SMOOTHNESS PROPERTIES OF GREEN'S FUNCTIONS

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July, 2014

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ABSTRACT

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Basic notions of potential theory are explained with illustrative examples. Many important properties, including the characteristic ones, of Green's functions that are defined by the help of equilibrium measures are given. It is discussed that for what kind of sets they are continuous. Then, it is analyzed how good their continuity can be, how smooth they can be. Examples are given for the optimal smoothness. Besides, many other examples with diverse moduli of continuity are considered. Recent developments and articles in this field are introduced in details. Finally, an open problem about finding a Cantor type set $K(\gamma)$ for preassigned smoothness of Green's function is presented.

Keywords: Green's functions, smoothness.

ÖZET

GREEN FONKSİYONLARININ PÜRÜZSÜZLÜK ÖZELLİKLERİ

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Potansiyel analizin en temel kavramları aydınlatıcı örneklerle açıklandı. Denge ölçüleri yardımıyla tanımlanan Green fonksiyonlarının karakteristik özelliklerini de içeren pek çok özelliği verildi. Ne tarz kümeler için sürekli olacakları ele alındı. Ardından süreklililiklerinin ne derece iyi olabiliceği, bir başka deyişle ne kadar pürüzsüz olabilecekleri incelendi. En pürüzsüz sürekliliğe sahip örnekler verildi. Ayrıca farklı türlerde süreklilik modülüne sahip örnekler de dikkate alındı. Bu alandaki son gelişmeler ve makaleler detaylı bir şekilde takdim edildi. Son olarak, pürüzsüzlüğü önceden belirlenmiş Green fonksiyonu için bir Cantor tipi küme olan $K(\gamma)$'yı bulma ile ilgili açık bir soru tanıtıldı.

Anahtar sözcükler: Green fonksiyonları, pürüzsüzlük.

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Chapter 1

Introduction

In applied mathematics, Green's functions are auxiliary functions in the solution of linear partial differential equations. The history of the Green's function dates back to 1828, when George Green (1793 - 1841) published a privately printed booklet in which he sought solutions of Poisson equation $\Delta u = f$ for the electric potential u defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined the "Green's function". This significant work was ignored until William Thomson (Lord Kelvin) discovered it, recognized its great value and had it published nine years after Green's death.

Green's functions are used to solve linear partial differential equations and defined as follows. A Green's function, $g(x, s)$, of a linear differential operator $L = L(x)$ is any solution of $Lg(x, s) = \delta(x-s)$ where δ is the Dirac delta function. This property of Green's function can be exploited to solve differential equations of the form $Lu(x) = f(x)$. Here is the motivation:

$$L\left(\int g(x,s)f(s)\,ds\right) = \int Lg(x,s)f(s)\,ds = \int \delta(x-s)f(s)\,ds = f(x) = Lu(x)$$

which suggests the solution $u(x) = \displaystyle\int g(x,s)f(s)\,ds$.

As shown above, Green's function is defined by a linear differential operator but the definition also relies on the *generalized function* δ. This brings to mind whether there is a possible relation between Green's functions and measures. Here we consider a Green's function for the Laplace operator in $\Omega \setminus \{z_0\}$, where Ω is a domain in $\mathbb{C}$, $z_0 \in \Omega$ is logarithmic pole of the Green's function. This function has a crucial role in numerous applications and in *Potential Theory*.

Potential theory originates from the study of gravitation by I. Newton, J. L. Lagrange, A. Legendre and P. S. Laplace in seventeenth and eighteenth centuries. The field of gravitational forces was called "potential field" by J. L. Lagrange. At last, this function was called only "potential" by C. F. Gauss. The term "potential theory" arose in 19th century physics, when it was realized that the fundamental forces of nature could be modeled using potentials which satisfy Laplace equation $\Delta u = 0$. From this point of view, potential theory is the study of harmonic functions. Since the close relationship between harmonic functions and real parts of complex analytic functions, potential theory of two variables on the plane is substantially a part of complex analysis.

Our aim is to analyze the smoothness of Green's functions. If a compact set $K \subset \mathbb{C}$ is *regular* with respect to the Dirichlet problem, then the *Green's function of K with pole at infinity* g_K is continuous throughout $\mathbb{C}$. We are interested in the analysis of the character of smoothness of g_K near the boundary of K. For example, if $K \subset \mathbb{R}$, then the monotonicity of the Green's function with respect to K implies that the best possible behaviour of g_K is $Lip\frac{1}{2}$ smoothness. Determining the character of smoothness sometimes needs a lot of work, hence we wish that Green's function has a rather simple form for calculations. However, there are only a few cases of that. Yet, some important properties of Green's functions make it possible to express them explicitly for some specific domains and, by conformal invariance of Green's functions, in conformal images of these domains. In addition, there are also special Cantor-type sets $K(\gamma)$ constructed in [1] as the intersection of the level domains for a certain sequence of polynomials depending on the parameter $\gamma = (\gamma_n)_{n=1}^{\infty}$ which provide a variety of Green's functions with diverse moduli of continuity.

Chapter 2

Introduction to Potential Theory

We will present the most basic concepts of potential theory which will provide us background information related to Green's functions. We follow here [2] and [3].

2.1 Harmonic Functions

Before going through potential theory, we need to study harmonic functions.

Definition 2.1.1. Let $D \subseteq \mathbb{R}^n$ be an open set and $u : D \to \mathbb{R}$ be a twice continuously differentiable function. Then u is said to be harmonic if it satisfies the Laplace equation in D

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

Example 2.1.2.

(a) $u(x) = |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ is not harmonic anywhere on $\mathbb{R}^n$ as $\Delta u = 2n$.

(b) $u(z) = x^2 - y^2$ is harmonic on $\mathbb{C}$.

(c) $u(z) = \log|z| = \log\sqrt{x^2 + y^2} = \frac{1}{2}\log(x^2 + y^2)$ is harmonic on $D = \mathbb{C} \setminus \{0\}$.

By the symmetric nature of the Laplace equation let look for a *radial* solution. That is, looking for a harmonic function u such that $u(x) = v(|x|) = v(r)$. Then

$$\frac{\partial u}{\partial x_i} = \frac{x_i}{|x|} v'(|x|), \quad |x| \neq 0,$$

which implies

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{1}{|x|} v'(|x|) - \frac{(x_i)^2}{|x|^3} v'(|x|) + \frac{(x_i)^2}{|x|^2} v''(|x|), \quad |x| \neq 0.$$

Therefore,

$$\Delta u = \frac{n-1}{|x|} v'(|x|) + v''(|x|) = \frac{n-1}{r} v'(r) + v''(r) = 0.$$

Hence

$$\frac{v''}{v'} = \frac{1-n}{r} \Rightarrow \log v' = (1-n)\log r + \log C \Rightarrow v' = \frac{C}{r^{n-1}},$$

which implies

$$v(r) = \begin{cases} c_1 \log r + c_2 & \text{if } n = 2 \\ \dfrac{c_1}{2-n} \dfrac{1}{r^{n-1}} + c_2 & \text{if } n \geq 3 \end{cases} \qquad r \neq 0;$$

equivalently

$$u(x) = \begin{cases} c_1 \log |x| + c_2 & \text{if } n = 2 \\ \dfrac{c_1}{2-n} \dfrac{1}{|x|^{n-1}} + c_2 & \text{if } n \geq 3 \end{cases} \qquad |x| \neq 0.$$

This indicates that the potential theory in two dimensions is different from the theory in higher dimensions, i.e., when we define *potentials*. Besides,

$$u(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x| & \text{if } n = 2, \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-1}} & \text{if } n \geq 3, \end{cases} \tag{2.1}$$

is called the *fundamental solution* of Laplace equation where $\alpha(n) = \dfrac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of n-dimensional unit ball, because (2.1) satisfies

$$-\Delta u = \delta$$

in the sense of distributions.

We will follow the two dimensional case in the rest exploiting the benefits of complex analysis.

Here are some classical results.

Theorem 2.1.3. *Let $f = u + iv$ be a holomorphic function on $D \subseteq \mathbb{C}$, then the real part $\Re(f) = u$ and the imaginary part $\Im(f) = v$ are harmonic on D.*

Sketch of the proof. Result follows from the Cauchy-Riemann equations.

Note that Example 2.1.2 (b) is the real part of $f(z) = z^2$. The converse of this theorem is also true if D is simply-connected.

Theorem 2.1.4. *Let D be a simply-connected open set in $\mathbb{C}$, and assume that h is harmonic on D. Then, there exists a holomorphic function f on D such that $\Re(f) = h$. Moreover, this function is unique up to a constant.*

Sketch of the proof. Existence follows from the Cauchy-Goursat theorem for the function $g := h_x - ih_y$ and uniqueness follows by the Cauchy-Riemann equations.

Note that the simple-connectedness condition on D is essential. Otherwise Example 2.1.2 (c) provides a counterexample when $D = \mathbb{C} \setminus \{0\}$.

Corollary 2.1.5. *If u is harmonic on an open set D, then $u \in C^\infty(D)$.*

Here is an important property of harmonic functions.

Theorem 2.1.6. *[2] [Mean Value Property] If u is harmonic in $|z - a| < r$ and continuous on its closure, then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

Sketch of the proof. This follows from Theorem 2.1.4 and the Cauchy Integral Formula.

This section ends with two further ways in which harmonic functions behave like holomorphic ones, an identity principle and a maximum principle. We deduce the harmonic versions of both these results from their holomorphic counterparts.

Theorem 2.1.7. *Let h and u be harmonic functions on a domain D in $\mathbb{C}$. If $h = u$ on a non-empty open subset of D, then $h = u$ throughout D.*

For holomorphic functions, a stronger form of identity principle holds: if two holomorphic functions agree on a set with a limit point in the domain, then they agree throughout the domain. However, this is not the case for harmonic functions. For instance, the function $h(z) = \Re(z)$ and $u(z) \equiv 0$ are both harmonic on $\mathbb{C}$ and agree on the imaginary axis without being equal on the whole of $\mathbb{C}$.

Theorem 2.1.8. *Let h be harmonic function on a domain D in $\mathbb{C}$.*

(a) *If h attains a local maximum on D, then h is constant.*

(b) *If h extends continuously to $\overline{D}$ and $h \leq 0$ on ∂D, then $h \leq 0$ on D.*

This is perhaps a timely moment for a reminder about our convention that all closures and boundaries are taken with respect to $\overline{\mathbb{C}}$ rather than $\mathbb{C}$. Indeed, part (b) would otherwise be false: consider, for example, the harmonic function $h(z) = \Re(z)$ on the domain $D = \{z \in \mathbb{C} : \Re(z) > 0\}$.

2.2 The Dirichlet Problem

The Dirichlet problem is to find a harmonic function on a domain with prescribed boundary values. It is one of the great advantages of harmonic functions over holomorphic ones that for "nice" domains, a solution always exists. This is a powerful tool with many application.

Here is the formal statement of the problem.

Definition 2.2.1. Let D be a subdomain of $\mathbb{C}$, and let $\phi : \partial D \to \mathbb{R}$ be a continuous function. The *Dirichlet problem* is to find a harmonic function h on D such that $\lim_{z \to \zeta} h(z) = \phi(\zeta)$ for all $\zeta \in \partial D$.

The question of uniqueness is easily settled.

Theorem 2.2.2. *The solution h to the Dirichlet problem is unique.*

Proof. Let h_1 and h_2 be solutions. Then $u = h_1 - h_2$ is harmonic on D, extends continuously to $\overline{D}$, and is zero on ∂D. Applying Theorem 2.1.8 to $\pm u$, we get $h_1 \equiv h_2$. $\square$

The question of existence of solutions to the Dirichlet problem is rather more delicate. However, there is one important special case that can be solved, namely when D is a disc. To this end, we make the following definition.

Definition 2.2.3.

(a) The *Poisson kernel* $P : \mathbb{D} \times \partial\mathbb{D} \to \mathbb{R}$ is defined by

$$P(z, \zeta) := \Re\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{1 - |z|^2}{|\zeta - z|^2}, \quad |z| < 1,\ |\zeta| = 1.$$

(b) If $D = B_\rho(w)$ and $\phi : \partial D \to \mathbb{R}$ is a Lebesgue-integrable function, then its *Poisson integral* $P_D\phi : D \to \mathbb{R}$ is defined by

$$P_D\phi(z) := \frac{1}{2\pi}\int_0^{2\pi} P\left(\frac{z - w}{\rho}, e^{i\theta}\right)\phi(w + e^{i\theta})\,d\theta, \quad z \in D.$$

More explicitly, if $r < \rho$ and $0 \le t < 2\pi$, then

$$P_D\phi(w + re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r cos(\theta - t) + r^2}\phi(w + \rho e^{i\theta})\,d\theta.$$

The Poisson kernel has several important properties.

Lemma 2.2.4. *The Poisson kernel P satisfies*

(i) $P(z, \zeta) > 0$,

(ii) $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})\,d\theta = 1$,

(iii) $\displaystyle\sup_{|\zeta - \zeta_0| \ge \delta} P(z, \zeta) \to 0$ *as* $z \to \zeta_0$ $\quad (|\zeta_0| = 1,\ \delta > 0)$.

The following result is fundamental.

Theorem 2.2.5. *[3] With the notation of the previous definition,*

(a) $P_D\phi$ *is harmonic on* D*,*

(b) *if* ϕ *is continuous at* $\zeta_0 \in \partial D$*, then* $\lim_{z\to\zeta_0} P_D\phi(z) = \phi(\zeta_0)$.

In particular, if ϕ *is continuous on the whole of* ∂D*, then* $h := P_D\phi$ *solves the Dirichlet problem on* D*.*

As an immediate consequence of this result, we obtain an analogue of the Cauchy integral formula for harmonic functions.

Corollary 2.2.6. *If* h *is harmonic on* $B_\rho(w)$ *and continuous on its closure, then for* $r < \rho$ *and* $0 \le t < 2\pi$*,*

$$h(w + re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r cos(\theta - t) + r^2} h(w + \rho e^{i\theta})\, d\theta.$$

Proof. Consider the Dirichlet problem on $D := B_\rho(w)$ with $\phi = h|_{\partial B_\rho(w)}$. Then h and $P_D h$ are both solutions, so by the uniqueness $h = P_D h$ on D. □

Note that this result is a generalization of the mean value property, which corresponds to the case $r = 0$. Moreover, it allows us to recapture the values of h everywhere on D, from knowledge of h on ∂D.

The mean value property actually characterizes harmonic functions.

