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Further stability results for a generalization of delayed feedback control

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Abstract In this paper, we consider the stabilization of unstable periodic orbits for one-dimensional and discrete time chaotic systems. Various control schemes for this problem are available and we consider a recent generalization of delayed control scheme. We prove that if a certain condition, which depends only on the period number, is satisfied then the stabilization is always possible. We will also present some simulation results.

Keywords Chaotic systems · Chaos control · Delayed feedback · Pyragas controller

1 Introduction

Chaotic behavior is a very interesting and fascinating phenomenon which is frequently observed in many physical systems; see, e.g., [1]. Mathematical models of such systems possess many interesting features whose investigations attracted the scientists from various disciplines; see, e.g., [2]. In particular, such systems generally possess many unstable periodic orbits embedded in their strange attractors; see, e.g., [3]. Stabilization of such unstable periodic orbits is an interesting and challenging problem which received considerable attention after the seminal work presented in [4]. Since then, various control schemes have been proposed to solve this problem. One of such schemes first proposed in [5], which is also called the Delayed Feedback Control (DFC), has received attention due to its many interesting features. However, it has been shown that this scheme has some limitations; see, e.g., [6, 7]. To eliminate these limitations, various generalizations of DFC have been proposed. One such generalizations which has some improvements over the classical DFC, has recently been proposed in [8]; for more information on the subject, see the references therein.

We note that there are various control schemes proposed in the literature for the stabilization of unstable periodic orbits of chaotic systems; see, e.g., [2], and the references therein. Our main aim is not to propose a novel scheme to solve this problem and compare it with the existing schemes, but to further investigate the stability properties of a particular scheme proposed in [8]. In the latter reference, a nonlinear DFC scheme was proposed and its stability was analyzed. In particular, it was shown in [8] that when a certain polynomial is stable, then the proposed controller solves the stabilization problem. We note that this polynomial depends both on the chaotic system to be controlled and the gain of the proposed controller. As a consequence, when the chaotic system is given, whether a stabilizing controller exits or not remains as an interesting question. In this paper, we will give a condition which provides an answer to this question. More precisely, we will give a simple condition which is re-

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lated to the given chaotic system such that when this condition is satisfied, there always exists a stabilizing controller. We will also give some bounds on the controller gain. Quite interestingly, this condition mainly depends on the period number of the chaotic system in question. Moreover, we will also provide some rigorous and novel stability results which were left either as conjectures or mentioned in [8] as observations based on extensive simulation results.

This paper is organized as follows. In the next section, we briefly introduce the notation used throughout the paper and summarize the basic results presented in [8]. In the following section, we will present the main results. Following some simulation results, we will give some concluding remarks.

2 Problem statement

Let us consider the following discrete-time system:

$$x(i+1) = f\bigl(x(i)\bigr), \tag{1}$$

where $i = 1, 2, \dots$ is the discrete time index, $x \in \mathbf{R}$, $f : \mathbf{R} \to \mathbf{R}$ is an appropriate function, which is assumed to be differentiable wherever required. We assume that the system given by (1) possesses a period T orbit characterized by the set

$$\Sigma_T = \bigl\{x_1^*, x_2^*, \dots, x_T^*\bigr\}, \tag{2}$$

i.e., for $x(1) = x_1^*$, the iterates of (1) yield $x(2) = x_2^*, \dots, x(T) = x_T^*$, $x(k) = x(k-T)$ for $k > T$.

For the notation, definition of various types of stability and stabilization results of Σ_T, see [7–10].

