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# On the Strain Feedback Control of a Flexible Robot Arm

Ömer Morgül

*Abstract*— **We consider a flexible robot arm modeled as a rigid hub which rotates in an inertial space; a light flexible link is clamped to the rigid body at one end and is free at the other. We assume that the flexible link performs only planar motion. We assume that the strain of the flexible link at the clamped end is measurable. We show that suitable control torques applied to the rigid hub stabilizes the system and achieves orientation under certain conditions. The proposed torque contains derivative, proportional and integral terms of the strain. The stability proofs depend on the passivity of the controller transfer function.**

**Keywords : Flexible robot arm, Boundary control, flexible structures, distributed parameter systems, Lyapunov functions, semigroup theory, direct strain control.**

## I. INTRODUCTION

The progress in space exploration and in fast rotating robot arms have resulted in the use of lightweight materials in similar mechanical structures. Such mechanical systems contain parts which can adequately be represented by partial differential equations due to flexibility. To achieve high performance requirements for such systems one has to take the effect of flexibility into account. Therefore in the last two decades there has been great interest in the modeling and control of such flexible structures.

Most of the flexible structures mentioned above contain both flexible and rigid parts. Hence, their motion is usually described by a set of coupled partial and ordinary differential equations. To analyze such systems, the common engineering approach is to obtain a finite dimensional model and to design a controller based on this model. Although such an approach simplifies the analysis, having established a control law based on such models does not always guarantee that the same control law will work on the original set of equations, e.g. due to the ignored "high frequency" dynamics, one might encounter the so-called "spillover" effects. Also, to represent the original dynamics adequately, the order of the finite dimensional model should be sufficiently large, and this increases the order of the controller.

In recent years, the boundary control of flexible systems, (i.e., controls applied to the *boundaries* of the flexible parts as opposed to the controls *distributed* over the flexible parts), has become an important research area. This idea was applied to the Euler-Bernoulli beam equation and it has been proven that, in a cantilever beam, a single actuator applied at the free end of the beam is sufficient to uniformly stabilize the beam deflections, [3]. Recently, the boundary control techniques has been applied to the stabilization of a flexible spacecraft performing planar motion, [17], [19], and three dimensional motion [16]. In the works cited above, the boundary controller is placed at the free end of the beam, which may not be easy in some applications. For rotating systems which consist of a rigid body and a flexible link clamped to it, an alternate approach would be to measure the strain of the flexible link at the clamped end and to apply a related control torque to the rigid body. This approach is called direct strain feedback, its implementation is quite easy and experimental results based on this approach are quite satisfactory, see e.g. [9], [10], [11], [14].For similar schemes, see e.g. [12], [13], [15], and the references therein .

In this paper we study the motion of a flexible robot arm clamped to a rigid hub at one end and is free at the other end. To control this structure, we assume that a control torque is applied to the rigid hub. Such a structure was investigated by many researchers, see e.g. [2], [5], [8], [9], [21], [23], [24]. Our approach here is closely related to that of the [9]. We apply various forms of direct strain feedback some of which were proposed in [9], and give stability results, which were not given in [9]. Our approach is based on the passivity of the proposed controller, see e.g. [18].

This paper is organized as follows. In the section 2, we give the equations of motion. For this system we pose certain problems related to the orientation and stabilization of the considered structure. To solve these problems, we propose a control law for the torque applied to the rigid hub, which is related to the strain of the link at the clamped end. This control law contains various strain terms, and depending on the coefficients multiplying these terms, the transfer function from the strain input to the torque generated will be positive real. By exploiting this property, in section 3 we prove various stability results. Finally we give some concluding remarks.

## II. PROBLEM STATEMENT

We consider a system which consists of a flexible link clamped to a rigid hub at one end and is free at the other end. For simplicity we assume that the center of mass of the rigid hub is fixed in an inertial frame and that the whole system performs planar motion. We assume that the link is initially straight and this configuration of the link is referred to as the reference configuration. Let $L$ be the length of the link , $Q$ be the point where the link is clamped to the rigid hub, $P$ be a link element whose distance from $Q$ in the reference configuration is $x$, $u$ be the vertical displacement of $P$. We assume that the link is inextensible and we use Euler-Bernoulli beam model. Neglecting gravitation, surface loads, rotatory inertia of the link cross-sections, nonlinear terms and

Ö. Morgül is with Department of Electrical Engineering, Bilkent University, 06800, Bilkent, Ankara, Turkey morgul@ee.bilkent.edu.tr

the dimension of the hub, the relevant equations of motion are :

$$\rho u_{tt} + EIu_{xxxx} + \rho \ddot{\theta} x = 0 \qquad 0 < x < L \quad , \tag{1}$$

$$I_R \ddot{\theta} = EIu_{xx}(0,t) + N(t) \quad , \tag{2}$$

$$u(0,t) = 0 \qquad , \qquad u_x(0,t) = 0 \quad , \tag{3}$$

$$EIu_{xxx}(L,t) = 0 \qquad , \qquad EIu_{xx}(L,t) = 0 \quad , \tag{4}$$

where a subscript letter denotes the partial differential with respect to the corresponding variable, a dot denotes time derivative, $\rho$ is the mass per unit length of the link, $EI$ is the flexural rigidity of the link, $\theta$ is the rotation angle of the rigid hub, $I_R$ is the moment of inertia of the rigid hub, $N(t)$ is the control torque applied to the rigid hub.