Theorem 2.2.7. *[3] [Converse to Mean Value Property] Let* $h : D \to \mathbb{R}$ *be a continuous function on an open subset* D *of* $\mathbb{C}$*, and suppose that it possesses the* local mean value property*, i.e given* $w \in D$*, there exists* $\rho > 0$ *such that*

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{i\theta})\, d\theta, \quad 0 \le r < \rho.$$

Then h *is harmonic on* D*.*

2.3 Basic Measure Theory

Definition 2.3.1. Let X be any set and $\mathcal{P}(X)$ represents its power set. Then $\mathscr{A} \subseteq \mathcal{P}(X)$ is called σ-algebra if the following three properties hold:

(A1) $\emptyset \in \mathscr{A}$,

(A2) If $A \in \mathscr{A}$ then $X \setminus A \in \mathscr{A}$,

(A3) If $A_1, A_2, A_3, \ldots$ are in $\mathscr{A}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$.

From this definition, $\mathscr{A}$ is closed under the usual set operations and also countable intersections.

Example 2.3.2. Let X be any set then $\{\emptyset, X\}$ and $\mathcal{P}(X)$ are trivial σ-algebras on X.

Lemma 2.3.3. *Let $\mathcal{C}$ be a collection of σ-algebras. Then $\bigcap_{\mathscr{A} \in \mathcal{C}} \mathscr{A}$ is a σ-algebra.*

Proposition 2.3.4. *Let X be any set and $S \subseteq \mathcal{P}(X)$. Then there exists a smallest σ-algebra that contains S.*

Proof. Let consider the collection $\mathcal{C} = \{\mathscr{A} : \mathscr{A}$ *is a σ-algebra and* $S \subseteq \mathscr{A}\}$. $\mathcal{C}$ is a non-empty collection since $\mathcal{P}(X)$ is in $\mathcal{C}$. Thus, by the previous lemma $\bigcap_{\mathscr{A} \in \mathcal{C}} \mathscr{A}$ is a σ-algebra and is the smallest one containing S by its very definition. $\square$

This smallest σ-algebra is called the *σ-algebra generated by S*.

Definition 2.3.5. Let (X, τ) be a topological space. The *Borel* σ-algebra is defined as the σ-algebra generated by τ. It is denoted by $\mathscr{B}(X)$ for the notational simplicity if the topology is known.

Definition 2.3.6. Let $\mathscr{A}$ be a σ-algebra. A function $\mu : \mathscr{A} \to [0, +\infty]$ is called *positive measure* if it is *σ-additive*:

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{i=n}^{\infty} \mu(A_n)$$

for any sequence $(A_n)_{n=1}^{\infty}$ of pairwise disjoint sets in $\mathscr{A}$ and $\mu(\emptyset) = 0$.

Example 2.3.7.

(a) $\delta(A) = \begin{cases} 1 & \text{if } 0 \in A, \\ 0 & \text{if } 0 \notin A. \end{cases}$ on $\mathscr{A} = \mathcal{P}([-1, 1])$.

(b) Lebesgue measure λ on $\mathscr{A} = \mathcal{L}[-1, 1]$, Lebesgue measurable sets in $[-1, 1]$.

(c) Arc length measure θ on $\mathscr{A} = \mathscr{B}(\partial\mathbb{D})$ where $\partial\mathbb{D}$ is the unit circle.

(d) Arcsine measure α on $\mathscr{A} = \mathcal{L}[-1, 1]$ where $d\alpha = \dfrac{1}{\sqrt{1 - t^2}}\, dt$.

Definition 2.3.8. Let (X, τ) be a topological space. Any measure μ defined on $\mathscr{B}(X)$ is called *Borel measure*.

Definition 2.3.9. Any measure μ defined on a σ-algebra of X with $\mu(X) = 1$ is called *unit measure* or *probability measure*.

Definition 2.3.10. Let μ be a positive Borel measure. The support of μ denoted by $supp(\mu)$ consists of all points z such that every open neighborhood of z has positive measure. That is $supp(\mu) = \{z : 0 < \mu(\mathcal{N}_z), \quad \forall \mathcal{N}_z\}$.

From these definitions, it is easy to see that support of a measure is a closed set in the corresponding topology and also considering Example 2.3.7:

(a) δ is a unit Borel measure with $supp(\mu) = \{0\}$,

(b) λ is a unit Borel measure with $supp(\lambda) = [-1, 1]$,

(c) θ is a unit Borel measure with $supp(\theta) = \partial\mathbb{D}$,

(d) α is a unit Borel measure with $supp(\alpha) = [-1, 1]$.

2.4 Potential and Energy

Definition 2.4.1. Let K be an arbitrary compact set in $\mathbb{C}$ and $\mathcal{M}(K)$ denote the collection of all positive unit Borel measures which are supported in K. Then, the *logarithmic potential* associated with $\mu \in \mathcal{M}(K)$ is given by

$$U^{\mu}(z) = \int \log \frac{1}{|z-t|}\, d\mu(t).$$

This definition relies on the fundamental solution of Laplace equation, so the potentials in higher dimensions would not be logarithmic as mentioned before.

Observe that $-\infty < U^{\mu}(z) \leq +\infty$ since K is compact.

Note that, the measures in Example 2.3.7 can be considered as unit Borel measures after multiplying them by normalizing factors. Here are some examples.

Example 2.4.2. Let $K = [-1,1]$ and $\mu = \delta$. Then

$$U^{\delta}(z) = \int \log \frac{1}{|z-t|}\, \delta(t) = \log \frac{1}{|z|}.$$

Example 2.4.3. Let $K = [-1,1]$ and $\mu = \dfrac{1}{2}\lambda$ be the normalized Lebesgue measure. Then

$$\begin{aligned} U^{\lambda}(z) &= \frac{1}{2}\int_{-1}^{1} \log \frac{1}{|z-t|}\, dt \\ &= \frac{1}{2}\int_{-1-z}^{1-z} \log \frac{1}{|-r|}\, dr \\ &= -\frac{1}{2}\int_{-1-z}^{1-z} \log|r|\, dr \\ &= -\frac{1}{2}(r\log|r| - r)\Big|_{-1-z}^{1-z} \\ &= 1 - \frac{1}{2}[\,(1+z)\log|1+z| + (1-z)\log|1-z|\,]. \end{aligned}$$

Example 2.4.4. Let $K = \partial B_r(0)$ and $d\mu = \dfrac{1}{2\pi r}\, d\theta$ be the normalized arc length measure. Then

$$U^\theta(z) = \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z - re^{i\theta}|}\, d\theta.$$

If $r < |z|$, $u(t) = \log\dfrac{1}{|z-t|}$ is harmonic for $|t| \le r$. By the Mean Value Property,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z - re^{i\theta}|}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta = u(0) = \log\frac{1}{|z|}. \tag{2.2}$$

If $|z| < r$,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z - re^{i\theta}|}\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|ze^{-i\theta} - r|}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|\bar{z}e^{i\theta} - r|}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|r - \bar{z}e^{i\theta}|}\, d\theta \\
&= \log\frac{1}{|r|}
\end{aligned}$$

again by the Mean Value Property, changing the roles $z \to r$ and $r \to \bar{z}$ in (2.2).

If $|z| = r$,

$$U^\theta(z) = \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z - re^{i\theta}|} d\theta = \lim_{\rho\to r^-} \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z - \rho e^{i\theta}|} d\theta = \log\frac{1}{r}$$

by the dominated convergence theorem.

Thus,

$$U^\theta(z) = \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z - re^{i\theta}|}\, d\theta = \begin{cases} \log\dfrac{1}{r} & \text{if } |z| \le r, \\ \log\dfrac{1}{|z|} & \text{if } |z| > r. \end{cases}$$

Example 2.4.5. [4] Let $K = [-1,1]$ and $d\mu = d\alpha = \dfrac{1}{\pi}\dfrac{1}{\sqrt{1-t^2}}\,dt$ be the normalised arcsine measure. Then

$$U^\alpha(z) = \frac{1}{\pi}\int_{-1}^{1} \log\frac{1}{|z-t|}\frac{1}{\sqrt{1-t^2}}\,dt;$$

$t = \cos\theta$ yields

$$U^\alpha(z) = \frac{1}{\pi}\int_{0}^{\pi} \log\frac{1}{|z-\cos\theta|}\,d\theta. \tag{2.3}$$

Since $\cos\theta = \cos(-\theta)$

$$U^\alpha(z) = \frac{1}{\pi}\int_{-\pi}^{0} \log\frac{1}{|z-\cos\theta|}\,d\theta. \tag{2.4}$$

Adding (2.3) and (2.4),

$$U^\alpha(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\frac{1}{|z-\cos\theta|}\,d\theta.$$

Now we apply Joukowski transformation

$$z = \frac{1}{2}(\zeta + \zeta^{-1})$$

which maps $|\zeta| > 1$ onto $\mathbb{C}\setminus[-1,1]$ and maps the unit circle $|\zeta| = 1$ onto $[-1,1]$ (covered twice). Its inverse is $z+\sqrt{z^2-1}$ with $\sqrt{z^2-1}$ denoting the branch that behaves like z near infinity. Noting that $2\cos\theta = e^{i\theta}+e^{-i\theta}$ we compute

$$|z-\cos\theta| = \left|\frac{1}{2}(\zeta+\zeta^{-1}) - \frac{1}{2}(e^{i\theta}-e^{-i\theta})\right| = \frac{1}{2}|\zeta-e^{i\theta}||\zeta^{-1}-e^{i\theta}|.$$

Thus,

$$U^\alpha(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\frac{2}{|\zeta-e^{i\theta}||\zeta^{-1}-e^{i\theta}|}\,d\theta = \log 2 + U^\theta(\zeta) + U^\theta(\zeta^{-1})$$

from the preceding Example 2.4.4. Consequently,

$$U^\alpha(z) = \log 2 + \log\frac{1}{\zeta} + \log 1 = \log 2 - \log|z+\sqrt{z^2-1}|.$$

In particular, for $z \in [-1,1] = supp(\alpha)$, we have $U^\alpha(z) = \log 2$.

Observe that U^α and U^λ are harmonic on $\mathbb{C} \setminus [-1, 1]$, U^θ on $\mathbb{C} \setminus \partial B_r(0)$, and U^δ on $\mathbb{C} \setminus \{0\}$. The following theorem explains the general case.

Theorem 2.4.6. *[2] The potential U^μ is harmonic on $\mathbb{C} \setminus supp(\mu)$.*

Proof. Let $z_0 \notin supp(\mu)$ be fixed. Then there is an open ball $B_r(z_0)$ such that $supp(\mu) \cap B_r(z_0) = \emptyset$. There exists a branch L of the logarithm in $B_r(z_0)$. Both $L(z)$ and $\dfrac{1}{z-t}$ are analytic on $B_r(z_0)$ for $t \in supp(\mu)$. Thus $\log \dfrac{1}{|z-t|}$ is harmonic on $B_r(z_0)$ as the real part of analytic function $L\left(\dfrac{1}{z-t}\right)$. So we have

$$\Delta U^\mu(z) = \int \Delta \log \frac{1}{|z-t|}\, d\mu(t) = 0,$$

because all partial derivatives of $\log \dfrac{1}{|z-t|}$ are continuous and we integrate on a compact set K. Hence, U^μ is harmonic on $\mathbb{C} \setminus supp(\mu)$ since z_0 was arbitrary. $\square$

Now we give the definition of the (logarithmic) energy.

Definition 2.4.7. Let K be a compact subset of $\mathbb{C}$. Then the *logarithmic energy* $I(\mu)$ for $\mu \in M(K)$ is defined as

$$I(\mu) = \iint \log \frac{1}{|z-t|}\, d\mu(z)\, d\mu(t).$$

Since K is compact, there is an $R > 0$ such that for all $z, t \in K$, $|z-t| \leq R$. Then, $\log \dfrac{1}{R} \leq I(\mu)$ as μ is unit, so we find a uniform lower bound for all $I(\mu)$. That is, $-\infty < \log \dfrac{1}{R} \leq I(\mu)$ for all μ. This fact will be exploited later.

Also, notice that the energy can be computed by $\displaystyle\int U^\mu(z)\, dz$ thanks to Fubini-Tonelli theorem and can take $+\infty$ value.

Example 2.4.8. Let $K = [-1, 1]$ and $\mu = \delta$. Then by Example 2.4.2

$$I(\delta) = \int U^\delta(z)\, \delta(z) = \int \log \frac{1}{|z|}\, \delta(z) = +\infty.$$

Example 2.4.9. Let $K = [-1,1]$ and $\mu = \dfrac{1}{2}\lambda$. Then by Example 2.4.3

$$
\begin{aligned}
I(\lambda) &= \int U^{\lambda}(z)\,\frac{1}{2}d\lambda(z) \\
&= \int_{-1}^{1}\left\{1-\frac{1}{2}[\,(1+z)\log|1+z|+(1-z)\log|1-z|\,]\right\}\frac{1}{2}dz \\
&= 1-\frac{1}{4}\int_{-1}^{1}\{[\,(1+z)\log(1+z)+(1-z)\log(1-z)\,]\}\,dz \\
&= 1-\frac{1}{4}\int_{-1}^{1}(1+z)\log(1+z)\,dz-\frac{1}{4}\int_{-1}^{1}(1-z)\log(1-z)\,dz \\
&= 1-\frac{1}{4}\int_{0}^{2}u\log u\,du+\frac{1}{4}\int_{2}^{0}v\log v\,dv \\
&= 1-\frac{1}{4}\int_{0}^{2}u\log u\,du-\frac{1}{4}\int_{0}^{2}v\log v\,dv \\
&= 1-\frac{1}{2}\int_{0}^{2}u\log u\,du \\
&= 1-\frac{1}{2}\lim_{\epsilon\to 0^{+}}\int_{\epsilon}^{2}u\log u\,du \\
&= 1-\frac{1}{2}\lim_{\epsilon\to 0^{+}}\left(\frac{u^{2}}{2}\log u-\frac{u}{2}\Big|_{\epsilon}^{2}\right) \\
&= 1-\frac{1}{2}\left[2\log 2-1-\lim_{\epsilon\to 0^{+}}\left(\frac{\epsilon^{2}}{2}\log\epsilon-\frac{\epsilon}{2}\right)\right] \\
&= 1-\frac{1}{2}\left[2\log 2-1-\lim_{\epsilon\to 0^{+}}\frac{\epsilon^{2}}{2}\log\epsilon+\lim_{\epsilon\to 0^{+}}\frac{\epsilon}{2}\right] \\
&= 1-\frac{1}{2}(2\log 2-1) \\
&= \frac{3}{2}-\log 2.
\end{aligned}
$$

Example 2.4.10. Let $K = [-1,1]$ and $d\mu = d\alpha = \dfrac{1}{\pi}\dfrac{1}{\sqrt{1-t^2}}\,dt$. Then by Example 2.4.5

$$I(\alpha) = \int U^{\alpha}(z)\,d\alpha(z) = \frac{1}{\pi}\int_{-1}^{1}\log 2\,\frac{1}{\sqrt{1-t^2}}\,dt = \log 2.$$

Note that $I(\alpha) = \log 2 < I(\lambda) = \dfrac{3}{2} - \log 2 < I(\delta) = +\infty$.

