COHOMOLOGICAL DIMENSION AND CUBIC SURFACES

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this thesis we give necessary and sufficient conditions for a curve C on a given cubic surface Q so that $Q - C$ is affine. We use this to give a simpler proof of $\mathrm{cd}(\mathbb{P}^3 - C) = 1$ by using Budach's method for these curves.We investigate the nature of curves on cubic surfaces such that the cubic surface minus these curves is an affine variety. We give combinatorial conditions for the existence of such curves.

Keywords: Cohomological dimension, Del Pezzo surfaces, Cubic surfaces, Intersection Theory.

ÖZET

KOHOMOLOJIK BOYUT VE ÜÇÜNCÜ DERECE YÜZEYLER

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Bu tezde $Q - C$'nin afin olması için, üçüncü derece bir yüzey Q üzerinde yatan C eğrisinin sağlaması gereken gerek ve yeter şartlar bulundu. Budach'ın metodu kullanılarak, bu eğriler için $\mathrm{cd}(\mathbb{P}^3 - C) = 1$ sonucunun basit bir ispatı verildi. Üçüncü derece bir yüzeyden bu eğrinin çıkarılmasıyla elde edilen uzayın afin olma koşulları incelendi ve bu çeşit eğriler için sayısal koşullar verildi.

Anahtar sözcükler: *Kohomolojik boyut, Del Pezzo yüzeyleri, Üçüncü derece yüzeyler, Kesişim Teorisi .*

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Contents

Chapter 1

Introduction

One of the most important aims in mathematical thinking is developing appropriate invariants to understand and classify the mathematical phenomena. Homology and cohomology groups are two such invariants which play great role in many areas of mathematics, but it is not easy to compute these groups in most cases. Here we discuss two invariants for an algebraic variety X in projective space, $\mathrm{cd}(X)$, the cohomological dimension of X, and $q(X)$. The cohomological dimension measures vanishing of cohomology groups and $q(X)$ measures the finite dimensionality.

We start with basic results and methods on cohomological dimension in Chapter 2, and try to the classify varieties of the form $\mathbb{P}^3 - X$ in terms of invariants cd and q. At the end of that chapter, we give an example to a method which is used to compute cohomological dimension generalized by Hartshorne [9] from an example of [2] and we state an open question emerging from this method: " Does there exist a nonsingular surface Q such that $Q-C$ is affine for given nonsingular irreducible curve C?" In this thesis we approach the problem backwards. We look for curves on cubic surfaces such that cubic surface minus these curves is an affine variety. We give combinatorial conditions for the existence of such curves.

A non-singular cubic surface is isomorphic to the projective plane with six points blown up. So our analysis of non-singular cubic surfaces can be reduced to the analysis of projective space with six points determined and the blow up map. So we use linear systems with base points as our setting. We use Harthshorne's work on intersection theory and cohomological dimension and Manin's work on cubic forms, see [9, 10, 12]. For details we refer to [1, 3].

Our main result is $\mathrm{cd}(\mathbb{P}^3 - C) = 1$ for curves satisfying certain combinatorial conditions on cubic surfaces, see Theorem 4.5. In our proof we apply the method given in Example 1 and combinatorial conditions on curves lying on cubic surfaces.

Chapter 2

Cohomological Dimension

2.1 Basic Definitions and Setting

Throughout this work, all schemes are noetherian, seperated and of finite Krull dimension over an algebraically closed field k of characteristic zero.

Definition 2.1. *We define the cohomological dimension of X, written* $\operatorname{cd} X$*, to be the smallest integer* $n \geq 0$ *such that* $H^i(X, \mathcal{F}) = 0$ *for all* $i > n$ *and for every quasi-coherent sheaf* $\mathcal{F}$ *on X.*

By a well known theorem of Grothendieck $\operatorname{cd} X \leq \dim X$, see [6].

Definition 2.2. *We will define another integer associated with a scheme X, denoted by* $q(X)$ *as the smallest integer* $n \geq -1$ *such that* $H^i(X, \mathcal{F})$ *is a finite dimensional* k *vector space for all* $i > n$ *and for every quasi-coherent sheaf* $\mathcal{F}$ *on X.*

From the definitions it is clear that $q(X) \leq \operatorname{cd}(X)$.