The equations (1)-(4) may model a robot arm with single flexible link, or a satellite with a flexible antenna, and have been studied in the past, see e.g. [2], [8], [9], [17] [24], etc. In [2], these equations are discretized and then a noncolocated control law is developed to control the structure and some experimental results are presented. In [8], a similar structure is considered and a control law based on LQR approach is given. In [17], simple feedback laws are proposed to control the structure. For further theoretical developments and experimental results, see e.g. [9], [23].

For the system given by (1)-(4) we now pose the following problems :

**orientation problem :** Consider the system given by (1)-(4). Let an angle $\theta_0 \in [0,2\pi)$ be given. Find appropriate control law for $N(t)$ such that the solutions $u(x,t), u_t(x,t)$ and $\theta(t)$ of (1)-(4) satisfy the following asymptotic relations :

$$\lim_{t\to\infty} u(x,t) = 0 \qquad , \qquad 0 \le x \le L \quad ,$$
$$\lim_{t\to\infty} u_t(x,t) = 0 \qquad , \qquad 0 \le x \le L \quad ,$$
$$\lim_{t\to\infty} \theta(t) = \theta_0 \quad ,$$
$$\lim_{t\to\infty} \dot{\theta}(t) = 0 \quad ,$$

where the angle $\theta_0$ is the desired orientation angle. □

**stabilization problem :** Consider the system given by (1)-(4). Find appropriate control law for $N(t)$ such that the solutions $u(x,t), u_t(x,t)$ and $\theta(t)$ of (1)-(4) satisfy the following asymptotic relations :

$$\lim_{t\to\infty} u(x,t) = 0 \qquad , \qquad 0 \le x \le L \quad ,$$
$$\lim_{t\to\infty} u_t(x,t) = 0 \qquad , \qquad 0 \le x \le L \quad ,$$
$$\lim_{t\to\infty} \dot{\theta}(t) = 0 \quad ,$$

□

We note that by definition, any solution to the orientation problem is also a solution to the stabilization problem, however the converse of this statement is not true. Hence, if our aim is only to solve the stabilization, but not the orientation problem, then a simpler control law may solve the problem.

The orientation problem stated above can be solved by using boundary controllers at the free end of the beam in addition to the torque control input $N(t)$. In [17], instead of (4), the following boundary controllers

$$EIu_{xxx}(L,t) = \alpha u_t(L,t) \quad , \quad -EIu_{xx}(L,t) = \beta u_{xt}(L,t) \quad , \tag{5}$$

and the following torque control laws were proposed :

$$\begin{aligned} N(t) \ &= L\alpha u_t(L,t) + \beta u_{xt}(L,t) - k_1 \dot{\theta}(t) \\ &\quad -k_2(\theta(t) - \theta_0) \quad . \end{aligned} \tag{6}$$

It was shown that for the system (1)-(3), (5),(6) and for $\alpha > 0,\ \beta \ge 0,\ k_1 > 0,\ k_2 > 0$, the orientation problem stated above is solved. If we set, $k_2 = 0$ then this control law solves the stabilization problem as well. This control law requires the use of boundary controllers at the free end of the beam. A more practical control law would be the use of the torque control only (i.e. $\alpha = \beta = 0$). It could be shown that for the system (1)-(4), (6) with $\alpha = \beta = 0$, the orientation problem is solved and the solutions decay to zero asymptotically; moreover if one assumes a damping for the beam (e.g. Kelvin-Voight damping), this decay is exponential. In this paper we consider a different torque control law which uses the direct strain feedback.

## III. STABILITY RESULTS

For the system given by (1)-(4), we propose the following control law :

$$\begin{aligned} N(t) \ &= k_d u_{xxt}(0,t) + (k_p - EI)u_{xx}(0,t) \\ &\quad + k_i \textstyle\int_0^t u_{xx}(0,s)ds - k_1 \dot{\theta}(t) - k_2(\theta(t) - \theta_0) \quad , \end{aligned} \tag{7}$$

where $k_d,\ k_p,\ k_i,\ k_1,\ k_2$ are positive constants. The term $u_{xx}(0,t)$ is called the strain and could be measured by using strain gauges. For the application of this type of control laws, see [9], [10], [13], [15].