To stabilize Σ_T for (1), we apply the following control law:

$$x(i+1) = f\bigl(x(i)\bigr) + u(i), \tag{3}$$

where $u(\cdot)$ is the control input. The control problem we consider is to find an appropriate control law for $u(\cdot)$ so that Σ_T becomes asymptotically stable. To solve this problem, various control schemes are proposed in the literature; see Remark 1 given below. The control law we consider is as given below

$$u(i) = \frac{K}{K+1}\bigl(x(i-m+1)\bigr) - f\bigl(x(i)\bigr), \tag{4}$$

where K is a constant gain to be determined and m is the period of the orbit. Here, we assume that at the discrete time index i, the state values $x(i)$ and $x(i-m+1)$ are available from the measurements. If we assume that these terms are the outputs of the system given by (1), then (4) represents a nonlinear output feedback law. Since the term $x(i-m+1)$ is $m-1$ unit delayed form of $x(i)$, the proposed control law is related to delayed feedback control laws. Indeed, if we use the linear term $x(i)$ instead of the nonlinear term $f(x(i))$ in (4), then the proposed control law would be quite similar to the classical DFC scheme; see Remark 2 given below. Due to the nonlinear term $f(x(i))$, which is computable since $x(i)$ is available from measurements, the proposed control law is nonlinear, and hence can be considered as a generalized version of DFC. Instead of the term $\frac{K}{K+1}$ in (4), we could use $\hat{K} = \frac{K}{K+1}$; however, the form given by (4) yields further interesting interpretations. For details, the reader is referred to [8].

Now we will give several remarks related to the control law given by (4).

Remark 1 Various control schemes are proposed in the literature for the stabilization problem given above; see, e.g., [5, 8, 11–13]. In fact, the literature is quite rich on the subject and interested reader may resort to, e.g., references cited above, [1, 2], and the references therein. Our main concern in this paper is not to provide a comparison or overview of these schemes, but to extend the results of a particular scheme proposed in [8].

Remark 2 In the classical DFC scheme as proposed in [5], the control law in (3) is given as $u(i) = K(u(i-T) - u(i))$, where K is the constant gain to be determined. Obviously, classical DFC is a linear control scheme. On the other hand, the control scheme given in (4) is nonlinear, which may be considered as a drawback of the proposed scheme. First note that although the linear control schemes are often preferred for their simplicity, many schemes proposed in the literature for the solution of a large amount of control problems related to complex systems are actually nonlinear; see, e.g., [14]. In fact, such schemes were also applied to the control of chaotic systems; see, e.g., [15], as well as to the stabilization problem considered here; see, e.g., [11, 12]. Secondly, note that although the control law given by (4) contains a term $f(\cdot)$, it is not based on cancellation, as opposed to many differential-geometric schemes proposed for nonlinear systems; see, e.g., [14]. In fact, on the periodic orbit Σ_T, the control law given by (4) vanishes. It can also be shown that if $u(i) \to 0$, then solutions of (3) converge to Σ_T. Hence, the proposed scheme enjoys the similar properties of the classical DFC; for details; see [8].

Remark 3 As noted in Remark 2, various nonlinear control schemes were proposed in the literature for the stabilization problem given above. Among these, the schemes proposed in [11] and [12] are somewhat related to the control law given by (4). It can easily be shown that for the case $T = 1$ (i.e., the stabilization of a fixed point), these schemes and the one given by (4) become equivalent. However, for higher order periods (i.e., for $T > 1$), the control laws given in [11] and [12] contain the map f^T (i.e., T-iterate of f), in their structure, whereas the control law given by (4) contains only f for any period. As a result, these schemes enjoy different stability properties for $T > 1$ case. The method proposed in [11] is based on prediction, and hence is called prediction-based control, and in the latter a combination of prediction-based schemes and classical DFC schemes is also proposed for the stabilization of Σ_T. We note that this combination also contains T iterate maps, and hence is different from the control scheme considered in this paper.

Let us assume $T = m$. For the system given by (1) and its periodic orbit Σ_m given by (2), we define $a_j = f'(x_j^*)$, $j = 1, 2, \ldots, m$, and $a = \prod_{j=1}^m a_j$. Associated with the system given by (3)–(4), we define the following polynomial:

$$p_m(\lambda) = \left(\lambda - \frac{K}{K+1}\right)^m - \frac{a}{(K+1)^m}\lambda^{m-1}. \qquad (5)$$

Main results of [8] can be summarized in the following theorem.