Example 2.4.11. Let $K = \partial B_r(0)$ and $d\mu = \dfrac{1}{2\pi r}\,d\theta$. Then by Example 2.4.4

$$I(\theta) = \int U^{\theta}(z)\,d\mu(z) = \frac{1}{2\pi}\int_{0}^{2\pi}\log\frac{1}{r}\,d\theta = \log\frac{1}{r}.$$

2.5 Minimal Energy and Equilibrium Measures

We mentioned that each energy $I(\mu)$ of compact set K has a common lower bound, $\log\dfrac{1}{R} \leq I(\mu)$ where $R = diam(K)$. Thus the infimum always exists as a finite real number or $+\infty$.

Definition 2.5.1. Let $K \subset \mathbb{C}$ be compact. Then $V_K := \inf\{I(\mu) : \mu \in \mathcal{M}(K)\}$ is called the *the minimal energy* for K.

If $K = [-1,1]$ then our best candidate for $V_{[-1,1]}$ is $I(\alpha) = \log 2$ so far as noted above and $\log\dfrac{1}{2} \leq V_{[-1,1]} \leq \log 2$. If $K = \partial B_r(0)$, then our only candidate is $I(\theta) = \log\dfrac{1}{r}$ and we have $\log\dfrac{1}{2r} \leq V_{\partial B_r(0)} \leq \log\dfrac{1}{r}$.

Definition 2.5.2. Let (μ_n) be a sequence of finite positive measures with $supp(\mu_n) \subseteq K$ for all n where K is a compact set of $\mathbb{C}$. Then we say that μ_n converges to μ in weak star sense if

$$\lim_{n\to\infty}\int f\,d\mu_n = \int f\,d\mu, \quad \forall f \in C(K)$$

denoted by $\mu_n \xrightarrow{*} \mu$.

Lemma 2.5.3. *[5] If a sequence of measures* $(\mu_n) \subset \mathcal{M}(K)$ *converges to a measure* $\mu \in \mathcal{M}(K)$ *in weak star sense, then* $I(\mu) \leq \liminf_n I(\mu_n)$.

Proof. First, let us define

$$k^\eta(z) = \begin{cases} \log \frac{1}{\eta}, & \text{if } |z| \leq \eta, \\ \log \frac{1}{|z|}, & \text{if } |z| > \eta, \end{cases}$$

which is called a truncated kernel. In fact, it is the arc length potential on $B_\eta(0)$.

It has the following properties:

(i) $k^\eta \in C(\mathbb{C})$,

(ii) for $z \in \mathbb{C}$, $k^\eta(z) \leq \log \frac{1}{|z|}$,

(iii) for $z \in \mathbb{C}$, $k^\eta(z) \nearrow \log \frac{1}{|z|}$ as $\eta \searrow 0$.

By (i) we have

$$\lim_n \iint k^\eta(z-t)\, d\mu_n(z)\, d\mu_n(t) = \iint k^\eta(z-t)\, d\mu(z)\, d\mu(t).$$

Using (ii) we get

$$\iint k^\eta(z-t)\, d\mu(z)\, d\mu(t) \leq \liminf_n \iint \log \frac{1}{|z-t|}\, d\mu_n(z)\, d\mu_n(t),$$

so

$$\iint k^\eta(z-t)\, d\mu(z)\, d\mu(t) \leq \liminf_n I(\mu_n).$$

If we let $\eta \searrow 0$, then using the property (iii) and monotone convergence theorem we reach the inequality

$$I(\mu) \leq \liminf_n I(\mu_n).$$

□

Theorem 2.5.4. *[4] [Helly's Selection Theorem] If (μ_n) is a sequence of measures on a compact set K with bounded total mass $|\mu_n|(K)$, then we can select a weak star convergent subsequence.*

Definition 2.5.5. $K \subset \mathbb{C}$ is called a *polar set* if $I(\mu) = +\infty$ for each $\mu \in \mathcal{M}(K)$. A property is said to hold *quasi-everywhere* if it holds except on a polar set.

Example 2.5.6. Let $K = \{0\}$. Then the only measure in $\mathcal{M}(K)$ is δ. We know from Example 2.4.8 that $I(\delta) = +\infty$, hence K is polar.

In fact, countable sets and countable unions of Borel polar sets are again polar. Polar sets are *exceptional* sets, *small* sets, in the sense of potential theory. They are also small in the sense of measure as we will mention at the end of the next chapter by providing an example of uncountable polar set.

Theorem 2.5.7. *[4] Suppose $K \subset \mathbb{C}$ is not polar. Then there exists a measure $\mu_K \in \mathcal{M}(K)$ such that $I(\mu_K) = V_K$.*

Proof. Since V_K is infimum, there is a sequence $I(\mu_n)$ so that $\lim_{n\to\infty} I(\mu_n) = V_K$. Let consider the corresponding measure sequence (μ_n). This is clearly bounded by total mass 1. By Theorem 2.5.4, there exists a weak star convergent subsequence, say $\mu_{n_k} \xrightarrow{*} \mu_K$. Then by Lemma 2.5.3, we have $I(\mu_K) \le \liminf_{n_k} I(\mu_{n_k}) = V_K$. However, as an infimum, $V_K \le I(\mu_K)$. Thus, $I(\mu_K) = V_K$. □

Note that if K is polar then $I(\mu) = +\infty$ for each $\mu \in \mathcal{M}(K)$ hence $V_K = +\infty$ as infimum! So, we can take any measure in $\mathcal{M}(K)$ which is the trivial case.

Definition 2.5.8. Let K be a non-polar compact set in $\mathbb{C}$. Then any measure μ_K satisfying $I(\mu_K) = V_K$ is called an *equilibrium measure* of K.

This definition is a little bit tricky because it leaves an open door to worry that there may be many equilibrium measures even if K is non-polar. However, this is not the case and now we give two lemmas to prove the uniqueness of equilibrium measure on non-polar compact sets.

Lemma 2.5.9. *[4] Let μ and ν be Borel measures with finite energy and also $\mu(K) = \nu(K)$. Then, $0 \leq I(\mu - \nu)$ and $I(\mu - \nu) = 0$ if and only if $\mu = \nu$.*

Proof. For the proof, please look at p.32 of [4]. □

Lemma 2.5.10. *[5] If $\mu, \nu \in \mathcal{M}(K)$ with finite energy, then*

$$I(\mu - \nu) = 2I(\mu) + 2I(\nu) - 4I\left(\frac{\mu+\nu}{2}\right) \tag{2.5}$$

and

$$I\left(\frac{\mu+\nu}{2}\right) \leq \frac{I(\mu) + I(\nu)}{2}. \tag{2.6}$$

Proof. We have

$$I(\mu - \nu) = \iint \log \frac{1}{|z-t|} \, [d\mu(z) - d\nu(z)][d\mu(t) - d\nu(t)]$$

$$= I(\mu) + I(\nu) - \iint \log \frac{1}{|z-t|} \, d\mu(z) \, d\nu(t) - \iint \log \frac{1}{|z-t|} \, d\nu(z) \, d\mu(t)$$

and

$$4I\left(\frac{\mu+\nu}{2}\right) = I(\mu) + I(\nu) + \iint \log \frac{1}{|z-t|} \, d\mu(z) \, d\nu(t) + \iint \log \frac{1}{|z-t|} \, d\nu(z) \, d\mu(t).$$

From these equalities, we reach the following equality easily:

$$I(\mu - \nu) = 2I(\mu) + 2I(\nu) - 4I\left(\frac{\mu+\nu}{2}\right).$$

We can rewrite it as

$$I\left(\frac{\mu+\nu}{2}\right) = \frac{I(\mu) + I(\nu)}{2} - \frac{I(\mu - \nu)}{4}.$$

The inequality (2.6) follows from the Lemma 2.5.9 since $\mu, \nu \in \mathcal{M}(K)$. □

Theorem 2.5.11. *Equlibrium measure of a non-polar compact set is unique.*

Proof. Let μ and ν be equilibrium measures. Then by (2.6), $\dfrac{\mu+\nu}{2}$ is also an equilibrium measure. So by (2.5), $I(\mu-\nu)=0$ implies $\mu=\nu$ by Lemma 2.5.9. $\square$

Theorem 2.5.12. *[3] [Frostman's theorem] Let K be a non-polar compact set, and μ_K be the equilibrium measure of K. Then,*

(a) $U^{\mu_K}(z) \leq V_K$ *for all* $z \in \mathbb{C}$,

(b) $U^{\mu_K}(z) = V_K$ *on* $K \setminus E$, *where E is an F_σ polar subset of* ∂K.

Theorem 2.5.13. *[4] Let $\mu \in \mathcal{M}(K)$ with finite energy. If $U^{\mu}(z)$ is constant quasi-everywhere on supp(μ) and it is at least as large as this constant on K, then μ is the equilibrium measure, $\mu = \mu_K$.*

Frostman's theorem is also called the fundamental theorem of potential theory due to its importance determining the equilibrium measure. Frostman's theorem and Theorem 2.5.13 give us a criterion to find the equilibrium measure in most cases.

Example 2.5.14. Let $K=[-1,1]$ and consider $\mu=\delta$. Then $U^{\delta}(z)=\log\dfrac{1}{|z|}$ and *supp*(δ) $=\{0\}$ by Example 2.4.2. So, U^{δ} is constant on *supp*(δ) even being $+\infty$ but is unbounded on $\mathbb{C}$. Thus, δ is not the equilibrium measure of $[-1,1]$.

Example 2.5.15. Let $K=[-1,1]$ and consider $\mu=\dfrac{1}{2}\lambda$. Then

$$U^{\lambda}(z) = 1 - \frac{1}{2}[\,(1+z)\log|1+z| + (1-z)\log|1-z|\,]$$

and *supp*(λ) $=[-1,1]$ by Example 2.4.3. Also, $U^{\lambda}(z)=U^{\lambda}(-z)$ and $U^{\lambda}(0)=1$. Differentiating U^{λ} on $[-1,1]$ gives $2(U^{\lambda})'(x) = \log(1-x) - \log(1+x)$. The derivative $(U^{\lambda})'$ is equal to zero only at $x=0$, hence U^{λ} is not constant. Actually, (U^{λ}) increases on $[-1,0]$ and decreases on $[0,1]$ so it can take same values only for two points. Thus, λ is not the equilibrium measure of $[-1,1]$.

Example 2.5.16. Let $K = [-1, 1]$ and consider $d\mu = \frac{1}{\pi}\frac{1}{\sqrt{1-t^2}}\,dt$. Then

$$U^{\alpha}(z) = \log 2 - \log|z + \sqrt{z^2 - 1}|.$$

Observe that $U^{\alpha} \equiv \log 2$ on $[-1, 1]$ and $supp(\alpha) = [-1, 1]$ by Example 2.4.5. Also, $I(\alpha) = \log 2 < +\infty$ by Example 2.4.10. Thus, arcsine measure is the equilibrium measure of $[-1, 1]$.

Example 2.5.17. Let $K = \partial B_r(0)$ and $d\mu = \frac{1}{2\pi r}\,d\theta$. Then

$$U^{\theta}(z) = \begin{cases} \log \frac{1}{r} & \text{if } |z| \le r, \\ \log \frac{1}{|z|} & \text{if } |z| > r, \end{cases}$$

and $supp(\theta) = \partial B_r(0)$ by Example 2.4.4. It is also clear that $U^{\theta} \equiv \log \frac{1}{r}$ on $\partial B_r(0)$ and $I(\theta) = \log \frac{1}{r} < +\infty$ by Example 2.4.11. Thus, arc length measure is the equilibrium measure of $\partial B_r(0)$.

These examples shows that $V_{[-1,1]} = \log 2$ and $V_{\partial B_r(0)} = \log \frac{1}{r}$. So, our lucky(!) guesses at the beginning of this section turned out to be true.

Chapter 3

Green's Functions

We are just one step away to define and analyze our main object, Green's function. Here we introduce an important class of functions to be well equipped when dealing with properties of Green's functions.

3.1 Subharmonic Functions

Definition 3.1.1. Let (X, τ) be a topological space. We say that a function $u : X \to [-\infty, +\infty)$ is *upper semicontinuous* if the set $\{x \in X : u(x) < \alpha\}$ is open in X for each $\alpha \in \mathbb{R}$. Also, $v : X \to (-\infty, +\infty]$ is *lower semicontinuous* if $-v$ is upper semicontinuous.

It is clear by definition that u is upper semicontinuous if and only if

$$\limsup_{y \to x} u(y) \leq u(x), \quad \forall x \in X.$$

So, u is continuous if and only if u is both upper and lower semicontinuous.

Theorem 3.1.2. *Let u be an upper semicontinuous function on X, and let K be a compact subset of X. Then u is bounded above on K and attains its maximum.*

Proof. The sets $A_n = \{x \in X : u(x) < n\}$ form an open cover of K, so there exists a finite subcover by compactness of K. Hence, u is bounded above on K. Let $M = sup_K u$. Then the open sets $\{x \in X : u(x) < M - 1/n\}$ cannot cover K, because they have no finite subcover. Hence $u(x) = M$ for at least one $x \in K$. □

Definition 3.1.3. Let D be an open subset of $\mathbb{C}$. A function $u : D \to [-\infty, \infty)$ is called *subharmonic* if it is upper semicontinuous and satisfies the *local submean inequality*, i.e. given $w \in D$ there exits $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta})\, d\theta, \quad 0 \leq r < \rho. \tag{3.1}$$

Also, $v : D \to (-\infty, \infty]$ is *superharmonic* if $-v$ is subharmonic.

Note that a function is harmonic if and only if it is both subharmonic and superharmonic. This follows from the mean value property of harmonic functions and Theorem 2.2.7. A stronger version of this inequality is also true.