We define the error angle $\theta_e$ and, following [10], a new variable $y(\cdot,t)$ as follows :

$$\theta_e(t) = \theta(t) - \theta_0 \quad , \quad y(\cdot,t) = u_{xx}(\cdot,t) \quad . \tag{8}$$

Note that since $\theta_0$ is a costant, we have $\dot{\theta} = \dot{\theta}_e$ and $\ddot{\theta} = \ddot{\theta}_e$. Let us define a new variable $r = L - x$. Then, assuming that $u$ is sufficiently differentiable, (1)-(4) and (7) could be written as :

$$\rho y_{tt}(r,t) + EIy_{rrrr}(r,t) = 0 \quad , \tag{9}$$

$$y(0,t) = 0 \quad , \quad y_r(0,t) = 0 \quad , \quad y_{rr}(L,t) = 0 \quad , \tag{10}$$

$$EIy_{rrr}(L,t) = \rho \ddot{\theta}_e(t) \quad , \tag{11}$$

$$\begin{aligned} I_R \ddot{\theta}_e \ &= k_d y_t(L,t) + k_p y(L,t) \\ &\quad + k_i \textstyle\int_0^t u_{xx}(0,s)ds - k_1 \dot{\theta}_e - k_2 \theta_e \quad . \end{aligned} \tag{12}$$

For details of derivation, see e.g. [10]. If our aim is not to control the orientation angle, but only to stabilize the beam deflections we could choose $k_p = k_i = k_1 = k_2 = 0$ in (12). Then by using (12) in (11), the latter becomes :

$$EIy_{rrr}(L,t) = \rho k_d / I_R y_t(L,t) \quad , \tag{13}$$

which is the standard boundary velocity feedback law, and it is known that the solutions of (9), (10) and (13) decay exponentially to zero, see [3], [18]. However, in the general case the relation between $y_{rrr}(L,t)$ and $y_t(L,t)$ is not as simple as (13). Following [18], we obtain this relation in frequency domain. By taking the Laplace transforms of (11)-(12), using zero initial conditions, we obtain

$$EI\hat{y}_{rrr}(L,s) = h(s)\hat{y}_t(L,s) \quad , \tag{14}$$

where a hat denotes the Laplace transform of the corresponding variable, $s$ is a complex variable and the transfer function $h(s)$ is :

$$h(s) = \rho \frac{k_d s^2 + k_p s + k_i}{I_R s^2 + k_1 s + k_2} \quad . \tag{15}$$

The stability of the closed loop system may be guaranteed if $h(s)$ is positive real (PR) or strictly positive real (SPR). Recall that a rational function $h(s)$ with real coefficients is said to be PR if

$$\mathscr{R}e\{h(s)\} \geq 0 \qquad \forall s, \quad \mathscr{R}e\{s\} \geq 0 \quad , \tag{16}$$

and is said to be SPR if $h(s-\varepsilon)$ is PR for some $\varepsilon > 0$. For details, see e.g. [22]. If $h(s)$ is SPR with $\mathscr{R}e\{h(j\omega)\} \geq \gamma > 0$, $\forall \omega \in \mathbf{R}$, then by a result of [18] it follows that the solutions of (9) exponentially decay to zero. Following this argument, we first find the conditions under which $h(s)$ is PR or SPR. By imposing the condition $\mathscr{R}e\{h(j\omega)\} > 0$, $\forall \omega \in \mathbf{R}$, we find that for the case $k_d > 0,\ k_p > 0,\ k_i > 0$, $h(s)$ is SPR if one of the following two conditions are satisfied :

$$k_1 k_p - k_i I_R - k_2 k_d > 0 \quad , \tag{17}$$

$$(k_1 k_p - k_i I_R - k_2 k_d)^2 < 4 k_i k_d k_2 I_R \quad . \tag{18}$$

We note that for $k_i = 0$, $h(s)$ could be at most PR provided that (17) is satisfied.

Note that we could analyze the system given by (9)-(12) directly, however finding an appropriate Lyapunov function turns out to be somehow complicated. An alternative approach would be to use the PR or SPR property of the transfer function $h(s)$ given by (15) and use the approach presented in [18]. The latter approach is natural since due to SPR property it will yield a natural Lyapunov function, see (29)-(31). Also note that since the controller given by (15) is finite dimensional, the results obtained by using this approach will be valid for the system given by (9)-(12), see e.g. [7]. Following the latter approach, now let us consider the system given by (9)-(10), (14)-(15). Let $(A,b,c,d)$ be a minimal realization of $h(s)$ given by (15), i.e. we have :

$$\dot{w} = Aw + b y_t(L,t) \quad , \tag{19}$$

$$f = c^T w + d y_t(L,t) \quad , \tag{20}$$

$$EI y_{rrr}(L,t) = f \quad , \tag{21}$$

where $w \in \mathbf{R}^n$ is the actuator state, $A \in \mathbf{R}^{n\times n}$ is a constant matrix, $b, c \in \mathbf{R}^n$ are constant column vectors, $d \in \mathbf{R}$ is a constant real number, the superscript $T$ stands for the transpose and $h(s) = d + c^T(sI - A)^{-1}b$. Since $h(s)$ is of second order, we have $n = 2$. Now consider the system given by (9), (10), (19)-(21). To analyze this system we first define the following spaces

$$\mathscr{H} := \{(u\ v)^T | u \in \mathbf{H}_0^2, v \in \mathbf{L}^2\} \quad , \tag{22}$$

where the spaces $\mathbf{L}^2$ and $\mathbf{H}_0^k$ are defined as follows :

$$\mathbf{L}^2 = \{f : [0,L] \to \mathbf{R} | \int_0^L f^2 dx < \infty\} \quad , \tag{23}$$

$$\mathbf{H}^k = \{f \in \mathbf{L}^2 | f, f', f'', \ldots, f^{(k)} \in \mathbf{L}^2\} \quad , \tag{24}$$

$$\mathbf{H}_0^k = \{f \in \mathbf{H}^k \mid \quad f(0) = f'(0) = 0\} \quad . \tag{25}$$