Theorem 1 *Let Σ_m given by (2) be a period $T = m$ orbit of (1) and set $a_j = f'(x_j^*)$, $j = 1, 2, \ldots, m$, $a = \prod_{j=1}^m a_j$. Consider the control scheme given by (3) and (4). Then:*

- i: *Σ_m is exponentially stable if and only if $p_m(\lambda)$ given by (5) is Schur stable, i.e., all of its roots are strictly inside the unit disc in the complex plane. This condition is only sufficient for asymptotic stability.*
- ii: *If $p_m(\lambda)$ has at least one unstable root, i.e., outside the unit disc, then Σ_m is unstable as well.*
- iii: *If $p_m(\lambda)$ is marginally stable, i.e., has at least one root on the unit disc which is simple while the rest of the roots are inside the unit disc, then the proposed method to test the stability of Σ_m is inconclusive.*

Proof See the proof of Theorem 2 in [8]. □

Based on Theorem 1, various observations/comments have been given in [8], and we list the important ones below:

Fact 1 For $m = 1$, stabilization is always possible provided that $a \neq 1$. This shows that the main limitation of DFC, namely the odd number limitation, does not hold for the proposed scheme for $m = 1$.

Fact 2 For $m \geq 2$, a necessary condition for stabilization is $a < 1$. In other words, the odd number limitation holds for the case $m \geq 2$.

Fact 3 For $m = 2$ and $a < 1$, stabilization is always possible. This can be considered as another improvement over classical DFC.

Observation 1 For $m \geq 3$ and $a < 1$, it was observed in [8] that there exists a number $a_{mcr} > 0$ such that when $|a| < a_{mcr}$, stabilization is possible. Moreover, by extensive numerical simulations some upper bounds for a_{mcr} for various m were found. In this paper, we will give an analytical expression for a_{mcr} and prove the observation stated above.

Observation 2 It was stated in [8] that $a_{mcr} \to 1$ as $m \to \infty$ as a conjecture. In this paper, we will show that this observation does not hold and we find $a_\infty = \lim_{m\to\infty} a_{mcr}$.

Observation 3 For stabilization, a necessary condition is $|\frac{K}{K+1}| < 1$, which implies $K > -0.5$. Let us define the following critical gain:

$$K_{cr} = -0.5 + 0.5(-a)^{1/m}. \qquad (6)$$

It was shown in [8] that for $K \leq K_{cr}$, stabilization is not possible and it was stated as a conjecture that if for $K = K_{cr}$, $p_m(\cdot)$ given by (5) is marginally stable, then stabilization is possible. In this paper, we will show that the latter observation holds.

3 Stabilization results

Let us consider the polynomial $p_m(\cdot)$ given by (5). First, we define the following polynomials:

$$q_1(\lambda) = \left(\lambda - \frac{K}{K+1}\right)^m,$$
$$q_2(\lambda) = -\frac{a}{(K+1)^m}\lambda^{m-1}. \tag{7}$$

Theorem 2 *Assume that* $|a| < 1$ *and* $m \geq 1$. *Then* $p_m(\cdot)$ *given by* (5) *is Schur stable for any* $K \geq 0$. (*Note that Schur stability means that the roots of the polynomial are strictly inside the unit disc.*)

Proof Since $|a| < 1$, stability for $K = 0$ is obvious from (5). Now assume $K > 0$. By using (5) and (7), we obtain the following:

$$\left|p_m(\lambda) - q_1(\lambda)\right| = \frac{|a|}{(K+1)^m}\lambda^{m-1}. \tag{8}$$

After straightforward calculations, we obtain:

$$\min_{|\lambda|=1} \left|q_1(\lambda)\right| = \frac{1}{(K+1)^m}. \tag{9}$$

From (7)–(9), it follows that for $|a| < 1$, we have

$$\left|p_m(\lambda) - q_1(\lambda)\right| < \left|q_1(\lambda)\right|, \quad |\lambda| = 1. \tag{10}$$

