LATTICE EMBEDDINGS AND K3 COVERS OF AN ENRIQUES SURFACE

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By

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Lattice Embeddings and K3 Covers of an Enriques Surface
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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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In this thesis, we establish necessary conditions for the existence of primitive embeddings of even lattices into lattices, and apply these criteria to characterize K3 surfaces admitting Enriques involutions. Utilizing Nikulin's theory of discriminant forms, we enumerate all such primitive embeddings into the lattice $\mathbf{\Lambda}^-$ and apply this framework to the problem of determining which K3 surfaces admit an Enriques involution. Furthermore, using the notion of idoneal genera, we classify all co-idoneal lattices as exceptional cases.

Keywords: K3 Surface, Enriques Surface, Lattice, Idoneal Genera.

ÖZET

LATİS GÖMMELERİ VE ENRIQUES YÜZEYLERİ ÖRTEN K3 YÜZEYLER

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Bu tezde, latislerin pirimitif yerleşimleri için gerekli koşulları belirliyor ve bu koşulları, bir Enriques yüzeyini örten cebirsel K3 yüzeylerinin karakterizasyonu problemine uyguluyoruz. Nikulin'in diskriminant biçimler kuramını ve idonel genera kavramını kullanarak tüm co-idonel latisleri belirliyoruz.

Anahtar sözcükler: K3 Yüzeyi, Enriques Yüzeyi, Latis, İdonel Genera.

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Every thesis has its most disquieting aspect, and without doubt, it is the part I am about to write. I neither have the space nor the words to fully express my gratitude to all those who have shaped my life and intellectual journey.

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"Run mad as often as you chuse, but do not faint!"
— Jane Austen

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Chapter 1

Introduction

The problem of determining which K3 surfaces admit Enriques involutions lies at the intersection of algebraic geometry and lattice theory. Keum's criterion provides a foundational lattice-theoretic characterization by relating the existence of an Enriques involution to the embedding behavior of the transcendental lattice T_X of a K3 surface X.

This line of work began with the characterization by Sertöz [1] for K3 surfaces with Picard number $\rho(X) = 20$. It was subsequently extended to $\rho(X) = 19$ by Lee [2], and to $\rho(X) = 18$ by Yörük [3], who adopted a computational approach. Ohashi [4] independently extended the criterion to the cases $\rho(X) = 10$ and $\rho(X) = 11$.

Keum's theorem characterizes precisely when a K3 surface admits an Enriques involution, thereby reducing the problem to one of lattice theory. In this thesis, we determine which K3 surfaces admit such involutions by examining the structure of their transcendental lattices through explicit analysis of their Gram matrices.

The methods developed here rely on two principal approaches. The first, presented in Chapter 4, adopts a global perspective by formulating general primitive embedding criteria for lattices. The second, developed in Chapter 5, employs local techniques based on Nikulin's theory of discriminant forms.

Chapter 2 primarily focuses on finite symmetric bilinear forms, finite quadratic forms, and lattices. Additionally, it examines lattices over a ring R, where R is either the ring of integers $\mathbb{Z}$ or the ring of p-adic integers $\mathbb{Z}_p$. Basic $\mathbb{Z}_p$-lattices, finite quadratic forms, and finite symmetric bilinear forms are presented as fundamental building blocks, and their relations are stated.

Chapter 3 is dedicated to the study of algebraic and complex K3 surfaces, along with their key sublattices: the Néron–Severi and transcendental lattices. We begin by introducing the definitions of algebraic and complex K3 surfaces. We also provide the definition of an Enriques surface, present Keum's criterion for the existence of Enriques involutions, with its extension by Ohashi to the cases $\rho(X) = 10$ and $\rho(X) = 11$.

In Chapter 4, we develop necessary conditions for lattice embeddings and apply them to the characterization of $K3$ surfaces that admit an Enriques involution. By refining existing criteria and providing a more elementary approach, we offer a new perspective on the structure of such surfaces. Our results apply to lattices, extending beyond specific cases and offering a comprehensive framework for understanding the embedding conditions in terms of Gram matrices. All the proven results presented in Chapter 4 are original contributions of the author. In particular, Theorem 4.9, Corollary 4.10, Lemma 4.12, Theorem 4.13, Corollary 4.14, Lemma 4.16, Lemma 4.17, Lemma 4.18, and Theorem 4.19 are solely due to the author of this thesis.

In Chapter 5, we apply Nikulin's theory of discriminant forms to classify the transcendental lattices of K3 surfaces that admit an Enriques involution. All the results presented in Chapter 5 are based on joint work by Brandhorst, Veniani, and Sonel [5]. In particular, Lemma 5.5, Theorem 5.6, and Theorem 5.7 are joint results.

In Chapter 6, we define the notions of co-idoneal lattices and idoneal genera, and describe the structural relationship between them in the context of lattice embeddings. Using the framework of idoneal genera, we present a complete enumeration of all co-idoneal lattices. All results in this chapter are based on joint work by Brandhorst, Veniani, and Sonel [5]. In particular, Lemma 6.1, Corollary 6.4, Theorem 6.5, and Theorem 6.6 arise from this collaboration.

Chapter 2

Bilinear Forms, Quadratic Forms and Lattices

The primary reference for this chapter is [6].

Chapter 2 lays the algebraic foundation for the study of lattices and their embeddings by introducing the relevant theory of bilinear and quadratic forms. We begin with finite symmetric bilinear forms and finite quadratic forms, and examine how these structures extend to lattices over $\mathbb{Z}$ and $\mathbb{Z}_p$. Particular attention is given to elementary lattices and their classification in terms of associated forms. We then turn to the structure of finite quadratic forms on 2-elementary abelian groups. The chapter concludes with a discussion of lattice genera, a key invariant in the classification of lattices up to isometry.

2.1 Finite Symmetric Bilinear Forms, Finite Quadratic Forms and Lattices

2.1.1 Finite Symmetric Bilinear and Finite Quadratic Forms

A *finite symmetric bilinear form* is a symmetric bilinear map

$$b : A \times A \to \mathbb{Q}/\mathbb{Z},$$

where A is a finite abelian group.

A *finite quadratic form* is a function

$$q : A \to \mathbb{Q}/2\mathbb{Z}$$

that satisfies the following conditions:

1. $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$ and $a \in A$,
2. $q(a + a') - q(a) - q(a') = 2b_q(a, a') \mod 2\mathbb{Z}$, where b_q is a finite symmetric bilinear form, which is called the *associated bilinear form* of q.

Definition 2.1. Let $q\colon A \to \mathbb{Q}/2\mathbb{Z}$ be a finite quadratic form on a finite abelian group A, and let $b_q\colon A \times A \to \mathbb{Q}/\mathbb{Z}$ denote its associated symmetric bilinear form defined by

$$b_q(a, a') := \frac{1}{2}\left(q(a + a') - q(a) - q(a')\right) \mod \mathbb{Z}.$$

Then q is called *nondegenerate* if either (equivalently, both) of the following hold:

1. The canonical group homomorphism

$$A \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}), \quad a \mapsto \big(a' \mapsto b_q(a, a')\big)$$

is an isomorphism.

2. The *radical* of b_q,

$$\mathrm{rad}(b_q) := \{a \in A \mid b_q(a, a') = 0 \text{ for all } a' \in A\},$$

is trivial; that is, $\mathrm{rad}(b_q) = \{0\}$.

If $H \subseteq A$ is a subgroup, the restriction of q to H is denoted by $q|_H$.

The abelian group A admits a unique (up to isomorphism) decomposition into the direct sum of its p-Sylow subgroups:

$$A \cong \bigoplus_p A_p,$$

where the sum runs over all prime divisors p of $|A|$, and A_p is the subgroup of elements of A whose order is a power of p.

If $q\colon A \to \mathbb{Q}/2\mathbb{Z}$ is a finite quadratic form, we denote by q_p the restriction of q to the p-component A_p. We define

$$\ell_p(q) := \ell(q_p),$$

where $\ell(q_p)$ denotes the length of the abelian group A_p, i.e., the minimum number of generators of A_p.

Proposition 2.2 (Nikulin [6, Proposition 1.2.1])**.** *Let q be a finite quadratic form defined on a finite abelian group A, and let $H \subset A$ be a subgroup. If the restriction $q|_H$ is nondegenerate, then*

$$q \cong q|_H \oplus q|_{H^\perp},$$

where $H^\perp = \{x \in A \mid b_q(x, h) = 0$ for all $h \in H\}$ and b_q is the symmetric bilinear form associated to q.

The analogous decomposition holds for finite symmetric bilinear forms.

Now, consider the associated algebraic structures arising from their bilinear and quadratic forms over finite abelian groups.

bil($\mathbb{Z}$): The semigroup of isomorphism classes of finite symmetric bilinear forms, under the operation of orthogonal direct sum.

qu($\mathbb{Z}$): The semigroup of isomorphism classes of finite quadratic forms, also under orthogonal direct sum.

Proposition 2.3 (Nikulin [6, Proposition 1.2.2])**.** *The finite quadratic form q and the bilinear form b can be decomposed as:*

$$q = \bigoplus_p q_p, \quad b = \bigoplus_p b_p,$$

where q_p and b_p are the restrictions of q and b to the p-component A_p of the finite group A. This decomposition provides a partition of the semigroups as follows:

$$\mathrm{qu}(\mathbb{Z}) = \bigoplus_p \mathrm{qu}(\mathbb{Z}_p), \quad \mathrm{bil}(\mathbb{Z}) = \bigoplus_p \mathrm{bil}(\mathbb{Z}_p),$$

where $\mathrm{qu}(\mathbb{Z}_p)$ *and* $\mathrm{bil}(\mathbb{Z}_p)$ *represent the isomorphism classes of finite quadratic forms and symmetric bilinear forms, respectively, defined over finite abelian p-groups.*

2.1.2 Lattices

Definition 2.4 (R-Lattice)**.** An R-*lattice* is a finitely generated free R-module M of finite rank, endowed with a nondegenerate symmetric R-valued bilinear form

$$b_M : M \times M \to R.$$

Definition 2.5 (Even R-Lattice)**.** An R-lattice M is called *even* if the associated bilinear form satisfies

$$b_M(x, x) \in 2R \quad \text{for all } x \in M,$$

where $2R = \{2r \mid r \in R\}$.

Remark 2.6. The terminology "even" is meaningful only for $R = \mathbb{Z}$ and $R = \mathbb{Z}_2$ because $2R$ is a proper subset of R in these cases. For $R = \mathbb{Z}_p$ with odd p, $2R = R$, making every lattice even.

Definition 2.7 (Gram Matrix)**.** Let M be an R-lattice, and let $b : M \times M \to R$ be a symmetric bilinear form. If $\{e_1, e_2, \dots, e_n\}$ is a basis of M, the *Gram matrix* of M with respect to this basis is the $n \times n$ matrix $G = (g_{ij})$, where

$$g_{ij} = b(e_i, e_j), \quad \text{for } 1 \le i, j \le n.$$

Definition 2.8 (Discriminant Bilinear Form)**.** The *discriminant bilinear form* of a lattice M is the bilinear form $b_M \colon A_M \times A_M \to \mathbb{Q}/\mathbb{Z}$, where $A_M = M^*/M$ is the discriminant group of the dual lattice $M^* = \mathrm{Hom}(M, \mathbb{Z})$ by M. It is defined as:

$$b_M(t_1 + M, t_2 + M) = t_1 \cdot t_2 + \mathbb{Z}, \quad \text{where } t_1, t_2 \in M^*.$$

Definition 2.9 (Discriminant Quadratic Form)**.** The *discriminant quadratic form* of a lattice M, when M is even, is the quadratic form $q_M \colon A_M \to \mathbb{Q}/2\mathbb{Z}$ associated with the bilinear form b_M. It is defined as:

$$q_M(t + M) = t^2 + 2\mathbb{Z}, \quad \text{where } t \in M^*.$$

Remark 2.10. For an integral lattice L,

$$q(x + L) \;:=\; B(x, x) \bmod \mathbb{Z}$$

is a well-defined *finite quadratic form* on $A_L = L^*/L$ iff L is **even**.