We can refine our setting by observing some properties of $\operatorname{cd} X$ and $q(X)$. First of all every quasi-coherent sheaf can be written as direct limit of coherent sheaves. Cohomology on noetherian schemes commute with direct limits so it is sufficient to consider only coherent sheaves in the definition of $\operatorname{cd} X$. Moreover if X is quasi-projective or non-singular, then we can even consider only locally free sheaves since every coherent sheaf can be written as a quotient of locally free sheaves [11]. The same considerations are also valid for the definition of $q(X)$.

Let X_{red} denote the reduced scheme associated to scheme X. Then any coherent sheaf X has a finite filtration with coherent sheaves on X_{red} as quotients. Conversely, any coherent sheaf on X_{red} can be considered as a coherent sheaf on X. Hence

$$\operatorname{cd}(X) = \operatorname{cd}(X_{red})$$
$$q(X) = q(X_{red})$$

We can similarly compare a scheme with its irreducible components. Let X a be scheme with irreducible components X_i, $i = 1, ..., r$, then

$$\operatorname{cd}(X) = \max(\operatorname{cd}(X_i))$$
$$q(X) = \max(q(X_i))$$

2.2 Classical Results

In this section we will give a brief summary of classical results in this area. We have already stated a well known theorem of Grothendieck; $\operatorname{cd} X \leq \dim X$. Another well known theorem is Serre's characterization of affine schemes [14]: "A scheme X is affine if and only if $\operatorname{cd} X = 0$". On the other side we have the "Lichtenbaum's Theorem", conjectured by Lichtenbaum, first proved by Groethendieck [8], and later proved by Kleiman [11], and Harthshorne [9] by different methods. Lichtenbaum's Theorem states that; for X irreducible of finite type over a field k $\operatorname{cd} X = \dim X$ if and only if X is proper. Projective varieties are examples of proper schemes while affine varieties are not. Since the other cases is either already done or can be reduced to this setting, we will study reduced, irreducible schemes with $0 < \operatorname{cd} X < \dim X$. Hence our main concern is on the schemes of the form $\mathbb{P}^3 - Y$ where Y is an algebraic variety.

On the other hand related with $q(X)$ there is a finiteness theorem of Serre and Groethendieck [EGA, III 3.2.1] which states that if X is proper over k, then $q(X) = -1$. Conversely, if $H^0(X,(F))$ is finite dimensional for every coherent sheaf $\mathcal{F}$, then none of the irreducible curves on X is affine, so X is proper. There is also another criterion developed by Harthshorne and Goodman [4], which states that "X is affine if and only if $q(X) = 0$ and X contains no complete curves".

2.3 Basic Results and Methods

After defining our setting in a clear way we can get some basic result on $\operatorname{cd} X$ and $q(X)$ by using properties of these invariants and sheaves. To start up we can bring an equivalent definition for $q(X)$:

Proposition 2.3 ([9], p 407)). *If X is proper over k, Y is a closed subset of X and $U = X - Y$, then $q(U)$ is the smallest integer n such that $H^i_Y(X, \mathcal{F})$ is finite dimensional for all $i > n+1$ and for all coherent sheaves $\mathcal{F}$. $H^i_Y(X, \mathcal{F})$ is the local cohomology defined by the right derived functor $\Gamma_Y(X, \cdot)$, where $\Gamma_Y(X, \mathcal{F})$ is the group of sections of $\mathcal{F}$ with support in Y for a given sheaf $\mathcal{F}$.*

Proof. Let X be proper over k, Y a closed subset of X and $U = X - Y$. Consider the local cohomology sequence [8], we get;

$$\cdots \longrightarrow H^i(X, \mathcal{F}) \longrightarrow H^i(U, \mathcal{F}) \longrightarrow H^{i+1}_Y(X, \mathcal{F}) \longrightarrow H^{i+1}(X, \mathcal{F}) \longrightarrow \cdots$$

Since X is proper,by finiteness theorem of Serre and Groethendick $q(X) = -1$ so $H^i(X, \mathcal{F})$ is finite dimensional for $i \geq 0$, two outside groups are finite dimensional. One of the middle groups is finite dimensional if and only if the other is. Since every coherent sheaf on U can be extended to a coherent sheaf of X, we have the proposition. □

Previous proposition states that the integer $q(U)$ depends only on local information around closed subset Y of X. On the other hand cohomological dimension $\operatorname{cd}(U)$ depends on X globally.