Theorem 3.1.4 (Global Submean Inequality)**.** *If u is a subharmonic function on an open set D in $\mathbb{C}$, and if $\overline{B}_\rho(w) \subset D$, then*

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + \rho e^{i\theta})\, d\theta.$$

Theorem 3.1.5. *If f is complex analytic on an open set D in $\mathbb{C}$, then* $\log|f|$ *is subharmonic on D.*

Proof. Evidently, $u := \log|f|$ is upper semicontinuous. Also, it satisfies the local submean inequality at each $w \in D$ for which $u(w) > -\infty$, because near such a point $\log|f|$ is actually harmonic. On the other hand, if $u(w) = -\infty$, then (3.1) is obvious anyway. □

Theorem 3.1.6. *Let K be compact set and $\mu \in \mathcal{M}(K)$. Then the corresponding potential $U^\mu(z) = \int \log \frac{1}{|z-t|}\, d\mu(t)$ is superharmonic on $\mathbb{C}$.*

Proof. By Theorem 3.1.5, $\log|z-t|$ is subharmonic on $\mathbb{C}$ and let fix $z \in \mathbb{C}$. Since K is compact, there exists $R > 0$ such that $|z-t| \le R$ for all $t \in K$. Then whenever $z_n \to z$, $\log R - \log|z_n - t|$ is non-negative for sufficiently large n thus by the Fatou's lemma,

$$\int (\log R - \log|z-t|)\, d\mu(t) \le \liminf_n \left(\int \log R - \log|z_n - t|\, d\mu(t) \right).$$

Since μ is unit and $\lim_n \int \log R\, d\mu(t)$ exists, we have

$$\log R + \int \frac{1}{\log|z-t|}\, d\mu(t) \le \log R + \liminf_n \left(\int \frac{1}{\log|z_n - t|}\, d\mu(t) \right),$$

so

$$U^\mu(z) \le \liminf_n U^\mu(z_n).$$

Hence, U^μ is lower semicontinuous. Let $w \in \mathbb{C}$ be fixed. Since $\log|z-t|$ is subharmonic on $\mathbb{C}$, for any $\rho > 0$, using the Fubini's Theorem and the Global Submean Inequality, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} -U^\mu(w+\rho e^{i\theta})\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \left(\int \log|w+\rho e^{i\theta} - t|\, d\mu(t) \right) d\theta \\
&= \int \left(\frac{1}{2\pi}\int_0^{2\pi} \log|w - t + \rho e^{i\theta}|\, d\theta \right) d\mu(t) \\
&\ge \int \log|w-t|\, d\mu(t) \\
&= -U^\mu(w).
\end{aligned}$$

So, $-U^\mu$ satisfies the local submean inequality. Thus, U^μ is superharmonic. □

Theorem 3.1.7 (Maximum Principle)**.** *Let u be a subharmonic function on a domain D in $\mathbb{C}$.*

(a) *If u attains a global maximum on D, then u is constant.*

(b) *If $\limsup_{z\to\zeta} u(z) \leq 0$ for all $\zeta \in \partial D$, then $u \leq 0$ on D.*

Note that in part (a), u can attain local maximum or a global minimum without constant on D. For example, $u(z) = \max\{\mathcal{R}(z), 0\}$ does both on $\mathbb{C}$. Also, the validity of part (b) depends on our convention that $\infty \in \partial D$ whenever D is unbounded.

Proof. (a) Suppose that u attains a maximum value M on D. Define

$$A = \{z \in D : u(z) < M\} \quad \text{and} \quad B = \{z \in D : u(z) = M\}.$$

Then A is open because u is upper semicontinuous. Also B is open, because if $u(w) = M$, then the local submean inequality forces u to be equal to M on all sufficiently small circles round w. Clearly A and B partition D, so since D is connected, either $A = D$ or $B = D$. By assumption $B \neq \emptyset$ hence $B = D$.

(b) Extend u to ∂D by defining $u(\zeta) = \limsup_{z\to\zeta} u(z)$ for $\zeta \in \partial D$. Then u is upper semicontinuous on $\overline{D}$, which is compact in $\overline{\mathbb{C}}$, so by Theorem 3.1.2 u attains the maximum at some $w \in \overline{D}$. If $w \in \partial D$, then by assumption $u(w) \leq 0$, hence $u \leq 0$ on D. On the other hand, if $w \in D$, then by part (a) u is constant on D, hence on $\overline{D}$, and so again $u \leq 0$ on D. □

Theorem 3.1.8. *[3] [Extended Maximum Principle] Let D be domain in $\mathbb{C}$, and let u be a subharmonic function on D which is bounded above.*

(a) *If ∂D is polar, then u is constant.*

(b) *If ∂D is non-polar and $\limsup_{z\to\zeta} u(z) \leq 0$ q.e. on ∂D, then $u \leq 0$ on D.*

3.2 Green's Function

Now we are ready to give the definition of Green's function.

Definition 3.2.1. Let K be a non-polar compact subset of $\mathbb{C}$. Then we define the *Green's function for* K *with pole at* ∞ as

$$g_K(z) = g_\Omega(z, \infty) = V_K - U^{\mu_K}(z)$$

where U^{μ_K} is the equilibrium potential with the equilibrium measure μ_K, V_K is the minimal energy and Ω is the unbounded component of $\overline{\mathbb{C}} \setminus K$.

Example 3.2.2. Let $K = [-1, 1]$. Then, μ_K is the arcsine measure by Example 2.5.16. Hence $U^\alpha(z) = \log 2 - \log |z + \sqrt{z^2 - 1}|$ by Example 2.4.5 and $V_{[-1,1]} = \log 2$ by Example 2.4.10. Thus,

$$g_{[-1,1]}(z) = \log |z + \sqrt{z^2 - 1}|$$

with $\sqrt{z^2 - 1}$ denoting the branch that behaves like z near infinity.

Example 3.2.3. Let $K = \partial B_r(0)$. Then, μ_K is the arc length measure by Example 2.5.17. Hence

$$U^\theta(z) = \begin{cases} \log \dfrac{1}{r} & \text{if } |z| \leq r, \\ \log \dfrac{1}{|z|} & \text{if } |z| > r, \end{cases}$$

by Example 2.4.4 and $V_{\partial B_r(0)} = \log \frac{1}{r}$ by Example 2.4.11. Thus,

$$g_{\partial B_r(0)}(z) = \begin{cases} 0 & \text{if } |z| \leq r, \\ \log \dfrac{|z|}{r} & \text{if } |z| > r. \end{cases}$$

These are the only examples we can give now, because we don't know the equilibrium measures for more complicated sets. But this won't stop us. We will study the properties of Green's function and obtain some characterizations without equilibrium measures.

Here are the basic properties of Green's function:

(i) g_K is non-negative on $\mathbb{C}$ by Frostman's Theorem.

(ii) g_K is harmonic on $\mathbb{C} \setminus K$ by Theorem 2.4.6 since $supp(\mu_K) \subset K$.

(iii) g_K is subharmonic on $\mathbb{C}$ by Theorem 3.1.6.

(iv) $\lim_{z\to\infty} g_K(z) - \log|z| = V_K$. Note that μ_K is unit, so

$$g_K(z) - \log|z| = V_K - U^{\mu_K}(z) - \log|z|$$

$$= V_K - \int \log \frac{1}{|z-t|}\, d\mu_K(t) - \int \log|z|\, d\mu_K(t)$$

$$= V_K + \int \log \left| \frac{z-t}{z} \right| \, d\mu_K(t).$$

Since $\log \left| \frac{z-t}{z} \right| \to 0$ uniformly as $z \to \infty$, we get the limit.

(v) $g_K(z) - \log|z|$ is bounded around ∞ by (iv).

(vi) g_K is bounded as z stays away from ∞. In fact, g_K is bounded on every compact subset of $\mathbb{C}$. g_K is non-negative by (i), so it is enough to show that g_K is bounded above. But it is clear from Theorem 3.1.2 since g_K is upper semicontinuous as being subharmonic.

(vii) $g_K \equiv 0$ quasi-everywhere on K by Frostman's Theorem.

(viii) $\lim_{z\to\zeta, z\in\Omega} g_\Omega(z,\infty) = 0$ for q.e. $\zeta \in \partial\Omega$. Note that U^{μ_K} is superharmonic, so

$$U^{\mu_K}(\zeta) \le \liminf_{z\to\zeta} U^{\mu_K}(z) \le \liminf_{z\to\zeta, z\in\Omega} U^{\mu_K}(z) \le \limsup_{z\to\zeta, z\in\Omega} U^{\mu_K}(z) \le V_K$$

for all $\zeta \in \partial\Omega$. It is clear that if $U^{\mu_K}(\zeta) = V_K$, then $\lim_{z\to\zeta, z\in\Omega} U^{\mu_K}(\zeta) = V_K$ and we get the limit. But we have already $U^{\mu_K}(\zeta) = V_K$ quasi-everywhere on $\partial\Omega$ by Frostman's Theorem (see Theorem 3.2.7).

We will see that some of the above properties are enough to define Green's function uniquely.

Theorem 3.2.4. *Let $\Omega \subset \overline{\mathbb{C}}$ be a proper subdomain containing the point at infinity and $\overline{\mathbb{C}} \setminus \Omega$ is non-polar. Then the Green's function of Ω with pole at ∞ is the unique function $g_\Omega(z,\infty)$ with the following properties:*

(i) $g_\Omega(z,\infty)$ *is harmonic on* $\Omega \setminus \{\infty\}$ *and is bounded as z stays away from ∞,*

(ii) $g_\Omega(z,\infty) - \log|z|$ *is bounded around ∞,*

(iii) $\lim\limits_{z\to\zeta, z\in\Omega} g_\Omega(z,\infty) = 0$ *for q.e.* $\zeta \in \partial\Omega$.

Proof. Let $K = \overline{\mathbb{C}} \setminus \Omega$. Then K is a non-polar compact set. Hence existence follows from g_K. Assume g_1 and g_2 satisfy the conditions, consider $h = g_1 - g_2$. Since $|h| \le |\, g_1 - \log|z|\,| + |\, g_2 - \log|z|\,|$ and $|h| \le |g_1| + |g_2|$, h is bounded and harmonic on $\Omega \setminus \{\infty\}$ with zero boundary limit quasi-everywhere. Hence by the Extended Maximum Principle (applied to h and $-h$) we obtain $g_1 \equiv g_2$. □

Since Green's functions are 0 quasi-everywhere on K by Frostman's Theorem, we consider their expressions outside the set K.

Example 3.2.5. Let K be a compact set such that $K = \{z \in \mathbb{C} : |P(z)| \le 1\}$ where $P(z) = a_n z^n + \cdots + a_0$. Then

$$g_K(z) = \frac{1}{n}\log|P(z)|\,, \quad \forall z \notin K.$$

(i) is satisfied since $P(z)$ is complex analytic and has all its zeros in K.

(ii) follows from $\frac{1}{n}\log|P(z)| = \log|z| + \frac{1}{n}\log|a_n| + o(z)$. In fact, $\lim\limits_{z\to\infty} \frac{1}{n}\log|P(z)| - \log|z| = \frac{1}{n}\log|a_n|$ implying $V_K = \frac{1}{n}\log|a_n|$.

(iii) holds since $\partial\Omega \subset \partial K = \{z \in \mathbb{C} : |P(z)| = 1\}$ and $P(z)$ is continuous.

Example 3.2.6. Let $P(z) = z$ then $K = \overline{\mathbb{D}}$ and $g_{\overline{\mathbb{D}}}(z) = \log|z|$.

Note that we also have $g_{\partial\mathbb{D}}(z) = \log|z|$ by Example 3.2.3. This is not a by chance as the following theorem shows.

Theorem 3.2.7. *[4] If K is a non-polar compact set and Ω is the unbounded component of $\overline{\mathbb{C}} \setminus K$, then the equilibrium measure μ_K of K is the same as the equilibrium measure $\mu_{\partial\Omega}$ of $\partial\Omega$. In particular, μ_K is supported on $\partial\Omega$.*

This theorem is related to another important concept of potential theory, *balayage measures*. Let $G \subset \overline{\mathbb{C}}$ be an open set such that ∂G is a non-polar compact subset of $\mathbb{C}$. Let ν be a finite Borel measure on G with $\nu(\overline{\mathbb{C}} \setminus G) = 0$. The *balayage* (or *sweeping out*) problem consists of finding another measure $\hat{\nu}$ supported on ∂G such that $||\hat{\nu}|| = ||\nu||$, where $||\cdot||$ denotes the total mass, and the potentials U^{ν} and $U^{\hat{\nu}}$ coincide on ∂G quasi-everywhere. Here $\hat{\nu}$ is called the *balayage measure* associated with ν when *sweeping out* ν from G onto ∂G.

Regarding Example 3.2.5, the more is true for the representation of Green's function.

Theorem 3.2.8. *[6] [Bernstein-Walsh] Let $K \subset \mathbb{C}$ be a non-polar compact set. Then, for any polynomial P of degree n, we have*

$$|P(z)| \leq \exp[ng_K(z)]\,||P||_K, \quad \forall z \in \mathbb{C}$$

where $||P||_K = \sup_{z \in K} |P(z)|$.

Corollary 3.2.9. *Let $K \subset \mathbb{C}$ be a non-polar compact set. Then,*

$$g_K(z) = \sup\left\{\frac{\log|P(z)|}{\deg P} : P \in \mathcal{P},\, \deg P \geq 1,\, ||P||_K \leq 1\right\}$$

where $\mathcal{P}$ is the class of all polynomials.

The notion of a Green's function with pole at some finite point a is similar. Let again $D \subset \overline{\mathbb{C}}$ be a domain such that ∂D is non-polar and a be a finite point in D. The Green's function $g_D(z,a)$ of D with pole at a is defined as the unique function on D satisfying the following properties:

(i) $g_D(z,a)$ is harmonic on $D \setminus \{a\}$ and is bounded as z stays away from a,

(ii) $g_D(z,a) - \log \dfrac{1}{|z-a|}$ is bounded in a punctured neighborhood of a,

(iii) $\lim\limits_{z \to \zeta, z \in D} g_D(z,a) = 0$ for q.e. $\zeta \in \partial D$.

Both the uniqueness and existence of $g_D(z,a)$ can be based on inversion with center a. If D' is the domain that we obtain from D under the mapping $z \to 1/(z-a)$, then consider the formula

$$g_D(z,a) := g_{D'}\left(\frac{1}{z-a}, \infty\right).$$

For simply connected domains, Green's function is related to conformal map. Here is the general principle.

