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The asymptotic zero distribution of sections and tails of classical Lindelöf functions

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We study the asymptotic (as $n \to \infty$) zero distribution of

$$I_n(z, \mu, \Gamma_\lambda) = (1 - \mu)s_n(z, \Gamma_\lambda) - \mu t_{n+1}(z, \Gamma_\lambda),$$

where $\mu \in \mathbb{C}$, s_n is n^{th} section, t_n is n^{th} tail of the power series of classical Lindelöf function Γ_λ of order λ. Our results generalize the results by A. Edrei, E. B. Saff, and R. S. Varga for the case $\mu = 0$.

1 Introduction

For a transcendental entire function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_0 > 0, \tag{1.1}$$

denote by

$$s_n(z, f) = \sum_{k=0}^{n} a_k z^k \quad \text{and} \quad t_n(z, f) = \sum_{k=n}^{\infty} a_k z^k \tag{1.2}$$

its n^{th} section and n^{th} tail respectively.

For some widely applicable concrete entire functions (such as the exponential function, the trigonometric functions and some others) elegant and sharp asymptotics (as $n \to \infty$) for zeros of $s_n(z, f)$ and $t_n(z, f)$ were obtained by G. Szegö [8], J. Dieudonné [1], P. C. Rosenbloom [7] and others. In the work of A. Edrei, E. B. Saff, and R. S. Varga [2] these asymptotics for zeros of $s_n(z, f)$ were extended to the Mittag-Leffler functions and to $\mathcal{L}$-functions.

Recall that $F(z)$ is called an $\mathcal{L}$-function if it satisfies the following two conditions.

(A) The function $F(z)$ is entire of order λ $(0 < \lambda < 1)$ and all its zeros are real and negative:

$$F(z) = F(0) \prod_{k=1}^{\infty} \left(1 + \frac{z}{x_k}\right) = \sum_{j=0}^{\infty} a_j z^j, \quad \text{where} \quad 0 < x_k, \quad \sum_{k=1}^{\infty} x_k^{-1} < +\infty, \quad F(0) > 0; \tag{1.3}$$

(B) Along the positive axis

$$\ln F(r) = \ln M(r, F) = B_1 r^\lambda (1 + o(1)), \quad B_1 > 0, \quad r \longrightarrow \infty. \tag{1.4}$$

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G. Szegö in [8] considered a more general problem of the asymptotic distribution of the zeros of the linear combination

$$I_n(z,\mu,f) = (1-\mu)s_n(z,f) - \mu t_{n+1}(z,f) \tag{1.5}$$

when $\mu \in \mathbb{C}$. Evidently, $I_n(z,0,f) = s_n(z,f)$ and $I_n(z,1,f) = -t_{n+1}(z,f)$. G. Szegö in [8] proved a remarkable theorem related to the asymptotic behavior of the roots of the equation

$$I_n(z,\mu,e^z) = 0\,.$$

It was discovered by G. Szegö that the set of all zeros of

$$I_n(z,\mu,e^z)\,, \quad \mu \neq 0,1,$$

is approximately equal to $\{nz : |ze^{1-z}| = 1\}$, the set of all zeros of $s_n(z,e^z)$ is approximately equal to $\{nz : |ze^{1-z}| = 1, |z| \leq 1\}$, the set of all zeros of $t_n(z,e^z)$ is approximately equal to $\{nz : |ze^{1-z}| = 1, |z| \geq 1\}$.

A survey of investigations prior to 1997 on several aspects of the distribution of zeros of sections and tails is given by I. V. Ostrovskii in [6].

In [9], the zero distribution of linear combinations (1.5) of Mittag-Leffler functions was considered. The results obtained in [9] extend some results of A. Edrei, E. B. Saff, and R. S. Varga [2] on the zero distribution of sections $s_n(z,f)$ of Mittag-Leffler functions.

The following problem seems to be of interest. Is it possible to extend the results of A. Edrei, E.B. Saff and R.S. Varga [2] on the zero distribution of sections $s_n(z,f)$ of $\mathcal{L}$-functions to the zero distribution of linear combinations (1.5) of $\mathcal{L}$-functions?

Below we present the main result of [2] on $\mathcal{L}$-functions (see [2], p. 21).

Theorem A. *Let $F(z)$ be an $\mathcal{L}$-function of order λ.*

I. Define the sequence $\{R_m\}_m$ by the conditions

$$a(R_m) = m \quad (m = 1,2,3,\ldots), \quad \textit{where} \quad a(r) = r\frac{F'(r)}{F(r)}. \tag{1.6}$$

Let erfc(ζ) denote the complementary error function

$$\mathit{erfc}(\zeta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\zeta e^{-v^2}dv = \frac{2}{\sqrt{\pi}}\int_\zeta^\infty e^{-v^2}dv\,.$$

Then, if ζ is an auxiliary complex variable, we have

$$\frac{s_m\left(R_m\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right),F\right)}{F(R_m)\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right)^m} \longrightarrow \frac{1}{2}\mathit{exp}(\zeta^2)\mathit{erfc}(\zeta), \tag{1.7}$$

uniformly on every compact set of the ζ-plane.

II. With every given ϕ $(0 < |\phi| < \pi)$ it is possible to associate a real sequence $\{\sigma_m(\phi)\}$ such that

$$\lim_{m\to\infty} \sigma_m(\phi) = \sigma(\phi),$$

where $\sigma = \sigma(\phi)$ is the unique solution in $(0,1)$ of the equation

(i)

$$\sigma^\lambda \cos(\phi\lambda) - 1 - \lambda \ln \sigma = 0;$$

(ii) write

$$\xi_m = \xi_m(\phi) = \sigma_m(\phi)e^{i\phi}, \quad \xi = \sigma(\phi)e^{i\phi}, \quad L_m = (2\pi\lambda m)^{1/2}\xi_m^{-m}\{F(R_m)\}^{-1};$$

then the polynomials in ζ

$$L_m s_m\left(R_m\xi_m\left(1+\frac{\zeta}{m(1-\xi^\lambda)}\right)\right) \tag{1.8}$$

are uniformly bounded on every compact set of the ζ*-plane.*

III. Every limit function of the polynomials in (1.8) *is of the form*

$$\exp\left(\frac{\zeta}{1-\xi^\lambda}\right)\left\{e^{i\chi}e^{-\zeta}-\frac{\xi}{1-\xi}\right\}=Z_\chi(\zeta),$$

where the real quantity χ *may depend on the particular sequence of integers through which* $m\to+\infty$*.*

For any $\mathcal{L}$-function F of order λ, let $\mathcal{M}_n(\mu,F)$, $\mu\in\mathbb{C}$, be the set of all roots of the equation

$$I_n(R_nz,\mu,F)=0,$$

where

$$I_n(R_nz,\mu,F)=(1-\mu)s_n(R_nz,F)-\mu t_{n+1}(R_nz,F).$$

In particular, $\mathcal{M}_n(0,F)$ ($\mathcal{M}_{n-1}(1,F)$) coincides with the zero set of $s_n(R_nz,F)$ ($t_n(R_{n-1}z,F)$). Define $\mathcal{M}(\mu,F)$ to be the set of all accumulation points of $\bigcup_{n=1}^\infty\mathcal{M}_n(\mu,F)$.

It follows from Theorem A, parts II and III, that

$$\{z=\sigma e^{i\phi}:\ \sigma^\lambda\cos(\phi\lambda)-1-\lambda\ln\sigma=0,\ 0<\sigma<1,\ 0<|\phi|<\pi\}\subset\mathcal{M}(0,F) \tag{1.9}$$

for any $\mathcal{L}$-function F.