Let $\mathscr{H}_1 = \mathscr{H} \times R^n$. Then the equations (9), (10), (19)-(21) could be written in the following form :

$$\dot{z} = \mathscr{A} z \quad , \quad z(0) \in \mathscr{H}_1 \quad , \tag{26}$$

where $z = (y\ y_t\ w)^T \in \mathscr{H}_1$, the operator $\mathscr{A}\ :\ \mathscr{H}_1 \to \mathscr{H}_1$ is a linear unbounded operator defined as :

$$\mathscr{A} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ -\frac{EI}{\rho} u_{rrrr} \\ Aw + bv(L) \end{pmatrix} \quad , \tag{27}$$

and the domain $D(\mathscr{A})$ of $\mathscr{A}$ is defined as :

$$\begin{aligned} D(\mathscr{A}) = \{ & (u\ v\ w)^T | u \in \mathbf{H}_0^4,\ v \in \mathbf{H}_0^2, \quad w \in \mathbf{R}^n; \\ & -EI u_{rrr}(L) + c^T w + dv(L) = 0; \quad u_{rr}(L) = 0\ \} \quad . \end{aligned} \tag{28}$$

If $h(s)$ is SPR, then by Kalman-Yakubovich lemma there exist symmetric, positive definite matrices $P, Q \in \mathbf{R}^{n\times n}$ and a vector $q \in \mathbf{R}^n$ and a constant $\varepsilon > 0$ such that the following holds :

$$A^T P + PA = -qq^T - \varepsilon Q \quad , \tag{29}$$

$$Pb - c = \sqrt{2d} q \quad . \tag{30}$$

In case $d = 0$, one can take $\varepsilon = 1$ and $q = 0$; moreover if $h(s)$ is only PR, then $Q$ in (29) is only semi-definite, see e.g. [22, p. 132-133].

Let $P$ be the solution of (29)-(30). In $\mathscr{H}_1$ we define the following "energy" norm :

$$\begin{aligned} E(t) &= \|z(t)\|^2 = \tfrac{1}{2}\int_0^L \rho y_t^2 dx \\ &+ \tfrac{1}{2}\int_0^L EI y_{rr}^2 dx + \tfrac{1}{2} w^T P w \quad . \end{aligned} \tag{31}$$

We note that one can define an "energy" inner product which induces the norm given by (31). Hence, without loss of generality we assume that $\mathscr{H}_1$ is a Hilbert space.

**Remark 1 :** In analogy with the PID type controllers, we can associate $k_p$ with $P$, $k_i$ with $I$ and $k_d$ with $D$ type controllers. In the following theorem, we will consider *PID* and *PI* type controllers separately, since in these cases the transfer function $h(s)$ will be an SPR function, provided that (17) or (18) is satisfied. □

**Theorem 1 :** Consider the system given by (26). Let $k_1 > 0$, $k_2 > 0$ and let the nonnegative coefficients $k_d,\ k_p,\ k_i$ satisfy (17) or (18).

**i :** The operator $\mathscr{A}$ generates a $C_0$ semigroup of contractions $T(t)$ in $\mathscr{H}_1$ for the *PID* ($k_d > 0,\ k_p > 0,\ k_i > 0$) and the *PI* ($k_d = 0,\ k_p > 0,\ k_i > 0$) controller cases.

**ii :** For the *PID* controller case, the semigroup $T(t)$ is exponentially decaying, i.e. for some $M > 0$ and $\delta > 0$ the following holds

$$\|T(t)\| \leq M e^{-\delta t} \qquad \forall t \geq 0 \quad . \tag{32}$$

For the *PI* controller case, the system is asymptotically stable, hence the solutions $z(t)$ of (26) asymptotically decay to zero, i.e. $\lim_{t\to\infty} \|z(t)\| = 0$.

**Proof :**

**i :** For the *PID* controller case, the transfer function $h(s)$ is SPR with $\mathscr{R}e\{h(j\omega)\} \geq \gamma > 0,\ \forall \omega \in \mathbf{R}$, for some $\gamma > 0$. That $\mathscr{A}$ generates a $C_0$ semigroup of contractions now follows from [18]. For the *PI* controller case, by comparing (15) and (19), (20) we see that $d = 0$. Hence we can take $d = 0,\ q = 0,\ \varepsilon = 1$ in (29), (30); moreover, $h(s)$ is SPR , hence $Q$ in (29) is positive definite.