Definition 2.11 (Discriminant of an R-Lattice)**.** The *discriminant* of an R-lattice M, denoted $\mathrm{discr}(M)$, is defined by

$$\mathrm{discr}(M) = \det(e_i \cdot e_j) \mod (R^\times)^2,$$

where $\{e_i\}$ is a basis of M, $R^\times$ is the multiplicative group of units in R, and $(R^\times)^2$ is the subgroup of squares.

Definition 2.12 (Unimodular Lattices)**.** An R-lattice M is called *unimodular* if

$$\mathrm{discr}(M) \in R^\times/(R^\times)^2.$$

Theorem 2.13 (Local Decomposition of Discriminant Forms)**.** *Let K be a lattice over $\mathbb{Z}$. Then for each prime p, the discriminant bilinear form of the localized lattice satisfies*

$$b_{K \otimes \mathbb{Z}_p} = (b_K)_p,$$

and the global discriminant form decomposes as a direct orthogonal sum over all primes:

$$b_K = \bigoplus_p b_{K\otimes\mathbb{Z}_p}.$$

If K is even, then the same holds for the discriminant quadratic form:

$$q_{K\otimes\mathbb{Z}_p} = (q_K)_p, \qquad q_K = \bigoplus_p q_{K\otimes\mathbb{Z}_p}.$$

Now, consider the following semigroups:

$\mathbf{Bil}(R)$: The semigroup of isomorphism classes of R-lattices equipped with a non-degenerate symmetric bilinear form, under the operation of orthogonal direct sum.

$\mathbf{Qu}(R)$: The semigroup of isomorphism classes of even R-lattices, meaning R-lattices equipped with a non-degenerate quadratic form, under the same operation of orthogonal direct sum.

Theorem 2.14. *The following maps are surjective:*

$$b : \mathbf{Bil}(\mathbb{Z}) \twoheadrightarrow \mathbf{bil}(\mathbb{Z}), \qquad q : \mathbf{Qu}(\mathbb{Z}) \twoheadrightarrow \mathbf{qu}(\mathbb{Z}),$$

where b sends an integral lattice to its discriminant bilinear form, and q sends an even lattice to its discriminant quadratic form.

$\mathbf{St}.\mathbf{Bil}(\mathbb{Z})$: The semigroup of stable equivalence classes of integral lattices equipped with a non-degenerate symmetric bilinear form, under the operation of orthogonal direct sum. Two lattices L_1 and L_2 are considered stably equivalent if there exist unimodular lattices U_1 and U_2 such that

$$L_1 \oplus U_1 \cong L_2 \oplus U_2.$$

$\mathbf{St}.\mathbf{Qu}(\mathbb{Z})$: The semigroup of stable equivalence classes of even integral lattices under orthogonal direct sum. Two even lattices L_1 and L_2 are stably equivalent if there exist even unimodular lattices U_1 and U_2 such that

$$L_1 \oplus U_1 \cong L_2 \oplus U_2.$$

Following early claims by Kneser and Puppe that the discriminant bilinear form determines a lattice up to stable equivalence in the case of odd order, complete proofs were later given by Wilkens [7] and Durfee [8]. The result was extended to quadratic forms by Durfee using p-adic methods. The corresponding theorem for quadratic forms was obtained by Wall [9], and later also by Miranda and Morrison [10].

Theorem 2.15. *The following maps are isomorphisms of semigroups:*

$$b : \mathbf{St.Bil}(\mathbb{Z}) \xrightarrow{\sim} \mathbf{bil}(\mathbb{Z}), \qquad q : \mathbf{St.Qu}(\mathbb{Z}) \xrightarrow{\sim} \mathbf{qu}(\mathbb{Z}),$$

where b sends an integral lattice to its discriminant bilinear form, and q sends an even lattice to its discriminant quadratic form.

Using the result above, Nikulin reproved the existence of a well-defined signature homomorphism modulo 8 on the semigroups of finite quadratic, thereby recovering a well-known result stated in the following theorem.

Theorem 2.16 (Nikulin [6, Theorem 1.3.3])**.** *On the semigroups* St.Qu($\mathbb{Z}$) *and* qu($\mathbb{Z}$)*, there exist canonical homomorphisms the* signature modulo 8*—mapping into the additive group $\mathbb{Z}/8\mathbb{Z}$.*

If L is an even lattice with signature (t_+, t_-), then

$$t_+ - t_- \equiv \operatorname{sign}(L) \equiv \operatorname{sign} q_L \pmod 8.$$

2.1.3 Elementary $\mathbb{Z}_p$-lattices, Elementary Finite Quadratic Forms, Elementary Finite Symmetric Bilinear Forms

In this subsection, we introduce the fundamental $\mathbb{Z}_p$-lattices, finite quadratic forms, and finite symmetric bilinear forms, which serve as the building blocks for the corresponding semigroups $\mathbf{Bil}(\mathbb{Z}_p)$, $\mathbf{qu}(\mathbb{Z})$, and $\mathbf{bil}(\mathbb{Z})$.

We now define the fundamental $\mathbb{Z}_p$-lattices, which are central to the study of $\mathbf{Bil}(\mathbb{Z}_p)$.

Definition 2.17. Let p be a prime, $k \in \mathbb{Z}_{>0}$, and $\varepsilon \in \{1,3,5,7\}$ for $p=2$ or $\varepsilon \in \{\pm 1\}$ for $p>2$. Define the following $\mathbb{Z}_p$-lattices:

$$W^{\varepsilon}_{p,k} := \left(\mathbb{Z}_p, (x,y) \mapsto a \cdot p^k \cdot xy\right), \quad \text{where } a \text{ satisfies } \left(\frac{2a}{p}\right) = \varepsilon, \text{ for } p>2, \tag{2.1}$$

$$W^{\varepsilon}_{2,k} := \left(\mathbb{Z}_2, (x,y) \mapsto \varepsilon \cdot 2^k \cdot xy\right), \quad \text{for } p=2, \tag{2.2}$$

$$U_k := \left(\mathbb{Z}_2 \times \mathbb{Z}_2, ((x_1,x_2),(y_1,y_2)) \mapsto 2^k(x_1y_2 + x_2y_1)\right), \tag{2.3}$$

$$V_k := \left(\mathbb{Z}_2 \times \mathbb{Z}_2, ((x_1,x_2),(y_1,y_2)) \mapsto 2^{k+1}x_1y_1 + 2^k(x_1y_2 + x_2y_1) + 2^{k+1}x_2y_2\right). \tag{2.4}$$

The Gram matrices of the fundamental $\mathbb{Z}_p$-lattices are as follows: For $p>2$, the lattice $W^{\varepsilon}_{p,k}$ of rank 1 (Eq. 2.1) has the Gram matrix $\begin{pmatrix} a \cdot p^k \end{pmatrix}$, where a satisfies $\left(\frac{2a}{p}\right) = \varepsilon$. For $p=2$, the lattice $W^{\varepsilon}_{2,k}$ of rank 1 (Eq. 2.2) has the Gram matrix $\begin{pmatrix} \varepsilon \cdot 2^k \end{pmatrix}$. The lattice U_k of rank 2 (Eq. 2.3) has the Gram matrix $\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}$, and the lattice V_k of rank 2 (Eq. 2.4) has the Gram matrix $\begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}$.

To establish the foundation of $\mathbf{qu}(\mathbb{Z})$, we define the finite quadratic forms associated with the fundamental $\mathbb{Z}_p$-lattices.

Definition 2.18. Let $k \in \mathbb{N}$, $\varepsilon \in \{1,3,5,7\}$ for $p=2$, or $\varepsilon \in \{\pm 1\}$ for odd primes p. Define the following finite quadratic forms:

$$\mathbf{w}^{\varepsilon}_{p,k} := \left(\mathbb{Z}/p^k\mathbb{Z}e, \quad q(xe) = x^2 \cdot a \cdot p^{-k}\right), \quad \text{where } a \text{ satisfies } \left(\frac{2a}{p}\right) = \varepsilon, \text{ for } p>2. \tag{2.5}$$

$$\mathbf{w}^{\varepsilon}_{2,k} := \left(\mathbb{Z}/2^k\mathbb{Z}e, \quad q(xe) = x^2 \cdot \varepsilon \cdot 2^{-k}\right), \quad \text{for } p=2. \tag{2.6}$$

$$\mathbf{u}_k := \left(\mathbb{Z}/2^k\mathbb{Z}e_1 \oplus \mathbb{Z}/2^k\mathbb{Z}e_2, \quad q(xe_1 + ye_2) = xy \cdot 2^{1-k}\right), \tag{2.7}$$

$$\mathbf{v}_k := \left(\mathbb{Z}/2^k\mathbb{Z}e_1 \oplus \mathbb{Z}/2^k\mathbb{Z}e_2, \quad q(xe_1 + ye_2) = (x^2 + xy + y^2) \cdot 2^{1-k}\right). \tag{2.8}$$

To establish the foundation of $\mathbf{bil}(\mathbb{Z})$, we introduce the finite symmetric bilinear forms derived from the same $\mathbb{Z}_p$-lattices.

Definition 2.19. Let $k \in \mathbb{N}$, $\varepsilon \in \{1, 3, 5, 7\}$ for $p = 2$, or $\varepsilon \in \{\pm 1\}$ for odd primes p. Define the following finite symmetric bilinear forms:

$$\underline{\mathbf{w}}_{p,k}^{\varepsilon} := \left(\mathbb{Z}/p^k\mathbb{Z}e, \quad b(xe, x'e) = xx' \cdot a \cdot p^{-k}\right), \quad \text{where } a \text{ satisfies } \left(\frac{2a}{p}\right) = \varepsilon, \text{ for } p > 2. \tag{2.9}$$

$$\underline{\mathbf{w}}_{2,k}^{\varepsilon} := \left(\mathbb{Z}/2^k\mathbb{Z}e, \quad b(xe, x'e) = xx' \cdot \varepsilon \cdot 2^{-k}\right), \quad \text{for } p = 2. \tag{2.10}$$

$$\underline{\mathbf{u}}_k := \left(\mathbb{Z}/2^k\mathbb{Z}e_1 \oplus \mathbb{Z}/2^k\mathbb{Z}e_2, \quad b(xe_1 + ye_2, x'e_1 + y'e_2) = (xy' + x'y) \cdot 2^{-k}\right), \tag{2.11}$$

$$\underline{\mathbf{v}}_k := \left(\mathbb{Z}/2^k\mathbb{Z}e_1 \oplus \mathbb{Z}/2^k\mathbb{Z}e_2, \quad b(xe_1 + ye_2, x'e_1 + y'e_2) = (2xx' + xy' + x'y + 2yy') \cdot 2^{-k}\right). \tag{2.12}$$

Theorem 2.20 ([6, Proposition 1.8.1])**.** *The semigroups $\mathbf{Bil}(\mathbb{Z}_p)$, $\mathbf{qu}(\mathbb{Z})$, and $\mathbf{bil}(\mathbb{Z})$ are generated as follows:*

- *The semigroup $\mathbf{Bil}(\mathbb{Z}_p)$ is generated by $\mathbf{W}_{p,k}^{\varepsilon}$ (Eq. 2.1) for odd p, and by $\mathbf{W}_{2,k}^{\varepsilon}$ (Eq. 2.2), $\mathbf{U}_k$ (Eq. 2.3) and $\mathbf{V}_k$ (Eq. 2.4) for $p = 2$, where $k \geq 0$.*
- *The semigroup $\mathbf{qu}(\mathbb{Z})$ is generated by the finite quadratic forms $\mathbf{w}_{p,k}^{\varepsilon}$ (Eq. 2.6) for odd p, and by $\mathbf{w}_{2,k}^{\varepsilon}$ (Eq. 2.8), $\mathbf{u}_k$ (Eq. 2.6), and $\mathbf{v}_k$ (Eq. 2.8) for $p = 2$, where $k \geq 1$.*
- *The semigroup $\mathbf{bil}(\mathbb{Z})$ is generated by the finite bilinear forms $\underline{\mathbf{w}}_{p,k}^{\varepsilon}$ (Eq. 2.9) for odd p, and by $\underline{\mathbf{w}}_{2,k}^{\varepsilon}$, $\underline{\mathbf{u}}_k$ (Eq. 2.11) and $\underline{\mathbf{v}}_k$ (Eq. 2.12) for $p = 2$, where $k \geq 1$.*

Definition 2.21 (Isometry of Finite Quadratic Forms)**.** Let (A, q) and $(\widetilde{A}, \widetilde{q})$ be *finite quadratic forms*, where A and $\widetilde{A}$ are finite abelian groups equipped with quadratic forms q and $\widetilde{q}$, respectively.