The following problem seems to be of interest. Does the embedding (1.9) remain in force if we replace $\mathcal{M}(0,F)$ by $\mathcal{M}(\mu,F)$ when $\mu\in\mathbb{C}$? In the present paper we study the zero distribution of the linear combination $I_n(R_nz,\mu,F)$ of the Lindelöf classical functions

$$\Gamma_\lambda(z)=\prod_{n=1}^\infty\left(1+\frac{z}{n^{1/\lambda}}\right),\quad 0<\lambda<1, \tag{1.10}$$

and show that for Lindelöf classical functions (not arbitrary $\mathcal{L}$-function) the embedding (1.9) can be extended to all μ in $\mathbb{C}$, and moreover, changed to equality. To our knowledge, for arbitrary $\mathcal{L}$-function the answer to the above question is still open.

2 Main curves and regions

To formulate the main result of the paper we need to introduce some curves and regions. For any λ satisfying $0<\lambda<1$, and h, being sufficiently small, denote

$$S(\lambda,h)=\{z=re^{i\phi}:r^\lambda\cos(\lambda\phi)-\lambda\ln r-1=h,\ |\phi|\le\pi\}.$$

Clearly, $S(\lambda,h)$ is symmetric with respect to the x-axis. We have, if $z=re^{i\phi}\in S(\lambda,h)$,

$$\cos(\lambda\phi)=g(r,h)=\frac{1+h+\lambda\ln r}{r^\lambda}.$$

Since $\frac{dg(r,h)}{dr}=-\frac{\lambda(h+\lambda\ln r)}{r^{\lambda+1}}$, then $g(r,h)$ increases when $r\in\left(0,e^{-h/\lambda}\right)$ and decreases when $r\in\left(e^{-h/\lambda},\infty\right)$. We give rough shapes of the curves $S(\lambda,h)$ in three different cases (when $h=0$, $h>0$ and $h<0$) in Fig. 1, Fig. 2 and Fig. 3.

Let us fix constants λ and h, $0<\lambda<1$, $h\ge0$. Note that the curve $S(\lambda,h)$ divides the complex plane $\mathbb{C}$ into three different regions. Denote by I_h and II_h two of these three regions. Namely, let I_h be the region containing $z=0$ and let II_h be the region that contains neither $z=0$ nor -1. Curve $S(\lambda,-h)$ divides the complex plane $\mathbb{C}$ into two different regions. Denote by III_h that region that does not contain $z=0$. We give rough sketches of the regions I_h, II_h and III_h in Fig.4.

If $0<\varepsilon_1<\pi$, we define

$$\Delta=\Delta(\varepsilon_1)=\{z=re^{i\phi}:|\phi|\le\pi-\varepsilon_1,\ r>0\}. \tag{2.1}$$

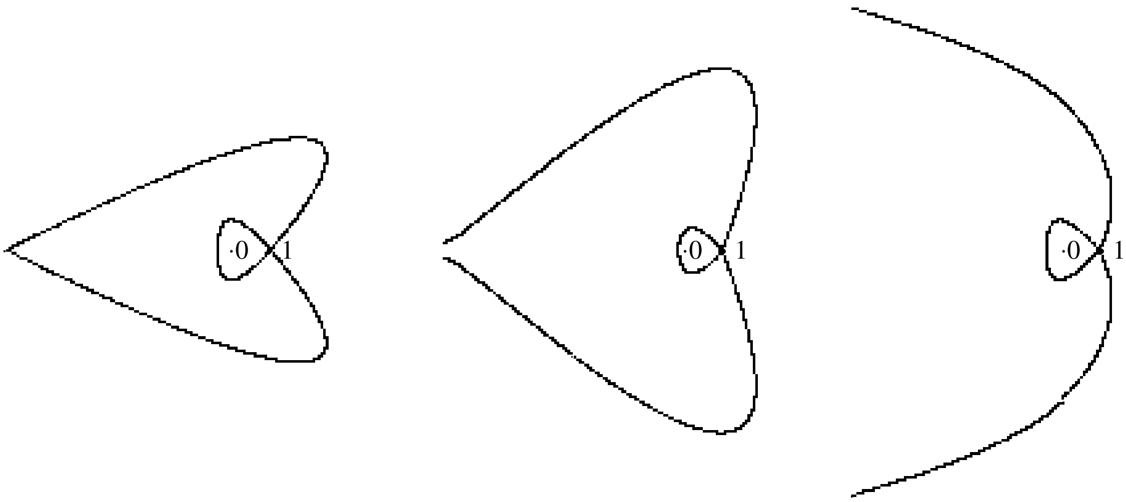

Fig. 1 Curves $S(\lambda, 0)$ for $0 < \lambda < 1/2$, $\lambda = 1/2$ and $1/2 < \lambda < 1$ respectively.

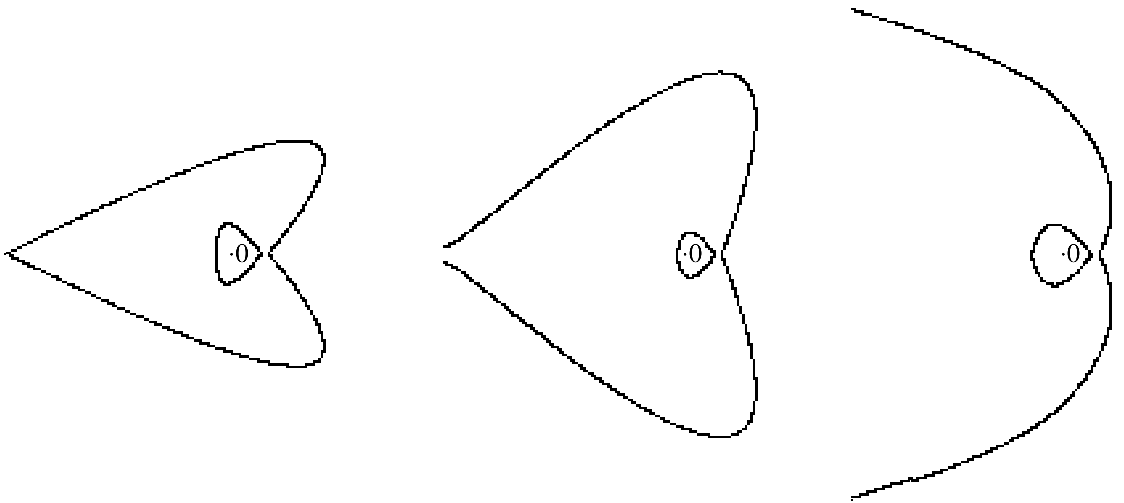

Fig. 2 Curves $S(\lambda, h)$ with $h > 0$ for $0 < \lambda < 1/2$, $\lambda = 1/2$ and $1/2 < \lambda < 1$ respectively

3 Results

The first theorem we prove shows regions where zeros of $I_m(R_m w, \mu, \Gamma_\lambda)$ may be.

Theorem 3.1 *Let $\Gamma_\lambda(z)$ be a Lindelöf classical function of order λ $(0 < \lambda < 1)$. Suppose that $\mu \neq 0$. Then, if δ, ε_1 and h are sufficiently small positive constants, $I_m(R_m w, \mu, \Gamma_\lambda)$ does not vanish in $(I_0 \cup II_0 \cup III_h) \cap \Delta$, for all sufficiently large m.*

Theorem 3.1 implies that the zeros of $I_m(R_m w, \mu, \Gamma_\lambda)$ may lie only in the vicinity of the curve $S(\lambda, 0)$ and the ray $\arg z = \pi$. The proof of Theorem 3.1 is given in Section 5. The case $\mu = 0$ was studied in [2] not only for the classical Lindelöf function $\Gamma_\lambda(z)$ but for any $\mathcal{L}$-function (see Theorem A above).

We define

$$\mathcal{M}_*(\mu, \Gamma_\lambda) = \mathcal{M}(\mu, \Gamma_\lambda) \backslash \{z : \arg z = \pi\}.$$

The following remark is a corollary of Theorem 3.1.