Then by Rouché's theorem (see e.g. [16]), it follows that $p_m(\cdot)$ and $q_1(\cdot)$ have the same number of roots inside the unit disc. Since the latter has all of its roots inside the unit disc for any $K > 0$, it follows that so does $p_m(\cdot)$. □

Since the proposed scheme does not achieve stabilization for $a > 1$ when $m \geq 2$, and achieves stabilization for $|a| < 1$, in the sequel we will consider the case $a < -1$. In the latter case, stabilization is always possible when $m = 2$, hence we will consider the case $m \geq 3$ as well. Also, note that for the case mentioned above, we have $K_{cr} > 0$, see (6). Next, we consider the case $0 < K < K_{cr}$.

Theorem 3 *Let* $m \geq 3$, $a < -1$ *and consider* $p_m(\cdot)$ *given by* (5). *For* $0 < K < K_{cr}$, $m - 1$ *roots of* p_m *are inside the unit disc and the remaining root is in the interval* $(a, -1)$.

Proof By using (5) and (7), we obtain the following:

$$\left|p_m(\lambda) - q_2(\lambda)\right| = \left|\lambda - \frac{K}{K+1}\right|^m. \tag{11}$$

It follows easily that maximum of (11) on the unit disc occurs at $\lambda = -1$, e.g., we have

$$\max_{|\lambda|=1} \left|\lambda - \frac{K}{K+1}\right|^m = \left(\frac{2K+1}{K+1}\right)^m. \tag{12}$$

Since $K < K_{cr}$, it follows from (6) that

$$(2K+1)^m < (2K_{cr}+1)^m = |a|. \tag{13}$$

Hence, by using (11)–(13), we obtain

$$\left|p_m(\lambda) - q_2(\lambda)\right| < \left|q_2(\lambda)\right|, \quad |\lambda| = 1. \tag{14}$$

Then by Rouché's theorem (see e.g. [16]), it follows that $p_m(\cdot)$ and $q_2(\cdot)$ have the same number of roots inside the unit disc. Since the latter has $m - 1$ roots inside the unit disc, it follows that so does $p_m(\cdot)$. Now consider the remaining root of $p_m(\cdot)$. It follows easily that

$$p_m(a) = (-1)^m \frac{[|a|(K+1)+1]^m - |a|^m}{(K+1)^m}, \tag{15}$$

$$p_m(-1) = (-1)^m \frac{(2K+1)^m - |a|}{(K+1)^m}. \tag{16}$$

From (15)–(16), it follows that $p_m(a)p_m(-1) < 0$, hence $p_m(\cdot)$ has a real root in the interval $(a, -1)$. □

Now we consider the case $K = K_{cr}$. By direct substitution $\lambda = -1$ and $K = K_{cr}$ in (5), we obtain $p_m(-1) = 0$. Next, we investigate the remaining roots of $p_m(\cdot)$. By using (5) from $p_m(\lambda) = 0$, we obtain

$$\left|\lambda - \frac{K}{K+1}\right|^m = \frac{|a|}{(K+1)^m}, \quad |\lambda| = 1. \tag{17}$$

It is easy to show that

$$\frac{1}{K+1} \leq \left|\lambda - \frac{K}{K+1}\right| \leq \frac{2K+1}{K+1}, \quad |\lambda| = 1, \tag{18}$$

here the upper and lower bounds occur at $\lambda = -1$ and $\lambda = 1$, respectively. By using (6) and (18) in (17), it follows that when $K = K_{cr}$, $p_m(\cdot)$ can have roots on the unit disc only at $\lambda = -1$, while the remaining roots are strictly inside the unit disc. Next, we give a condition for which this root is simple.