Another simple but useful result follows from considering the morphisms between algebraic varieties

Proposition 2.4 ([9], p 406). *Let $f : X' \to X$ be a finite morphism. Then*

$$\operatorname{cd} X' \leq \operatorname{cd} X \quad \textit{and} \quad q(X') \leq q(X).$$

If furthermore f is surjective, then we have equality in both cases.

After discussing basic properties of cohomological dimension and $q(X)$, we can look at the classification of varieties in $\mathbb{P}^2$. Note that any irreducible non-complete curve is affine. For a complete curve X, $q(X) = -1$ and $\operatorname{cd} X = 1$. For a non-complete curve X, $q(X) = 0$ and $\operatorname{cd} X = 0$.

The situation is more complicated in $\mathbb{P}^3$, so to start up let us work out with some examples [9];

X	$q(X)$	$\operatorname{cd}(X)$
complete surface	-1	2
affine surface	0	0
affine surface with point blown up	0	1
surface minus a point	1	1

The table above summarizes all the possibilities for a non-singular surface X. The first two lines of the table just follows from the theorem of Lichtenbaum; " for X irreducible of finite type over a field k $\operatorname{cd} X = \dim X$ if and only if X is proper"and Groethendick's characterization of affine schemes; "A scheme X is affine if and only if $\operatorname{cd} X = 0$".

For the proof of fourth line, consider an affine surface A minus a point p; $A - p = U$. Choose $\mathcal{F} = \mathcal{O}_U$, then $H^1(U, \mathcal{F}) = \infty$. U is not proper so $\operatorname{cd}(U) < \dim U = 2$. U is not proper so $q(U) \neq -1$ hence $q(U) = \operatorname{cd}(U) = 1$.

An affine surface with one point blown up is neither affine nor proper hence $\operatorname{cd}(X) = 1$ and $q(X) \neq -1$, moreover there is not a bijective morphism between an affine surface with one point blown up and affine surface minus a point, so $q(X) \neq 1$. The only possibility left is $q(X) = 0$.

Let Y be a closed subset of codimension p in projective n-space $\mathbb{P}^n$. We say that Y is a set-theoretic complete intersection if it is the intersection of p hypersurfaces $H_1, \cdots, H_p$, that is, $Y = H_1 \cap \cdots \cap H_p$. Then, $\mathbb{P}^n - Y = \bigcup_{i=1}^{p} \mathbb{P}^n - H_i$, that is, $\mathbb{P}^n - Y$ is the union of p open affine subsets $\mathbb{P}^n - H_i$.

By computing Čech cohomology, ([9], p 408) one can prove that

$$q(\mathbb{P}^n - Y) = \mathrm{cd}(\mathbb{P}^n - Y) = p - 1.$$

In fact to show that $\mathrm{cd}(\mathbb{P}^n - Y) \leq p - 1$ is trivial. Consider the open cover $\mathcal{U} = \bigcup_{i=1}^{p} \mathbb{P}^n - H_i$. We define a Čech complex for the open covering $\mathcal{U}$ as follows:

$$C^p(\mathcal{U}, \mathcal{F}) = \Pi_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0, \cdots, i_p})$$

where $U_{i_0, \cdots, i_p}$ denotes the intersection $U_{i_o} \cap \cdots \cap U_{i_p}$. Hence an element $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ can be determined by an element $\alpha_{i_0, \cdots, i_p} \in \mathcal{F}(U_{i_0, \cdots, i_p})$ for each (p+1) tuple $i_0 < \cdots < i_p$. We define the coboundary map $d : \; C^p \longrightarrow C^{p+1}$ by the following equation

$$(d\alpha)_{i_0, \cdots, i_{p+1}} = \Sigma_{k=0}^{p+1} (-1)^k \alpha_{i_0, \cdots, \hat{i}_k, \cdots, i_{p+1}}$$

where $\hat{i}_k$ denotes i_k is omitted. By definition, $\hat{H}^p(\mathcal{U}, \mathcal{F}) = H^p(C^*(\mathcal{U}, \mathcal{F}))$. Since the cover $\mathcal{U}$ consists of p open affine subsets, $\hat{H}^i(\mathcal{U}, \mathcal{F}) = 0$ for $i \geq p$.

