lemma 5.2.11 of Vidyasagar (1985), which states that the set of X for which $A + BX$ is non-singular, is generic.

Let U be a unimodular matrix such that $UE = [\bar{E}' \; 0]'$ where $\bar{E}$ is full row rank. Then, there exists a unimodular matrix V such that

$$UAV = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

Clearly A_{11} and A_{12} are non-singular. Also let $UB = [B_1' \; B_2']'$ and $XV = [X_1 \; X_2]$. Since $[A \; B]$ is left unimodular, for any X_1, $(A_{11} + B_1X_1, B_1)$ and $(A_{22}, A_{21} + B_2X_1, B_2)$ are left coprime. This shows that if $[A_{21} \; B_2] = 0$ then A_{22} is unimodular. Now define $\hat{A}_{11} := A_{11} + B_1X_1$, $\hat{A}_{21} := A_{21} + B_2X_1$, and $\hat{A}_{22} := A_{22} + B_2X_2$.

Case 1: $[A_{21} \;\; B_2] = 0$

In this case A_{22} is unimodular. Also, from Lemma 2.1 of Özgüler (1990), for almost all X_1 $(\hat{A}_{11}, \bar{E})$ is left coprime. Fix one such X_1. Let $X = [X_1 \;\; X_2]V^{-1}$, where X_2 is arbitrary. By unimodular operations, it appears that $[A + BX \;\; E]$ is left unimodular if and only if so is

$$\begin{bmatrix} \hat{A}_{11} & 0 & \bar{E} \\ 0 & A_{22} & 0 \end{bmatrix}$$

which is clearly left unimodular. Since X_1 is almost arbitrary, X_2 is arbitrary and $X = [X_1 \;\; X_2]V^{-1}$, we have that for almost all X $(A + BX, E)$ is left coprime.

Case 2: $[A_{21} \;\; B_2] \neq 0$

Then, it is easy to verify that $A_{21} + B_2 X_1 \neq 0$ for almost all X_1. So, for almost all X_1 (i) $(\hat{A}_{11}, \bar{E})$ is left coprime, and (ii) $\hat{A}_{21} \neq 0$. Choose one such X_1. There exist matrices K, L, $\bar{A}_{11}$, $\bar{B}_1$, Ψ_1, Ψ_2, Ψ_3, Ψ_4, Ψ_5, Ψ_6 such that

$$[\hat{A}_{11} \;\; B_1]\begin{bmatrix} K & -\bar{B}_1 \\ L & \bar{A}_{11} \end{bmatrix} = [I \;\; 0] \tag{A 1}$$

$$\begin{bmatrix} \hat{A}_{11} & \bar{E} \\ \Psi_5 & \Psi_6 \end{bmatrix}\begin{bmatrix} \Psi_1 & \Psi_3 \\ \Psi_2 & \Psi_4 \end{bmatrix} = I \tag{A 2}$$

It can be verified that $[A \;\; B]$ is equivalent over $\mathbf{S}$ to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & A_{22} & B_2\hat{A}_{11} - \hat{A}_{21}\bar{B}_1 \end{bmatrix}$$

which implies that $(A_{22}, B_2\bar{A}_{11} - \hat{A}_{21}\bar{B}_2)$ is left coprime. This shows that $(A_{22}, (B_2\bar{A}_{11} - \hat{A}_{21}\bar{B}_1) + \hat{A}_{21}\Psi_3\Psi_5\bar{B}_1, \hat{A}_{21}\Psi_3)$ is left coprime. From (A 1) and (A 2), $(B_2\bar{A}_{11} - \hat{A}_{21}\bar{B}_1) + \hat{A}_{21}\Psi_3\Psi_5\bar{B}_1 = (B_2 - \hat{A}_{21}\Psi_1 B_1)\bar{A}_{11}$. This implies that $(A_{22}, B_2 - \hat{A}_{21}\Psi_1 B_1, \hat{A}_{21}\Psi_3)$ is left coprime.