Theorem 4 *Assume that* $K = K_{cr}$, $a < -1$ *and* $m \geq 3$. *Let us define*

$$a_{mcr} = \left(\frac{m}{m-2}\right)^m. \tag{19}$$

If

$$|a| < a_{mcr}, \tag{20}$$

then $p_m(\lambda)$ *has a single root at* $\lambda = -1$, *while the remaining roots are strictly inside the unit circle.*

Proof From the above discussions, it is clear that when $K = K_{cr}$, $p_m(\cdot)$ has at least one root at $\lambda = -1$, while the remaining roots are strictly inside the unit disc. Next, we will show that $\lambda = -1$ cannot be a multiple root if (20) is satisfied; hence, $p_m(\cdot)$ is marginally stable. We prove this result by using contradiction. Assume that $\lambda = -1$ is a multiple root. Then $p'_m(-1) = 0$ must hold. From (5), we obtain

$$p'_m(-1) = (-1)^{m-1} m \frac{(2K_{cr}+1)^{m-1}}{(K_{cr}+1)^{m-1}} + (-1)^{m-2}(m-1)\frac{|a|}{(K_{cr}+1)^m}. \tag{21}$$

By using (6) in (21), it follows from $p'_m(-1) = 0$ that

$$m(K_{cr}+1)|a| - |a|(m-1)(2K_{cr}+1) = 0. \tag{22}$$

By rearranging (22) and using (6), after straightforward calculations, we obtain

$$|a|^{\frac{1}{m}}\left(1 - \frac{m}{2}\right) + \frac{m}{2} = 0, \tag{23}$$

which implies

$$|a| = \left(\frac{m}{m-2}\right)^m. \tag{24}$$

Hence, it follows that if $|a| \neq a_{mcr}$, then we have $p'_m(-1) \neq 0$, hence $p_m(\cdot)$ cannot have a multiple root at $\lambda = -1$. If we rewrite $p_m(\cdot)$ as $p_m(\lambda) = (\lambda + 1)g(\lambda)$, it easily follows that for the stability of $g(\cdot)$, we must have $|a| < a_{cr}$. Then the result follows from the discussions given above. □

Remark 4 It follows from (22) that if $K = K_{cr}$ and $|a| = a_{mcr}$, then $p_m(\cdot)$ has a double root at $\lambda = -1$, hence in this case $p_m(\cdot)$ is unstable. On the other hand, if $K = K_{cr}$ and $|a| < a_{mcr}$ holds, then $p_m(\cdot)$ is marginally stable.

Next, we consider the case $K > K_{cr}$.

Theorem 5 *Assume that $K > K_{cr}$ and $m \geq 3$. If (20) holds, then there exists a constant $K_m > K_{cr}$ such that for $K_{cr} < K < K_m$, $p_m(\cdot)$ is Schur stable.*

Proof First note that $p_m(\cdot)$ has m roots, which depend continuously on K for $K \geq 0$. Let us denote these roots as $r_1(K), \ldots, r_m(K)$. From Theorem 4, it follows that, say $r_m(K_{cr}) = -1$, and $|r_j(K_{cr})| < 1$, $j = 1, \ldots, m-1$. Hence, there exists a $\delta > 0$ such that $|r_j(K_{cr})| < 1 - \delta$, $j = 1, \ldots, m-1$. By continuity, given a sufficiently small $\delta_1 > 0$, there exists a $\varepsilon_1 > 0$ such that for $K_{cr} < K < K_{cr} + \varepsilon_1$, we have $|r_j(K)| < 1 - \delta_1$, $j = 1, \ldots, m-1$. Next, we show that the remaining root $r_m(K)$ will also be inside the unit disc. First note that if (20) holds then by using (6), (19), and (20), after some straightforward calculations, we obtain

$$-m|a|(K_{cr}+1) + (m-1)|a|(2K_{cr}+1) < 0. \tag{25}$$

Next, note that for $K = K_{cr}$, we have $p_m(-1) = 0$ and if (20) holds we have $p'_m(-1) \neq 0$. By using (6), (21), and $K = K_{cr}$, we obtain