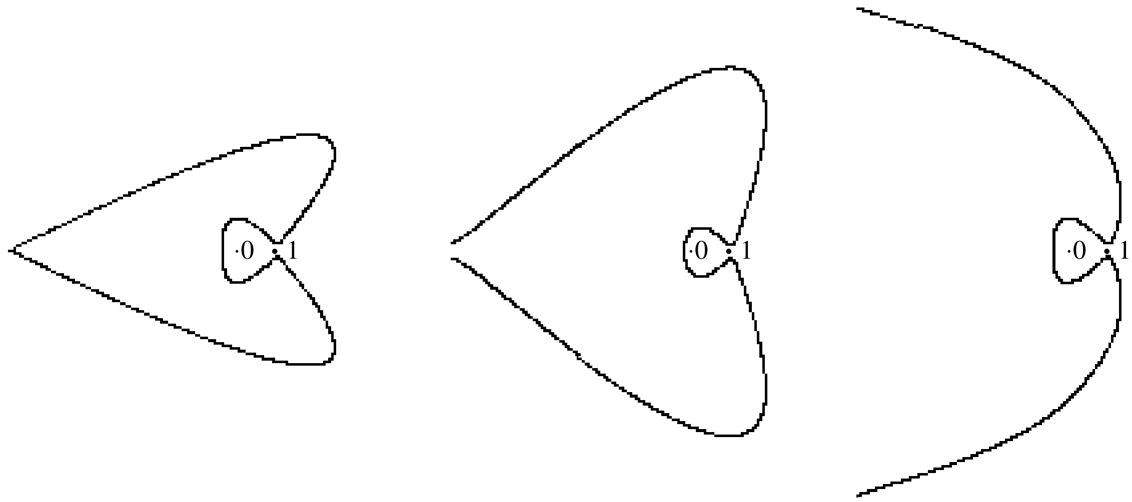

Fig. 3 Curves $S(\lambda, h)$ with $h < 0$ for $0 < \lambda < 1/2$, $\lambda = 1/2$ and $1/2 < \lambda < 1$ respectively

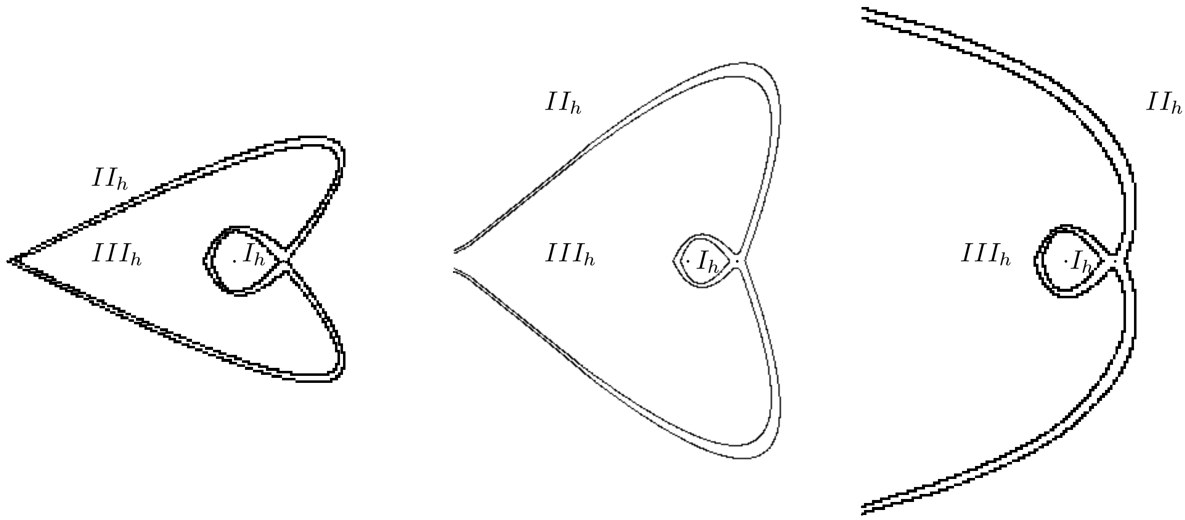

Fig. 4 Regions I_h, II_h and III_h for $0 < \lambda < 1/2$, $\lambda = 1/2$ and $1/2 < \lambda < 1$ respectively

Remark 3.2

$$\mathcal{M}_*(\mu, \Gamma_\lambda) \subset S(\lambda, 0). \tag{3.1}$$

The next theorem shows that each point on the curve $S(\lambda, 0)$ is an accumulation point of zeros of $I_m(R_m z, \mu, \Gamma_\lambda)$ when $\mu \in \mathbb{C}\backslash\{0, 1\}$.

Theorem 3.3 *Let $\xi = \xi(\phi) = |\xi|e^{i\phi}$, $0 < |\phi| < \pi$, be a fixed point on the curve $S(\lambda, 0)$. We define $\tau = |\xi|^\lambda \sin(\lambda\phi) - \lambda\phi$, and let the sequences $\{\tau_m\}_{m=1}^\infty$ and $\{\varepsilon_m(\zeta)\}_{m=1}^\infty$ be defined by the conditions*

$$\tau_m \equiv \frac{\tau}{\lambda} m (mod \quad 2\pi), \quad -\pi < \tau_m \le \pi,$$

and

$$\varepsilon_m(\zeta) = \frac{\log m}{2(1 - \xi^\lambda)m} - \frac{\zeta - i\tau_m}{(1 - \xi^\lambda)m}.$$

Then, as $m \to \infty$,

$$I_m(R_m\xi(1+\varepsilon_m(\zeta)),\mu,\Gamma_\lambda)\left(\frac{2m^{1+\lambda}\sin(\pi\lambda)}{\lambda^{1-\lambda}e}\right)^{\frac{1}{2\lambda}}\frac{(2\pi)^{\frac{1}{2}}}{\xi^m(1+\varepsilon_m(\zeta))^m e^{\frac{m}{\lambda}}}$$

$$\longrightarrow \begin{cases} \alpha(\xi)e^{\zeta} - \dfrac{\xi}{1-\xi}, & \textit{if} \quad |\xi|<1; \\ \beta(\xi)e^{\zeta} - \dfrac{\xi}{1-\xi}, & \textit{if} \quad |\xi|>1 \end{cases}$$

uniformly on every compact set of the ζ*-plane, where*

$$\alpha(\xi) = (1-\mu)\left(\frac{2\pi\lambda}{\xi}\right)^{\frac{1}{2}} e^{\frac{\xi^\lambda - 1}{2\lambda}}, \quad \beta(\xi) = -\mu\left(\frac{2\pi\lambda}{\xi}\right)^{\frac{1}{2}} e^{\frac{\xi^\lambda - 1}{2\lambda}}.$$

The proof of Theorem 3.3 is given in Section 6. To prove Theorem 3.3 we repeat the proof of Theorem 2 from [2] with slight modifications.

The next result is a corollary of Theorems 3.1 and 3.3.

Corollary 3.4 *One has:*
(i) $\mathcal{M}_*(0,\Gamma_\lambda) = S(\lambda,0)\cap\{z:|z|\le 1\}$,
(ii) $\mathcal{M}_*(1,\Gamma_\lambda) = S(\lambda,0)\cap\{z:|z|\ge 1\}$,
(iii) $\mathcal{M}_*(\mu,\Gamma_\lambda) = S(\lambda,0)$ *for* $\mu \ne 0,1$.

The next result shows how quickly the zeros of $I_m(R_m w,\mu,F)$ approach the point $w = 1$ for arbitrary $\mathcal{L}$-function F.