On the other hand, let $X = [X_1 \;\; X_2]V^{-1}$, where X_2 is arbitrary. Unimodular operations yield that $[A + BX \;\; E]$ is left unimodular if and only if so is $(A_{22} + (B_2 - \hat{A}_{21}\Psi_1 B_1)X_2, \hat{A}_{21}\Psi_3)$ is left unimodular. Let $D_l :=$ gclf $(A_{22}, B_2 - \hat{A}_{21}\Psi_1 B_1)$, such that $A_{22} = D_l\tilde{A}$ and $B_2 - \hat{A}_{21}\Psi_1 B = D_l\tilde{B}$ for a left coprime pair of matrices $(\tilde{A}, \tilde{B})$. Since A_{22} is non-singular, D_l and $\tilde{A}$ are non-singular. Let $D_l^{-1}\hat{A}_{21}\Psi_3 = \tilde{E}\tilde{D}^{-1}$ for a right coprime pair of matrices $(\tilde{E}, \tilde{D})$. Since $\bar{E}$ is full row rank, so is Ψ_3. This, and the fact that $\hat{A}_{21} \neq 0$ imply $\tilde{E} \neq 0$. Also $(A_{22} + (B_2 - \hat{A}_{21}\Psi_1 B_1)X_2, \hat{A}_{21}\Psi_3)$ is left coprime if and only if $(\tilde{A} + \tilde{B}X_2, \tilde{E})$ is left coprime. This is the same type of equation as the one we started with, except that now the number of rows of A is reduced at least by one. Applying the same arguments repeatedly, we either terminate at Case 1, at some step, or terminate at Case 2, with the number of rows of $\hat{A}$ being 1. In this case $\tilde{E}$ is full row rank and applying Lemma 2.1 of Özgüler (1990) completes the proof. □

Proof of Lemma A.2

This is a straightforward generalization of Proposition 7.6.15 in Vidyasagar (1985). □

Proof of Lemma A.3

It is enough to prove the Lemma when $E \in \mathbf{S}^{k \times 1}$. If $B = 0$ we can obtain the solution by using Lemma A.2, because in this case A is unimodular and the lemma reduces to showing that the set of X for which (X, E) is left coprime, is open and dense in $\mathbf{S}^{k \times k}$. Now assume that $B \neq 0$. It can be shown, by using Lemma A.2 that the set of X_1 for which (AX_1, B) is left coprime is open and dense in $\mathbf{S}^{k \times k}$. Fix one such X_1. Then, from Lemma A.1, the set of X_2 for which $(AX_1 + BX_2, E)$ is left coprime, is open and dense in $\mathbf{S}^{c \times k}$. So, the set of $[X_1' : X_2']'$ for which $(AX_1 + BX_2, E)$ is left coprime is open and dense in $\mathbf{S}^{k+c \times k}$. □

Proof of Lemma A.4

First consider the following fact, which can be verified by using standard results on the properness of rational matrices. Let $G \in \mathbf{R}^{k \times k}$. Assume that $G = ND^{-1}$ where $N, D \in \mathbf{R}^{k \times k}[s]$ and N is column reduced. Then, G is biproper if and only if $\delta_{cj}(N) = \delta_{cj}(D)$, $j = 1, ..., k$, where $\delta_{cj}(\cdot)$ denotes the jth column degree of the matrix.

Assume that $X_o \in \mathbf{S}^{k \times k}$ and is not biproper. Let $d \in \mathbf{S}$ denote the least common multiple of the denominators of X_o. Then, $X_o = (1/d)N$, where $N \in R^{k \times k}[s]$. Consider $X_o = N(dI)^{-1}$. First notice that $\delta_{cj}(N) \leqslant \delta(d)$, $j = 1, ..., k$. We can write $N = C_1 z^{\delta(d)} I + \bar{N}$, where $\bar{N} \in \mathbf{R}^{k \times k}[s]$ and having entries with degree strictly less than $\delta(d)$, and $C_1 \in \mathscr{R}^{k \times k}$. Observe that C_1 is singular, because otherwise it would be the highest column degree coefficient matrix and N would be column reduced. However, there exists $\Delta \in \mathscr{R}^{k \times k}$ with arbitrarily small spectral norm such that $C_1 + \Delta$ is non-singular. Also, $X_o + \Delta = (N + \Delta d)(dI)^{-1}$. It is easy to see that $N + \Delta d$ is now column reduced, and $\delta_{cj}(N) = \delta_{cj}(D)$, $j = 1, ..., k$. Hence, $X_o + \Delta$ is biproper. Since the norm of Δ can be chosen arbitrarily small, we conclude that every neighbourhood of X_o contains a biproper matrix in $\mathbf{S}^{k \times k}$. This completes the proof. □

Proof of Lemma A.5

To show that the set of such Z is open let $Z = ND^{-1} \in \mathbf{R}^{c \times k}$, with (N, D) is right coprime and $(AD + BN, E)$ is left coprime. From Lemma A.3, we know that there exists $\delta > 0$, such that

$$\left\| \begin{matrix} D - X_1 \\ N - X_2 \end{matrix} \right\| < \delta$$

implies that $(AX_1 + BX_2, E)$ is left coprime.