By differentiating (31), integrating by parts and by using (9), (10), (19)-(21), (29), (30) we obtain

$$\dot{E} = -w^T Q w \quad , \tag{33}$$

see [18] for similar calculations. This shows that $\mathscr{A}$ is dissipative. It is known that $\lambda I - \mathscr{A} : \mathscr{H}_1 \to \mathscr{H}_1$ is onto for $\lambda > 0$, see [18]. Hence by Lumer-Phillips theorem it follows that the operator $\mathscr{A}$ generates a $C_0$ semigroup of contractions $T(t)$ in $\mathscr{H}_1$ for these cases, see [15].

**ii :** For the *PID* controller case, the results follows from [18] since we have $\gamma > 0$. Now consider the *PI* controller case. It is known that the operator $(\lambda I - \mathscr{A})^{-1} : \mathscr{H}_1 \to \mathscr{H}_1$ is a compact operator for $\lambda > 0$, see [18]. Since by (33) the solutions of (26) are bounded in $\mathscr{H}_1$, it follows from LaSalle's invariance theorem that all solutions of (26) asymptotically tend to the maximal invariant subset of the following set :

$$\mathscr{S} = \{z \in \mathscr{H}_1 \mid \dot{E} = 0\} \quad , \tag{34}$$

where $\dot{E}$ is given by (33), see [15].

Since the matrix $Q$ in (33) is positive definite, it follows from $\dot{E} = 0$ that we have $w = 0$. From (19) and (20) it follows that $y_t(L,t) = 0$ and $f = 0$. By using separation of variables, it can easily be shown that with these boundary conditions, the only possible solution of (9), (10) and (21) is the zero solution. Therefore the set $\mathscr{S}$ given by (34) contains only the zero solution, and by LaSalle's invariance theorem, all solutions of (26) asymptotically tend to zero. □

**Remark 2 :** For the *P* controller ($k_d = 0,\ k_i = 0,\ k_p > 0$) and the *PD* controller ($k_d > 0,\ k_i = 0,\ k_p > 0$) cases, the approach given above could be still used in the same way. However, in these cases, since $h(s)$ is only PR, the matrix $Q$ in (33) is only positive semi-definite. Therefore, to determine the set $\dot{E} = 0$ given by (33), we need the structure of $Q$. This could be done by using a particular realization of $h(s)$ given by (15) for *P* and *PD* controller cases. Note that this will not change the generality of the results, since all minimal realizations are equivalent, see e.g. [7]. □

**Remark 3 :** It is well known that $h(s)$ is PR if and only if it is an impedance function of a passive RLC electrical circuit. In the realizations for the *P* and *PD* controller cases given below, we considered $y_t(L,t)$ as the input current, $f$ as the input voltage and obtained a passive RLC circuit realization of $h(s)$. The state space realizations which will be given in the following Theorem are the state equations for the corresponding "equivalent" electrical circuit. For details, see e.g. [4]. Accordingly, in Theorem 2, both the actuator realizations and the various coefficients actually obtained by constructing an electrical equivalent circuit whose input impedance function is equal to $h(s)$ and then by obtaining the state equations of the resulting electrical circuit. The coefficients L, R, C, etc., actually refers to various inductance, resistance and capacitance values in that circuit, see e.g. (39)-(44). □

Now we consider the *P* and *PD* controller cases mentioned in Remark 2.

**Theorem 2 :** Consider the system given by (26). Let $k_1 > 0,\ k_2 > 0$ and let the nonnegative coefficients $k_d,\ k_p,\ k_i$ satisfy (17) or (18).

**i :** The operator $\mathscr{A}$ generates a $C_0$ semigroup of contractions $T(t)$ in $\mathscr{H}_1$ for the *P* ($k_d = 0,\ k_p > 0,\ k_i = 0$) and the *PD* ($k_d > 0,\ k_p > 0,\ k_i = 0$) controller cases.

**ii :** For the *P* and *PD* controller cases, the system is asymptotically stable, hence the solutions $z(t)$ of (26) asymptotically decay to zero, i.e. $\lim_{t\to\infty} \|z(t)\| = 0$.

**Proof :**

**i :** Note that since $h(s)$ is PR, (29)-(30) still holds, but $Q$ is only positive semi-definite. By following the proof of Theorem 1, it follows that (33) also holds. Hence, following the arguments made in Theorem 1, it follows the operator $\mathscr{A}$ generates a $C_0$ semigroup of contractions $T(t)$ in $\mathscr{H}_1$ for the considered cases as well.

**ii :** Since $Q$ in (33) is only positive semi-definite, to conclude asymptotic stability by using LaSalle's invariance theorem, we need the structure of $Q$. For this, we need a special representation for $h(s)$. Note that this will not change the generality of the results, see Remark 2.