$$C p'_m(-1) = (-1)^{m-1}|a|\big[m(K_{cr}+1) - (m-1)(2K_{cr}+1)\big], \tag{26}$$

where $C = (K_{cr}+1)^m(2K_{cr}+1) > 0$. It follows from (25) and (26) that if m is even then $p'_m(-1) < 0$ and if m is odd then $p'_m(-1) > 0$. By continuity, there exists a sufficiently small $\varepsilon_2 > 0$ such that for $K_{cr} < K < K_{cr} + \varepsilon_2$, this property still holds. In the latter case, it can easily be shown that $p_m(-1) > 0$ if m is even and $p_m(-1) < 0$ if m is odd. It follows from these that if $\varepsilon_2 > 0$ is sufficiently small, then for $K_{cr} < K < K_{cr} + \varepsilon_2$ the remaining root $r_m(K)$ satisfies $|r_m(K)| < 1$. Since $r_m(K_{cr}) = -1$ and $r_m(K)$ depends continuously on K, it follows that for sufficiently small $\varepsilon_2 > 0$, we have $|r_m(K)| < 1 - \delta_1$ for $K_{cr} < K < K_{cr} + \varepsilon_2$. Hence, if we choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then for $K_{cr} < K < K_{cr} + \varepsilon$ we will have $|r_j(K)| < 1 - \delta_1$, $j = 1, \ldots, m$. □

Several remarks are now in order.

Remark 5 The existence of a_{mcr} was mentioned and some upper bounds were found through extensive simulations in [8]. For example, for $m = 3$ the upper bound was found as 27 in [8] which is exactly the same as given by (19). On the other hand, the upper bounds for $m = 4, 5, 6$ were found as 15, 11.5, 9.8, respectively, in [8], and it turns out that these estimates are rather conservative, since by using (19) a_{mcr} can be found for these values of m as 16, 12.86, 11.39, respectively. Theorem 5 also justifies the numerical simulation results given in [8], where a periodic orbit with $m = 10$ and $a = -7.74$, and another one with $m = 16$ and $a = -1.629$ were stabilized with the proposed scheme; indeed in these cases by using (19), we

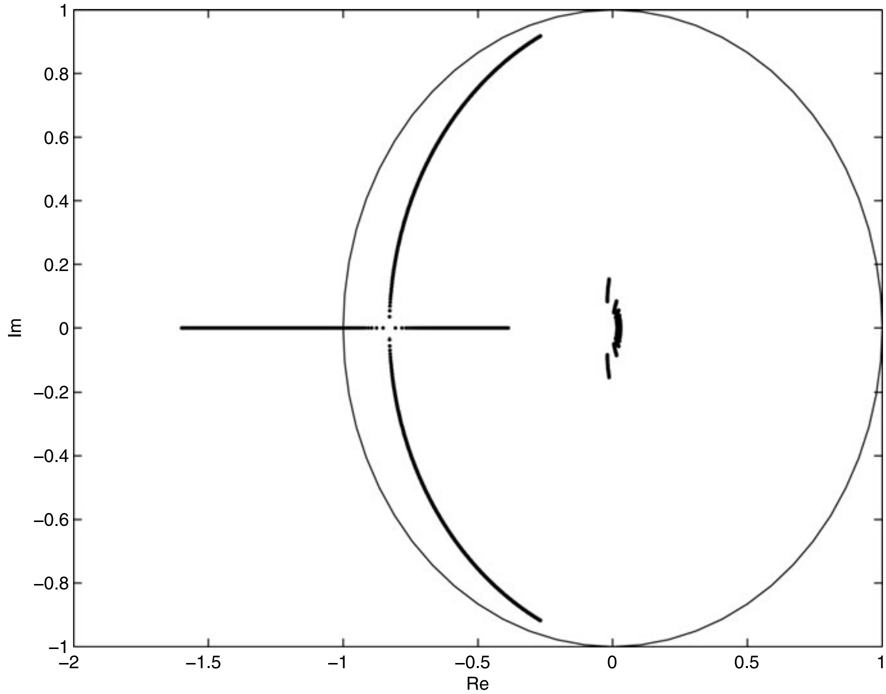

Fig. 1 Location of the roots of $p_m(\lambda)$

find $a_{10cr} = 9.81$ and $a_{16cr} = 8.64$. See also Observation 1.