Consider any basic neighbourhood of Z over $\mathbf{R}^{c \times k}$ defined as

$$\left\{ \bar{X}_2 \bar{X}_1^{-1} \;\middle|\; \left\| \begin{matrix} D - \bar{X}_1 \\ N - \bar{X}_2 \end{matrix} \right\| < \varepsilon \right\}, \quad \varepsilon < \mu(N, D)$$

Then, the set

$$\mathscr{T} := \left\{ \bar{N}\bar{D}^{-1} \in \mathbf{R}^{c \times k} \;\middle|\; \left\| \begin{matrix} D - \bar{D} \\ N - \bar{N} \end{matrix} \right\| < \min(\varepsilon, \delta) \right\}$$

is nothing but an open set in the subspace topology of $\mathbf{R}^{c \times k}$, containing ND^{-1}. It is also true that if $\bar{N}\bar{D}^{-1} \in \mathscr{T}$, then $(A\bar{D} + B\bar{N}, E)$ is left coprime. This shows that the set of such Z is open.

To show that the set of such Z is dense in $\mathbf{R}^{c\times k}$, consider $Z = ND^{-1} \in \mathbf{R}^{c\times k}$, (N, D) is right coprime, and $(AD + BN, E)$ is not left coprime. For any $\delta > 0$, there exists a basic neighbourhood of ND^{-1} over $\mathbf{R}^{c\times k}(s)$ defined as

$$\mathscr{T} = \left\{ \bar{X}_2 \bar{X}_1^{-1} \;\middle|\; \left\| \begin{matrix} D - \bar{X}_1 \\ N - \bar{X}_2 \end{matrix} \right\| < \varepsilon \right\}, \quad \varepsilon < \min(\mu(N, D), \delta)$$

From Lemma A.3, on the other hand, the above set contains some $X_2 X_1^{-1}$ such that $(AX_1 + BX_2, E)$ is left coprime. There also exists $\alpha > 0$ such that for all $\bar{X}_1$, $\bar{X}_2$ such that

$$\left\| \begin{matrix} D - \bar{X}_1 \\ N - \bar{X}_2 \end{matrix} \right\| < \alpha, \quad (A\bar{X}_1 + B\bar{X}_2, E)$$

is left coprime. We can assume that $\alpha < \varepsilon/2$. So,

$$\mathscr{T}' := \left\{ \bar{X}_2 \bar{X}_1^{-1} \;\middle|\; \left\| \begin{matrix} X_1 - \bar{X}_1 \\ X_2 - \bar{X}_2 \end{matrix} \right\| < \alpha \right\} \subseteq \mathscr{T}$$

From Lemma A.4 there exists $\hat{X}_1$ such that $X_2 \hat{X}_1^{-1} \in \mathbf{R}^{c\times k}$ and $\|X_1 - \hat{X}_1\|$ can be made arbitrarily small. Hence, we can assume $X_2 \hat{X}_1^{-1} \in \mathscr{T}' \subseteq \mathscr{T}$. But then,

$$\left\{ \bar{X}_2 \bar{X}_1^{-1} \in \mathbf{R}^{c\times k} \;\middle|\; \left\| \begin{matrix} D - \bar{X}_1 \\ N - \bar{X}_2 \end{matrix} \right\| < \varepsilon \right\}$$

is open in $\mathbf{R}^{c\times k}$ and contains $X_2 \hat{X}_1^{-1}$, for which $(A\hat{X}_1 + BX_2, E)$ is left coprime. Since the choice of $\mathscr{T}$ is possible for arbitrary $\delta > 0$, this shows that the set of such Z is dense in $\mathbf{R}^{c\times k}$. □