An *injective group homomorphism*

$$\sigma : A \to \widetilde{A}$$

is called an *isometry* if it preserves the quadratic form, that is,

$$q(x) = \widetilde{q}(\sigma(x)) \quad \text{for all } x \in A.$$

If σ is also *bijective*, then (A, q) and $(\widetilde{A}, \widetilde{q})$ are said to be *isometric* finite quadratic forms.

2.1.4 Finite Quadratic Forms on 2-Elementary Abelian Groups

To develop further results on finite quadratic forms over 2-elementary abelian groups, we begin with the simplest case $A \cong \mathbb{Z}/2\mathbb{Z}$. Such forms can be either non-degenerate or degenerate, and we classify all isomorphism types in each case.

2.1.4.1 Non-Degenerate Finite Quadratic Forms on $\mathbb{Z}/2\mathbb{Z}$

For $k = 1$ and the underlying group $\mathbb{Z}/2\mathbb{Z}$, the finite quadratic forms are given by:

$$\mathbf{w}_{2,1}^{\varepsilon}(xe) = \frac{\varepsilon}{2}x^2 \quad \text{where} \quad \varepsilon \in \{1, 3, 5, 7\}.$$

Here, $x \in \mathbb{Z}/2\mathbb{Z}$, so x can be 0 or 1. Evaluating the forms:

- For $x = 0$: $\mathbf{w}_{2,1}^{\varepsilon}(0) = 0$.
- For $x = 1$: $\mathbf{w}_{2,1}^{\varepsilon}(e) = \frac{\varepsilon}{2}$.

Since the values lie in $\mathbb{Q}/2\mathbb{Z}$, we observe that

$$\frac{\varepsilon}{2} \equiv \frac{\varepsilon + 4k}{2} \mod 2\mathbb{Z} \quad \text{for any } k \in \mathbb{Z}.$$

Thus, the distinct forms are classified by $\varepsilon \bmod 4$:

- $\varepsilon \equiv 1, 5 \mod 4$: $\frac{1}{2}$ and $\frac{5}{2} \equiv \frac{1}{2} \mod 2\mathbb{Z}$,
- $\varepsilon \equiv 3, 7 \mod 4$: $\frac{3}{2}$ and $\frac{7}{2} \equiv \frac{3}{2} \mod 2\mathbb{Z}$.

Therefore,

$$\mathbf{w}_{2,1}^1 \cong \mathbf{w}_{2,1}^5, \qquad \mathbf{w}_{2,1}^3 \cong \mathbf{w}_{2,1}^7,$$

as the forms evaluate identically on $\mathbb{Z}/2\mathbb{Z}$ for these values of ε.

2.1.4.2 Degenerate Finite Quadratic Forms on $\mathbb{Z}/2\mathbb{Z}$

For $\varepsilon \in \{0, 2, 4, 6\}$, define quadratic forms on $\mathbb{Z}/2\mathbb{Z}$ by:

$$\mathbf{w}_{2,1}^\varepsilon(xe) = \frac{\varepsilon}{2}x^2.$$

As before, $x \in \mathbb{Z}/2\mathbb{Z}$ takes values 0 or 1, and the evaluation yields:

- For $x = 0$: $\mathbf{w}_{2,1}^\varepsilon(0) = 0$,
- For $x = 1$: $\mathbf{w}_{2,1}^\varepsilon(e) = \dfrac{\varepsilon}{2}$.

Since the values lie in $\mathbb{Q}/2\mathbb{Z}$, and

$$\frac{\varepsilon}{2} \equiv \frac{\varepsilon + 4k}{2} \mod 2\mathbb{Z} \quad \text{for any } k \in \mathbb{Z}.$$

, we compute:

- $\varepsilon \equiv 0, 4 \mod 4$: $\dfrac{0}{2} = 0$ and $\dfrac{4}{2} = 2 \equiv 0 \mod 2\mathbb{Z}$,
- $\varepsilon \equiv 2, 6 \mod 4$: $\dfrac{2}{2} = 1$ and $\dfrac{6}{2} = 3 \equiv 1 \mod 2\mathbb{Z}$.

Thus, the degenerate forms group into two isomorphism classes:

$$\mathbf{w}_{2,1}^0 \cong \mathbf{w}_{2,1}^4, \qquad \mathbf{w}_{2,1}^2 \cong \mathbf{w}_{2,1}^6.$$

We denote the corresponding isomorphism classes as:

$$\langle 0 \rangle := \text{the form with } q(1) = 0, \qquad \langle 1 \rangle := \text{the form with } q(1) = 1.$$

Let (A, q) be a finite quadratic form, where $A \cong (\mathbb{Z}/2)^r$ is a 2-elementary abelian group. Define the *radical* of b as

$$q^\perp := \{x \in A \mid b(x, y) = 0 \text{ for all } y \in A\}.$$

Then the short exact sequence of abelian groups

$$0 \longrightarrow q^\perp \longrightarrow A \longrightarrow A/q^\perp \longrightarrow 0$$

splits. Moreover, the quadratic form q decomposes orthogonally as

$$(A, q) \cong (q^\perp, q|_{q^\perp}) \perp (W, q|_W),$$

where $W \subset A$ is a subspace such that $A = q^\perp \oplus W$ and the restriction $b|_{W \times W}$ is non-degenerate.

Hence, the quadratic form q can be expressed as an orthogonal direct sum of elementary indecomposable forms:

$$q \cong \bigoplus^a \mathbf{u}_1 \oplus \bigoplus^b \mathbf{v}_1 \oplus \bigoplus^c \mathbf{w}^1_{2,1} \oplus \bigoplus^d \mathbf{w}^3_{2,1} \oplus \bigoplus^e \langle 0 \rangle \oplus \bigoplus^f \langle 1 \rangle, \qquad a, b, c, d, e, f \in \mathbb{Z}_{\geq 0}.$$

We now recall the notion of the signature of a finite quadratic form and a basic parity relation between this signature and the length.

Lemma 2.22. *Let q be any finite quadratic form. Then the length $\ell_2(q)$ satisfies*

$$\ell_2(q) \equiv \operatorname{sign}(q) \pmod 2.$$

Definition 2.23. A (possibly degenerate) finite quadratic form q is said to be *odd* if it can be expressed as

$$q \cong \mathbf{w}^\varepsilon_{2,1} \oplus q',$$

for some finite quadratic form q' where

$$\varepsilon \in \{1, 3, 5, 7\}$$

. If q cannot be written in this form, it is called *even*.

Lemma 2.24 ([5, Lemma 3.11]). *Let L be a non-degenerate lattice with discriminant form $q(L)$. Then*

$$\ell_2(q(L)) = \operatorname{rank}(L)$$

if and only if $L \cong L'(2)$ for some lattice L'.

Moreover, L' is even if and only if the discriminant form $q(L)$ is even.

2.2 Genus

We now recall the notion of the *genus* of an integral lattice, which classifies lattices up to local isometry at all completions of $\mathbb{Q}$.

Definition 2.25 (Genus of an integral lattice). Let L be a non-degenerate integral lattice. Two integral lattices L and M are said to be *in the same genus* if they are isometric over every completion of $\mathbb{Q}$:

$$L \otimes_{\mathbb{Z}} \mathbb{R} \;\cong\; M \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad L \otimes_{\mathbb{Z}} \mathbb{Z}_p \;\cong\; M \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \text{for all primes } p.$$

We now state fundamental results due to Nikulin, which characterize the genus of integral lattices in terms of their signature and discriminant form, and provide sufficient conditions under which a genus consists of a unique isomorphism class.

Theorem 2.26 (Nikulin [6, Corollary 1.9.4]). *Let L be an* even *non-degenerate integral lattice of signature (s_+, s_-). Let $A_L := L^*/L$ for its discriminant group and $q_L\colon A_L \to \mathbb{Q}/2\mathbb{Z}$ for the finite quadratic form. Then the genus of L is completely determined by the triple*

$$\big(s_+, s_-, (A_L, q_L)\big).$$

Equivalently, two even lattices L and M lie in the same genus if and only if they have the same signature and their discriminant quadratic forms are isometric.

Theorem 2.27 (Nikulin [6, Corollary 1.16.3]). *Let L be a non-degenerate integral lattice of signature (s_+, s_-). Let $A_L := L^*/L$ and let $b_L\colon A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ be the* *associated finite symmetric bilinear form. Then the genus of L is completely determined by the triple*

$$\big(s_+, s_-, (A_L, b_L)\big).$$

, and parity. In particular, two lattices L and M belong to the same genus if and only if they share the same signature, parity and their discriminant bilinear forms are isometric.

Theorem 2.28 (Nikulin [6, Theorem 1.14.2])**.** *Let T be an even, indefinite lattice. Assume the following conditions are satisfied:*

(a) $\operatorname{rank} T \geq \ell_p(q(T)) + 2$ *for every prime $p \neq 2$,*

(b) if $\operatorname{rank} T = \ell_2(q(T))$, *then the discriminant form $q(T)$ is isometric to either $\mathbf{u}_1 \oplus q'$ or $\mathbf{v}_1 \oplus q'$ for some quadratic form q'.*

Then the genus of T contains exactly one isometry class.

Remark 2.29. The *parity*, *rank*, *signature*, and *determinant* of a genus g refer to the respective properties of any lattice L within g.

Examples of Lattices with Their Discriminant Forms

- $\mathbf{U}$: The hyperbolic lattice of rank 2, with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  The discriminant form of $\mathbf{U}$ is trivial.

- $\mathbf{U}(2)$: The scaled hyperbolic lattice, with Gram matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

  The discriminant form of $\mathbf{U}(2)$ is $\mathbf{u}_1$.

- $\mathbf{E_8}$: The E_8 lattice, an even unimodular negative-definite lattice of rank 8, with Gram matrix

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

  The discriminant form of $\mathbf{E_8}$ is trivial.

- $\mathbf{E_8}(2)$: The scaled $\mathbf{E_8}$ lattice, with Gram matrix

$$\begin{pmatrix} -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -4 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 \end{pmatrix}.$$

  The discriminant form of $\mathbf{E_8}(2)$ is $4\mathbf{u}_1$.

- $\tilde{\mathbf{E}}$: A positive definite lattice of rank 8, with Gram matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 4 & 2 \\ 1 & 1 & 1 & 1 & -1 & 1 & 2 & 4 \end{pmatrix}.$$

  The discriminant form of $\tilde{\mathbf{E}}$ is $3\mathbf{u}_1$.

Chapter 3

K3 Surfaces and Enriques Surfaces

The primary references for this chapter are [11, 12, 13].

3.0.1 K3 Surfaces and Their Lattices

In this thesis, a *K3 surface* refers to a smooth, compact, complex surface that is simply connected and admits a nowhere vanishing holomorphic 2-form. These properties are equivalent to the vanishing of the irregularity, $H^1(X, \mathcal{O}_X) = 0$, and the triviality of the canonical bundle, $\Omega_X^2 \cong \mathcal{O}_X$.

The *Picard rank* of a K3 surface X, denoted ρ_X, is defined as the rank of the Néron–Severi group, i.e., the number of linearly independent divisor classes or, equivalently, the number of algebraically independent line bundles. When X is projective, one always has $\rho_X \geq 1$, but the converse does not necessarily hold.