Remark 6 It was conjectured in [8] that $a_{mcr} \to a_\infty = 1$ as $m \to \infty$. However, Theorem 5 shows that this conjecture is false. In fact, from (19), it follows that $a_\infty = e^2$. Moreover, we have $a_{mcr} > e^2$ for any $m \geq 3$. See also Observation 2.

Remark 7 The upper bound given by (19)–(20) is interesting in the sense that it neither depends on the periodic orbit itself nor to the particular chaotic system in question; indeed it only depends on the period number m. Note that for the classical DFC, a similar stability condition would depend on periodic orbit, chaotic system in question, and m; see, e.g., [7, 9, 10].

Remark 8 It was also conjectured in [8] that for $K = K_{cr}$, $a < -1$ and $|a| < a_{mcr}$, if $p_m(\cdot)$ is marginally stable, then stabilization is possible. Theorem 5 proves that this conjecture holds. See Observation 3.

4 Simulation results

For simulations, we will use the logistic map given as

$$x(i+1) = rx(i)\big(1 - x(i)\big), \tag{27}$$

which is well known for its chaotic behavior and studied extensively in the literature. Stabilization of various periodic orbits of (27) by using the control scheme given in Sect. 2 were considered in [8]. Here, as another example we consider (27) with $r = 3.579$. For this case, (27) has a period 20 orbit Σ_{20} for which $a = -5.6363$. Note that from (20) we find $a_{20cr} = 8.2253$, hence stabilization is possible with the proposed scheme. By using (6), the critical gain K_{cr} can be found as $K_{cr} = 0.0451$. By using extensive computations, we find that stabilization is possible for $K_{cr} < K < K_{\max}$ where $K_{\max} = 0.08276$. Indeed, the location of the roots of (5) for $0.04 \leq K \leq 0.07$ is given in Fig. 1. As seen from Fig. 1, for certain values of K, all of the roots are strictly inside the unit disc. For further simulation results, consider the system given by (27), (3), and (4) with $K = 0.05$ and $x(0) = 0.82$. The plot of $x(i-1)$ versus $x(i)$ for $i \geq 1000$ is shown in Fig. 2. As can be seen, the trajectory of $x(\cdot)$ converges to Σ_{20}. Finally, the control input $u(i)$ as given by (4) is plotted in Fig. 3. As can be seen, $u(i) \to 0$.

5 Conclusion

In this paper, we considered a generalization of DFC as given in [8]. We proved certain stability results