The following theorem says that Čech cohomology groups are isomorphic to cohomology groups in general:

Theorem 2.5 ([10], Theorem 4.5 p 222). *Let X be a noetherian separated scheme, let $\mathcal{U}$ be an open affine cover of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$, we have $\hat{H}^i(\mathcal{U}, \mathcal{F}) \cong H^i(X, \mathcal{F})$.*

Hence $H^i(\mathbb{P}^n - Y, \mathcal{F}) = 0$ for all $i \geq p$, which means $\mathrm{cd}(\mathbb{P}^n - Y) \leq p - 1$.

This observation gives a necessary condition for Y to be a set theoretic complete intersection. Thus it can be used to prove that certain subvarieties of projective space are not complete intersections. In particular, we will use this condition to prove an old theorem of Hartshorne in a more simple way, see [7]. For classical proof which uses the Cohen-Macaulay property, for an exposition of the theorem with applications and examples see [13].

Theorem 2.6 ([9], Theorem 1.3 p 408). *If Y in $\mathbb{P}^n$ is a set theoretic complete intersection of* $\dim \geq 1$*, then it is connected in codimension 1, that is, Y can not be disconnected by removing any closed subset $Y' \subseteq Y$ of codimension ≤ 2 in Y.*

Proof. Let $Y = H_1 \cap \cdots \cap H_p$ be of codimension p. Suppose such Y' exists, then we can find a suitable linear space $\mathbb{P}^{p+1} \subseteq \mathbb{P}^n$ which does not meet Y', but meets Y in a disconnected curve. This curve $Y \cap \mathbb{P}^{p+1}$ is again a complete intersection, so we will prove that a disconnected curve is not a complete intersection. For this it is enough to prove the following:

Let $Y_1,\ Y_2$ be curves in $\mathbb{P}^n$ of codimension $p = n-1$ such that $Y_1 \cap Y_2 = \emptyset$, then $H^{n-1}(\mathbb{P}^n - (Y_1 \cup Y_2), \mathcal{F}) \neq 0$ for some $\mathcal{F}$.

This will show that

$$\operatorname{cd}(\mathbb{P}^n - (Y_1 \cup Y_2)) > n-1 = p > p-1.$$

Hence $Y_1 \cup Y_2$ can not be a complete intersection.

Choose $\mathcal{F} = \mathcal{O}_p(-n-1)$, so $H^n(\mathbb{P}^n, \mathcal{F}) \cong k$. Applying local cohomology sequence, we get

$$\cdots \longrightarrow H^n_{Y_1}(\mathcal{F}) \longrightarrow H^n(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^n(\mathbb{P}^n - Y_1, \mathcal{F}) = 0.$$

The last term is zero by Lichtenbaum theorem; $\operatorname{cd} X = \dim X$ if and only if X is proper, and $\mathbb{P}^n - Y_1$ is not proper. Hence $\dim H^n_{Y_1}(\mathcal{F}) \geq 1$. Similarly $\dim H^n_{Y_2}(\mathcal{F}) \geq 1$.

Since $Y_1 \cap Y_2 = \emptyset$, we have $H^n_{Y_1 \cup Y_2}(\mathcal{F}) = H^n_{Y_1}(\mathcal{F}) \oplus H^n_{Y_2}(\mathcal{F})$, and hence $\dim H^n_{Y_1 \cup Y_2}(\mathcal{F}) \geq 2$.