The second cohomology group $H_X^2 := H^2(X, \mathbb{Z})$ carries a natural lattice structure via the cup product, and contains the following sublattices:

$$\begin{array}{lll} H^2_X = H^2(X,\mathbb{Z}) & \text{full second cohomology,} & \mathrm{sgn}(H^2_X) = (3,19), \\ T_X \subset H^2_X & \text{transcendental lattice,} & \mathrm{sgn}(T_X) = (2, 20-\rho_X), \\ NS_X \subset H^2_X & \text{Néron–Severi lattice,} & \mathrm{sgn}(NS_X) = (1, \rho_X - 1), \end{array}$$

where the signatures for T_X and NS_X are valid under the assumption that X is projective.

By Poincaré duality, the lattice H^2_X is unimodular, and by Wu's formula, its pairing is even. Since even indefinite unimodular lattices are uniquely determined up to isometry by their signature, we conclude that

$$H^2_X \cong 3\mathbf{U} \oplus 2\mathbf{E}_8,$$

where U denotes the hyperbolic plane and E_8 is the negative-definite E_8 lattice. This lattice is referred to as the *K3 lattice*, denoted by $\mathbf{\Lambda}$.

The Néron–Severi and transcendental lattices form orthogonal, primitive sublattices of H^2_X, and their direct sum sits inside H^2_X as a sublattice of finite index:

$$T_X \oplus NS_X \subset H^2_X.$$

3.0.2 Enriques Surfaces

A smooth projective surface Y is called an *Enriques surface* if it satisfies the following conditions:

$$2K_Y \sim 0, \quad H^1(Y, \mathcal{O}_Y) = 0, \quad H^2(Y, \mathcal{O}_Y) = 0.$$

Here, K_Y denotes the canonical divisor of Y, $\sim$ denotes linear equivalence, and $\mathcal{O}_Y$ is the structure sheaf of Y.

Theorem 3.1 (Kondo [13, Proposition 9.3])**.** *Let Y be an Enriques surface. Then:*

1. The fundamental group of Y is $\mathbb{Z}/2\mathbb{Z}$.

2. The universal cover of Y is a K3 surface.

Conversely, if X is a K3 surface with a fixed-point-free involution σ, then the quotient surface $X/\langle\sigma\rangle$ is an Enriques surface.

3.0.3 Keum's criterion

Keum proved the following theorem under the additional assumption that $\ell(T_X)+2 < \rho(X)$. This assumption is valid if $\rho(X) \geq 12$. In his proof, Keum utilized relied on Torelli-type theorems for K3 and Enriques surfaces to establish the following result.

For the remaining cases where $\rho(X) = 10$ and $\rho(X) = 11$, H.Ohashi proved in [4].

Theorem 3.2 (Keum [14, Theorem 1])**.** *A $K3$ surface X with transcendental lattice T_X covers an Enriques surface if and only if there exists a primitive embedding of T_X into $\mathbf{\Lambda}^- := \mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{E}_8(2)$ such that there exists no vector $v \in T_X^\perp$ with $v^2 = -2$.*

Chapter 4

Lattice Embedding and K3-Cover of Enriques Surfaces

Definition 4.1 (Lattice Morphism)**.** Let (L, b) and (M, b') be integral lattices, i.e., finitely generated free $\mathbb{Z}$-modules equipped with symmetric bilinear forms $b\colon L \times L \to \mathbb{Z}$ and $b'\colon M \times M \to \mathbb{Z}$, respectively.

A homomorphism $\phi\colon L \to M$ is called a *lattice morphism* if

$$b(x, y) = b'\big(\phi(x), \phi(y)\big) \quad \text{for all } x, y \in L.$$

Definition 4.2 (Lattice Embedding)**.** A lattice morphism $\phi\colon L \to M$ is called a *lattice embedding* if ϕ is injective.

Definition 4.3 (Primitive Sublattice)**.** Let L be a sublattice of a lattice M. The sublattice L is called a *primitive sublattice* of M if the quotient M/L is a free $\mathbb{Z}$-module, i.e., it has no torsion elements. *Equivalently*, L is a primitive sublattice of M if for any $v \in M$ and any non-zero integer m, whenever $m \cdot v \in L$, it follows that $v \in L$.

Definition 4.4 (Primitive Embedding)**.** Let L and M be two lattices. A lattice embedding $\phi : L \hookrightarrow M$ is called a *primitive embedding* if the image $\phi(L)$ is a primitive sublattice of M. That is, the quotient $M/\phi(L)$ is a free $\mathbb{Z}$-module.

Equivalently, the embedding $\phi : L \hookrightarrow M$ is a primitive embedding if for any $v \in M$ and any non-zero integer m, whenever $m \cdot v \in \phi(L)$, it follows that $v \in \phi(L)$.

Definition 4.5. Two matrices T_1 and T_2 are said to be $\mathbb{Z}$-equivalent if there exists an element $g \in GL(n, \mathbb{Z})$ such that

$$T_2 = g^\tau T_1 g.$$

The following theorem characterizes the primitive embedding of lattices.

Theorem 4.6 ([1, Lemma 3])**.** *An embedding is primitive if and only if the greatest common divisor of the maximal minors of the embedding matrix with respect to any choice of basis is 1.*

4.1 Necessary Conditions for Embeddings of Lattices

In this section, we provide the necessary conditions for the embedding of lattices.

From now on, an embedding is assumed to be primitive unless stated otherwise.

Lemma 4.7. *The map $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2\mathbb{Z})$ is surjective.*

Proof. Indeed, $GL_n(\mathbb{Z}/2\mathbb{Z}) \cong SL_n(\mathbb{Z}/2\mathbb{Z})$ is generated by transvections (elementary matrices), and these obviously lift to $GL_n(\mathbb{Z})$. □

Definition 4.8 (Congruence over $\mathrm{GL}(\mathbb{Z}/2)$)**.** Let $A, B \in \mathrm{Mat}_n(\mathbb{Z}/2)$ be two symmetric matrices. We say that A and B are *congruent over* $\mathbb{Z}/2$ (or $\mathrm{GL}(\mathbb{Z}/2)$-congruent) if there exists an invertible matrix $P \in \mathrm{GL}_n(\mathbb{Z}/2)$ such that

$$B = P^T AP.$$

Theorem 4.9. *Let F be a finite field of characteristic 2. Then, every symmetric matrix M_n over F with a zero diagonal is congruent to*

$$\bigoplus_{i=1}^{q} H \oplus \bigoplus_{i=1}^{n-2q} [0] \quad or \quad \bigoplus_{i=1}^{n} [0],$$

where $q \in \mathbb{N}^+$ and

$$H \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Furthermore, two symmetric matrices with a zero diagonal over the field F are congruent if and only if they have the same rank.

Proof. Suppose $M = 0$; the result is trivial.

Now, suppose $M \neq 0$. In this case, there exists some nonzero element a_{ij} in M. Since $GL_n(\mathbb{Z}/2) \cong SL_n(\mathbb{Z}/2)$ is generated by the elementary matrices $E_{ij}(\alpha)$, where $\alpha \in \mathbb{Z}/2$, we can perform elementary congruent transformations to write

$$M \cong \begin{bmatrix} H & B^t \\ B & A \end{bmatrix}, \quad \text{where } H \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

By performing successive elementary congruent transformations of $E_{ij}(\alpha)$, with i varying from 3 to n, we obtain a symmetric matrix such that

$$M \cong \begin{bmatrix} H & 0 \\ 0 & A_1 \end{bmatrix}, \quad \text{where } H \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here, A_1 is an $(n-2)$-row symmetric matrix with a zero diagonal. Therefore, we can proceed recursively to obtain a symmetric matrix of the form

$$\bigoplus_{i=1}^{q} H \oplus \bigoplus_{i=1}^{n-2q} [0]$$

that is congruent to the given matrix M.

Since matrix congruence is an equivalence relation that satisfies symmetry and transitivity, and based on the above reasoning, two symmetric matrices M_n with a zero diagonal over the field F are congruent if and only if they have the same rank. This completes the proof. $\square$

As a direct consequence, we obtain the following.

Corollary 4.10. *The number $o(n)$ of orbits of the even symmetric matrix M_n of size $n \times n$ over $\mathbb{Z}/2$ with a zero diagonal, under the action of $GL_n(\mathbb{Z}/2)$ by transposition, is given by*

$$o(n) = \begin{cases} \frac{1}{2}(n+1), & \text{if } n \text{ is odd,} \\ \frac{1}{2}(n+2), & \text{if } n \text{ is even.} \end{cases}$$

The map $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2)$ is surjective by Lemma 4.7; therefore, we can consider the action of $GL_n(\mathbb{Z}/2)$ by transposition on the set of even symmetric matrices M_n of size $n \times n$ over $\mathbb{Z}/2$.

We will characterize associated Gram matrices of lattices with respect to their ranks over $\mathbb{Z}/2$ by the following.

Definition 4.11. Let L be an integral even lattice of rank λ and G_L be its associated Gram matrix. Let L' be an induced $\mathbb{Z}/2$-module of L by restriction of the ring of integers $\mathbb{Z}$ to $\mathbb{Z}/2$ and its associated Gram matrices $G_L = (a_{ij})$ of L and $G_{L'} = (a_{ij} \mod 2)$ of L'. The rank of G_L over the field of characteristic 2 will be the rank of $G_{L'}$, denoted by $\mathrm{rank}_2(L)$.

Lemma 4.12. *Let L be an integral even lattice of rank λ and G_L be its associated Gram matrix, with $\mathrm{rank}_2(L) = 2q$. Then,*

$$G_L \cong \begin{pmatrix} 2a_{11} & a_{12} & \cdots & a_{1\lambda} \\ a_{12} & 2a_{22} & \cdots & a_{2\lambda} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1\lambda} & \cdots & \cdots & 2a_{\lambda\lambda} \end{pmatrix},$$

such that $a_{2k-1,2k}$ is odd for each $1 \leq k \leq q$, and the remaining off-diagonal entries are even.

Proof. Let L be an even lattice. The Gram matrix G_L of L is symmetric, and its diagonal entries are even.

Reduction of G_L modulo 2 yields a symmetric matrix over $\mathbb{Z}/2$. The diagonal entries, being even, reduce to zero, so the reduced matrix has zero diagonal entries, and the off-diagonal entries are either 0 or 1.

By Lemma 4.7, the map $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2)$ is surjective. Therefore, we can use the action of $GL_n(\mathbb{Z}/2)$ to bring $G_L \mod 2$ into a canonical form.

Over $\mathbb{Z}/2$, by Theorem 4.9, every symmetric matrix with zero diagonal can be reduced, via congruence, to a block diagonal form consisting of 2×2 blocks of the type:

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and possibly zero blocks.

The number of such blocks corresponds to the rank of the matrix modulo 2. Let $\operatorname{rank}_2(L) = 2q$, which means $G_L \mod 2$ can be transformed into a block diagonal matrix with q blocks of the form H, while the remaining part of the matrix consists of zero blocks.

Once we have the canonical form of $G_L \mod 2$, we lift this back to an integral matrix over $\mathbb{Z}$. For each block H in the reduced matrix, the corresponding entries in the lifted matrix G_L will be odd. That is, the off-diagonal terms $a_{2k-1,2k}$ are odd for each $k = 1, 2, \ldots, q$. The remaining off-diagonal terms, which were zero modulo 2, will be even in G_L. This completes the proof. □

Theorem 4.13. *Let L and M be even integral lattices of* $\operatorname{rank}(L)$ *and* $\operatorname{rank}(M)$, *and let* $\operatorname{rank}_2(L)$ *and* $\operatorname{rank}_2(M)$ *denote their ranks over* $\mathbb{Z}/2$. *Let ϕ be an embedding of L into M. Then, one of the following conditions holds:*

I *If* $\operatorname{rank}_2(M) = 0$, *then there exists a lattice T such that* $L \cong T(2)$.