Theorem 3.5 *Let* $F(z)$ *be an* $\mathcal{L}$*-function of order* λ $(0<\lambda<1)$. *Then, as* $m\to\infty$,

$$\frac{I_m\left(R_m\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right),\mu,F\right)}{F(R_m)\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right)^m} \longrightarrow \mathit{exp}(\zeta^2)\left(\frac{\mathit{erfc}(\zeta)}{2}-\mu\right) \tag{3.2}$$

uniformly on every compact set of the ζ*-plane.*

Theorem 3.5 can be viewed as an extension of part I of Theorem A and is an easy corollary of part I of Theorem A. The proof of Theorem 3.5 is given in Section 7.

4 Preliminaries

Let the functions $F(z)$ and $\Gamma_\lambda(z)$ be given by (1.3) and (1.10) respectively. We mention without proof properties of the functions $F(z)$ and $\Gamma_\lambda(z)$ which the reader can find in [2], [3] and [4].

1) It is known that (see [2], p. 90)

$$\ln F(z) = B_1 z^\lambda(1+\eta(z)) \tag{4.1}$$

and

$$z\frac{F'(z)}{F(z)} = B_1\lambda z^\lambda(1+\eta(z)), \tag{4.2}$$

where $\eta(z)\to 0$ uniformly in Δ, as $z\to\infty$.

Also (see [3], p. 158)

$$\Gamma_\lambda(z) = \frac{e^{\frac{\pi z^\lambda}{\sin\pi\lambda}+\nu(z)}}{z^{\frac{1}{2}}(2\pi)^{\frac{1}{2\lambda}}}, \tag{4.3}$$

where $z\nu(z)$ is uniformly bounded in Δ, as $z \to \infty$.

2) Using (4.3) we can easily calculate the indicator function of the classical Lindelöf function Γ_λ (see [5], p. 53). It is

$$h_{\Gamma_\lambda}(\theta) = \frac{\pi}{\sin \pi\lambda} \cos \lambda\theta, \quad -\pi < \theta < \pi.$$

Since the indicator function of an entire function of finite order and finite type is a continuous function then

$$h_{\Gamma_\lambda}(\theta) = \frac{\pi}{\sin \pi\lambda} \cos \lambda\theta, \quad \theta \in [-\pi, \pi].$$

It follows (see [5], p. 56) that

$$|\Gamma_\lambda(re^{i\theta})| < e^{(\frac{\pi \cos \lambda\theta}{\sin \pi\lambda} + \alpha) r^\lambda} \tag{4.4}$$

for all $r > r(\alpha)$ and when $\theta \in [-\pi, \pi]$, where α is sufficiently small.

3) Let the sequence $\{R_m\}_m$ be defined by conditions (1.6). Then (see [2], p. 93)

$$R_m = \left\{\frac{m}{B_1\lambda}\right\}^{1/\lambda} (1 + o(1)), \quad m \longrightarrow \infty, \tag{4.5}$$

and

$$a_m {R_m}^m = \frac{F(R_m)}{(2\pi\lambda m)^{1/2}} (1 + o(1)), \quad m \longrightarrow \infty. \tag{4.6}$$

Using (1.6) and (4.3), for $\Gamma_\lambda(z)$ we have,

$$R_m^\lambda = \frac{(m + (1/2)) \sin(\pi\lambda)}{\pi\lambda} + o(1), \quad m \longrightarrow \infty, \tag{4.7}$$

$$R_m^{1/2} = \left(\frac{m \sin(\pi\lambda)}{\pi\lambda}\right)^{1/(2\lambda)} (1 + o(1)), \quad m \longrightarrow \infty. \tag{4.8}$$

It follows from (4.3), (4.7) and (4.8) that

$$\Gamma_\lambda(R_m w) = \frac{e^{\frac{m w^\lambda}{\lambda}}}{m^{\frac{1}{2\lambda}}} \left(\frac{\lambda e^{w^\lambda}}{2 w^\lambda \sin(\pi\lambda)}\right)^{\frac{1}{2\lambda}} (1 + o(1)), \quad m \longrightarrow \infty. \tag{4.9}$$

It follows from (4.6) and (4.9) that for $\Gamma_\lambda(z)$ we have

$$a_m R_m = \frac{e^{\frac{m}{\lambda}}}{(2\pi)^{\frac{1}{2}}} \left(\frac{\lambda^{1-\lambda} e}{2 m^{1+\lambda} \sin(\pi\lambda)}\right)^{\frac{1}{2\lambda}} (1 + o(1)), \quad m \longrightarrow \infty\,. \tag{4.10}$$

4) Let w satisfy $|w - 1| \le \eta < 1/2$. Then (see [2], p. 96), as $m \to \infty$,

$$F(R_m w) = F(R_m) \exp\left\{(w - 1)m + \frac{(w - 1)^2}{2} m(\lambda - 1 + o(1)) + (w - 1)^3 m\eta(m, w)\right\}, \tag{4.11}$$

where the sequence $\{\eta(m, w)\}_m$ is uniformly bounded in $\{w : |w - 1| \le \eta\}$.

5 Proof of Theorem 3.1

To prove Theorem 3.1 we will find the asymptotic behavior of $I_m(R_m w, \mu, \Gamma_\lambda)$ in regions I_h, II_h and III_h. We rewrite $I_m(R_m w, \mu, \Gamma_\lambda)$ as

$$I_m(R_m w, \mu, \Gamma_\lambda) = (1-\mu)\Gamma_\lambda(R_m w) - t_{m+1}(R_m w, \Gamma_\lambda). \tag{5.1}$$

Since the asymptotic behavior of $\Gamma_\lambda(R_m w)$ is known (see (4.9)), then the problem of finding the asymptotic behavior of $I_m(R_m w, \mu, \Gamma_\lambda)$ is reduced to the problem of finding the asymptotic behavior of $t_{m+1}(R_m w, \Gamma_\lambda)$.

Suppose that $w \in \Delta \cap \{w : |w| \le C\}$ for some constant C. By Cauchy's integral formula,

$$t_{m+1}(R_m w, \Gamma_\lambda) = \frac{R_m^{m+1} w^{m+1}}{2\pi i} \oint_{|\xi|=2R_m|w|} \frac{\Gamma_\lambda(\xi)}{\xi^{m+1}(\xi - R_m w)}\, d\xi.$$

Since

$$\frac{1}{\xi - R_m w} = -\frac{1}{R_m w} + \frac{\xi}{R_m w(\xi - R_m w)},$$

then

$$\begin{aligned} t_{m+1}(R_m w, \Gamma_\lambda) &= -\frac{R_m^m w^m}{2\pi i} \oint_{|\xi|=2R_m|w|} \frac{\Gamma_\lambda(\xi)}{\xi^{m+1}}\, d\xi + \frac{R_m^m w^m}{2\pi i} \oint_{|\xi|=2R_m|w|} \frac{\Gamma_\lambda(\xi)}{\xi^m(\xi - R_m w)}\, d\xi \\ &=: A_1 + \frac{R_m^m w^m}{2\pi i} \oint_{|\xi|=2R_m|w|} \frac{\Gamma_\lambda(\xi)}{\xi^m(\xi - R_m w)}\, d\xi, \end{aligned} \tag{5.2}$$

where, due to (4.10),

$$A_1 = -a_m R_m^m w^m = -\frac{w^m e^{\frac{m}{\lambda}}}{(2\pi)^{\frac{1}{2}}} \left(\frac{\lambda^{1-\lambda} e}{2m^{1+\lambda}\sin(\pi\lambda)}\right)^{\frac{1}{2\lambda}} (1 + o(1)), \quad m \longrightarrow \infty. \tag{5.3}$$

Further, we study separately two different cases:

Case 1): $w \in G_1 = \{|w| > 1 - \frac{\delta}{2}, \quad |w-1| > \delta\} \cap \Delta$,

Case 2): $w \in G_2 = \{|w| \le 1 - \frac{\delta}{4}, \quad |w-1| > \delta\} \cap \Delta$.