Writing the local cohomology sequence for $Y_1 \cup Y_2$, we get

$$\cdots \longrightarrow H^{n-1}(\mathbb{P}^n - (Y_1 \cup Y_2), \mathcal{F}) \longrightarrow H^n_{Y_1 \cup Y_2}(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^n(\mathbb{P}^n, \mathcal{F}) \longrightarrow 0$$

The right hand side term is isomorphic to k, hence has dimension 1. For the middle term, we have $\dim H^n_{Y_1 \cup Y_2}(\mathbb{P}^n, \mathcal{F}) \geq 2$. Thus $H^{n-1}(\mathbb{P}^n - (Y_1 \cup Y_2), \mathcal{F}) \neq 0$. □

2.4 A Method for computing cohomological dimension of $\mathbb{P}^3 - C$

The following proposition provides us another technique for calculating cohomological dimension of some algebraic varieties. In particular, we will focus on calculating the cohomological dimension of $\mathbb{P}^3$ minus an irreducible curve.

Proposition 2.7 ([9], Proposition 1.4 p409). *Let X be a scheme, and let Y be a closed subscheme of X. Assume*

1. $\operatorname{cd} Y = l$
2. $H^i_Y(\mathcal{F}) = 0$ *for all $i > m$ and for all coherent sheaves $\mathcal{F}$ on X*
3. $\operatorname{cd}(X - Y) = n$

   or $q(X - Y) = n$.

Then

$$\operatorname{cd}(X) \leq \max(l + m, n) \quad \text{or}$$

$$q(x) \leq \max(l + m, n)$$

To illustrate the techniques involved, we consider the following example.

Example 1. Let C be a non-singular quartic curve in $\mathbb{P}^3$. This is a non-singular curve C in $\mathbb{P}^3$, of degree 4, which is abstractly isomorphic to $\mathbb{P}^1$. We will calculate

$\mathrm{cd}(\mathbb{P}^3 - C)$. By Castelnuevo type argument one can prove that such a curve C lies on a quadric surface. In fact, there are two classes of curves of degree 4 in $\mathbb{P}^3$; the rational quartic curves and elliptic quartic curves. Elliptic ones are the complete intersections of two quadric surfaces. We know that $\mathrm{cd}(\mathbb{P}^n - X) = p-1$ for a complete intersection variety of codimension p in $\mathbb{P}^n$, hence $\mathrm{cd}(\mathbb{P}^3 - C) = 1$ for an elliptic quartic curve C

Let X be $\mathbb{P}^3 - C$ and Y be $Q - C$ in the Proposition 2.7.

We claim that Y is affine. In order to prove that Y is affine we will use the following criteria [5]:

Let U be a scheme satisfying the following conditions

(i) $\dim_k H^1(U, \mathcal{F}) < \infty$ for every coherent sheaf $\mathcal{F}$ on U,

(ii) U contains no complete curves.

Then U is affine, and conversely.

Note:$(i) \Leftrightarrow q(X) < 1$ in general. By [[9], p 408], it is enough to check that $C^2 > 0$ in order to prove that $q(X) = 0$. After defining the intersection theory on quadric in following paragraph, we will show that $C^2 > 0$ for rational quartic curve.

In order to prove these two conditions, we need information on the intersection theory of the quadric surface. The intersection theory on quadric surface Q can be summarized as follows: $Pic\ Q \cong \mathbb{Z} \oplus \mathbb{Z}$ and we can take as generators lines $\mathbf{a}$ of type (1,0) and $\mathbf{b}$ of type (0,1). Then $\mathbf{a}^2 = 0$, $\mathbf{b}^2 = 0$, $\mathbf{a} \cdot \mathbf{b} = 1$. Clearly two lines in the same family are skew and two lines from different families intersect at a single point. Hence the intersection pairing on Q is given as follows: If C has type (a,b) and C' has type (a', b') then $C \cdot C' = ab' + a'b$.

If C is a nonhyperelliptic curve of genus $g \geq 3$, the embedding $C \to \mathbb{P}^{g-1}$ determined by the canonical linear system is the *canonical embedding* of C, and its image, which is a curve of degree $2g-2$ is a canonical curve. Back to our example, rational quartic curve C is of type (1,3). In fact if C is a nonhyperelliptic curve of genus 3, then its canonical embedding is a quartic curve. It has strictly positive intersection with any other curve lying on Q and $C^2 = 6$. So $q(Q-C) = 0$ and $Q - C$ contains no complete curves hence it is affine.