First we consider the *P* controller case. Such a special representation can be given as follows:

$$\dot{w}_1 = w_2 \quad , \tag{35}$$

$$\dot{w}_2 = -\frac{k_2}{I_R} w_1 - \frac{k_1}{I_R} w_2 + y_t(L,t) \quad , \tag{36}$$

$$f = \rho \frac{k_p}{I_R} w_2 \quad . \tag{37}$$

For this case, simple calculations show that (29) and (30) are satisfied by diagonal matrices $P = diag\{c_1, c_2\}$ and $Q = diag\{0, c_3\}$ where $c_1 = \rho k_p k_2 / I_R^2$, $c_2 = \rho k_p / I_R$ and $c_3 = 2\rho k_1 k_p / I_R^2$. Hence from (33) we have $w_2 = 0$ for $\dot{E} = 0$. It follows from (35)-(37) that $f = 0,\ w_1 = w_1(\infty),\ y_t(L,t) = k_2/I_R w_1(\infty)$ where $w_1(\infty)$ is a constant. By using separation of variables, it follows from (9), (10), (21) that only possible solution is zero solution. Hence it follows that $y_t(L,t) = 0$, and from (36) that $w_1(\infty) = 0$ as well. Therefore $\mathscr{S}$ contains only the zero solution, and by LaSalle's invariance theorem all solutions of (26) asymptotically tend to zero for the *P* controller case.

Next, we consider the *PD* controller case. This case is considered in [9], however our approach is different than the one used in there. Similar to the argument given above, we first give a special realization of for $h(s)$. Note that this will not change the generality of the results, see Remark 2. For this aim, we first define the following constants :

$$a = \frac{k_1 k_p - k_2 k_d}{k_p I_R} \qquad , \qquad b = \frac{k_p}{k_d} \quad . \tag{38}$$

It follows from (17) that $a > 0$. We consider the following cases :

**1 :** $a > b$. In this case, we first define the following quantities :

$$L_1 = \frac{\rho k_p}{k_2} \quad , \quad L_2 = \frac{\rho k_d}{I_R(a-b)} \tag{39}$$

$$R_1 = \frac{\rho k_d}{I_R} \quad , \quad R_2 = \frac{\rho k_d b}{I_R(a-b)} \quad . \tag{40}$$

Then a minimal realization of (19)-(21) is the following

$$\dot{w}_1 = -\frac{R_1}{L_1}w_1 - \frac{R_1}{L_1}w_2 + \frac{R_1}{L_1}y_t(L,t) \quad , \tag{41}$$

$$\dot{w}_2 = -\frac{R_1}{L_2}w_1 - \frac{R_1+R_2}{L_2}w_2 + \frac{R_1}{L_2}y_t(L,t) \quad , \tag{42}$$

$$f = -R_1 w_1 - R_1 w_2 + R_1 y_t(L,t) \quad . \tag{43}$$

In this case, we choose the energy function $E(t)$ given by (31) with a diagonal matrix $P$ as $P = diag\{L_1, L_2\}$, where $L_1$ and $L_2$ are given by (39). By differentiating (31), using (9), (10), (21), (41)-(43), and integrating by parts we obtain :

$$\dot{E} = -R_1(y_t(L,t) - w_1 - w_2)^2 - R_2 w_2^2 \quad , \tag{44}$$

where $R_1$ and $R_2$ are given by (40). Note that $\dot{E} = 0$ implies that $w_2 = 0$, $y_t(L,t) = w_1$. It follows from (43) that $f = 0$ and from (41) that $\dot{w}_1 = 0$, hence $w_1$ is constant. Similar to the theorem 1, it can be shown that the only possible solution of (9)-(10), (21) with these boundary conditions is the zero solution. Hence $w_1 = 0$ as well and the set $\mathscr{S}$ contains only the zero solution.

**2 :** $b > a$. In this case, we first define the following quantities :

$$C_2 = \frac{I_R}{\rho k_d(b-a)} \quad , \quad G_2 = \frac{I_R a}{\rho k_d(b-a)} \quad . \tag{45}$$

Then a minimal realization of (19)-(21) is the following :

$$\dot{w}_1 = -\frac{R_1}{L_1}w_1 + \frac{1}{L_1}w_2 + \frac{R_1}{L_1}y_t(L,t) \quad , \tag{46}$$

$$\dot{w}_2 = -\frac{1}{C_2}w_1 - \frac{G_2}{C_2}w_2 + \frac{1}{C_2}y_t(L,t) \quad , \tag{47}$$

$$f = -R_1 w_1 + w_2 + R_1 y_t(L,t) \quad . \tag{48}$$

In this case we choose the energy function $E(t)$ given by (31) with a diagonal matrix $P$ as $P = diag\{L_1, C_2\}$, where $L_1$ and $C_2$ are given by (39) and (45), respectively. By differentiating (31), using (9), (10), (21), (46)-(48), and integrating by parts we obtain :

$$\dot{E} = -R_1(y_t(L,t) - w_1)^2 - G_2 w_2^2 \quad , \tag{49}$$

where $R_1$ and $G_2$ are given by (39) and (45), respectively. by using similar arguments it can easily be shown that the set $\mathscr{S}$ contains only the zero solution. Therefore, from LaSalle's invariance theorem it follows that in all cases, the solutions of (26) asymptotically decay to zero.