Proof of Lemma 3.1

First note that (11) is complete if and only if

$$\left([T_2 \quad 0], \begin{bmatrix} Q_{11} & S_1 \\ -\bar{R}_c T_1 & \bar{Q}_c \end{bmatrix}, \begin{bmatrix} S_2 \\ 0 \end{bmatrix} \right) \tag{A 3}$$

is complete, where $P_c Q_C^{-1} = \bar{Q}_c^{-1} \bar{R}_c$ for some left coprime pair of matrices $(\bar{Q}_c, \bar{R}_c)$. Let U and V be unimodular matrices such that

$$U \begin{bmatrix} Q_{11} & S_2 \\ -T_2 & 0 \end{bmatrix} V = \begin{bmatrix} \Lambda & 0 \\ 0 & \Psi \end{bmatrix} \tag{A 4}$$

where the matrix on the right hand side is the Smith normal form of the matrix at the left. Partition U and V as $U = [U_{ij}]$, $V = [V_{ij}]$, $i, j = 1, 2$. It now follows that (11) is complete if and only if

$$\left([0 \quad -U_{21} S_1 P_c], \begin{bmatrix} \Lambda & U_{11} S_1 P_c \\ -T_1 V_{11} & Q_c \end{bmatrix}, \begin{bmatrix} 0 \\ -T_1 V_{12} \end{bmatrix}, \Psi \right) \tag{A 5}$$

is complete. Similarly (18) is complete if and only if

$$\left([0 \quad -U_{21} S_1], \begin{bmatrix} \Lambda & U_{11} S_1 \\ -\bar{R}_c T_1 V_{11} & \bar{Q}_c \end{bmatrix}, \begin{bmatrix} 0 \\ -\bar{R}_c T_1 V_{12} \end{bmatrix}, \Psi \right) \tag{A 6}$$

is complete. Using a generalization of Lemma 2.5 in Özgüler (1990), we can assume that $(\Lambda, U_{11} S_1)$ is left coprime and $(T_1 V_{11}, \Lambda)$ is right coprime. So, there exist matrices Φ_1, Φ_2, Φ_3, Φ, $\bar{\Phi}_4$, $\bar{\Phi}_3$ and Θ_1, Θ_2, Θ_3, Θ, $\bar{\Theta}_4$, $\bar{\Theta}_3$, with Θ and Φ

non-singular, such that

$$\begin{bmatrix} \Theta_1 & -\Theta_2 \\ \Theta_3 & \Theta \end{bmatrix} \begin{bmatrix} \Lambda & \bar{\Theta}_4 \\ -T_1 V_{11} & \bar{\Theta}_3 \end{bmatrix} = I \qquad (A\,7)$$

and

$$\begin{bmatrix} \Lambda & U_{11} S_1 \\ -\bar{\Phi}_4 & \bar{\Phi}_3 \end{bmatrix} \begin{bmatrix} \Phi_1 & -\Phi_3 \\ \Phi_2 & \Phi \end{bmatrix} = I \qquad (A\,7)$$

Unimodular operations yield that (A 5) is complete if and only if

$$\left([0 \quad -U_{21} S_1 P_c], \begin{bmatrix} I & 0 \\ 0 & \Theta_3 U_{11} S_1 P_c + \Theta Q_c \end{bmatrix}, \begin{bmatrix} 0 \\ -\Theta T_1 V_{12} \end{bmatrix}, \Psi \right) \qquad (A\,9)$$

is complete, and (A 6) is complete if and only if

$$\left([0 \quad -U_{21} S_1 \Phi], \begin{bmatrix} I & 0 \\ 0 & \bar{R}_c T_1 V_{11} \Phi_3 + \bar{Q}_c \Phi \end{bmatrix}, \begin{bmatrix} 0 \\ -\bar{R}_c T_1 V_{12} \end{bmatrix}, \Psi \right) \qquad (A\,10)$$

is complete.

Now, let $\bar{Z}_{11} = TQ^{-1}S + \bar{W}_{11}$ be a bicoprime fractional representation of $\bar{Z}_{11}$. Using (6) and (7) define

$$[A : B] := [\Theta_3 U_{11} S_1 : \Theta] \begin{bmatrix} S_r & D_r \\ T_r & -N_r \end{bmatrix} \qquad (A\,11)$$

and

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} := \begin{bmatrix} S_l & T_l \\ D_l & -N_l \end{bmatrix} \begin{bmatrix} T_1 V_{11} \Phi_3 \\ \Phi \end{bmatrix} \qquad (A\,12)$$

From (A 7) and (A 8), it follows that (A, B) is left coprime and $(\tilde{A}, \tilde{B})$ is right coprime. Moreover

$$\Theta_3 U_{11} S_1 P_c + \Theta Q_c = AX_1 + BX_2$$
$$\bar{R}_c T_1 V_{11} \Phi_3 + \bar{Q}_c \Phi = Y_1 \tilde{A} + Y_2 \tilde{B}$$

where (8) and (9) hold for X_1, X_2, Y_1 and Y_2.