Note that $G_1 \cap G_2 = \mathbb{C} \cap \Delta \cap \{w : |w-1| > \delta\}$.

Case 1). Suppose that $w \in G_1$. By (5.2),

$$\begin{aligned} t_{m+1}(R_m w, \Gamma_\lambda) &= A_1 + \frac{R_m^m w^m}{2\pi i} \oint_{|\xi|=R_m|w|/2} \frac{\Gamma_\lambda(\xi)}{\xi^m(\xi - R_m w)}\, d\xi + \Gamma_\lambda(R_m w) \\ &=: A_1 + A_2 + A_3, \end{aligned} \tag{5.4}$$

where

$$A_2 = \frac{R_m^m w^m}{2\pi i} \oint_{|\xi|=R_m|w|/2} \frac{\Gamma_\lambda(\xi)}{\xi^m(\xi - R_m w)}\, d\xi, \quad A_3 = \Gamma_\lambda(R_m w). \tag{5.5}$$

It follows from (4.9), as $m \to \infty$, that

$$\begin{aligned} A_3 &= \frac{e^{\frac{m w^\lambda}{\lambda}}}{m^{\frac{1}{2\lambda}}} \left(\frac{\lambda e^{w^\lambda}}{2w^\lambda \sin(\pi\lambda)}\right)^{\frac{1}{2\lambda}} (1 + o(1)) \\ &= \frac{w^m e^{\frac{m}{\lambda}} e^{\frac{m}{\lambda}(w^\lambda - \lambda \ln w - 1)}}{m^{\frac{1}{2\lambda}}} \left(\frac{\lambda e^{w^\lambda}}{2w^\lambda sin(\pi\lambda)}\right)^{\frac{1}{2\lambda}} (1 + o(1)). \end{aligned} \tag{5.6}$$

We will find the asymptotic expression for the integral A_2 in in the following three steps:

Step 1): change the contour of integration of A_2;
Step 2): show that the main contribution to A_2 comes from the neighborhood of the point $\zeta = R_m$.
Step 3): find an asymptotic expression for A_2 by using Laplace's Method for contour integrals.
Consider the curve (see Fig. 5)

$$T(\lambda) = \left\{ z = re^{i\phi} : \ r^\lambda = \frac{\lambda\phi}{\sin(\lambda\phi)}, \ |\phi| \le \pi \right\}$$

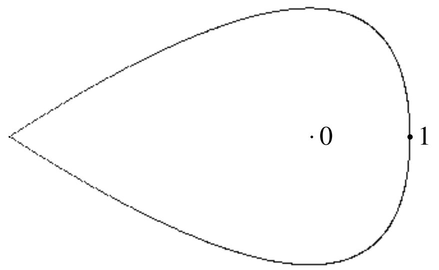

Fig. 5 Curve $T(\lambda)$

Case 1, Step 1. Curves $S\big(\lambda, -\frac{h}{2}\big)$ and $T(\lambda)$ have two points of intersection, say z_1 and z_2. We have, $z_1 = de^{i\gamma}$ and $z_2 = de^{-i\gamma}$, where $\gamma \sim \frac{\sqrt{h}}{\lambda}$ and $d^\lambda \sim \frac{\sqrt{h}}{\sin\sqrt{h}}$, as $h \to 0$. Define
$l_1 = T(\lambda) \cap \{z : |z| \le d\}$,
$l_2 = S(\lambda, -\frac{h}{2}) \cap \{z : |z| \le d\}$,
For sufficiently small positive h, we have

$$\begin{aligned} A_2 &= \frac{R_m^m w^m}{2\pi i} \oint\limits_{|\xi| = R_m(1-\delta/2)} \frac{\Gamma_\lambda(\xi)}{\xi^m(\xi - R_m w)}\, d\xi \\ &= \frac{R_m^m w^m}{2\pi i} \oint\limits_{R_m l_1 \cup R_m l_2} \frac{\Gamma_\lambda(\xi)}{\xi^m(\xi - R_m w)}\, d\xi \\ &= \frac{w^m}{2\pi i} \oint\limits_{l_1 \cup l_2} \frac{\Gamma_\lambda(R_m t)}{t^m(t-w)}\, dt \\ &= \frac{w^m}{2\pi i} \left(\int\limits_{l_1} + \int\limits_{l_2} \right) \frac{\Gamma_\lambda(R_m t)}{t^m(t-w)}\, dt \\ &=: A_{21} + A_{22}, \end{aligned} \tag{5.7}$$

where

$$A_{21} = \frac{w^m}{2\pi i} \int\limits_{l_1} \frac{\Gamma_\lambda(R_m t)}{t^m(t-w)}\, dt \quad \text{and} \quad A_{22} = \frac{w^m}{2\pi i} \int\limits_{l_2} \frac{\Gamma_\lambda(R_m t)}{t^m(t-w)}\, dt.$$

Case 1, Step 2. It follows from (4.4) and (4.7) that for $t = |t|e^{i\phi}$ we have,

$$\begin{aligned} |\Gamma_\lambda(R_m t)| &= |\Gamma_\lambda(R_m |t| e^{i\phi})| \le e^{\left(\frac{\pi\cos\lambda\phi}{\sin\pi\lambda} + \alpha\right) R_m^\lambda |t|^\lambda} \\ &= e^{\left(\frac{\pi\cos\lambda\phi}{\sin\pi\lambda} + \alpha\right)\left(\frac{m\sin\pi\lambda}{\pi\lambda} + \frac{\sin\pi\lambda}{2\pi\lambda} + o(1)\right)|t|^\lambda} = e^{\frac{m}{\lambda}(\cos\lambda\phi + \beta)|t|^\lambda}, \end{aligned} \tag{5.8}$$

where β is sufficiently small, $m \geq m(\beta)$. Therefore, since $\Re(t^\lambda - \lambda \ln t - 1) = -\frac{h}{2}$ for $t \in l_2$, we have

$$\begin{aligned} |A_{22}| &= \left| \frac{w^m}{2\pi i} \int\limits_{l_2} \frac{\Gamma_\lambda(R_m t)}{t^m (t-w)}\, dt \right| \\ &\leq \mathit{Const} |w|^m \int\limits_{l_2} e^{\frac{m}{\lambda}(|t|^\lambda \cos \lambda\phi - \lambda \ln|t| + \beta |t|^\lambda)} d|t| \\ &= |w|^m o\left(e^{\frac{m}{\lambda}(1-\frac{h}{4})}\right). \end{aligned} \tag{5.9}$$

Case 1, Step 3. The estimation of A_{21} is more complicated than that of A_{22}. By (4.9),

$$\frac{\Gamma_\lambda(R_m t)}{t^m} = \frac{e^{\frac{m}{\lambda}} e^{\frac{m}{\lambda}(t^\lambda - \lambda \ln t - 1)}}{m^{\frac{1}{2\lambda}}} \left(\frac{\lambda e^{t^\lambda}}{2t^\lambda \sin(\pi\lambda)} \right)^{\frac{1}{2\lambda}} (1 + o(1)), \quad m \longrightarrow \infty,$$

where $t \in \Delta$. Thus, since $\Im(t^\lambda - \lambda \ln t - 1) = 0$ for $t \in l_1$ we have

$$A_{21} = \frac{w^m e^{\frac{m}{\lambda}} \lambda^{1/(2\lambda)}}{2\pi i (2 \sin(\pi\lambda) m)^{1/(2\lambda)}} \int\limits_{l_1} \left(\frac{e^{t^\lambda}}{t^\lambda} \right)^{1/(2\lambda)} \frac{e^{\frac{m}{\lambda}(|t|^\lambda \cos(\lambda \arg t) - \lambda \ln|t| - 1)} (1 + o(1))}{t - w}\, dt. \tag{5.10}$$

Further we use the following lemma.