Theorem 3.2.10. *[3] [Subordination Principle] Let D_1 and D_2 be domains in $\overline{\mathbb{C}}$ with non-polar boundaries, and let $f : D_1 \to D_2$ be a meromorphic function. Then*

$$g_{D_1}(z,a) \leq g_{D_2}\big(f(z), f(a)\big)$$

with equality if f is a conformal mapping of D_1 onto D_2.

Corollary 3.2.11. *Let $D_1 \subset D_2$ be domains in $\overline{\mathbb{C}}$ with non-polar boundaries. Then*

$$g_{D_1}(z,a) \leq g_{D_2}(z,a).$$

Proof. Take $f : D_1 \to D_2$ to be the inclusion map. □

In fact, g_D increases continuously with D, in the following sense.

Theorem 3.2.12. *Let D be a domain in $\overline{\mathbb{C}}$ such that ∂D is non-polar, and let (D_n) be subdomains of D such that $D_1 \subset D_2 \subset D_3 \subset \cdots$ and $\bigcup_n D_n = D$. Then*

$$\lim_{n\to\infty} g_{D_n}(z,a) = g_D(z,a), \quad z,a \in D.$$

With the help of Subordination principle, we can obtain many examples of Green's function with pole at finite point. Here are the most basic ones.

Example 3.2.13. Let $D_1 = \mathbb{D}$ unit disk and $D_2 = \overline{\mathbb{C}} \setminus D_1$. Then $f(z) = \frac{1}{z}$ and $g_{D_2}(z,\infty) = \log|z|$ so $g_{\mathbb{D}}(z,0) = \log|f(z)| = \log\frac{1}{|z|}$.

Example 3.2.14. Let $D_1 = D_2 = \mathbb{D}$ and $a \in \mathbb{D}$. Then $f(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$, where $\theta = arg(a)$, and $g_{D_2}(z,0) = \log\frac{1}{|z|}$, so $g_{\mathbb{D}}(z,a) = \log\left|\frac{1-\overline{a}z}{z-a}\right|$.

Example 3.2.15. Let $D_1 = \mathbb{H}$ be the upper half plane, $a \in \mathbb{H}$ and $D_2 = \mathbb{D}$. Then $f(z) = \frac{z-a}{z-\overline{a}}$ and $g_{D_2}(z,0) = \log\frac{1}{|z|}$, so $g_{\mathbb{H}}(z,a) = \log\left|\frac{z-\overline{a}}{z-a}\right|$.

Example 3.2.16. Let D_1 be the right half plane, $a \in D_1$ and $D_2 = \mathbb{D}$. Then $f(z) = \frac{z-a}{z+\overline{a}}$ and $g_{D_2}(z,0) = \log\frac{1}{|z|}$, so $g_{D_1}(z,a) = \log\left|\frac{z+\overline{a}}{z-a}\right|$.

Example 3.2.17. Let $D_1 = \overline{\mathbb{C}} \setminus [-1,1]$ and $D_2 = \mathbb{D}$. Then $f(z) = z + \sqrt{z^2-1}$ and $g_{\mathbb{D}}(z,a) = \log\left|\frac{1-\overline{a}z}{z-a}\right|$, so $g_{D_1}(z,a) = \log\left|\frac{1-\overline{f(a)}f(z)}{f(z)-f(a)}\right|$.

As a final example, we try to find the Green's function of $[a,b]$.

Example 3.2.18. Let $K = [a,b]$, $D_1 = \mathbb{C} \setminus K$ and $D_2 = \mathbb{C} \setminus [-1,1]$. Consider $f : D_1 \to D_2$ such that

$$f(z) = \frac{2z - (a+b)}{b-a}.$$

Note that $f(\infty) = \infty$ and f is a conformal bijection. In fact,

$$f'(z) = \frac{2}{b-a} \neq 0.$$

Remembering $g_{D_2}(z) = \log|z + \sqrt{z^2-1}|$ by Example 3.2.2 we obtain

$$g_{D_1}(z,\infty) = g_{D_2}(f(z),\infty) = \log\left| \frac{2z-(a+b)}{b-a} + \sqrt{\left(\frac{2z-(a+b)}{b-a}\right)^2 - 1} \right|$$

$$= \log\frac{2}{b-a} + \log\left| z + \sqrt{(z-a)(z-b)} - \frac{a+b}{2} \right|.$$

Also,

$$\begin{aligned} V_{[a,b]} &= \lim_{z\to\infty} (g_{D_2}(z,\infty) - \log|z|) \\ &= \log\frac{2}{b-a} + \lim_{z\to\infty} \log\left| 1 + \sqrt{\left(1-\frac{a}{z}\right)\left(1-\frac{b}{z}\right)} - \frac{a+b}{2z} \right| \\ &= \log\frac{2}{b-a} + \log 2 \\ &= \log\frac{4}{b-a}. \end{aligned}$$

Here the equilibrium measure $\mu_{[a,b]}$ of $[a,b]$ is $d\mu_{[a,b]} = \dfrac{1}{\pi}\dfrac{1}{\sqrt{(t-a)(b-t)}}\,dt$.

3.3 Capacity

Even though polar sets have played a prominent role, we still lack an effective means of determining whether or not a given set is polar. In case of compact set, we will see how the minimal energy V_K can be used as an indicator.

Definition 3.3.1. The *logarithmic capacity* of a compact set $K \subset \mathbb{C}$ is given by

$$c(K) = \sup_{\mu} e^{-I(\mu)}$$

where the supremum is taken over $\mathcal{M}(K)$. In particular, if K is a compact set with equilibrium measure μ_K, then

$$c(K) = e^{-I(\mu_K)} \quad \text{or equivalently} \quad c(K) = e^{-V_K}.$$

Here it is understood that $e^{-\infty} = 0$, so $c(K) = 0$ precisely when K is polar.

Example 3.3.2. Let $K = [-1, 1]$. Then by Example 2.5.16, $V_{[-1,1]} = \log 2$. Hence $c([-1, 1]) = \frac{1}{2}$.

Example 3.3.3. Let $K = \partial B_r(0)$. Then by Example 2.5.17, $V_{\partial B_r(0)} = \log \frac{1}{r}$. Hence $c(\partial B_r(0)) = r$.

Example 3.3.4. Let $K = [a, b]$. Then by Example 3.2.18, $V_{[a,b]} = \log \dfrac{4}{b-a}$. Hence $c([a, b]) = \dfrac{b-a}{4}$.

Example 3.3.5. Let $K = \{z \in \mathbb{C} : |P(z)| \leq 1\}$ where $P(z) = a_n z^n + \cdots + a_0$. Then by Example 3.2.5 $V_K = \frac{1}{n} \log |a_n|$. Hence $c(K) = \dfrac{1}{\sqrt[n]{|a_n|}}$.

We list some elementary properties.

Theorem 3.3.6. *Let $K_1 \subset K_2$ and K be compact. Then*

(a) $c(K_1) \leq c(K_2)$.

(b) $c(\alpha E + \beta) = |\alpha| c(E)$ *for all* $\alpha, \beta \in \mathbb{C}$.

(c) $c(K) = c(\partial K)$.

Note that (c) follows from Theorem 3.2.7.

Theorem 3.3.7. *If $K_1 \supset K_2 \supset K_3 \supset \cdots$ are compact and $K = \bigcap_n K_n$, then*

$$c(K) = \lim_{n \to \infty} c(K_n).$$

Proof. We certainly have by Theorem 3.3.6 (a)

$$c(K_1) \geq c(K_2) \geq c(K_3) \geq \cdots \geq c(K). \tag{3.2}$$

For the other direction, let ν_n be the equilibrium measures of K_n. Then by Helly's Selection Theorem there exists a weak star convergent subsequence, say $\nu_{n_k} \stackrel{*}{\to} \nu$. Applying Lemma 2.5.3 we deduce that

$$\limsup_{k \to \infty} I(\nu_{n_k}) \leq I(\nu).$$

Moreover, since $supp(\nu_n) \subset K_n$ it follows that $supp(\nu) \subset K$ and so $e^{I(\nu)} \leq c(K)$. Thus we obtain

$$\limsup_{k \to \infty} c(K_{n_k}) \leq c(K)$$

and combining this with (3.2) yields the desired conclusion. □

It is also true that if $B_1 \subset B_2 \subset B_3 \subset \cdots$ are Borel and $B = \bigcup_n B_n$. Then

$$c(B) = \lim_{n \to \infty} c(B_n).$$

The subordination principle gives rise to a useful inequality for capacity.

Theorem 3.3.8. *Let K_1 and K_2 be compact subsets of $\mathbb{C}$, and let D_1 and D_2 be the components containing ∞ of $\overline{\mathbb{C}} \setminus K_1$ and $\overline{\mathbb{C}} \setminus K_2$, respectively. If there is a meromorphic function $f : D_1 \to D_2$ such that*

$$f(z) = z + O(1) \quad \text{as } z \to \infty, \tag{3.3}$$

then

$$c(K_2) \leq c(K_1)$$

with equality if f is a conformal mapping of D_1 onto D_2.

Example 3.3.9. The function $f(z) = z + \dfrac{1}{z}$ maps $\overline{\mathbb{C}} \setminus B_1(0)$ conformally onto $\overline{\mathbb{C}} \setminus [-1, 1]$ and satisfies (3.3), so

$$c\left([-2, 2]\right) = c\left(B_1(0)\right) = 1.$$

Note that this is consistent with Example 3.3.4.

Capacity also behaves well under taking inverse images by polynomials.

Theorem 3.3.10. *Let K be compact set, and let $P(z) = a_n z^n + \cdots + a_0$. Then*

$$c\left(P^{-1}(K)\right) = \left(\frac{c(K)}{|a_n|}\right)^{\frac{1}{n}}.$$

Note that if we take $K = \overline{\mathbb{D}}$ closed unit disk, then we get Example 3.3.5.

Example 3.3.11. Let $0 \leq a \leq b$ and $K = [a^2, b^2]$. Take $P(z) = z^2$, then $P^{-1}(K) = [-b, -a] \cup [a, b]$. Hence

$$c\left([-b, -a] \cup [a, b]\right) = c\left(P^{-1}[a^2, b^2]\right) = c\left([a^2, b^2]\right)^{\frac{1}{2}} = \frac{\sqrt{b^2 - a^2}}{2}.$$

This also shows that capacity is not *subadditive* as a set function. That is, $c(A \cup B) \leq c(A) + c(B)$ is not always true, unless $A \cup B$ is connected see [7].

Here are some extra examples and properties of capacity referring to [3].

Example 3.3.12. Let K be an ellipse with semi axes a, b. Then $c(K) = \dfrac{a+b}{2}$.

Example 3.3.13. Let $K = B_1(0) \cup [0, R]$ where $R \geq 1$. Then $f(z) = z + \dfrac{1}{z}$ maps $\overline{\mathbb{C}} \setminus K$ conformally onto $\overline{\mathbb{C}} \setminus [-2, R + 1/R]$, hence $c(K) = \dfrac{R + 2 + 1/R}{4}$.

Theorem 3.3.14. *Let μ be a Borel measure on $\mathbb{C}$ with compact support and $I(\mu)$ be finite. Then*

$$c(K) = 0 \implies \mu(K) = 0.$$

Corollary 3.3.15. *Borel polar sets are of Lebesgue measure zero.*

Theorem 3.3.16. *Let (B_n) be sequence of Borel sets, $B = \bigcup_n B_n$ and $d > 0$.*

(a) *If $diam(B) \leq d$, then $c(B) \leq d$ and*

$$\frac{1}{\log d/c(B)} \leq \sum_n \frac{1}{\log d/c(B_n)}.$$

(b) *If $dist(B_j, B_k) \geq d$ whenever $j \neq k$, then*

$$\frac{1}{\log^+ d/c(B)} \geq \sum_n \frac{1}{\log^+ d/c(B_n)}.$$

f^+ is the positive part of f, that is, $f^+ = \max\{f, 0\}$.

Theorem 3.3.17. *Let K be compact and $T : K \to \mathbb{C}$ be a map satisfying*

$$|T(z) - T(w)| \leq A|z - w|^\alpha, \quad \forall z, w \in K$$

where A and α are positive constants. Then

$$c(T(K)) \leq A\, c(K)^\alpha.$$

Theorem 3.3.18. *Let K be compact subset of $\mathbb{C}$.*

(a) *If K is connected and has diameter d, then $d/4 \leq c(K) \leq d/2$.*

(b) *If K is a rectifiable curve of length l, then $c(K) \leq l/4$.*

(c) *If $K \subset \mathbb{R}$ is of Lebesgue measure m, then $c(K) \geq m/4$.*

(d) *If $K \subset \partial\mathbb{D}$ of arc length measure α, then $c(K) \geq \sin(\alpha/4)$.*

Theorem 3.3.19. *If K is a compact set of area A, then $c(K) \geq \sqrt{A/\pi}$.*

Let $s := (s_n)$ such that $0 < s_n < 1$. Define $C(s_1)$ to be the set obtained from $[0, 1]$ by removing an open interval of length s_1 from the center. At the nth stage, let $C(s_1, ..., s_n)$ be the set obtained by removing from the middle of each interval in $C(s_1, ..., s_{n-1})$ an open subinterval whose length is proportion s_n of the whole interval. We then obtain a decreasing sequence of compact sets $(C(s_1, ..., s_n))$ and the corresponding *generalized Cantor set* is defined to be

$$C(s) := \bigcap_n C(s_1, ..., s_n).$$

It is readily be checked that $C(s)$ is a compact, perfect, totally disconnected uncountable set of Lebesgue measure $\prod_n (1 - s_n)$.

Theorem 3.3.20. *With the notation above,*

$$\frac{pq}{2} \leq c\,(C(s)) \leq \frac{q}{2}$$

where $p = \prod_n (1 - s_n)^{1/2^n}$ and $q = \prod_n s_n^{1/2^n}$.

Thus, the standard Cantor set obtained by taking $s_n = 1/3$ has capacity at least $1/9$, in particular it is non-polar set of Lebesgue measure zero. On the other hand, if we let $s_n = 1 - (1/2)^{2^n}$, then $C(s)$ is polar, thereby providing the long-promised example of an uncountable polar set.

Chapter 4

Smoothness of Green's Functions

Here we discuss continuity and smoothness of Green's function.

4.1 Continuity

We know from Theorem 2.4.6 that the potential U^μ is harmonic on $\mathbb{C} \setminus supp(\mu)$, hence continuous there. But what about in $supp(\mu)$?