**Case 3 :** $b = a$. This case can be treated similar to the ones given above and we obtain similar results. For brevity, it will not be included here. □

**Remark 4 :** The stability results given above are valid for the system (9)-(12). The original system (1)-(4) and (7) is related to this system by (8). It can easily be shown that similar results hold for the original system as well. However, since $y = u_{xx}$ and it is required that $(y(\cdot,0),\ y_t(\cdot,0))^T \in \mathscr{H}$, it follows that $u$ should be twice many differentiable in space variables, more precisely we require $(u(\cdot,0),\ u_t(\cdot,0))^T \in \hat{\mathscr{H}}$, where $\hat{\mathscr{H}}$ is

$$\hat{\mathscr{H}} := \{(u\ v)^T | u \in \mathbf{H}_0^4, v \in \mathbf{H}^2\} \quad ,$$

and the results of the theorems 1, 2 and the corollaries 1,2 will be valid for the original system provided that the function spaces are changed accordingly. □

**Remark 5 :** Theorems 1 and 2 show that for *PID*, *PD*, *PI* and *P* controller cases, the flexible vibrations for the system given by (9), (10), (19)-(21), hence for the original system given by (9)-(12), decay at least asymptotically to zero. However, from these results we cannot directly deduce the asymptotic behaviour of $\dot{\theta}_e$ and $\theta_e$. This will be done in the sequel. Note that $D$ controller case (i.e. $k_d > 0,\ k_p = k_i = 0$) cannot be analyzed with our approach, since in this case the transfer function $h(s)$ is not SPR or PR, see (17)-(18). □

Before analyzing the asymptotic behaviour of $\dot{\theta}_e$ and $\theta_e$, we first give the following simple corollary.

**Corollary 1 :** Consider the system given by (26). Let $k_1 > 0$, $k_2 > 0$ and let the nonnegative coefficients $k_d$, $k_p$, $k_i$ satisfy (17) or (18). Let $z(0) \in \mathscr{H}_1$, and $z(t) \in \mathscr{H}_1$ be the corresponding solution of (18). For the *PID*, *PD*, *PI* and *P* controller cases we have the following :

$$\int_0^\infty z(t)dt \in D(\mathscr{A}). \tag{50}$$

Hence $x(\infty)$ defined by the following limit exists :

$$x(\infty) = \lim_{t\to\infty} \int_0^t y(L,s)ds = \int_0^\infty y(L,t)dt \ < \ \infty. \tag{51}$$

**Proof :** This result follows easily from Theorem 1, 2, and from the fact that the resulting systems are asymptotically stable, see e.g. [20]. □

The next result is on the asymptotic behaviour of $\dot{\theta}_e$ and $\theta_e$.

**Corollary 2 :** Consider the system given by (9)-(12). Let the conditions in Theorem 1 and 2 be satisfied. Then :

**i :** For *PID* and *PI* controller cases, $\dot{\theta}_e(t)$ converges to zero and $\theta_e(t)$ converges to a constant.

**ii :** For the *P* and *PD* controller cases, both $\dot{\theta}_e(t)$ and $\theta_e(t)$ converges to zero.

**Proof :** Note that due to Theorem 1 and 2, we have $y_t(L,t) \to 0$ and $y(L,t) \to 0$ as $t \to \infty$. From (12) and (51), one can easily show that in all cases $\dot{\theta}_e(t)$ converges to zero and $\theta_e(t)$ converges to a constant. Moreover, we have

$$k_i x(\infty) = k_2 \theta_e(\infty) \quad . \tag{52}$$

In particular, when $k_i = 0$ (i.e. $I$ control is not present), then from the same analysis we conclude that $\theta_e(t)$ converges to zero as well. □

**Remark 6 :** From the above analysis we conclude that the asymptotical relation (52) holds as long as $k_i > 0$. Note that from (52) we expect that by choosing $k_i$ and $k_2$ appropriately, we may reduce $\mid \theta_e(\infty) \mid$. From Theorem 1, we see that *PID* controller case is the only one among the controllers considered here which guarantees exponential decay. On the other hand, in all our simulations we observed that $x(\infty) = \theta_e(\infty) = 0$. Whether this is always true or not remains as an interesting question. Also note that in $P$ control case, if we choose $k_p = EI$, it follows from (7) that the strain term does not appear in the expression of the control torque, i.e. it may be possible to asymptotically stabilize the system without measuring the strain and by using only orientation angle and angular velocity as feedback terms.

Finally we note that the coefficients have to satisfy the positive real conditions (17) or (18). □

## IV. Conclusion

In this paper we study the motion of a flexible robot arm modeled as a flexible link clamped to a rotating rigid hub at one end and is free at the other end. For simplicity we assumed that the system performs only planar motion. The system is controlled by a torque applied to the rigid hub. For this system we considered orientation and stabilization problems. To solve these problems we assumed that the strain of the flexible link at the clamped end can be measured and we apply a torque to the rigid hub which is related to the strain. Our approach is closely related to that of [9]. We considered various forms of strain feedback and proved various stability results. The stability proofs depends on the passivity of the controller used, see [18]. We note that such stability proofs are not given in [9]. The proposed control torque contains the derivative, proportional and integral terms of the strain of the link at the clamped end. The parameters multiplying these terms have to be positive and should satisfy some inequalities to ensure that the corresponding controller transfer function is positive real. We showed that if the integral term is not included then the orientation problem is asymptotically solved. However, if the integral term is present, then a steady state error for the orientation angle may occur. This may look like a disadvantage, but with the integral term it may be possible to prove the exponential stability for the flexible link (i.e. the energy of the flexible vibrations exponentially decay to zero). Also it may be possible to make the steady state error as small as desired by choosing the coefficient multiplying integral term sufficiently small. We also note that in our simulations this steady state error appears to be zero. Whether this is always true or not remains as an interesting question.