II *If* $\operatorname{rank}_2(M) > 0$ *and* $\operatorname{rank}_2(L) = 0$*, then*

$$\operatorname{rank}(L) \leq \operatorname{rank}(M) - \frac{1}{2}\operatorname{rank}_2(M),$$

and there exists a lattice T *such that* $L \cong T(2)$*.*

III *If* $\operatorname{rank}_2(M) > 0$ *and* $\operatorname{rank}_2(L) > 0$*, then*

$$\operatorname{rank}(L) \leq \operatorname{rank}(M) - \frac{1}{2}\operatorname{rank}_2(M) + \frac{1}{2}\operatorname{rank}_2(L),$$

and there exists an even lattice T *such that* $L \cong T$*, and its associated Gram matrix must have the form:*

$$G_T = \begin{pmatrix} 2a_{11} & a_{12} & \cdots & a_{1\lambda} \\ a_{12} & 2a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{1\lambda} & \cdots & \cdots & 2a_{\lambda\lambda} \end{pmatrix},$$

where $a_{2k-1,2k}$ *is odd for each* $1 \leq k \leq \frac{1}{2}\operatorname{rank}_2(L)$*, and the remaining off-diagonal entries are even.*

IV *If* $\operatorname{rank}(L) = \operatorname{rank}(M)$*, then* $L \cong M$*.*

Proof. Let L have a Gram matrix G_L and let M have a Gram matrix G_M. The induced $\mathbb{Z}/2$-modules are given by:

$$L' \cong \bigoplus_{i=1}^{l}(\mathbb{Z}/2)x_i \quad \text{and} \quad M' \cong \bigoplus_{i=1}^{m}(\mathbb{Z}/2)u_i,$$

where $\{x_i\}_i$ and $\{u_i\}_i$ are basis for L and M of ranks l and m, respectively.

By Theorem 4.9 and Corollary 4.10, we know that:

$$G_{L'} \cong \bigoplus_{i=1}^{p} H \oplus \bigoplus_{i=1}^{l-2p}[0] \text{ or } \bigoplus_{i=1}^{l}[0],$$

where $\text{rank}_2(L) = 2p$. Similarly,

$$G_{M'} \cong \bigoplus_{i=1}^{q} H \oplus \bigoplus_{i=1}^{m-2q} [0] \text{ or } \bigoplus_{i=1}^{m} [0],$$

where $\text{rank}_2(M) = 2q$. They are uniquely determined by their ranks over $\mathbb{Z}$ and $\mathbb{Z}/2$ by Theorem 4.9 and Corollary 4.10.

Consider the embedding $\phi : L \to M$ and the induced embedding $\phi' : L' \to M'$. Since ϕ and ϕ' are embeddings, the necessary conditions for the embedding of L' into M' are that:

$$\text{rank}(L) \leq \text{rank}(M), \quad \text{rank}_2(L) \leq \text{rank}_2(M).$$

Therefore, each embedding ϕ of L into M must satisfy:

$$\text{rank}(L) \leq \text{rank}(M), \quad \text{rank}_2(L) \leq \text{rank}_2(M).$$

Now we can analyze each case.

Case I: $\text{rank}_2(M) = 0$

Since $\text{rank}_2(M) = 0$, then $G_{M'} \cong \bigoplus_{i=1}^{m} [0]$. It implies that

$$G_{L'} \cong \bigoplus_{i=1}^{l} [0]$$

where $l \leq m$.

By Lemma 4.12, there exists a lattice T such that $L \cong T(2)$. Thus, we have:

$$L \cong T(2)$$

for some lattice T. This completes the proof for Case I.

Case II: $\text{rank}_2(M) > 0$ **and** $\text{rank}_2(L) = 0$

Since $\text{rank}_2(L) = 0$, we conclude that $G_{L'} \cong \bigoplus_{i=1}^{l}[0]$. Suppose that

$$\text{rank}(L) > \text{rank}(M) - \frac{1}{2}\text{rank}_2(M).$$

By Theorem 4.9, Corollary 4.10, we have

$$G_{L'} \cong \bigoplus_{i=1}^{t} H \oplus \bigoplus_{i=1}^{l-2t}[0] \tag{4.1}$$

for some $t \in \mathbb{N}^+$. This contradicts the condition that $\text{rank}_2(L) = 0$. Therefore,

$$\text{rank}(L) \leq \text{rank}(M) - \frac{1}{2}\text{rank}_2(M).$$

The existence of a lattice T such that $L \cong T(2)$ follows from Lemma 4.12. This completes the proof for Case II.

Case III: $\text{rank}_2(M) > 0$ **and** $\text{rank}_2(L) > 0$

Suppose that

$$\text{rank}(L) > \frac{1}{2}\text{rank}_2(L) + \text{rank}(M) - \frac{1}{2}\text{rank}_2(M).$$

By Theorem 4.9 and Corollary 4.10, we have

$$G_{L'} \cong \bigoplus_{i=1}^{t} H \oplus \bigoplus_{i=1}^{l-2t}[0] \tag{4.2}$$

for some $t \in \mathbb{N}^+$ such that $t > \text{rank}_2(L)$. This leads to a contradiction with the uniqueness of its canonical form. Hence, we obtain the following rank condition:

$$\text{rank}(L) \leq \text{rank}(M) - \frac{1}{2}\text{rank}_2(M) + \frac{1}{2}\text{rank}_2(L).$$

By Lemma 4.12, we can express the Gram matrix G_L of L in the specified form:

$$G_L \cong \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix}$$

where $a_{2k-1,2k}$ is odd for each $1 \leq k \leq \frac{1}{2}\operatorname{rank}_2(L)$, and the remaining off-diagonal entries are even.

This completes the proof for Case III.

Case IV: $\operatorname{rank}(L) = \operatorname{rank}(M)$

if $\operatorname{rank}(L) = \operatorname{rank}(M), L \cong M$. The claim trivially follows by the definition of a primitive embedding.

By addressing each case, we have proved that for any embedding ϕ of L into M, one of the conditions stated in the theorem must hold. Therefore, the proof is complete.

□

4.2 K3-Covers of Enriques Surfaces: An Application

As a consequence of the foregoing theorem, we derive the following:

Corollary 4.14. *Let T_X be a transcendental lattice of signature $(2, \lambda - 2)$. If there exists an embedding of T_X into $\mathbf{\Lambda}^-$, then* $\operatorname{rank}_2(T_X) = 0$ *or* 2. *In particular,* *the associated Gram matrix of each embedding of T_X into $\mathbf{\Lambda}^-$ must be of one of the following types:*

1. $T_X \cong T(2)$, *where T is an even lattice,*
2. $T_X \cong T(2)$, *where T is an odd lattice,*
3. $\mathrm{rank}_2(T_X) = 2$,

$$G_{T_X} \cong \begin{pmatrix} 2a_{11} & a_{12} & \cdots & a_{1\lambda} \\ a_{12} & 2a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{1\lambda} & \cdots & \cdots & 2a_{\lambda\lambda} \end{pmatrix},$$

such that a_{ij} is even for each $1 \leq i, j \leq \lambda$ except a_{11} and a_{12}.

Proof. Let $\{x_i\}$ be a basis of T_X, and $\mathbf{U} = \langle u_1, u_2 \rangle$ and $\mathbf{U}(2) = \langle v_1, v_2 \rangle$ with $\{u_1, u_2\}$, $\{v_1, v_2\}$ their standard bases.

Suppose $\phi \colon T_X \hookrightarrow \mathbf{\Lambda}^-$ is a lattice embedding given by ;

$$\phi(x_i) = a'_{i1}u_1 + a'_{i2}u_2 + a'_{i3}v_1 + a'_{i4}v_2 + w_i, \tag{4.3}$$

where a'_{ij} are integers and $w_i \in \mathbf{E_8}(2)$ for $1 \leq i \leq \lambda$ and $1 \leq j \leq 4$ then, we have that,

$$\phi(x_i) \cdot \phi(x_i) = 2a'_{i1}a'_{i2} + 4a'_{i3}a'_{i4} + w_1^2 = 2a_{ii}, \tag{4.4}$$

for $1 \leq i \leq \lambda$ and

$$\phi(x_i) \cdot \phi(x_k) = a'_{i1}a'_{k2} + a'_{i2}a'_{k1} + 2a'_{i3}a'_{k4} + 2a'_{i4}a'_{k4} + w_i w_k = a_{ik} \tag{4.5}$$

for $1 \leq i < k \leq \lambda$.

By Theorem 4.13, the equations 4.4 and the equations 4.5 are solvable over $\mathbb{Z}/2$ if and only if $\mathrm{rank}_2(T_X) \leq \mathrm{rank}_2(\mathbf{\Lambda}^-)$. Since $\mathrm{rank}_2(\mathbf{\Lambda}^-) = 2$, $\mathrm{rank}_2(T_X)$ is either 0 or 2.

Case I: $\mathrm{rank}_2(T_X) = 0$

Since $\mathrm{rank}_2(T_X) = 0$, then $T'_X \cong \bigoplus_{i=1}^{\lambda}[0]$, $\lambda \leq 11$ using Theorem 4.13. By Lemma 4.12, there are two types of associated Gram matrix of T_X arouse after lifting up to $\mathbb{Z}$:

1. $T_X \cong T(2)$, where T is an even lattice,
2. $T_X \cong T(2)$, where T is an odd lattice.

Case II: $\mathrm{rank}_2(T_X) = 2$

If $\mathrm{rank}_2(T_X) = 2$, by Theorem 4.13 and Lemma 4.12,

$$G_{T_X} \cong \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix},$$

such that a_{12} is odd and the remaining off-diagonal entries are even.

To determine the parities of diagonal entries of T_X, we need to consider the equations 4.4 and the equations 4.5. If both a_{11} and a_{22} are odd, it contradicts with the equations 4.4 and the equations 4.5. Suppose both a_{11} and a_{22} are even. Then, under the action of element $g \in \mathrm{GL}(\lambda, \mathbb{Z})$ where all diagonal entries are 1, the $(2,1)$-entry is 1, and all other off-diagonal entries are 0, both a'_{11} and a'_{12} will be odd, the parities of remaining entries of T_X will be invariant. Hence, without loss of generality, we can assume that a_{11} is odd and a_{22} is even. Since a_{11} is odd, it enforces that a'_{11} and a'_{12} are odd by the equations 4.4. a_{12} is odd, it enforces also that a'_{21} and a'_{22} have a different parity by the equations 4.5. Thus, by the equations 4.4, 4.5, both a'_{i1} and a'_{i2} are even for $3 \leq i \leq \lambda$, it implies that $a_{ii} \in 2\mathbb{Z}$ for $3 \leq i \leq \lambda$. This completes the proof.

□

We will now present and prove the subsequent theorems that broaden the criteria for K3-covers an Enriques surface, showing that for each embedding of T_X into $\mathbf{\Lambda}^-$ or an induced embedding T_X into $\mathbf{\Lambda}^-$, there exists no $v \in T_X^\perp$ with $v^2 = -2$.

Remark 4.15. This notion of induced embedding is used to transform an embedding of T_X into a new embedding that satisfies certain parity or valuation constraints on its coefficients in $\mathbf{U}$.

Lemma 4.16. *Let T_X be an even lattice of signature $(2, \lambda-2)$ with $\operatorname{rank}_2(T_X) = 2$. Then, for each embedding of T_X into $\mathbf{\Lambda}^-$, there exists no vector $v \in T_X^\perp$ such that $v^2 = -2$.*

Proof. Let $\{x_i\}$ be a basis of T_X, and $\mathbf{U} = \langle u_1, u_2 \rangle$ and $\mathbf{U}(2) = \langle v_1, v_2 \rangle$ with $\{u_1, u_2\}$, $\{v_1, v_2\}$ their standard bases.

Consider a primitive embedding $\phi : T_X \hookrightarrow \mathbf{\Lambda}^-$ given by

$$\phi(x_i) = a_{i1}u_1 + a_{i2}u_2 + a_{i3}v_1 + a_{i4}v_2 + w_i,$$

where $a_{ij} \in \mathbb{Z}$ and $w_i \in E_8(2)$ for all $1 \leq i \leq \lambda$ and $1 \leq j \leq 4$.