Let us define

$$\Gamma := -T_1 V_{12}, \quad \Omega := U_{21} S_1 \qquad (A\,13)$$

With this new notation, we remind that (11) is complete if and only if

$$(-\Omega(S_r X_1 + D_r X_2), AX_1 + BX_2, \Theta\Gamma, \Psi) \qquad (A\,14)$$

is complete, and (A 3) is complete if and only if

$$(-\Omega\Phi, Y_1 \tilde{A} + Y_2 \tilde{B}, (Y_1 S_l + Y_2 D_l)\Gamma, \Psi) \qquad (A\,15)$$

is complete. Also, notice that (11) is complete for almost all $Z_c \in \mathscr{X}_c(\bar{Z}_{11})$, if and only if for almost all X_2 (A 14) is complete, with $X_1 = I$. This can be verified by using the definition of the topology over $\mathscr{X}_c(\bar{Z}_{11})$ and (10). As a dual result, (A 3) is complete for almost all $Z_c \in \mathscr{X}(\bar{Z}_{11})$, if and only if for almost all Y_2 (A 15) is complete, with $Y_1 = I$. On the other hand, (11) is complete for almost all $Z_c \in \mathbf{R}^{m \times p}$, if and only if for almost all $Z \in \mathbf{R}^{m \times p}$, with $Z = X_2 X_1^{-1}$ for some right coprime pair of matrices (X_2, X_1), (A 14) is complete. Similarly, (A 3) is complete for almost all $Z_c \in \mathbf{R}^{m \times p}$, if and only if for almost all $Z \in \mathbf{R}^{m \times p}$, with $Z = Y_1^{-1} Y_2$ for some left coprime pair of matrices (Y_1, Y_2), (30) is complete. These results can also be verified by using the topology on $\mathbf{R}^{m \times p}$ and (8) and (9).

We now proceed by investigating three cases.

Case 1: *At least one of* Γ *and* Ω *is non-zero.*

If Γ is non-zero, since Θ is non-singular, $\Theta\Gamma$ is non-zero. Then, applying Lemma A.1 gives us that for almost all X_2, $(A + BX_2, \Theta\Gamma)$ is left coprime. This implies that for almost all $Z_c \in \mathscr{Z}_c(\bar{Z}_{11})$ (A 14) is complete. Also applying Lemma A.5 yields that for almost all $Z_c \in \mathbf{R}^{m \times p}$ (A 14) is complete. If Ω is non-zero, on the other hand, then $\Omega\Phi$ is non-zero, because of the non-singularity of Φ. So, applying the dual of Lemma A.1 we observe that for almost all Y_2, $(\Omega\Phi, \tilde{A} + Y_2\tilde{B})$ is right coprime. This implies that for almost all $Z_c \in \mathscr{Z}_c(\bar{Z}_{11})$ (A 15) is complete. Also, applying the dual of Lemma A.5 yields that for almost all $Z_c \in \mathbf{R}^{m \times p}$ (A 15) is complete.

Case 2: $\Gamma = 0$, $\Omega = 0$, $\Psi \neq 0$

In this case (A 14) is complete if and only if $(0, AX_1 + BX_2, K\Psi, \Psi)$ is complete. Clearly, there exists a matrix K over $\mathbf{S}$ of appropriate size such that $K\Psi$ is non-zero and $(0, AX_1 + BX_2, 0, \Psi)$ is equivalent to $(0, AX_1 + BX_2, K\Psi, \Psi)$ over $\mathbf{S}$. Repeating Case 1 yields that for almost all $Z_c \in \mathscr{Z}_c(\bar{Z}_{11})$ and for almost all $Z_c \in \mathbf{R}^{m \times p}$ (A 14) is complete.