## References


[1] J. Baillieul and M. Levi, "Rotational Elastic Dynamics," *Physica D,* 27 D, 1987, pp.43-62.

[2] R. A. Cannon, and E. Schmitz, "Initial experiments on the end-point control of a flexible one-link robot," *Int. J. Robotics Res.,*, vol. 3, no. 3, 1984, pp. 62-75.

[3] G. Chen, M. C. Delfour, A. M. Krall and G. Payre, "Modelling, Stabilization and Control of Serially Connected Beams," *SIAM J. of Contr. and Opt.*, vol.25, 1987, pp.526-546.

[4] C. A. Desoer, and E. S. Kuh, *Basic Circuit Theory*, McGraw-Hill, 15th ed., 1985.

[5] H. Fujii, T. Ohtsuka and S. Udou, "Mission Function Control for a Slew Maneuver Experiment," *J. Guidance, Contr. Dynamics*, vol. 16, no. 5, 1991, pp. 986-992.

[6] D. Greenspan, and V. Casulli , *Numerical Analysis for Applied Mathematics, Science and Engineering*, Addison-Wesley, Reading, MA, 1988.

[7] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, 1980.

[8] B. B. King, "Modeling and Control of a Multiple Component Structure," *Journal of Mathematical Systems, Estimation and Control*, 4(4), 1994, pp. 1-36.

[9] Z. H. Luo, "Direct Strain Feedback Control of Flexible Robot Arms : New Theoretical and Experimental Results,", *IEEE Trans. on Auto. Contr.*, vol. 38, no. 11, 1993, pp. 1610-1623.

[10] Z. H. Luo and B. Z. Guo, "Exponential Stability of Some Differential Equations Arising from Control of Flexible Arms," *Proc. of 33rd CDC*, vol. 1, Florida, USA, 1994, pp. 757-762.

[11] Z. H. Luo and B. Z. Guo, "Further theoretical Results on Direct Strain Feedback Control of Flexible Robot Arms,", *IEEE Trans. on Auto. Contr.*, vol. 40, no. 4, 1995, pp. 747-751.

[12] Z. H. Luo, N. Kitamura, and B. Z. Guo, "Shear Force Feedback Control of Flexible Robot Arms ", *IEEE Trans. on Robot. Auto.*, vol. 11, no. 5, 1995, pp. 760-765.

[13] Z. H. Luo and B. Z. Guo, "Shear Force Feedback of a Single Link Flexible Robot with a Revolute Joint,", *IEEE Trans. on Auto. Contr.*, vol. 42, no. 1, 1997, pp. 53-65.

[14] Z. H. Luo and D. X. Feng, "Nonlinear Torque Control of a Single Link Flexible Robot,", *Journal of Robotic Systems* ., vol. 16, no. 1, 1999, pp. 25-35.

[15] Z. H. Luo, B. Z. Guo, Ö. Morgül, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer-Verlag, London, 1999.

[16] Ö. Morgül, " Control and Stabilization of a Flexible Beam Attached to a Rigid Body," *Int. J. Contr.*, vol. 51, 1990, pp. 11-33.

[17] Ö. Morgül, "Orientation and Stabilization of a Flexible Beam Attached to a Rigid Body : Planar Motion," *IEEE Trans. on Auto. Contr.*, vol 36, No. 8, 1991, pp. 953-963.

[18] Ö. Morgül, "Dynamic Boundary Control of a Euler-Bernoulli Beam," *IEEE Trans. on AC*, vol. 37, no. 5, 1992, pp. 639-642.

[19] Ö. Morgül, "Control and Stabilization of a Rotating Flexible Structure," *Automatica*, vol. 30, no. 2, 1994, pp. 351-356.

[20] V. Q. Phong, "On the Stability of $C_0$ Semigroups," *Proc. American Math. Soc.*, vol. 129, No. 10, 2001, pp. 2871-2879.

[21] Y. Sakawa, F. Matsuno and S. Fukushima, "Modelling and feedback control of a flexible arm," *J. Robotic Sys.,* vol. 2, no. 4, 1985, pp. 453-472.

[22] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*, Englewood Cliffs, Prentice Hall, 1991.

[23] D. Wang and M. Vidyasagar, "Control of a flexible beam for optimum step response," *Proc. IEEE Conf. Robotics and Automation,* 1987, pp. 1567-1572.

[24] C. Z. Xu and J. Baillieul, "Stabilizability and stabilization of a rotating body-beam system with torque control," *IEEE Trans. on Auto. Contr.,*, vol. 38, no. 12 1993, pp. 1754-1765.