By Corollary 4.14, the Gram matrix entry $a_{11} \equiv 2 \mod 4$ implies that both a_{11} and a_{12} are odd. Similarly, the condition $a_{12} \equiv 2 \mod 4$ implies that a_{21} and a_{22} have opposite parity.

To show that the orthogonal complement of $\operatorname{Im}(\phi)$ in $\mathbf{\Lambda}^-$ contains no vector of norm -2, suppose that

$$z = au_1 + bu_2 + cv_1 + dv_2 + e \in (\operatorname{Im}(\phi))^\perp,$$

where $a, b, c, d \in \mathbb{Z}$, and $e \in E_8(2)$ with $e \cdot e = -4k$ for some $k \in \mathbb{Z}_{\geq 0}$.

Orthogonality to $\phi(x_2)$ yields:

$$\begin{aligned} 0 = z \cdot \phi(x_2) &= (au_1 + bu_2 + cv_1 + dv_2 + e) \cdot (a_{21}u_1 + a_{22}u_2 + a_{23}v_1 + a_{24}v_2 + w_2) \\ &= aa_{22} + ba_{21} + 2ca_{24} + 2da_{23} + e \cdot w_2. \end{aligned}$$

Reducing modulo 2 and using $e \cdot w_2 \equiv 0 \mod 2$, we obtain:

$$aa_{22} + ba_{21} \equiv 0 \mod 2.$$

Since a_{21} and a_{22} have opposite parity, this forces $a \equiv b \equiv 0 \mod 2$.

Now compute the norm of z:

$$z \cdot z = 2ab + 4cd + e \cdot e = 2ab + 4cd - 4k.$$

As a and b are even, we have $2ab \equiv 0 \mod 4$, and hence:

$$z \cdot z \equiv 0 \mod 4.$$

In particular, $z \cdot z \neq -2$, since $-2 \not\equiv 0 \mod 4$. Hence, there exists no vector $z \in (\mathrm{Im}(\phi))^{\perp} \subset \mathbf{\Lambda}^{-}$ such that $z \cdot z = -2$, completing the proof. □

Lemma 4.17. *Let T_X be an even lattice of signature $(2, \lambda - 2)$ and $T_X \cong T(2)$, where T is an even lattice. Then, for each embedding of T_X into $\mathbf{\Lambda}^{-}$, there exists an induced embedding such that there exists no $v \in T_X^{\perp}$ with $v^2 = -2$.*

Proof. Let $\{x_i\}$ be a basis of T_X, and $\mathbf{U} = \langle u_1, u_2 \rangle$ and $\mathbf{U}(2) = \langle v_1, v_2 \rangle$ with $\{u_1, u_2\}$, $\{v_1, v_2\}$ their standard bases.

Consider an embedding $\phi \colon T_X \hookrightarrow \mathbf{\Lambda}^{-}$ defined by

$$\phi(x_i) = a_{i1}u_1 + a_{i2}u_2 + a_{i3}v_1 + a_{i4}v_2 + w_i, \tag{4.6}$$

where $a_{ij} \in \mathbb{Z}$, and $w_i \in \mathbf{E_8}(2)$ for all $1 \leq i \leq \lambda$ and $1 \leq j \leq 4$.

Assume $T_X \cong T(2)$, where T is an even lattice. Then, by Corollary 4.14, the Gram matrix entries a_{ij} are even for all $1 \leq i, j \leq \lambda \leq 11$.

Since T_X is even of signature $(2, \lambda - 2)$, there exist coefficients a_{ij} with $a_{i1} \neq 0$ or $a_{i2} \neq 0$ for all $1 \leq i \leq \lambda$. Assume that a_{i1} and a_{i2} have different parity for some fixed i; then, by the same argument as in Lemma 4.16, the orthogonal complement $\mathrm{Im}(\phi)^{\perp} \subset \mathbf{\Lambda}^{-}$ contains no vector of norm -2.

Suppose instead that all nonzero a_{ij} for $j = 1, 2$ are of the form

$$a_{ij} = 2^{k_{ij}} m_{ij}, \quad \text{with } k_{ij} \in \mathbb{Z}_{>0} \text{ and } m_{ij} \notin 2\mathbb{Z}.$$

Let $k := \min\{v_2(a_{i1}) \mid a_{i1} \neq 0\}$, and assume without loss of generality that this minimum is achieved at $i = 1$, i.e., $k = k_{11}$.

Define a new embedding $\phi \colon T_X \hookrightarrow \mathbf{\Lambda}^{-}$ by rescaling:

$$a'_{i1} := 2^{k_{i1}-k} m_{i1}, \qquad a'_{i2} := 2^{k_{i2}+k} m_{i2},$$

for all $1 \leq i \leq \lambda$, where $a_{i1}, a_{i2} \neq 0$. Then ϕ' satisfies the same quadratic form conditions (i.e., preserves the equations in (4.4) and (4.5)), and if the original embedding ϕ is primitive, then so is ϕ' by Theorem 4.6.

In this new embedding, we have that a'_{11} and a'_{12} are of different parity, hence by Theorem 4.16, the orthogonal complement of $\mathrm{Im}(\phi')$ in $\mathbf{\Lambda}^{-}$ contains no vector of norm -2, as desired. □

Let X be a complex K3 surface with transcendental lattice T_X. According to Keum's criterion, X does not admit an Enriques involution (i.e., does not cover any Enriques surface) if and only if at least one of the following conditions holds:

(i) $\nexists$ primitive embedding $T_X \hookrightarrow \mathbf{\Lambda}^{-}$,

(ii) $\forall$ primitive embeddings $\phi \colon T_X \hookrightarrow \mathbf{\Lambda}^{-}$, $\exists\, v \in \phi(T_X)^{\perp}$ such that $v \cdot v = -2$.

A lattice T_X satisfying condition (ii) is called a *co-idoneal lattice*, following the terminology of [5]. Explicitly,

$$T_X \text{ is co-idoneal} \iff \forall\, \phi \colon T_X \hookrightarrow \mathbf{\Lambda}^{-} \text{ primitive}, \exists\, v \in \phi(T_X)^{\perp} \text{ with } v^2 = -2.$$

Lemma 4.18. *If T_X is a co-idoneal lattice, then $T_X \cong T(2)$, where T is an odd lattice.*

Proof. The proof of this theorem is a direct consequence of Corollary 4.14, Theorem 4.16, Theorem 4.17. ☐

Theorem 4.19. *Let X be a K3 surface with $10 \leq \rho(X) \leq 20$, and let T_X denote its transcendental lattice. Then X admits an Enriques involution (i.e., covers an Enriques surface) if and only if one of the following holds:*

I $11 \leq \rho(X) \leq 20$*, and* $T_X \cong T(2)$ *for some even lattice* T*.*

II $11 \leq \rho(X) \leq 20$*, and* $T_X \cong T(2)$ *for some odd lattice* T*, provided* T_X *is not co-idoneal.*

III $11 \leq \rho(X) \leq 20$*, and* $T_X \cong T$ *for some even lattice* T *with Gram matrix*

$$G_T = \begin{pmatrix} 2a_{11} & a_{12} & \cdots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \cdots & \cdots & 2a_{\lambda\lambda} \end{pmatrix}$$

where all $a_{ij} \in 2\mathbb{Z}$*, except* a_{11} *and* a_{12}*, which are odd.*

IV $\rho(X) = 10$*, and* $T_X \cong \mathbf{\Lambda}^-$*.*

Proof. Let X be a K3 surface with $11 \leq \rho(X) \leq 20$, and let T_X denote its transcendental lattice of rank λ and signature $(2, \lambda - 2)$.

Each primitive embedding of T_X into $\mathbf{\Lambda}^-$ is, in particular, an embedding. By Corollary 4.14, the associated Gram matrix of each embedding of T_X into $\mathbf{\Lambda}^-$ must be of one of the following types:

1. $T_X \cong T(2)$, where T is an even lattice,
2. $T_X \cong T(2)$, where T is an odd lattice,

3. $\text{rank}_2(T_X) = 2$,

$$G_{T_X} \cong \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{1\lambda} & \cdots & \cdots & 2a_{\lambda\lambda} \end{pmatrix},$$

such that a_{ij} is even for all $1 \le i, j \le \lambda$ except a_{11} and a_{12}.

If $T_X \cong T(2)$, where T is an even lattice, then by Lemma 4.17, for each embedding of T_X into $\mathbf{\Lambda}^-$, there exists an induced embedding such that no vector $v \in T_X^\perp$ satisfies $v^2 = -2$. Hence, by Theorem 3.2, the claim follows.

If $T_X \cong T(2)$, where T is an odd lattice and T_X is not a co-idoneal lattice, then by the definition of a co-idoneal lattice and Theorem 3.2, the claim again follows.

If $\text{rank}_2(T_X) = 2$, then by Lemma 4.16, for each embedding of T_X into $\mathbf{\Lambda}^-$, there exists no vector $v \in T_X^\perp$ such that $v^2 = -2$. Thus, by Theorem 3.2, the claim holds in this case as well.

Finally, if $\rho(X) = 10$, then $\text{rank}(T_X) = \text{rank}(\mathbf{\Lambda}^-)$, so $T_X \cong \mathbf{\Lambda}^-$, and the claim is trivial. $\square$

Chapter 5

Nikulin's Embedding Criterion and K3-Covers of Enriques Surfaces

The primary references for this chapter are the works [6, 5].

5.1 Existence of Primitive Embeddings in the Non-Unimodular Case

Definition 5.1. An *even overlattice* of an even lattice L is an even lattice $L' \supseteq L$ for which the quotient $H_{L'} := L'/L$ is finite.

The canonical inclusions

$$L \subseteq L' \subseteq (L')^* \subseteq L^*$$

imply

$$H_{L'} \subseteq L^*/L = A_L, \qquad (L^*/L)\big/H_{L'} \cong A_{L'}.$$

Since $A_L = L^*/L$ is equipped with a finite quadratic form q_L, we can also consider the orthogonal complement $H^{\perp}_{L'} \subset A_L$ with respect to q_L.

Proposition 5.2 (Nikulin [6, Proposition 1.4.1])**.** *Let L be an even lattice with discriminant group $A_L := L^{\vee}/L$ and discriminant form $q_L\colon A_L \to \mathbb{Q}/2\mathbb{Z}$. Then the correspondence*

$$L' \longmapsto H := L'/L \subset A_L$$

defines a bijection between the set of even overlattices $L' \supset L$ and the set of isotropic subgroups $H \subset A_L$, that is, subgroups satisfying

$$q_L|_H = 0.$$

Moreover, if $H = L'/L$, then the discriminant group and discriminant form of L' satisfy:

$$A_{L'} \cong H^{\perp}/H \subset A_L, \quad \textit{and} \quad q_{L'} = q_L|_{H^{\perp}/H}.$$

5.1.1 Primitive Embeddings and Their Orthogonal Extensions

Recall that an embedding $L \hookrightarrow M$ of lattices is called *primitive* if M/L is torsion-free.

Let $L \hookrightarrow M$ be a primitive embedding, and define $K := L^{\perp}_M$, the orthogonal complement of L in M. Then $L \oplus K \subset M$, and M is an overlattice of $L \oplus K$.

By Proposition 5.2, the corresponding isotropic subgroup is given by

$$H_M := M/(L \oplus K) \subset A_L \oplus A_K,$$

and the associated discriminant form is

$$q_M = (q_L \oplus q_K)\big|_{(H_M)^{\perp}/H_M}.$$

The primitiveness of the embeddings $L \hookrightarrow M$ and $K \hookrightarrow M$ is equivalent to requiring that the natural projections

$$p_L\colon H_M \to A_L, \qquad p_K\colon H_M \to A_K$$

are injective.