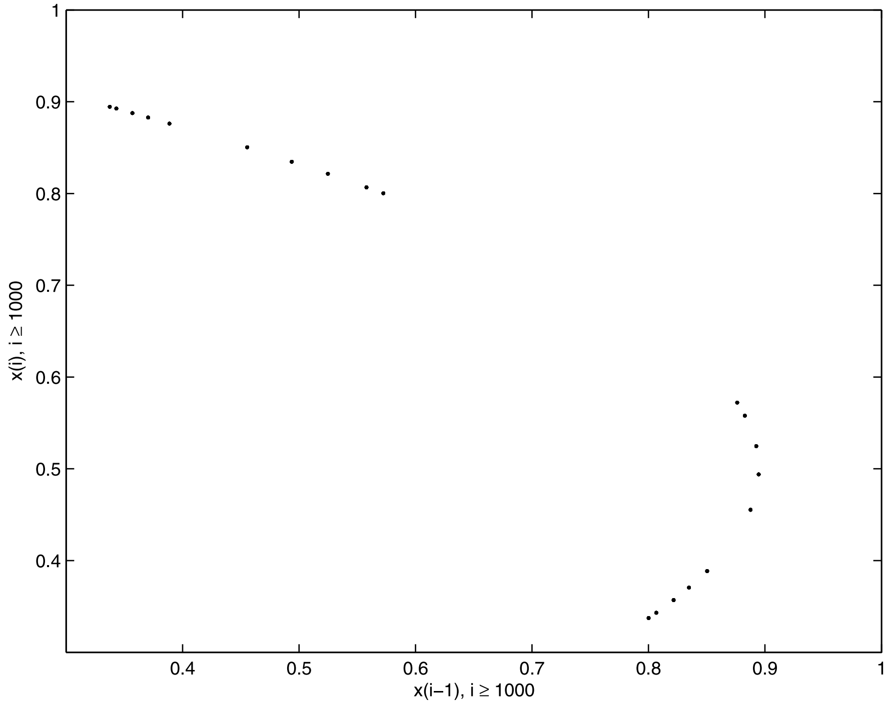

Fig. 2 $x(i-1)$ vs. $x(i)$ for $i \geq 1000$

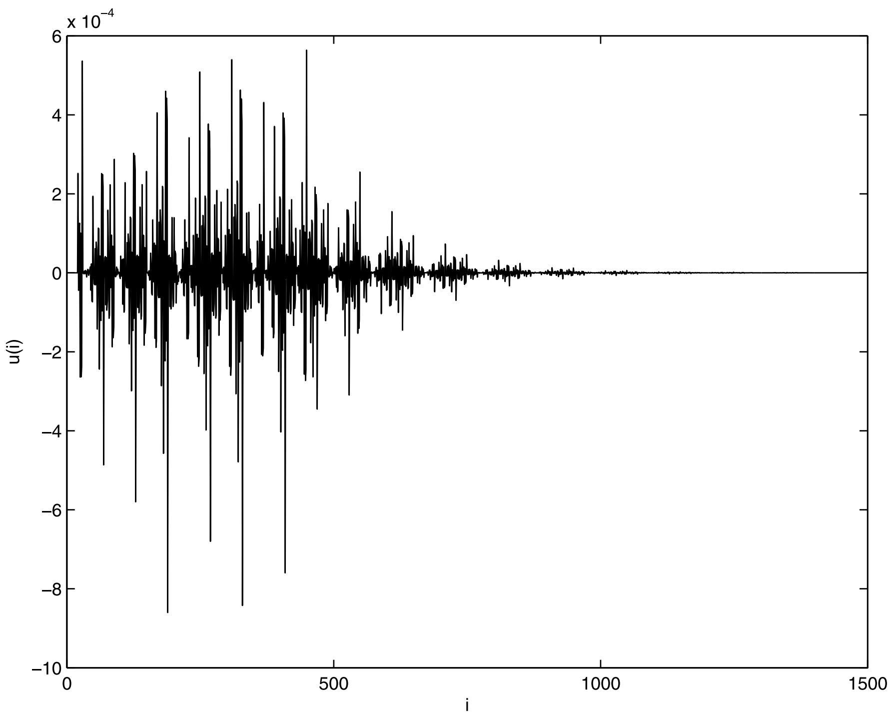

Fig. 3 Control input $u(i)$

which were not proven but mentioned as conjectures and/or observations in [8]. In particular, we have shown that when the periodic orbit satisfies a condition, which mainly depends on the period number, then the stabilization is always possible with the proposed scheme.

Various generalizations of the proposed scheme may be possible. An interesting problem may be the generalization to higher dimensional case. Finding an upper bound as given in Theorem 5 might be an interesting and open problem. Another possible generalization might be the combination of the double period scheme as given in [13] with the proposed scheme. However, these points require further research.

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