Lemma 5.1 *Suppose that* $|w - 1| \geq \delta$ *and let* $p(t)$ *be analytic in some neighborhood of* $t = 1$. *Then for sufficiently small positive* h,

$$\int\limits_{l_1} \frac{e^{\frac{m}{\lambda}(t^\lambda - \lambda \ln t - 1)} p(t)}{t - w}\, dt = \frac{i\sqrt{2\pi} p(1)(1 + o(1))}{\lambda^{\frac{1}{2}} (1 - w) m^{1/2}}, \quad m \longrightarrow \infty.$$

P r o o f. Note that the function $v = -t^\lambda + \lambda \ln t + 1$ maps the region

$$\{t : |t - 1| < 1/2\} \cap \{t : |t|^\lambda \sin(\lambda \arg(t)) - \lambda \arg(t) > 0\}$$

conformally onto some neighborhood of 0 in the v-plane cut along the positive ray. Denote this neighborhood by U. In particular, the image of the curve l_1 is the segment $\left[0, \frac{h}{2}\right]$ traced twice, since

$$\Im(t^\lambda - \lambda \ln t - 1) = 0$$

for $t \in l_1$ and the end points z_i, $i = 1, 2$, of l_1 satisfy the condition $\Re(z_i^\lambda - \lambda \ln z_i - 1) = -\frac{h}{2}$, $i = 1, 2$. Rewrite

$$\int\limits_{l_1} \frac{e^{\frac{m}{\lambda}(t^\lambda - \lambda \ln t - 1)} p(t)}{t - w}\, dt = \int_{\mathcal{D}} e^{-\frac{m}{\lambda} v} f(v)\, dv,$$

where $v = -t^\lambda + \lambda \ln t + 1$, $f(v)\, dv = \frac{p(t)}{t-w}\, dt$, or equivalently, $f(v) = \frac{tp(t)}{\lambda(t-w)(1-t^\lambda)}$, and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, where $\mathcal{D}_1$ is the upper side of the segment $[0; h/2]$ following the direction of the decrease of v and $\mathcal{D}_2$ is the lower side of the segment $[0; h/2]$ following the direction of the increase of v.

The transformation $\chi = \sqrt{v}$ maps U onto $\{\chi : Im\chi > 0\} \cap V$ for some neighborhood V of the origin. We have

$$\chi^2 = -t^\lambda + \lambda \ln t + 1 = -\frac{\lambda^2}{2}(t-1)^2 \psi(t),$$

where $\psi(t)$ is an analytic function in some neighborhood of $t = 1$ and $\psi(1) = 1$. Then

$$\chi = \frac{\lambda}{\sqrt{2}} i(t-1)\psi_1(t),$$

where $\psi_1(t)$ is an analytic function in some neighborhood of $t = 1$ and $\psi_1(1) = 1$. Since χ is analytic in a neighborhood of $t = 1$ and $\chi'(1) = \frac{\lambda i}{\sqrt{2}} \neq 0$, the inverse function $t(\chi)$ is analytic in a neighborhood of $\chi = 0$, and hence the following function

$$g(\chi) := \chi f(\chi^2) = \frac{i(t-1)\psi_1(t)tp(t)}{\sqrt{2}(t-w)(1-t^\lambda)} = -\frac{i\psi_1(t)tp(t)}{\sqrt{2}\lambda(t-w)}(1+o(1)), \quad |t| \longrightarrow 1,$$

is analytic in some neighborhood of $\chi = 0$, say $|\chi| \leq C$, where C is a constant not depending on w. If $|\chi| < C/2$, then

$$\begin{aligned} g(\chi) &= \frac{1}{2\pi i} \int\limits_{|\zeta|=C} \frac{g(\zeta)}{\zeta - \chi}\, d\zeta \\ &= \frac{1}{2\pi i} \int\limits_{|\zeta|=C} \frac{g(\zeta)}{\zeta} d\zeta + \frac{\chi}{2\pi i} \int\limits_{|\zeta|=C} \frac{g(\zeta)}{\zeta(\zeta-\chi)} d\zeta \\ &= g(0) + \chi\alpha(\chi) \\ &= -\frac{ip(1)}{\sqrt{2}\lambda(1-w)} + \chi\alpha(\chi), \end{aligned}$$

where $\alpha(\chi)$ is a function analytic in $|\chi| < C/2$, and

$$|\alpha(\chi)| \leq \frac{2\pi C \max\limits_{|\zeta|=C} |g(\zeta)|}{2\pi C^2/2} \leq C_3,$$

where C_3 is a constant. This implies that $f(v) = g(0)v^{-1/2} + \alpha(v^{1/2})$ in some neighborhood of $v = 0$ cut along the positive ray. Let h be so small that $h/2 < \frac{C^4}{16}$. Then

$$\int\limits_{\mathcal{D}} e^{-\frac{m}{\lambda}v} f(v)\, dv = g(0) \int\limits_{\mathcal{D}} e^{-\frac{m}{\lambda}v} v^{-1/2}\, dv + \int\limits_{\mathcal{D}} e^{-\frac{m}{\lambda}v} \alpha(v^{1/2})\, dv =: g(0)J_1 + J_2.$$

Note that

$$J_2 = \int\limits_{\mathcal{D}} e^{-\frac{m}{\lambda}v} \alpha(v^{1/2})\, dv = O\left(\frac{1}{m}\right), \quad m \longrightarrow \infty,$$

and

$$J_1 = \int\limits_{\mathcal{D}} e^{-\frac{m}{\lambda}v} v^{-1/2}\, dv = \frac{1}{(m/\lambda)^{1/2}} \int\limits_{\frac{m}{\lambda}\mathcal{D}} e^{-u} u^{-1/2}\, du = -\frac{2\Gamma\left(\frac{1}{2}\right)(1+o(1))}{(m/\lambda)^{1/2}}, \quad m \longrightarrow \infty.$$

Thus,

$$\int\limits_{\mathcal{D}} e^{-\frac{m}{\lambda}v} f(v)\, dv = \frac{i\sqrt{2\pi}p(1)(1+o(1))}{\lambda(1-w)\left(\frac{m}{\lambda}\right)^{1/2}}, \quad m \longrightarrow \infty.$$

□

It follows from (5.10) and Lemma 5.1 that as $w \in G_1$,

$$A_{21} = \frac{w^m e^{\frac{m}{\lambda}}}{(2\pi)^{1/2}(1-w)} \left(\frac{\lambda^{1-\lambda} e}{2m^{\lambda+1}\sin(\pi\lambda)} \right)^{1/(2\lambda)} (1+o(1)), \quad m \longrightarrow \infty. \tag{5.11}$$

Therefore, by (5.7), (5.9) and (5.11), as $w \in G_1$,

$$A_2 = \frac{w^m e^{\frac{m}{\lambda}}}{(2\pi)^{1/2}(1-w)} \left(\frac{\lambda^{1-\lambda} e}{2m^{\lambda+1}\sin(\pi\lambda)} \right)^{1/(2\lambda)} (1+o(1)), \quad m \longrightarrow \infty,$$

and hence, due to (5.4), (5.3), (5.6), as $w \in G_1$

$$\begin{aligned} t_{m+1}(R_m w, \Gamma_\lambda) &= A_1 + A_2 + A_3 \\ &= \frac{w^{m+1} e^{\frac{m}{\lambda}}}{(2\pi)^{1/2}(1-w)} \left(\frac{\lambda^{1-\lambda} e}{2m^{\lambda+1}\sin(\pi\lambda)} \right)^{1/(2\lambda)} (1+o(1)) \\ &\quad + \frac{w^m e^{\frac{m}{\lambda}} e^{\frac{m}{\lambda}(w^\lambda - \lambda \ln w - 1)}}{m^{\frac{1}{2\lambda}}} \left(\frac{\lambda e^{w^\lambda}}{2w^\lambda sin(\pi\lambda)} \right)^{\frac{1}{2\lambda}} (1+o(1)), \quad m \longrightarrow \infty. \end{aligned} \tag{5.12}$$