Theorem 4.1.1. *[3] [Continuity Principle] Let μ be a finite Borel measure on $\mathbb{C}$ with compact support K.*

(a) *If $\zeta_0 \in K$, then* $\liminf_{z \to \zeta_0} U^\mu(z) = \liminf_{\substack{\zeta \to \zeta_0 \\ \zeta \in K}} U^\mu(\zeta)$.

(b) *If further* $\lim_{\substack{\zeta \to \zeta_0 \\ \zeta \in K}} U^\mu(\zeta) = U^\mu(\zeta_0)$, *then* $\lim_{z \to \zeta_0} U^\mu(z) = U^\mu(\zeta_0)$.

Note that part (b) follows from part (a) and shows that if the potential is continuous in K with respect to K then it is continuous in K with respect to $\mathbb{C}$.

Example 4.1.2. Let consider $\mu = \delta$. Then by Example 2.4.2 $supp(\delta) = \{0\}$ and $U^{\delta}(z) = \log \frac{1}{|z|}$, hence part (a) and part (b) are satisfied trivially.

It is also easy to check the following examples.

Example 4.1.3.

(a) Let $\mu = \frac{1}{2}\lambda$ on $[-1, 1]$. Then by Example 2.4.3 $supp(\lambda) = [-1, 1]$ and

$$U^{\lambda}(z) = 1 - \frac{1}{2}[\,(1+z)\log|1+z| + (1-z)\log|1-z|\,].$$

(b) Let $\mu = \alpha$ on $[-1, 1]$. Then by Example 2.4.5 $supp(\alpha) = [-1, 1]$ and

$$U^{\alpha}(z) = \log 2 - \log|z + \sqrt{z^2-1}|.$$

(c) Let $\mu = \theta$ on $\partial B_r(0)$. Then by Example 2.4.4 $supp(\theta) = \partial B_r(0)$ and

$$U^{\theta}(z) = \begin{cases} \log \frac{1}{r} & \text{if } |z| \le r, \\ \log \frac{1}{|z|} & \text{if } |z| > r. \end{cases}$$

Note that the continuity principle also holds for the Green's function with pole at ∞ by its very definition with the equilibrium potential.

Example 4.1.4. $g_{[-1,1]}(z) = \log|z + \sqrt{z^2-1}|$.

Example 4.1.5.

$$g_{\partial B_r(0)}(z) = \begin{cases} 0 & \text{if } |z| \le r, \\ \log \frac{|z|}{r} & \text{if } |z| > r. \end{cases}$$

As indicated above, the continuity points of g_Ω and U^{μ_K} coincide. Since μ_K is supported on $\partial\Omega$, both of these functions are continuous away from $\partial\Omega$. It is clear that $\zeta \in \partial\Omega$ is a continuity point if and only if

$$g_\Omega(\zeta, \infty) = 0,$$

which is equivalent to

$$U^{\mu_K}(\zeta) = V_K.$$

In particular, the set of discontinuity points of g_Ω is a F_σ polar set by Frostman's Theorem.

There is a close relationship between the continuity points of Green's function and the *Generalized Dirichlet problem.*

4.2 Generalized Dirichlet Problem

Recall from Definition 2.2.1 that, given a domain D and a continuous function $\phi : \partial D \to \mathbb{R}$, the Dirichlet problem is to find a harmonic function on D such that $\lim_{z\to\zeta} h(z) = \phi(\zeta)$ for all $\zeta \in \partial D$. By Theorem 2.2.2, if such a solution h exists, it is unique. Also, if the domain D is a disc, then a solution always does exist, and Theorem 2.2.5 even gives a formula for it.

For a general domain D, the situation is more complicated. In this case, the Dirichlet problem may well have no solution. For example, consider the punctured unit disk $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$, and let $\phi : \partial D \to \mathbb{R}$ be given by

$$\phi(\zeta) = \begin{cases} 0, & |\zeta| = 1, \\ 1, & \zeta = 0. \end{cases}$$

Then any solution h would have $h(z) \leq 0$ by the extended maximum principle, violating the condition that $\lim_{z\to 0} h(z) = \phi(0) = 1.$

Thus, it is convenient to extend the Dirichlet problem in two ways. First, we allow D to be any proper subdomain of $\overline{\mathbb{C}}$. Since the Dirichlet problem is invariant under conformal mapping of the sphere, this is really no more general than working on a subdomain of $\mathbb{C}$. Second way will be to consider arbitrary bounded functions $\phi : \partial D \to \mathbb{R}$ rather than just continuous ones. Although certainly no solution to the Dirichlet problem is possible if ϕ is discontinuous, it is nevertheless useful to allow this extra freedom, as will become clear.

Definition 4.2.1. Let D be a proper subdomain of $\overline{\mathbb{C}}$, and $\phi : \partial D \to \mathbb{R}$ be a bounded function. The associated *Perron function* $H_D\phi : D \to \mathbb{R}$ is defined by

$$H_D\phi = \sup_{u \in \mathcal{U}} u,$$

where $\mathcal{U}$ denotes the family of all subharmonic functions u on D such that $\limsup_{z \to \zeta} u(z) \leq \phi(\zeta)$ for each $\zeta \in \partial D$.

The motivation for this definition is that, if the Dirichlet problem has a solution at all, then $H_D\phi$ is it! Indeed, if h is such a solution, then certainly $h \in \mathcal{U}$, and so $h \leq H_D\phi$. On the other hand, by the maximum principle, if $u \in \mathcal{U}$, then $u \leq h$ on D, and so $H_D\phi \leq h$. Therefore $H_D\phi = h$.

Our first result is that, regardless of whether the Dirichlet problem has a solution, $H_D\phi$ is always a bounded harmonic function.

Theorem 4.2.2. *Let D be proper subdomain of $\overline{\mathbb{C}}$, and let $\phi : \partial D \to \mathbb{R}$ be a bounded function. Then $H_D\phi$ is harmonic on D, and*

$$\sup_D |H_D\phi| \leq \sup_{\partial D} |\phi|.$$

As we have considered in the punctured unit disk, some boundary points may behave so *irregular* that prevent to solve Dirichlet problem. To distinguish such points we make the following definition.

Definition 4.2.3. Let D be a proper subdomain of $\overline{\mathbb{C}}$. Then $\zeta \in \partial D$ is said to be a *regular point with respect to Dirichlet problem* if for every bounded boundary function on ∂D which is continuous at ζ, the solution of the Dirichlet problem in D is continuous at ζ. Also, if every point of ∂D is regular, then we call D *regular with respect to the Dirichlet problem.*

Example 4.2.4. The punctured unit disk is not regular as we have shown that the origin is not a regular point.

Theorem 4.2.5. *Let D be a proper subdomain of $\overline{\mathbb{C}}$, and let $\zeta_0 \in \partial D$ be a regular boundary point. If $\phi : \partial D \to \mathbb{R}$ is a bounded function which is continuous at ζ_0, then*

$$\lim_{z \to \zeta_0} H_D \phi(z) = \phi(\zeta_0).$$

Now we can solve the Dirichlet problem.

Theorem 4.2.6. *Let D be a regular domain in $\overline{\mathbb{C}}$, and let $\phi : \partial D \to \mathbb{R}$ be a continuous function. Then there exists a unique harmonic function h on D such that $\lim_{z \to \zeta} h(z) = \phi(\zeta)$ for all $\zeta \in \partial D$.*

In fact, regularity is also necessary to solve the Dirichlet problem. Thus, the theorem is, in some sense, the best possible result. However, deciding whether a given domain is regular by using its definition is inconvenient many times. Here, the celebrated theorem of Wiener characterize the regular points as follows:

Theorem 4.2.7. *[4][Wiener's Theorem] Let $D \subset \overline{\mathbb{C}}$ be a domain such that ∂D is non-polar, $0 < \lambda < 1$ and for $\zeta \in \partial D$, set*

$$A_n(\zeta) := \{z \notin D : \lambda^n \leq |z - \zeta| \leq \lambda^{n-1}\}.$$

Then $\zeta \neq \infty$ is regular with respect to the Dirichlet problem in D if and only if

$$\sum_{n=1}^{\infty} \frac{n}{\log\left(\frac{1}{c\left(A_n(\zeta)\right)}\right)} = \infty.$$

As an immediate consequence of Wiener's theorem, we have that every simply connected domain $D \in \overline{\mathbb{C}}$ is regular. For example, the unit disk $\mathbb{D}$ is regular.

Example 4.2.8. Let $K = [-1, 1]$. Then K is a regular set. To see this we will show that all of its point is regular. Let us start with the point -1. Choosing $\lambda = \frac{1}{2}$ we have

$$A_n(-1) = \{z \in K : \frac{1}{2^n} \le |z+1| \le \frac{1}{2^{n-1}}\},$$

so

$$A_n(-1) = [-1 + \frac{1}{2^n}, -1 + \frac{1}{2^{n-1}}];$$

hence

$$c(A_n(-1)) = \frac{1}{2^{n+2}}$$

by Example 3.3.4. From this,

$$\log \frac{1}{c(A_n(-1))} = (n+2)\log 2.$$

Since

$$\sum_{n=1}^{\infty} \frac{n}{(n+2)\log 2} = \infty,$$

-1 is a regular point of K. Similarly for the point 1,

$$A_n(1) = [1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}]$$

and all the computations are the same. Thus 1 is also a regular point of K. Now let $\zeta \in (-1, 1)$. Then choose λ such that $[\zeta - \lambda, \zeta + \lambda] \subset (-1, 1)$. So

$$A_n(\zeta) = [\zeta - \lambda^{n-1}, \zeta - \lambda^n] \cup [\zeta + \lambda^n, \zeta + \lambda^{n-1}].$$

By the translation invariance of capacity and Example 3.3.11

$$c(A_n(\zeta)) = c\left([-\lambda^{n-1}, -\lambda^n] \cup [\lambda^n, \lambda^{n-1}]\right) = \frac{\sqrt{1-\lambda^2}}{2\lambda}\lambda^n,$$

so

$$\log \frac{1}{c(A_n(\zeta))} = \log \frac{2\lambda}{\sqrt{1-\lambda^2}} + n \log \frac{1}{\lambda}.$$

Since

$$\sum_{n=1}^{\infty} \frac{n}{\log \frac{2\lambda}{\sqrt{1-\lambda^2}} + n \log \frac{1}{\lambda}} = \infty,$$

ζ is a regular point of K. Because ζ is arbitrary, $K = [-1, 1]$ is regular.

Here is the bridge that connects the continuity of g_K and regularity of Ω.

Theorem 4.2.9. *[4] Let K be a non-polar compact set and Ω be the unbounded component of $\overline{\mathbb{C}} \setminus K$. Then g_K is continuous at $\zeta \in \partial\Omega$ if and only if ζ is a regular point with respect to the Dirichlet problem in Ω.*

Corollary 4.2.10 (Kellogg's Theorem)**.** *Let D be proper subdomain of $\overline{\mathbb{C}}$. Then the set of* irregular *boundary points is an F_σ polar set.*

Proof. By first performing a conformal mapping, we can suppose that $\infty \in D$. Set $K = \overline{\mathbb{C}} \setminus D$. If K is polar, then by the extended maximum principle every point of ∂D will be not regular, and the result follows. If K is non-polar, then by the previous theorem the set of irregular points is exactly the discontinuity points of g_K and hence the result follows by the Frostman's theorem. □

Now we can solve the generalized Dirichlet problem.

Theorem 4.2.11. *[Solution of the Generalized Dirichlet Problem] Let D be a domain in $\overline{\mathbb{C}}$ such that ∂D is non-polar, and let $\phi : \partial D \to \mathbb{R}$ be a bounded function which is continuous quasi-everywhere on ∂D. Then there exists a unique bounded harmonic function h on D such that $\lim_{z\to\zeta} h(z) = \phi(\zeta)$ for q.e. $\zeta \in \partial D$.*

Proof. Set $h = H_D\phi$. Then by Theorem 4.2.2 h is harmonic and bounded on D. Also, from Theorem 4.2.5

$$\lim_{z\to\zeta} h(z) = \phi(\zeta), \quad \zeta \in \partial D \setminus (E_1 \cup E_2)$$

where E_1 is the set of irregular boundary points of D and E_2 is the set of discontinuity points of ϕ. Now E_1 is polar by Kellogg's theorem and E_2 is polar by hypothesis. Also both sets are Borel, so their union $E_1 \cup E_2$ is a polar set. Hence $\lim_{z\to\zeta} h(z) = \phi(\zeta)$ for q.e. $\zeta \in \partial D$. This proves existence.

For uniqueness, suppose h_1 and h_2 are two solutions. Then $u = h_1 - h_2$ is a bounded harmonic function on D with zero boundary limit quasi-everywhere. Applying the Extended Maximum Principle to $\pm u$ we conclude $h_1 \equiv h_2$ on D. □

Let return the punctured unit disk $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and the function $\phi : D \to \mathbb{R}$ defined as

$$\phi(\zeta) = \begin{cases} 0, & |\zeta| = 1, \\ 1, & \zeta = 0. \end{cases}$$

We know that there is no solution for the classical Dirichlet problem as indicated but what about the generalized one? Since $\partial D = \partial \mathbb{D} \cup \{0\}$ is not polar, Theorem 4.2.11 says that there exists a unique solution to the generalized Dirichlet problem for D and is given by $H_D\phi$. Then $H_D\phi$ is a bounded harmonic function on D such that $\lim_{z \to \zeta} H_D\phi(z) = \phi(\zeta) = 0$ q.e. on ∂D. Thus, by the extended maximum principle applied to $\pm H_D\phi$ we find that $H_D\phi \equiv 0$.

We close this section with an example of a compact set whose Green's function is discontinuous because of an irregular boundary point. Let $K = \{0\} \cup \bigcup_{n=1}^{\infty} I_n$ such that $I_n = [2^{-n}, 2^{-n} + 4a_n]$ where $0 < a_n < 2^{-n-2}$ and also let Ω be the unbounded component of $\overline{C} \backslash K$. Clearly K is bounded, and closed as a union of disjoint closed intervals so it is compact. K is also not polar since $c(K) > c(I_1) = a_1 > 0$. Then, K has a unique equilibrium measure μ_K implying existence of its Green's function g_K. To check the continuity of g_K, we consider the regularity of Ω. Note that $\partial\Omega = K$ so if $\zeta \in \partial\Omega$ and $\zeta \neq 0$ then $\zeta \in I_{n_0}$ for some n_0. Then ζ is a regular point by the Wiener's Criterion as shown in Example 4.2.8. Hence we are in a position that the set K is regular if and only if 0 is a regular point. Choosing $\lambda = 2^{-1}$ in the Wiener's Theorem, we see that $A_n(0) = I_n$ so $c\big(A_n(0)\big) = c(I_n) = a_n$. Therefore, K is regular if and only if

$$\sum_{n=1}^{\infty} \frac{n}{\log \frac{1}{a_n}} = \infty.$$

Hence, if we take, for example, $a_n = \exp(-n^3)$ then the series is convergent that makes the set K irregular. Thus, the corresponding Green's function g_K, even we do not know it explicitly, is discontinuous at 0 by Theorem 4.2.9.