The next theorem characterizes, in terms of explicit invariants,

$$T \hookrightarrow M \text{ primitively}, \quad \text{with } M \in g$$

where T is an even lattice and g is a fixed genus.

Theorem 5.3 (Nikulin [6, Proposition 1.15.1])**.** *The primitive embeddings of a lattice S into an even lattice with invariants $(m_{(+)}, m_{(-)}, q)$ are determined by the data $(H_S, H_q, \gamma, K, \gamma_K)$, where:*

- *$H_S \subset A_S$ and $H_q \subset A_q$ are subgroups of the discriminant groups of S and of a given form q, respectively.*
- *$\gamma \colon q_S|_{H_S} \xrightarrow{\sim} q|_{H_q}$ is an isomorphism of quadratic forms on the subgroups, preserving the restriction of the forms.*
- *K is an even lattice with invariants*

  $$\operatorname{sign}(K) = \left(m_{(+)} - t_{(+)},\ m_{(-)} - t_{(-)}\right),$$

  where $\operatorname{sign}(S) = (t_{(+)}, t_{(-)})$.
- *The discriminant form q_K satisfies*

  $$q_K \cong (q_S \oplus (-q))\,|_{\Gamma^\perp/\Gamma},$$

  where $\Gamma := \{(x, \gamma(x)) \mid x \in H_S\} \subset A_S \oplus A_q$ is the graph of γ, and $\Gamma^\perp \subset A_S \oplus A_q$ is its orthogonal complement with respect to $q_S \oplus (-q)$.
- *$\gamma_K \colon q_K \xrightarrow{\sim} (-\delta)$ is an isomorphism of forms, where*

  $$\delta \cong (q_S \oplus (-q))|_{\Gamma^\perp/\Gamma}.$$

5.2 Existence Results for Even Lattices of Signature (s_+, s_-) and Discriminant Form q

Given integers s_+, s_-, we define the following conditions on the finite quadratic form q:

$\mathbf{C}_1(s_+)$: The signature of q satisfies

$$\text{sign } q \equiv s_+ \pmod{8}.$$

$\mathbf{C}_2(s_+, s_-)$: For every odd prime p, the length of q satisfies $\ell_p(q) \leq s_+ + s_-$. If equality holds, i.e., $\ell_p(q) = s_+ + s_-$, then

$$|A_q| \equiv (-1)^{s_-} \cdot \text{discr } K_p(q) \pmod{(\mathbb{Z}_p^\times)^2}.$$

$\mathbf{C}_3(s_+)$: The 2-adic length satisfies $\ell_2(q) \leq s_+$. In the case of equality and when q is even, we require

$$|A_q| \equiv \pm\text{discr } K_2(q) \pmod{(\mathbb{Z}_2^\times)^2}.$$

Theorem 5.4 (Nikulin [6, Theorem 1.10.1])**.** *Let $l_+, l_- \in \mathbb{Z}_{\geq 0}$, and let q be a finite quadratic form. Then there exists an even lattice L with* $\text{sign}(L) = (l_+, l_-)$ *and discriminant form $q(L) \cong q$ if and only if the triple*

$$(l_+, l_-, q)$$

satisfies $\mathrm{C}_1(l_+ - l_-)$*,* $\mathrm{C}_2(l_+, l_-)$*, and* $\mathrm{C}_3(l_+ + l_-)$*.*

5.3 K3 Covers of Enriques Surfaces: An Application

This section applies Nikulin's theory of discriminant forms, and conditions for primitive embeddings to study the lattice-theoretic relationships between K3 surfaces and their Enriques quotients.

Lemma 5.5 ([5, Lemma 3.9]). *Let q_1 and q_2 be finite quadratic forms such that*

$$q_1 \cong \bigoplus_{i=1}^{n} \mathbf{u}_1.$$

Suppose there exist subgroups $H \leq A_{q_1}$, $K \leq A_{q_2}$, and an isometry of quadratic forms

$$\varphi\colon q_1|_H \xrightarrow{\sim} q_2|_K.$$

Let $\Gamma \subset A_{q_1} \oplus A_{q_2}$ denote the push-out of φ, i.e., the graph

$$\Gamma = \{(h, \varphi(h)) \mid h \in H\}.$$

Then the following hold:

1. *There is an isometry of quadratic forms:*

$$\ell_2(H) \cdot \mathbf{u}_1 \oplus (q_1 \oplus (-q_2))\big|_{\Gamma^\perp/\Gamma} \cong q_1 \oplus (-q_2),$$

   where $\ell_2(H)$ denotes the minimum number of generators of H as a 2-group.

2. *The 2-adic lengths satisfy the identity:*

$$\ell_2(q_1) + \ell_2(q_2) = \ell_2\left(\Gamma^\perp/\Gamma\right) + 2\ell_2(H).$$

Let T be the transcendental lattice of a K3 surface, and let $H \subset A_T$ be the subgroup of the discriminant group of $q(T)$ arising from Theorem 5.3.

To summarize, the possible values for $\ell_2(q(T))$, $\ell_2(H)$, and $\ell_2(q(T^\perp))$, along with the parity of $q(T)$ using Lemma 5.5, Lemma 2.24, are:

Case	$\ell_2(q(T))$	$\ell_2(H)$	$\ell_2(q(T^\perp))$	Parity of $q(T)$
λ_a	$\lambda - 2$	$\lambda - 2$	$12 - \lambda$	even
λ_b	λ	$\lambda - 1$	$12 - \lambda$	even or odd
λ_c	λ	λ	$10 - \lambda$	even

Table 5.1: Values of λ, $\ell_2(q(T))$, $\ell_2(H)$, $\ell_2(q(T^\perp))$, and the parity of $q(T)$.

Theorem 5.6 ([5, Proposition 3.10])**.** *Let* T *be an even lattice with* $\operatorname{sign}(T) = (2, \lambda - 2)$*. Then a primitive embedding*

$$T \hookrightarrow \Lambda^-$$

exists if and only if the discriminant form $q(T)$ *is isomorphic to one of the forms listed in the second column of Table 5.2.*

λ_{case}	$q(T)$	$q(T^{\perp})$	$\ell_2(q)$	Conditions on q
2_a	q	$5\mathbf{u_1} \oplus q(-1)$	0	$C_3(0)$, even
2_b	q	$4\mathbf{u_1} \oplus q(-1)$	2	even or odd
2_c	q	$3\mathbf{u_1} \oplus q(-1)$	2	even
3_a	q	$4\mathbf{u_1} \oplus q(-1)$	3	$C_3(1)$
3_b	q	$3\mathbf{u_1} \oplus q(-1)$	3	even or odd
3_c	q	$2\mathbf{u_1} \oplus q(-1)$	3	even
4_a	q	$3\mathbf{u_1} \oplus q(-1)$	2	$C_3(2)$, even
4_b	q	$2\mathbf{u_1} \oplus q(-1)$	4	even or odd
4_c	q	$\mathbf{u_1} \oplus q(-1)$	4	even
5_a	q	$2\mathbf{u_1} \oplus q(-1)$	3	$C_3(3)$, even
5_b	q	$\mathbf{u_1} \oplus q(-1)$	5	even or odd
5_c	q	$q(-1)$	5	even
6_a	q	$\mathbf{u_1} \oplus q(-1)$	4	$C_3(4)$, even
6_b	q	$q(-1)$	6	even or odd
6_c	$\mathbf{u_1} \oplus q$	$q(-1)$	4	even
7_a	q	$q(-1)$	5	$C_2(5,0), C_3(5)$, even
7_b	$\mathbf{u_1} \oplus q$	$q(-1)$	5	$C_2(5,0)$, even or odd
7_c	$2\mathbf{u_1} \oplus q$	$q(-1)$	5	$C_2(5,0)$, even
8_a	$\mathbf{u_1} \oplus q$	$q(-1)$	4	$C_2(4,0), C_3(4)$, even
8_b	$2\mathbf{u_1} \oplus q$	$q(-1)$	4	$C_2(4,0)$, even or odd
8_c	$3\mathbf{u_1} \oplus q$	$q(-1)$	4	$C_2(4,0)$, even
9_a	$2\mathbf{u_1} \oplus q$	$q(-1)$	3	$C_2(3,0), C_3(3)$, even
9_b	$3\mathbf{u_1} \oplus q$	$q(-1)$	3	$C_2(3,0)$, even or odd
9_c	$4\mathbf{u_1} \oplus q$	$q(-1)$	3	$C_2(3,0)$, even
10_a	$3\mathbf{u_1} \oplus q$	$q(-1)$	2	$C_2(2,0), C_3(2)$, even
10_b	$4\mathbf{u_1} \oplus q$	$q(-1)$	2	$C_2(2,0)$, even or odd
10_c	$5\mathbf{u_1} \oplus q$	$q(-1)$	2	$C_2(2,0)$, even
11_a	$4\mathbf{u_1} \oplus q$	$q(-1)$	1	$C_2(1,0), C_3(1)$, even
11_b	$5\mathbf{u_1} \oplus q$	$q(-1)$	1	$C_2(1,0)$, even or odd
12	$5\mathbf{u_1}$	–	–	–

Table 5.2: Table of λ cases, $q(T)$, $q(T^{\perp})$, $\ell_2(q)$, and conditions on q.

Recall that a transcendental lattice T_X is called *co-idoneal* if

$$\forall\, \varphi : T_X \hookrightarrow \mathbf{\Lambda}^- \text{ primitive, } \exists\, v \in \varphi(T_X)^\perp \subset \mathbf{\Lambda}^- \text{ such that } v^2 = -2.$$

Theorem 5.7. *Let X be a K3 surface, and let T be its transcendental lattice with $\operatorname{rank}(T) = \lambda$. The surface X admits a fixed-point-free involution (i.e., an Enriques cover) if and only if one of the following conditions holds:*

(i_1) $2 \le \lambda \le 6 \ \wedge\ \exists T'$ *even such that* $T \cong T'(2)$,

(i_2) $2 \le \lambda \le 6 \ \wedge\ \exists T'$ *odd such that* $T \cong T'(2) \ \wedge\ T$ *is not co-idoneal,*

(i_3) $2 \le \lambda \le 6 \ \wedge\ \exists\, (a_{ij}) \in \operatorname{Mat}_\lambda(\mathbb{Z})$ *such that*

T *admits Gram matrix* (a_{ij}) *with*

$a_{11} \equiv 2 \pmod 4, \quad a_{ii} \equiv 0 \pmod 4$ *for all* $i \ge 2$,

$a_{12} \equiv 1 \pmod 2, \quad a_{ij} \equiv 0 \pmod 2$ *for all* $i \ne j,\ (i,j) \ne (1,2)$.

(ii) $\lambda = 7 \ \wedge\ \exists T'$ *even such that* $T \cong \mathbf{U} \oplus T'(2)$,

(iii) $\lambda = 7 \ \wedge\ \exists T'$ *with* $T \cong \mathbf{U}(2) \oplus T'(2) \ \wedge\ T$ *is not co-idoneal,*

(iv) $\lambda = 8 \ \wedge\ \exists T'$ *even such that* $T \cong \mathbf{U} \oplus \mathbf{U}(2) \oplus T'(2)$,

(v) $\lambda = 8 \ \wedge\ \exists T'$ *with* $T \cong 2\mathbf{U}(2) \oplus T'(2) \ \wedge\ T$ *is not co-idoneal,*

(vi) $\lambda = 9 \ \wedge\ \exists T'$ *even such that* $\mathbf{U}(2) \oplus T \cong \tilde{\mathbf{E}}_8(-1) \oplus T'(2)$,

(vii) $\lambda = 9 \ \wedge\ \exists T'$ *with* $\mathbf{U} \oplus T \cong \tilde{\mathbf{E}}_8(-1) \oplus T'(2) \ \wedge\ T$ *is not co-idoneal,*

$(viii)$ $\lambda = 10 \ \wedge\ \exists T'$ *even such that* $T \cong \tilde{\mathbf{E}}_8(-1) \oplus T'(2)$,

(ix) $\lambda = 10 \ \wedge\ \exists T'$ *with* $T \cong \mathbf{E}_8(2) \oplus T'(2) \ \wedge\ T$ *is not co-idoneal,*

(x) $\lambda = 11 \ \wedge\ \exists\, n > 0$ *such that* $T \cong \mathbf{U} \oplus \mathbf{E}_8(2) \oplus [4n]$,

(xi) $\lambda = 11 \ \wedge\ \exists\, n > 0$ *such that* $T \cong \mathbf{U}(2) \oplus \mathbf{E}_8(2) \oplus [2n] \ \wedge\ T$ *is not co-idoneal,*

(xii) $\lambda = 12 \ \wedge\ T \cong \mathbf{\Lambda}^-$.