It follows from (5.1), (4.9) and (5.12) that

$$I_m(R_m w, \mu, \Gamma_\lambda) = \begin{cases} -\dfrac{w^{m+1} e^{\frac{m}{\lambda}}}{(2\pi)^{1/2}(1-w)} \left(\dfrac{\lambda^{1-\lambda} e}{2m^{\lambda+1}\sin(\pi\lambda)} \right)^{1/(2\lambda)} (1+o(1)), & w \in III_h \cap G_1; \\ -\mu \dfrac{w^m e^{\frac{m}{\lambda}} e^{\frac{m}{\lambda}(w^\lambda - \lambda \ln w - 1)}}{m^{\frac{1}{2\lambda}}} \left(\dfrac{\lambda e^{w^\lambda}}{2w^\lambda sin(\pi\lambda)} \right)^{\frac{1}{2\lambda}} (1+o(1)), & w \in II_0. \end{cases} \tag{5.13}$$

Case 2). Suppose that $w \in G_2$. By (5.2),

$$\begin{aligned} t_{m+1}(R_m w, \Gamma_\lambda) &= A_1 + \frac{R_m^m w^m}{2\pi i} \oint\limits_{|\xi| = 2R_m|w|} \frac{\Gamma_\lambda(\xi)}{\xi^m(\xi - R_m w)}\, d\xi \\ &= A_1 + \frac{w^m}{2\pi i} \oint_{|t|=2|w|} \frac{\Gamma_\lambda(R_m t)}{t^m(t-w)} dt \\ &:= A_1 + A_4. \end{aligned} \tag{5.14}$$

To find an asymptotic expression for A_4, we will follow the same three steps that we did to find the asymptotic expression for integral A_2.

Case 2, Step 1. Note that the curve $S\big(\lambda, -\frac{h}{2}\big)$ intersects the circle $\big\{z : |z| = 1 - \frac{\delta}{8} =: d_2\big\}$ at two points, say $z_5 = d_2 e^{i\gamma_1}$ and $z_6 = d_2 e^{-i\gamma_1}$. We write

$$l_3 = S\big(\lambda, -\tfrac{h}{2}\big) \cap \{z : d_2 \le |z| \le d\}, \quad l_4 = \{z = d_2 e^{i\phi}, \gamma_1 \le \phi \le 2\pi - \gamma_1\},$$

where d is the same constant that we introduced while considering the curves l_1 and l_2.

We have,

$$\begin{aligned} A_4 &= \frac{w^m}{2\pi i} \oint\limits_{l_1 \cup l_3 \cup l_4} \frac{\Gamma_\lambda(R_m t)\, dt}{t^m(t-w)} \\ &= \frac{w^m}{2\pi i} \left(\int\limits_{l_1} + \int\limits_{l_3} + \int\limits_{l_4} \right) \frac{\Gamma_\lambda(R_m t)\, dt}{t^m(t-w)} \\ &=: A_{21} + A_{43} + A_{44}, \end{aligned} \tag{5.15}$$

where

$$A_{43} = \frac{w^m}{2\pi i} \int\limits_{l_3} \frac{\Gamma_\lambda(R_m t)\, dt}{t^m(t-w)} \quad \text{and} \quad A_{44} = \frac{w^m}{2\pi i} \int\limits_{l_4} \frac{\Gamma_\lambda(R_m t)\, dt}{t^m(t-w)}.$$

Case 2, Steps 2 and 3. Recall that the asymptotic expression for A_{21} was found in (5.11). The same arguments that we used to estimate integral A_{22} show that

$$A_{43} = w^m o\left(e^{\frac{m}{\lambda}(1-\frac{h}{4})}\right), \quad m \longrightarrow \infty. \tag{5.16}$$

Using the inequality (5.8), we have

$$|A_{44}| \le \mathit{Const}|w|^m \frac{e^{\frac{m}{\lambda}(\cos\lambda\gamma_1+\beta)|1-\frac{\delta}{8}|^\lambda}}{|1-\frac{\delta}{8}|^m} = |w|^m o\left(e^{\frac{m}{\lambda}(1-\frac{h}{4})}\right), \quad m \longrightarrow \infty, \tag{5.17}$$

for sufficiently small δ and h.

It follows from (5.15), (5.11), (5.16) and (5.17) that as $w \in G_2$,

$$A_4 = \frac{w^m e^{\frac{m}{\lambda}}}{(2\pi)^{1/2}(1-w)} \left(\frac{\lambda^{1-\lambda} e}{2m^{\lambda+1}\sin(\pi\lambda)}\right)^{1/(2\lambda)} (1+o(1)), \quad m \longrightarrow \infty, \tag{5.18}$$

and hence, by (5.14), (5.3), (5.18), as $w \in G_2$

$$\begin{aligned} t_{m+1}(R_m w, \Gamma_\lambda) &= A_1 + A_4 \\ &= \frac{w^{m+1} e^{\frac{m}{\lambda}}}{(2\pi)^{1/2}(1-w)} \left(\frac{\lambda^{1-\lambda} e}{2m^{\lambda+1}\sin(\pi\lambda)}\right)^{1/(2\lambda)} (1+o(1)), \quad m \longrightarrow \infty. \end{aligned} \tag{5.19}$$

It follows from (5.1), (4.9) and (5.19) that, as $m \to \infty$,

$$I_m(R_m w, \mu, \Gamma_\lambda) = \begin{cases} -\dfrac{w^{m+1} e^{\frac{m}{\lambda}}}{(2\pi)^{1/2}(1-w)} \left(\dfrac{\lambda^{1-\lambda} e}{2m^{\lambda+1}\sin(\pi\lambda)}\right)^{1/(2\lambda)} (1+o(1)), & w \in III_h \cap G_2; \\ (1-\mu)\dfrac{w^m e^{\frac{m}{\lambda}} e^{\frac{m}{\lambda}(w^\lambda - \lambda \ln w - 1)}}{m^{\frac{1}{2\lambda}}} \left(\dfrac{\lambda e^{w^\lambda}}{2w^\lambda sin(\pi\lambda)}\right)^{\frac{1}{2\lambda}} (1+o(1)), & w \in I_0. \end{cases} \tag{5.20}$$

Theorem 3.1 is an immediate corollary of (5.13) and (5.20).

It follows from (5.12) and (5.19) that $t_{m+1}(R_m w, \Gamma_\lambda)$ does not have zeros in $\{w : |w| \ge 1, |w-1| \ge \delta\}$, as $m \to \infty$. □

6 Proof of Theorem 3.3

We suppose that $\xi \in S(0,\lambda)$ and that $|\xi| < 1$. By (5.1), (4.9) and (5.19) we have,