Case 3: $\Gamma = 0$, $\Omega = 0$, $\Psi = 0$

In this case (A 5) (and, therefore (11)) is complete if and only if

$$\begin{bmatrix} \Lambda & U_{11} S_1 P_c \\ -T_1 V_{11} & Q_c \end{bmatrix} \tag{A 16}$$

is unimodular. It can be verified that, in this case $\bar{Z}_{11} = T_1 Q_{11}^{-1} S_1 = T_1 V_{11} \Lambda^{-1} U_{11} S_1$. Since the right hand side of the equation is bicoprime, this implies that (31) is unimodular if and only if $Z_c \in \mathscr{Z}_c(\bar{Z}_{11})$. Noting that Γ, Ω and Ψ are all zero if and only if Z_{12}, Z_{21} and Z_{22} are all zero, the proof of (1) of Lemma 3.1 is thus completed. In this case to complete the proof of (2), just observe that $\mathscr{Z}_c(\bar{Z}_{11})$ is not dense in $\mathbf{R}^{m \times p}$ (see the proof of Theorem 4.1).

Proof of Lemma 4.2

We omit the 'Only if' part of the proof as it is straightforward. For the 'If' part observe that (14) holds for some P_c, Q_c described by (8), if

$$\operatorname{rk}\left(\begin{bmatrix} Q_{11} & S_1 P_c & S_2 \\ -T_1 & Q_c & 0 \\ -T_2 & 0 & 0 \end{bmatrix}\right) \geqslant q + p + 1 \tag{A 17}$$

where $q := \operatorname{size}(Q)$. Repeating the arguments in the proof of Lemma (3.1), (A 17) holds if and only if

$$\operatorname{rk}\left(\begin{bmatrix} AX_1 + BX_2 & \Theta\Gamma \\ \Omega(S_r X_1 + D_r X_2) & \Psi \end{bmatrix}\right) \geqslant p + 1 \qquad \text{(A 18)}$$

Writing (A 18) explicitly, we have that (A 18) holds if and only if

$$\operatorname{rk}\left(\begin{bmatrix} \Theta_3 U_{11} S_1 & \Theta & \Theta\Gamma \\ \Omega & 0 & \Psi \end{bmatrix}\begin{bmatrix} S_r & D_r & 0 \\ T_r & -N_r & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} X_1 & 0 \\ X_2 & 0 \\ 0 & I \end{bmatrix}\right) \geqslant p + 1 \qquad \text{(A 19)}$$

The hypothesis implies that $[\Omega : \Psi]$ and $[\Gamma' : \Psi']'$ are non-zero. This fact and the fact that Θ is non-singular imply that the first matrix in (A 19) has rank no less than $p + 1$.

Write $C := \Theta\Gamma$, $D := \Omega S_r$, $E := \Omega D_r$. The conclusion above and the fact that the middle matrix in (A 19) is unimodular, imply

$$\operatorname{rk}\left(\begin{bmatrix} A & B & C \\ D & E & \Psi \end{bmatrix}\right) \geqslant p + 1 \qquad \text{(A 20)}$$

Let $\tilde{U}$ be a unimodular matrix such that

$$\begin{bmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ \tilde{U}_{21} & \tilde{U}_{22} \end{bmatrix}\begin{bmatrix} C \\ \Psi \end{bmatrix} = \begin{bmatrix} \hat{C} \\ 0 \end{bmatrix} \qquad \text{(A 21)}$$

where $\hat{C}$ is a full row rank matrix. Also let

$$\begin{bmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ \tilde{U}_{21} & \tilde{U}_{22} \end{bmatrix}\begin{bmatrix} A & B \\ D & E \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{D} & \hat{E} \end{bmatrix}$$

for some matrices $\hat{A}$, $\hat{B}$, $\hat{D}$, $\hat{E}$. It follows from (A 20) and (A 21) that the rank of $[\hat{D} : \hat{E}]$ is no less than $p + 1 - c$, where $c := \operatorname{size}(\hat{C}) \geqslant 1$. Observe that (A 17) holds if and only if

$$\operatorname{rk}[\hat{D} : \hat{E}]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \geqslant p + 1 - c \qquad \text{(A 22)}$$

Now, it is not difficult to show by straightforward manipulations that the set of X_1, X_2 for which (A 22) and thus (A 17) holds is generic in $\{X_1 \in \mathbf{S}^{p \times p}$ and non-singular, $X_2 \in \mathbf{S}^{m \times p} \mid X_2 X_1^{-1} \in \mathbf{R}^{m \times p}\}$. This completes the proof. □

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