4.3 Smoothness

As indicated above, if a non-polar compact set $K \subset \mathbb{C}$ is regular with respect to the Dirichlet problem, then the Green's function g_K with pole at ∞ is continuous on $\mathbb{C}$. Now we analyze how *good* its continuity can be, how *smooth* it can be.

Definition 4.3.1. Given a function f, the modulus of continuity of f is a function $\omega(f, \delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|$, where $x, y \in \mathrm{Dom}(f)$.

Example 4.3.2. Let $f(x) = x^\alpha$ be given on $[0, 1]$ with $\alpha > 0$. If $\delta \geq 1$, then by the monotonicity $|f(x) - f(y)|$ is maximized when $x = 1$ and $y = 0$ with $|f(1) - f(0)| = 1$. Hence $\omega(f, \delta) \equiv 1$ for $\delta \geq 1$. So let fix $\delta < 1$. On the intervals $[c, c + \delta] \subset [0, 1]$, again by the monotonicity, $|f(x) - f(y)|$ is maximized when $x = c + \delta$ and $y = c$ with $|f(c + \delta) - f(c)| = (c + \delta)^\alpha - c^\alpha$. Now consider this as a function of c on $[0, 1-\delta]$. Differentiating with respect to c gives $\alpha[(c+\delta)^{\alpha-1} - c^{\alpha-1}]$. Thus, the derivative is positive if $\alpha > 1$ and is negative if $\alpha < 1$ which implies the function is increasing and decreasing respectively. Hence $(c + \delta)^\alpha - c^\alpha$ is maximized when $c = 1 - \delta$ if $\alpha > 1$ and it is maximized when $c = 0$ if $\alpha < 1$. Consequently, $\omega(f, \delta) = 1 - (1 - \delta)^\alpha$ if $\alpha > 1$ and $\omega(f, \delta) = \delta^\alpha$ if $\alpha < 1$. It is also clear that if $\alpha = 1$, then $\omega(f, \delta) = \delta$.

It is not always possible to find the exact modulus of continuity for a given function. But it is useful to estimate the modulus of continuity.

Definition 4.3.3. Let f be a real or complex valued function on Euclidean space. We say f satisfies Hölder condition if there exists non-negative real constant C and α such that

$$|f(x) - f(y)| \leq C\,|x - y|^\alpha$$

for all points x and y in the domain of f.

Note that if $\alpha = 0$, then f is just bounded and we cannot be sure its continuity. Also if $\alpha > 1$ then f will be constant by the zero derivative. So it is interesting when $0 < \alpha \leq 1$. In this case, we say that f is Hölder continuous of order α or f belongs to Lipschitz class α, denoted by $f \in Lip\alpha$.

Let $K \subset \mathbb{C}$ be non-polar regular compact set. Since $g_K \equiv 0$ on K and is infinitely differentiable on $\mathbb{C}\setminus K$, because of being harmonic there, it is interesting to figure out what kind of continuity g_K has near the boundary of K. We say that the point $p = p(\delta)$ realizes the modulus of continuity of g_K if $dist(p, K) \leq \delta$, and $g_K(z) \leq g_K(p)$ for all z with $dist(z, K) \leq \delta$. Then $\omega(g_K, \delta) = g_K(p)$.

Let us start with our known examples of Green's function.

Example 4.3.4. Let $K = [-1, 1]$ then $g_K(z) = \log|z + \sqrt{z^2 - 1}|$ where $\sqrt{z^2 - 1}$ denotes the branch that behaves like z near infinity. The points $-1-\delta$ and $1+\delta$ realize the modulus of continuity and

$$g_K(-1-\delta) = g_K(1+\delta) = \log|1 + \delta + \sqrt{2\delta + \delta^2}| \leq \log\left|1 + \sqrt{3\delta} + \frac{3\delta}{2}\right| \leq \sqrt{3\delta}$$

by the Taylor expansion of $e^{\sqrt{3\delta}}$. Also,

$$g_K(-\delta i) = g_K(\delta i) = \log|\delta i + \sqrt{-\delta^2 - 1}| = \log|\delta + \sqrt{1 + \delta^2}| \leq \log\left|1 + \delta + \frac{\delta^2}{2}\right| \leq \delta$$

by the Taylor expansion of e^{δ}. For all remaining z such that $0 < dist(z, K) \leq \delta$, $g_K(z) \leq C_z\, \delta^{\alpha}$, where $\frac{1}{2} < \alpha \leq 1$ and C_z depends on z. Thus, $g_{[-1,1]} \in Lip\frac{1}{2}$.

This smoothness is the best possible for $K \subset \mathbb{R}$ due to the following argument: Take $a = \min K$ and $b = \max K$. Then $K \subset [a, b]$ and, by monotonicity of g_K with respect to K, we have

$$g_K(z) \geq g_{[a,b]}(z) = \log\left|\frac{2z - (a+b)}{b-a} + \sqrt{\left(\frac{2z - (a+b)}{b-a}\right)^2 - 1}\right|$$

by Example 3.2.18. Then

$$\omega(g_K, \delta) \geq g_K(b+\delta) \geq g_{[a,b]}(b+\delta) = \log\left|\frac{2\delta + b - a}{b-a} + \sqrt{\left(\frac{2\delta + b - a}{b-a}\right)^2 - 1}\right|$$

and we cannot make $\omega(g_K, \delta)$ smaller than $C\,\delta^{\frac{1}{2}}$ see [8].

Example 4.3.5. Let $K = \overline{\mathbb{D}}$. Then

$$g_{\overline{\mathbb{D}}}(z) = \begin{cases} 0 & \text{if } |z| \leq 1, \\ \log|z| & \text{if } |z| > 1. \end{cases}$$

Hence for all $z \in \overline{B_{1+\delta}(0)} \setminus \overline{\mathbb{D}}$,

$$g_{\overline{\mathbb{D}}}(z) = \log|z| \leq \log(1+\delta) \leq \delta.$$

Thus, $g_{\overline{\mathbb{D}}} \in Lip1$ which is the best possible smoothness as $K \subset \mathbb{C}$, by a similar argument as above.

To achieve the optimal smoothness in $\mathbb{R}$, we do not need to consider intervals such as $[-1,1]$. Indeed, connectedness can be so relaxed that we can consider totally disconnected sets. In [8], Totik constructed a set E of zero linear measure whose Green's function satisfies the optimal smoothness. In fact, let $0 \leq \varepsilon_j < 1$ and $C\{\varepsilon_j\}$ be the Cantor type set of the corresponding sequence. The classical Cantor ternary set corresponds to the sequence $\varepsilon_j = 1/3$ for all $j = 1, 2, 3, \ldots$. The set $C(\varepsilon_j)$ is of zero linear measure if and only if $\sum \varepsilon_j = \infty$. It is known that $C(\varepsilon_j)$ is of positive capacity if and only if

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{1-\varepsilon_j} < \infty.$$

The Green's function of $C(\varepsilon_j)$ is in a $Lip\alpha$ class for some $\alpha > 0$ if and only if

$$\sum_{j=1}^{k} \log \frac{1}{1-\varepsilon_j} = O(k).$$

The next theorem shows for all $C(\varepsilon_j)$ of positive measure, the Green's function $g_{C(\varepsilon_j)}$ is $Lip\frac{1}{2}$ smooth.

Theorem 4.3.6. *Let $\{\varepsilon_j\}$ be a sequence of numbers from the interval $[0,1)$. Then the Green's function of $C(\varepsilon_j)$ is in the $Lip\frac{1}{2}$ class if and only if $\sum \varepsilon_j^2 < \infty$.*

Corollary 4.3.7. *The compact set $C(1/2, 1/3, 1/4, \ldots)$ is of zero linear measure but its complement has a $Lip\frac{1}{2}$ Green's function.*

Corollary 4.3.8. *The classical Cantor set K_0 is of zero linear measure and also the corresponding Green's function does not have the optimal smoothness, but it is still in a $Lip\alpha_0$ class for some $0 < \alpha_0 < 1/2$.*

The exact value of $\alpha_0 = \sup\{\alpha : g_{K_0} \in Lip\alpha\}$ is not known. There were numerous attempts to find it. In [9], Ransford and Rostand made considerable computations. Also by using their earlier result about the capacity of the Cantor set, which is approximately 0.2209 and is correct up to four decimal places, they arrived at the estimate $\alpha = 0.3413$. Thus, we have $0.3413 \leq \alpha_0 < 0.5$ and it is a rather tight bound.

In [10], Andrievskii considered a non-polar regular compact set $E \subset [-1, 1]$ such that $\pm 1 \in E$ and $\Omega = \overline{\mathbb{C}} \setminus E$. He was also interested in studying the metric properties of E such that g_Ω satisfies the Hölder condition with $\alpha = 1/2$:

$$|g_\Omega(z_1) - g_\Omega(z_2)| \leq c\,|z_1 - z_2|^{1/2}, \qquad z_1, z_2 \in \Omega \setminus \{\infty\}. \tag{4.1}$$

He noted that since the monotonicity of Green's function yields

$$g_E(1 + r) \geq g_{[-1,1]}(1 + r) > \frac{\sqrt{r}}{2}, \qquad 0 < r < 1,$$

the right-hand side of (4.1) appears to be the best suited for the theory. He analyzed how sparse E can be such that it satisfies (4.1) in terms of its Hausdorff dimension $\dim E$. First he showed that

$$\dim E \geq \frac{1}{2}$$

and then proved the following.

Theorem 4.3.9. *There exists a regular set $E_0 \subset \mathbb{R}$ with the following properties:*

(i) *g_{E_0} satisfies (4.1),*

(ii) *$\dim E_0 = 1/2$.*

He also considered the problem of how sparse the set E can be such that the following local version of (4.1) is valid:

$$g_\Omega(z) = g_\Omega(z) - g_\Omega(-1) \le c_2\,|z+1|^{1/2}, \qquad z \in \Omega \setminus \{\infty\}, \tag{4.2}$$

where $c_2 \ge 0$ is a constant and proved the theorem given below.

Theorem 4.3.10. *There exists a regular set $E_1 \subset \mathbb{R}$ with the following properties:*

(i) g_{E_1} *satisfies* (4.2),

(ii) $\dim E_1 = 0$.

He concluded with the following remark. One of the natural ways to construct sparse sets with Hölder continuous Green function is to consider (nowhere dense) Cantor-type sets. According to [8], Theorem 5.1 (here Theorem 4.3.6) and the reasoning in the same monograph the following three conditions are equivalent:

(i) g_C satisfies (4.1),

(ii) g_C satisfies (4.2),

(iii) $\sum_j \varepsilon_j^2 < \infty$.

Meanwhile, by [11] Theorem 10.5 each Cantor type set $C(\varepsilon_j)$ with the property

$$\lim_{j\to\infty} \varepsilon_j = 0$$

has Hausdorff dimension 1. Therefore, Andrievskii constructed non Cantor type sets with the optimal smoothness to prove Theorems 4.3.9 and 4.3.10.

Before continuing examples on how smooth Green's function can be, we need a general notion of modulus of continuity.

Definition 4.3.11. A modulus of continuity is any real-extended valued function $\omega : [0, +\infty] \to [0, +\infty]$ vanishing at 0 and continuous at 0. That is

$$\lim_{\delta \to 0} \omega(\delta) = \omega(0) = 0.$$

We say that a function $f : \mathbb{C} \to \mathbb{R}$ admits ω as a modulus of continuity if

$$|f(x) - f(y)| \leq \omega(|x - y|)$$

for all x and y.

Note that if we take $\omega(x) = kx^{\alpha}$ with $0 < \alpha \leq 1$ and $k > 0$, then we get the Hölder continuity. If a function f admits ω as a modulus of continuity and $\omega_1 \geq \omega$ then clearly f also admits ω_1 as a modulus of continuity. Hence it is natural to ask a *minimal modulus of continuity* or referred as the *optimal modulus of continuity*. It turns out that $\omega(f, \delta)$ is the optimal modulus of continuity for f.

Some authors require additional properties such as ω being increasing or continuous. However, if f admits a modulus of continuity in the weaker definition above, it also admits a modulus of continuity which is increasing:

(i) $\omega_1(t) := sup_{s \leq t} \omega(s)$ is increasing and $\omega_1 \geq \omega$,

(ii) $\omega_2(t) := \dfrac{1}{t} \displaystyle\int_t^{2t} \omega_1(s)\, ds$ is also continuous and $\omega_2 \geq \omega_1$.

In fact, a suitable variant of the preceding definition also makes ω_2 infinitely differentiable on $(0, +\infty)$.

There are also special moduli of continuity such as *concave*, *subadditive*, *uniform continuous*, *sublinear*, *dominated by a concave modulus*. But we will not go through them.

Now we are ready to continue giving examples.

In [12], Altun and Goncharov considered Cantor type sets $K^{(\alpha)}$ with "lowest smoothness" of the corresponding Green's function. Let $1 < \alpha$, $0 < l_1 < \frac{1}{2}$, and $2l_1^{\alpha-1} < 1$. Then $K^{(\alpha)} = \bigcap_{s=1}^{\infty} E_s$, where $E_0 = I_{1,0} = [0,1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,s}$ of length $l_s = l_{s-1}^{\alpha}$, and E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,s}$ with $j = 1, 2, \ldots, 2^s$. The set $K^{(\alpha)}$ is not polar if and only if $\alpha < 2$. In addition, by Pleśniak [13], in the case of the Cantor type set, the corresponding set is regular if and only if it is not polar. Thus, for $1 < \alpha < 2$, the Green's function $g_{\mathbb{C}\setminus K^{(\alpha)}}$ is continuous. They showed that its modulus of continuity can be estimated in terms of the function $\varphi(\delta) = (1/\log \frac{1}{\delta})$, which is used in the definition of the logarithmic measure. Here are the theorems referring to [12] with $\gamma = \frac{\log 2}{\log \alpha} - 1$.