$$\begin{aligned} &I_m(R_m \xi(1+\varepsilon_m(\zeta)), \mu, \Gamma_\lambda) \\ &= (1-\mu)\frac{e^{\frac{m\xi^\lambda(1+\varepsilon_m(\zeta))^\lambda}{\lambda}}}{m^{\frac{1}{2\lambda}}} \left(\frac{\lambda e^{\xi^\lambda(1+\varepsilon_m(\zeta))^\lambda}}{2\xi^\lambda(1+\varepsilon_m(\zeta))^\lambda \sin(\pi\lambda)}\right)^{\frac{1}{2\lambda}} (1+o(1)) \\ &\quad - \frac{\xi(1+\varepsilon_m(\zeta))\xi^m(1+\varepsilon_m(\xi))^m e^{\frac{m}{\lambda}}}{(1-\xi-\xi\varepsilon_m(\zeta))(2\pi)^{\frac{1}{2}}} \left(\frac{\lambda^{1-\lambda} e}{2m^{\lambda+1}\sin(\pi\lambda)}\right)^{\frac{1}{2\lambda}} (1+o(1)) \\ &= \frac{\xi^m(1+\varepsilon_m(\zeta))^m e^{\frac{m}{\lambda}}}{(2\pi)^{\frac{1}{2}}} \left(\frac{\lambda^{1-\lambda} e}{2m^{1+\lambda}\sin(\pi\lambda)}\right)^{\frac{1}{2\lambda}} (1+o(1)) \\ &\quad \times \left((1-\mu)\frac{e^{\frac{m}{\lambda}(\xi^\lambda(1+\varepsilon_m(\zeta))^\lambda - \lambda\ln(\xi(1+\varepsilon_m(\zeta))-1)} m^{\frac{1}{2}} e^{\frac{\xi^\lambda(1+\varepsilon_m(\zeta))^\lambda}{2\lambda}} (2\pi\lambda)^{\frac{1}{2}}}{\xi^{\frac{1}{2}}(1+\varepsilon_m(\zeta))^{\frac{1}{2}} e^{\frac{1}{2\lambda}}} - \frac{\xi(1+\varepsilon_m(\zeta))}{1-\xi-\xi\varepsilon_m(\zeta)}(1+o(1))\right) \\ &= \left((1-\mu)\left(\frac{2\pi\lambda}{\xi}\right)^{\frac{1}{2}} e^{\frac{\xi^\lambda-1}{2\lambda}} e^{\frac{m}{\lambda}(\xi^\lambda(1+\varepsilon_m(\zeta))^\lambda - \lambda\ln(\xi(1+\varepsilon_m(\zeta))-1+\frac{\lambda\ln m}{2m})}(1+o(1)) - \frac{\xi}{1-\xi}(1+o(1))\right) C, \end{aligned}$$

where

$$C = \frac{\xi^m (1+\varepsilon_m(\zeta))^m e^{\frac{m}{\lambda}}}{(2\pi)^{\frac{1}{2}}} \left(\frac{\lambda^{1-\lambda} e}{2m^{1+\lambda} \sin(\pi\lambda)} \right)^{\frac{1}{2\lambda}} .$$

Since $\xi = |\xi| e^{i\phi} \in S(\lambda, 0)$ and $\tau_m = \frac{m}{\lambda}\tau + 2\pi k$ for some k in $\mathbb{Z}$, we have

$$\begin{aligned}
&\xi^\lambda (1+\varepsilon_m(\zeta))^\lambda - \lambda \ln \xi - \lambda \ln(1+\varepsilon_m(\zeta)) - 1 + \frac{\lambda \ln m}{2m} \\
&\quad = \xi^\lambda - \lambda \ln \xi - 1 + \xi^\lambda ((1+\varepsilon_m(\zeta))^\lambda - 1) - \lambda \ln(1+\varepsilon_m(\zeta)) + \frac{\lambda \ln m}{2m} \\
&\quad = i(|\xi|^\lambda \sin \lambda\phi - \lambda\phi) + \xi^\lambda \left(\frac{\lambda \ln m}{2(1-\xi^\lambda)m} - \frac{\lambda\zeta - \lambda i \tau_m}{(1-\xi^\lambda)m} + o\left(\frac{1}{m}\right) \right) \\
&\qquad - \frac{\lambda \ln m}{2(1-\xi^\lambda)m} + \frac{\lambda\zeta - i\lambda\tau_m}{(1-\xi^\lambda)m} + \frac{\lambda \ln m}{2m} \\
&\quad = \frac{\lambda\zeta}{m} - i\frac{2\lambda\pi k}{m} + o\left(\frac{1}{m}\right)
\end{aligned}$$

for some integer k. Therefore, for $\xi \in S(\lambda, 0)$ and $|\xi| < 1$, as $m \to \infty$

$$\frac{I_m(R_m \xi(1+\varepsilon_m(\zeta)), \mu, \Gamma_\lambda)}{C} \longrightarrow (1-\mu) \left(\frac{2\pi\lambda}{\xi} \right)^{\frac{1}{2}} e^{\frac{\xi^\lambda - 1}{2\lambda}} e^\zeta - \frac{\xi}{1-\xi} .$$

We suppose that $\xi \in S(\lambda, 0)$ and that $|\xi| > 1$. Then, by (5.1), (4.9) and (5.12),

$$\begin{aligned}
&I_m(R_m \xi(1+\varepsilon_m(\zeta)), \mu, \Gamma_\lambda) \\
&\quad = -\mu \frac{e^{\frac{m\xi^\lambda (1+\varepsilon_m(\zeta))^\lambda}{\lambda}}}{m^{\frac{1}{2\lambda}}} \left(\frac{\lambda e^{\xi^\lambda (1+\varepsilon_m(\zeta))^\lambda}}{2\xi^\lambda (1+\varepsilon_m(\zeta))^\lambda \sin(\pi\lambda)} \right)^{\frac{1}{2\lambda}} (1+o(1)) \\
&\qquad - \frac{\xi(1+\varepsilon_m(\zeta)) \xi^m (1+\varepsilon_m(\xi))^m e^{\frac{m}{\lambda}}}{(1-\xi-\xi\varepsilon_m(\zeta))(2\pi)^{\frac{1}{2}}} \left(\frac{\lambda^{1-\lambda} e}{2m^{\lambda+1} \sin(\pi\lambda)} \right)^{\frac{1}{2\lambda}} (1+o(1)) .
\end{aligned}$$

The expression for

$$I_m(R_m \xi(1+\varepsilon_m(\zeta)), \mu, \Gamma_\lambda)$$

with $|\xi| > 1$ differs from the expression for

$$I_m(R_m \xi(1+\varepsilon_m(\zeta)), \mu, \Gamma_\lambda)$$

with $|\xi| < 1$ only by one coefficient, namely, instead of $(1-\mu)$ we have μ. The same calculations that were done for $I_m(R_m \xi(1+\varepsilon_m(\zeta)), \mu, \Gamma_\lambda)$ with $|\xi| < 1$ show that, as $m \to \infty$,

$$\frac{I_m(R_m \xi(1+\varepsilon_m(\zeta)), \mu, \Gamma_\lambda)}{C} \longrightarrow -\mu \left(\frac{2\pi\lambda}{\xi} \right)^{\frac{1}{2}} e^{\frac{\xi^\lambda - 1}{2\lambda}} e^\zeta - \frac{\xi}{1-\xi} .$$

This completes the proof of Theorem 3.3. □

7 Proof of Theorem 3.5

By (4.11), as $m \to \infty$,

$$\ln F\left(R_m \left(1 + \left(\frac{2}{\lambda m} \right)^{1/2} \zeta \right) \right) - \ln F(R_m) = \left(\frac{2}{\lambda m} \right)^{1/2} \zeta m + \frac{\zeta^2 (\lambda - 1 + o(1))}{\lambda} + o(1).$$

Hence,

$$\ln \frac{F\left(R_m\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right)\right)}{\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right)^m F(R_m)} = \zeta^2 + o(1), \quad m \longrightarrow \infty,$$

and then, by (1.7), as $m \to \infty$,

$$\begin{aligned}
&\frac{t_{m+1}\left(R_m\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right), F\right)}{\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right)^m F(R_m)} \\
&\quad = \frac{F\left(R_m\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right)\right)}{\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right)^m F(R_m)} - \frac{s_m\left(R_m\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right), F\right)}{\left(1+\left(\frac{2}{\lambda m}\right)^{1/2}\zeta\right)^m F(R_m)} \\
&\quad \longrightarrow \exp\{\zeta^2\} - \frac{1}{2}\exp\{\zeta^2\}\mathrm{erfc}(\zeta),
\end{aligned} \tag{7.1}$$

uniformly on every compact set of the ζ-plane. Theorem 3.5 follows immediately from (1.5), (1.7) and (7.1). □

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