Proof. The cases $2 \le \lambda \le 6$ and $\lambda = 12$ are already proven in Theorem 4.19.

By Theorem 3.2, whenever T fails to be co-idoneal, the surface X admits a fixed-point-free involution if and only if $T \hookrightarrow \mathbf{\Lambda}^-$ with primitive image. If we prove that conditions in Theorem 5.6 are shown to be equivalent to conditions

(ii) through (xi), Theorem 5.7 follows from Keum's Criterion 3.2. Thus, it is necessary to analyze all the cases presented in Table 5.2.

Since the arguments for integer values λ between 7 and 11 are analogous, we present only the case $\lambda = 11$ as a representative example.

Case $\lambda = 11$:

Assume T satisfies case $(\mathbf{x})$. Then:

$$q(T) \cong 4\mathbf{u}_1 \oplus q,$$

where $q = q(T')$ for an even finite quadratic form q such that

$$C_1(1)(q), \quad C_2(1,0)(q), \quad C_3(1)(q)$$

hold, by Lemma 2.24 and Theorem 5.4.

$$\text{By Theorem 5.6,} \quad \exists\, \varphi\colon T \hookrightarrow \mathbf{\Lambda}^- \text{ with } \operatorname{im}(\varphi) \text{ primitive.}$$

Conversely, assume that T satisfies case $\boldsymbol{\lambda}_{\mathrm{a}}$, i.e.,

$$q(T) = 4\mathbf{u}_1 \oplus q,$$

where q is even and fulfills $C_2(1,0)(q)$ and $C_3(1)(q)$. Since T exists, Theorem 5.4 implies that $C_1(1)(q)$ must also hold.

Then, there exists an even lattice T' such that

$$\operatorname{sign}(T') = (1,0) \quad \text{and} \quad q(T'(2)) \cong q,$$

by Theorem 5.4.

Moreover, by Theorem 2.28, the genus of T contains a unique isomorphism class. Therefore,

$$T \cong \mathbf{U} \oplus \mathbf{E}_8(2) \oplus [4n],$$

and thus case $(\mathbf{x})$ applies.

Case $\lambda = 11$:

Assume case (**xi**) applies. Then

$$q(T) \cong 5\mathbf{u}_1 \oplus q,$$

where $q = q(T')$ is a finite quadratic form fulfilling $C_2(1,0)(q)$, as ensured by Theorem 5.4.

$$\text{By Theorem 5.6,} \quad \exists\, \varphi \colon T \hookrightarrow \mathbf{\Lambda}^- \text{ with } \operatorname{im}(\varphi) \text{ primitive.}$$

Conversely, assume that case $\boldsymbol{\lambda}_{\mathrm{b}}$ or $\boldsymbol{\lambda}_{\mathrm{c}}$ applies. Then again

$$q(T) \cong 5\mathbf{u}_1 \oplus q,$$

where q fulfills $C_2(1,0)(q)$. Since T exists, Theorem 5.4 implies that q also satisfies $C_1(1)(q)$ and $C_3(1)(q)$.

Hence, by Theorem 5.4, there exists a lattice T' with

$$\operatorname{sign}(T') = (1,0) \quad \text{and} \quad q(T'(2)) \cong q.$$

Then, by Theorem 2.28, the genus of T contains a unique isomorphism class. Therefore,

$$T \cong \mathbf{U}(2) \oplus \mathbf{E}_8(2) \oplus [2n],$$

and thus case (**xi**) applies.

□

Chapter 6

Co-idoneal Lattices and Idoneal Genera

The primary references for this chapter is the work [5].

Lemma 6.1. *Let $T \cong T'(2)$ with T' an odd lattice, and let $T \hookrightarrow \mathbf{\Lambda}^-$ be a primitive embedding. Then:*

1. *There exists an odd lattice T'' such that*

$$T^\perp \cong T''(2).$$

2. *There exists a unique genus g such that*

$$\forall L \in g, \quad \exists \text{ primitive } \phi\colon T \hookrightarrow \mathbf{\Lambda}^- \text{ with } L \cong (\operatorname{im}\phi)^\perp.$$

Proof. Suppose that T is of the form $T = T'(2)$ for some lattice T'. Then, by Lemma 2.24, the lattice T satisfies $\ell_2(q(T)) = \operatorname{rank}(T)$. Furthermore, since T' is an odd lattice, it follows from Lemma 2.24 that $q(T)$ is also odd. Under the assumption that T embeds primitively into $\mathbf{\Lambda}^-$, Theorem 5.6 implies that $q(T)$ satisfies the condition $\boldsymbol{\lambda}_{\mathrm{b}}$, or more generally, the parity of $q(T^\perp)$ determined by the parity of $q(T)$, and vice versa. Consequently, $q(T^\perp)$ is odd.

On the other hand, by Theorem 5.6, the lattice $T^\perp$ satisfies the relation $\ell_2(q(T^\perp)) = \mathrm{rank}(T^\perp)$. Since $q(T^\perp)$ is odd, it follows, by Lemma 2.24 again, that $T^\perp \cong T''(2)$, where T'' is an odd lattice.

Now, let $T \hookrightarrow \mathbf{\Lambda}^-$ be given with primitive image and $T^\perp = S$. By Theorem 5.6,

$$q(T) \text{ determines } q(T^\perp) \quad \text{and vice versa.}$$

Furthermore, by Theorem 2.27, the triple $(\mathrm{sign}(L), \epsilon(L), b_q)$ uniquely determines the genus of the lattice L, where $\mathrm{sign}(L)$ is the signature, $\epsilon(L)$ denotes the parity, and b_q is the discriminant bilinear form. Therefore, it determines the genus of S uniquely. By Theorem 5.3, every lattice L in the genus of S arises as the orthogonal complement $\phi(T)^\perp \subset \mathbf{\Lambda}^-$ for some primitive embedding $\phi : T \hookrightarrow \mathbf{\Lambda}^-$. Since such orthogonal complements are isometric to lattices of the form $T''(2)$, where T'' is an odd lattice, it follows that every $L \in \mathrm{genus}(S)$ is isometric to a rescaled odd lattice $T''(2)$.

□

Definition 6.2. A genus g is called *idoneal* if

$$\forall L \in g, \quad \text{where } L \text{ is positive definite, then } \exists\, L' \text{ such that } L \cong L' \oplus \langle 1 \rangle.$$

Definition 6.3. A transcendental lattice T_X is called *co-idoneal* if

$$\forall\, \varphi : T_X \hookrightarrow \mathbf{\Lambda}^- \text{ primitive}, \quad \exists\, v \in \varphi(T_X)^\perp \subset \mathbf{\Lambda}^- \text{ such that } v^2 = -2.$$

$$\text{By Lemma 4.18}, \quad T_X \text{ is co-idoneal} \quad \Longrightarrow \exists\, T \text{ odd with } T_X \cong T(2).$$

The following result links co-idoneal lattices to idoneal genera.

Corollary 6.4 ([5, Proposition 3.15])**.** *If T is a co-idoneal lattice, then there exists a unique idoneal genus g such that*

$$\forall L \in g,\ \exists\, \varphi : T \hookrightarrow \mathbf{\Lambda}^- \ \textit{primitive},\ \textit{with } L(-2) \cong \varphi(T)^\perp.$$

Proof. By Lemma 6.1, there exists a unique genus g' such that

$$\forall L' \in g',\ \exists\, \varphi : T \hookrightarrow \mathbf{\Lambda}^- \text{ primitive, with } L' \cong \varphi(T)^\perp.$$

Since T is co-idoneal, we have

$$\forall \varphi : T \hookrightarrow \mathbf{\Lambda}^- \text{ primitive, } \exists\, v \in \varphi(T)^\perp \text{ with } v^2 = -2.$$

Therefore, every $L' \in g'$ contains a vector v with $v^2 = -2$. Let g be the genus defined by $g = \{L \mid L(-2) \in g'\}$.

$$\forall L \in g, \quad \exists\, L' \in g' \text{ such that } L(-2) \cong L', \quad \Rightarrow \quad \exists\, v \in L \text{ with } v^2 = 1.$$

Thus, g is idoneal by definition. □

Theorem 6.5 ([5, Theorem 1.4])**.** *The set of idoneal genera is finite; that is,*

$$\#\{\, g \mid g \text{ is an idoneal genus}\,\} < \infty.$$

Theorem 6.6 ([5, Theorem 1.10])**.** *The set of co-idoneal lattices is finite; that is,*

$$\#\{\, T \mid T \text{ is a co-idoneal lattice}\,\} < \infty.$$

Example 6.7. We apply the algorithm described in [5] to the case $\lambda = 2$. The procedure begins by constructing the set of all idoneal genera having rank 10, as provided in the file `idoneal.genera.txt` [15]. This list consists of six genera, each represented by a lattice:

$$L_1 = \mathbf{E}_8 \oplus 2[1], \quad L_2 = \mathbf{E}_8 \oplus [2] \oplus [1], \quad L_3 = \mathbf{A}_2 \oplus 8[1],$$
$$L_4 = \mathbf{D}_9 \oplus [1], \quad L_5 = \mathbf{A}_3 \oplus 7[1], \quad L_6 = \mathbf{D}_8 \oplus [2] \oplus [1].$$

Next, the algorithm executes the following forms

$$x_i := q\big(L_i(-2)\big), \quad \text{for } i = 1, \ldots, 6,$$

yielding:

$$x_1 = 4\mathbf{u}_1 \oplus 2\mathbf{w}^3_{2,1}, \quad x_2 = 4\mathbf{u}_1 \oplus \mathbf{w}^3_{2,1} \oplus \mathbf{w}^7_{2,2},$$
$$x_3 = 3\mathbf{u}_1 \oplus \mathbf{v}_1 \oplus \mathbf{w}^1_{2,1} \oplus \mathbf{w}^3_{2,1} \oplus \mathbf{w}^{-1}_{3,1}, \quad x_4 = 4\mathbf{u}_1 \oplus \mathbf{w}^3_{2,1} \oplus \mathbf{w}^7_{2,3},$$
$$x_5 = 3\mathbf{u}_1 \oplus \mathbf{v}_1 \oplus \mathbf{w}^1_{2,1} \oplus \mathbf{w}^1_{2,3}, \quad x_6 = 2\mathbf{u}_1 \oplus \mathbf{v}_1 \oplus \mathbf{w}^1_{2,1} \oplus \mathbf{u}_2 \oplus \mathbf{w}^1_{2,2}.$$

Among these, the forms x_1, x_2, x_4 match the structure

$$x_i \cong 4\mathbf{u}_1 \oplus q_i(-1),$$

as described in Case 2_{b} of Theorem 5.6. The associated forms

$$y_i := q(T), \quad \text{for } T^{\perp} \cong L_i(-2),$$

are taken from the column $q(T)$ in Table 5.2, yielding:

$$y_1 = 2\mathbf{w}^1_{2,1}, \quad y_2 = \mathbf{w}^1_{2,1} \oplus \mathbf{w}^1_{2,2}, \quad y_4 = \mathbf{w}^1_{2,1} \oplus \mathbf{w}^1_{2,3}.$$

The resulting lattices are:

$$T_1 = 2[2], \qquad T_2 = [2] \oplus [4], \qquad T_4 = [2] \oplus [8],$$

each of which is unique in its genus.

This computation reproduces the characterization previously obtained by Sertöz [1].

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