Theorem 4.3.12. *For every $0 < \varepsilon < \gamma$, there exist constants δ_0, C_0, depending on α and ε, such that $g_{\mathbb{C}\setminus K^{(\alpha)}}(z) \le C_0 \varphi^{\gamma-\varepsilon}(\delta)$ for $z \in \mathbb{C}$ with $dist(z, K^{(\alpha)}) = \delta \le \delta_0$.*

Theorem 4.3.13. *There are constants δ_0 and C_1 depending only on α such that $C_1 \varphi^{\gamma}(\delta) \le g_{\mathbb{C}\setminus K^{(\alpha)}}(-\delta)$ for $\delta \le \delta_0$.*

Note that the arbitrariness of ε in the first theorem can be tempting, but we cannot conclude that $g_{\mathbb{C}\setminus K^{(\alpha)}}$ admits φ^{γ} as a modulus of continuity. However, the second theorem implies that one cannot hope a better smoothness.

In [14], Çelik and Goncharov considered the set $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$ such that $I_k = [a_k, b_k]$ where $b_k = \exp(-2^k)$ and $a_k = b_k - b_{k+1}$. By the Wiener criterion, the point 0 is regular. Thus, $g_{\mathbb{C}\setminus K}$ is continuous throughout $\mathbb{C}$. Let $\varphi(\delta) = (1/\log \frac{1}{\delta})$ for $0 < \delta < 1$ and let $\gamma = -\frac{2-\sqrt{2}}{2}$. They gave the modulus of continuity of $g_{\mathbb{C}\setminus K}$ in terms of the function φ. Here is the theorem in [14].

Theorem 4.3.14. *Let $dist(z, K) = \delta \le b_1$. Then $g_{\mathbb{C}\setminus K}(z) \le C\,\varphi^{\gamma}(\delta)$, where the constant C does not depend on δ. On the other hand, $g_{\mathbb{C}\setminus K}(-\delta) > \varphi^{\gamma}(\delta)$.*

Now we consider one of the most recent articles in this field and introduce an open problem in it.

In [1], A. P. Goncharov introduced a new class of Cantor-type sets depending on $\gamma = (\gamma_s)_{s=1}^{\infty}$ which possess interesting properties in terms of potential theory. Before mentioning these properties let us show how the sets are defined. Given a sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < 1/4$. Let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. Define inductively a sequence of real polynomials: let $P_2(x) = x(x-1)$ and $P_{2^{s+1}} = P_{2^s}(P_{2^s} + r_s)$ for $s \in \mathbb{N}$. It is easy to check by induction that the polynomial P_{2^s} has 2^{s-1} points of minimum with equal values $-r_{s-1}^2/4$. By that we have a geometric procedure to define new (with respect to P_{2^s}) zeros of $P_{2^{s+1}}$: they are the abscissas of points of intersection of the line $y = -r_s$ with the graph $y = P_{2^s}$. Let E_s denote the set $\{x \in \mathbb{R} : P_{2^{s+1}}(x) \leq 0\}$. Since $r_s < r_{s-1}^2/4$, the set E_s consists of 2^s disjoint closed *basic intervals* $I_{j,s}$. In general, the lengths $l_{s,j}$ of the intervals of the same level are different, however, by the construction of $K(\gamma)$, we have $\max_{1\leq j\leq 2^s} l_{j,s} \to 0$ as $s \to \infty$. Clearly, $E_{s+1} \subset E_s$. Set $K(\gamma) = \bigcap_{s=1}^{\infty} E_s$. The level domains $D_s = \{z \in \mathbb{C} : |P_{2^s}(z) + r_s/2| < r_s/2\}$, $s = 1, 2, \ldots$ is a nested family and $\overline{D_s} \searrow K(\gamma)$.

Now we give some properties of $K(\gamma)$ referring to [1] for all theorems, lemmas, examples and the open problem at the end.

Theorem 4.3.15. *The set $K(\gamma)$ is polar if and only if $\sum_{k=1}^{\infty} 2^{-k} \log \frac{1}{\gamma_k} = \infty$ or equivalently $\lim_{s\to\infty} \log \frac{2}{r_s} = \infty$. If this is finite and $z \notin K(\gamma)$, then*

$$g_{\mathbb{C}\setminus K(\gamma)}(z) = \lim_{s\to\infty} 2^{-s} \log \frac{|P_{2^s}(z)|}{r_s}.$$

From now on let us make the assumption that

$$\gamma_s \leq 1/32 \quad \text{for} \quad s \in \mathbb{N}, \tag{4.3}$$

which makes the polynomial P_{2^s} convex on $I_{j,s-1}$.

The concept of *uniformly perfect sets* introduced in [15]. A dozen equivalent descriptions are suggested in [16]. We use the following: a compact set $K \subset \mathbb{C}$ is uniformly perfect if K has at least two points and there exists $\varepsilon_0 > 0$ such that for any $z_0 \in K$ and $0 < r < diam(K)$, $K \cap \{z : \varepsilon_0 r < |z - z_0| < r\} \neq \emptyset$.

Theorem 4.3.16. *The set $K(\gamma)$, with (4.3), is uniformly perfect if and only if $\inf \gamma_s > 0$.*

Given $s \in \mathbb{N}$, let us uniformly distribute the mass 2^{-s} on each interval $I_{j,s}$ for $1 \leq j \leq 2^s$. We will denote λ_s the normalized (in this sense) Lebesgue measure on the set E_s, so $d\lambda_s = (2^s l_{j,s})^{-1} dt$ on $I_{j,s}$.

Theorem 4.3.17. *Suppose γ satisfies (4.3) and $c\big(K(\gamma)\big) > 0$. Then $\lambda_s \xrightarrow{*} \mu_{K(\gamma)}$.*

Remark 4.3.18. Clearly, any compact set K with non-empty interior cannot be equilibrium regarding 4.3.17 since $supp(\mu_K) \subset \partial K$. Then the set K in [14] is not equilibrium. Nor are geometrically symmetric Cantor type sets of positive capacity. Also the set $K^{(\alpha)}$ in [12] is not equilibrium.

Let define $\delta_s = \gamma_1 \gamma_2 \ldots \gamma_s$ and $\rho_s = \sum_{k=s+1}^{\infty} 2^{-k} \log \frac{1}{2\gamma_k}$. These parameters will be related to the smoothness of $g_{\mathbb{C}\setminus K(\gamma)}$. Clearly we have $\gamma_s = \frac{1}{2} \exp[2^s(\rho_s - \rho_{s-1})]$ and $\delta_s = 2^{-s} \exp[2^s \rho_s - \sum_{k=1}^{s-1} 2^k \rho_k - 2\rho_0]$. From this,

$$2^{-s} \log \delta_s \to 0 \quad \text{as} \quad s \to \infty.$$

In addition, (4.3) implies $\rho_s \geq 2^{-s} \log 16$ and $c\big(K(\gamma)\big) \leq 1/32$.

We proceed to evaluate the modulus of continuity of the Green's function corresponding to the set $K(\gamma)$. Let define function ω by the following conditions: $\omega(0) = 0$, $\omega(\delta) = \rho_1$ for $\delta \geq \delta_1$. If $s \geq 2$ then $\omega(\delta) = \rho_s + 2^{-s} \log \frac{\delta}{\delta_s}$ for $\delta_s \leq \delta \leq \delta_{s-1}/16$ and $\omega(\delta) = \rho_{s-1} - k_s(\delta_{s-1} - \delta_s)$ for $\delta_{s-1}/16 < \delta < \delta_{s-1}$ with $k_s = \frac{16}{15} \cdot 2^{-s} \delta_{s-1}^{-1} \log 8$.

In what follows the symbol $\sim$ denotes strong equivalence: $a_s \sim b_s$ means that $a_s = b_s\big(1 + o(1)\big)$ for $s \to \infty$.

Lemma 4.3.19. *The function ω is a concave modulus continuity. If $\gamma_s \to 0$ then for any positive constant C we have $\omega(\delta) \sim \rho_s + 2^{-s} \log \frac{C\delta}{\delta_s}$ as $\delta_s \to 0$ with $\delta_s \leq \delta < \delta_{s-1}$.*

Theorem 4.3.20. *Let γ satisfy (4.3) and $c\big(K(\gamma)\big) > 0$. If $\delta_s \le \delta < \delta_{s-1}$ then $\rho_s + 2^{-s}\log\frac{\delta}{\delta_s} < \omega(g_{K(\gamma)},\delta) < \rho_s + 2^{-s}\log\frac{16\delta}{\delta_s}$. If $\gamma_s \to 0$ then $\omega(g_{K(\gamma)},\delta) \sim \omega(\delta)$ as $\delta \to 0$.*

Here are some model examples of smoothness.

If a set K is uniformly perfect, then $g_{\mathbb{C}\setminus K}$ is Hölder continuous (see [16], p.119). By theorem 4.3.16, $g_{\mathbb{C}\setminus K(\gamma)}$ is Hölder continuous provided $\gamma_s = constant$. Now we can control the order α of Hölder continuity.

Example 4.3.21. Let choose $\gamma_s = \gamma_1 \le 1/32$ for all s. Then $\delta_s = \gamma_1^s$ and $\rho_s = 2^{-s}\log\frac{1}{2\gamma_1}$. Setting $\alpha = -\frac{\log 2}{\log\gamma_1}$ we have $g_{\mathbb{C}\setminus K} \in Lip\alpha$. Hence if we are given $0 < \alpha \le 1/5$ then choosing $\gamma_s = 2^{-1/\alpha}$ for all s, provides $g_{\mathbb{C}\setminus K} \in Lip\alpha$ and $g_{\mathbb{C}\setminus K} \notin Lip\beta$ for $\beta > \alpha$.

The next example is related to the function $h(\delta) = \big(\log\frac{1}{\delta}\big)^{-1}$ that defines the logarithmic measure of sets. Let us write $g_{\mathbb{C}\setminus K} \in Lip_h\alpha$ if for some positive constant C we have

$$g_{\mathbb{C}\setminus K}(z) \le C\, h^{\alpha}\big(dist(z,K)\big) \quad \text{for all} \quad z \in \mathbb{C}.$$

Example 4.3.22. Let $1/2 < \rho < 1$ and choose $\rho_s = \rho^s$ for $s > s_0$ where $\frac{\rho}{1-\rho}\log 16 < (2\rho)^{s_0}$. This condition provides $\gamma_s < 1/32$ for $s \ge s_0$. Suppose $\gamma_s = 1/32$ for $s \le s_0$. For large s we have $\delta_s = c\,2^{-s}\,\eta^{(2\rho)^s}$ with $\eta = \exp\left(\frac{2\rho-2}{2\rho-1}\right)$ and some constant c. Let us take $\alpha = \frac{\log 1/\rho}{\log 2\rho}$. Then $g_{\mathbb{C}\setminus K} \in Lip_h\alpha$ and $g_{\mathbb{C}\setminus K} \notin Lip_h\beta$ for $\beta > \alpha$. So given $\alpha > 0$, choosing $\rho = 2^{-\frac{\alpha}{1+\alpha}}$ provides the corresponding Green's function of the exact class $Lip_h\alpha$ (compare this to [12] and [14]).

Example 4.3.23. Let $\rho_s = 1/s$. Then $\gamma_s = \frac{1}{2}\exp\left(\frac{-2^s}{s^2-s}\right) < 1/32$ for $s \ge 8$. As above, all previous values of γ_s are 1/32. Here, $\delta_s = c\,2^{-s}\exp\Big[\frac{2^s}{s} - \sum_{k=1}^{s-1}\frac{2^k}{k}\Big]$. Summation by parts yields $\delta_s = c\,2^{-s}\exp\big[-2^{s+1}\big(s^{-2} + o(s^{-2})\big)\big]$. From this, $\omega(g_{\mathbb{C}\setminus K(\gamma)},\delta) \sim \frac{1}{s} \sim \frac{\log 2}{\log\log 1/\delta_s}$.

Example 4.3.24. Given $N \in \mathbb{N}$, let define $F_N(t) = \log\log\ldots\log t$ be the N-th iteration of the logarithmic function. Let $\rho_s = \big[F_N(s)\big]^{-1}$ for large enough s. Here $\rho_{k-1} - \rho_k \sim \big[k \cdot \log k \cdot F_2(k) \cdots F_{n-1}(k) \cdot F_N^2(k)\big]^{-1}$. Since $\delta_s = c\, 2^{-s} \exp\big[-\sum_{k=1}^{s} 2^k(\rho_{k-1} - \rho_k)\big]$ we have, as above, $s \sim \frac{\log 2}{\log\log 1/\delta_s}$. Thus $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \delta) \sim \big[F_{N+2}(1/\delta)\big]^{-1}$.

We see that a slower decrease of (ρ_s) implies a less smooth $g_{\mathbb{C}\setminus K(\gamma)}$ and conversely. If, in examples above, we take $\gamma_s = 1/32$ for $s < s_0$ with rather large s_0, then the set $K(\gamma)$ will have logarithmic capacity as close to $1/32$ as we wish.

Here is the open problem we mentioned before.

Problem. Given a modulus of continuity ω, to find $(\gamma_s)_{s=1}^{\infty}$ such that $\omega(g_{\mathbb{C}\setminus K(\gamma)}, \cdot)$ coincides with ω at least on some null sequence.

Chapter 5

Conclusion

First of all, Example 4.3.21 implies that the problem is not solvable unless ω has at most $Lip\frac{1}{5}$ smoothness. On the other hand, last example makes us to think there may be no worst smoothness condition on ω. Our hypothesis is that the problem is solvable for smooth enough ω, at least for concave or subadditive modulus of continuity. We concentrated on the twice differentiable concave ones. Main reason is the concave modulus of continuity in Lemma 4.3.19 which is an underpinning argument for our hypothesis. Another reason is that since ω is concave, it is differentiable except on countably many points. So, there is no danger to assume that ω is differentiable everywhere. But then, continuity of its derivative comes free because of the concavity that makes ω be a C^1 function. Moreover, again by the concavity, ω has also second derivative except on a set of Lebesgue measure zero by Rademacher's theorem and Alexandrov theorem [17]. Thus, assuming ω is twice differentiable is not a big deal in this sense.

We obtained some classifications with respect to the given examples but of course they are not sufficient for the general case, the preassigned smoothness.

All in all, we attacked the problem from every side we found its Achilles heel with different types of weapons ranging from trial and error to asymptotic estimation. But, unfortunately it resisted to be solved and remained unsolved.

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