We have shown that $Y = Q-C$ is affine, so $l = 0$ in the setting of Proposition 2.7. Y is locally defined by a single equation so $H^i_Y(\mathcal{F}) = 0$ for all $i > 1$,hence $m = 1$ and $X - Y = \mathbb{P}^3 - Q$ is affine then $n = 0$. Hence by Proposition 2.7 $\operatorname{cd} \mathbb{P}^3 - C \leq 1$. Since $\mathbb{P}^3 - C$ is not affine, $\operatorname{cd}(\mathbb{P}^3 - C) = 1$.

2.5 Conclusion

The technique of subtracting a curve form a surface will work for any curve C in $\mathbb{P}^3$ which lies on surface Q such that $Q - C$ is affine. For any irreducible curve C the existence of such surface Q is not known. We approach the problem backwards. We aim to find such irreducible curves C for given surface Q. In order to prove $Q - C$ is affine we must know the intersection theory on the surface Q. The first natural candidate for such surfaces are the Del Pezzo surfaces. We will try to find out which curves on Del Pezzo surfaces will satisfy the setting we have described in this chapter. In other words given a Del Pezzo surface Q, which curves will satisfy the condition that $Q - C$ is affine? As the first case, we will look at cubic surfaces. Therefore next chapter is a summary of related information on cubic surfaces.

Chapter 3

The cubic surface in $\mathbb{P}^3$

In this chapter we will review what is known about the non-singular cubic surfaces in $\mathbb{P}^3$, which are isomorphic to the projective plane with six points blown up. We will use this isomorphism to study the geometry of curves on the cubic surface. In order to prove this isomorphism we will use the linear system of plane cubic curves with six base points so we start with some general background about linear system withs base points. For the classical results on surfaces we refer to [10, 8, 12]. For the Del Pezzo surfaces we refer to [3, pp 23-69].

3.1 Linear Systems With Base Points

This section closely follows [10, pp 395-397]. Let X be a surface, $|D|$ a complete linear system of curves on X, and let $P_1, \cdots, P_r$ be distinct points of X. Consider the sublinear system δ consisting of divisors $D \in |D|$ which pass through the points $P_1, \cdots, P_r$. We denote it by $|D - P_1 - \cdots - P_r|$. We say that $P_1, \cdots, P_r$ are the assigned base points of δ.

Let $\pi : X' \to X$ be the morphism obtained by blowing up $P_1, \cdots, P_r$ and let $E_1, \cdots, E_r$ be the exceptional curves. Then by the map $D \mapsto \pi^* D - E_1 - \cdots - E_r$ we get a natural one-to-one correspondence between the elements of δ on X and the elements of the complete linear system $\delta' = |\pi^* D - E_1 - \cdots - E_r|$. Note that the divisor $\pi^* D - E_1 - \cdots - E_r$ is effective on X' if and only if D passes through the points $P_1, \cdots, P_r$. The linear system δ' on X' may or may not have base points. Any base point of δ', considered as an infinitely near point of X, is called an unassigned base point of δ. These concepts are also well defined if some of the P_i are infinitely near point of X, or if they have multiplicities greater than 1.

After fixing our definitions and language, we can talk about linear systems on different blown up models of X in terms of suitable linear systems with suitable assigned base points on X.

Remark 3.1. *A complete linear system $|D|$ is very ample if and only if*

(a) $|D|$ has no base points

(b) for every $P \in X$, $|D - P|$ has no unassigned base points

For any distinct closed points Q and P there exists a divisor $D \in |D|$ such that $P \in \operatorname{Supp} D$ and $Q \notin \operatorname{Supp} D$ which means that$Q$ is not a base point of $|D - P|$. Given a closed point and a tangent vector $v \in (m_P/m_{P^2})$, there is a $D \in |D|$ such that $P \in \operatorname{Supp} D$, but $P \notin T_P(D)$ which means that $|D - P|$ has no unassigned base points infinitely near to P.

Remark 3.2. *We can interpret Remark 3.1 in terms of the dimension drops when we assign a base point which was not already unassigned base point. We can bring another condition on $|D|$ to be very ample as follows: $|D|$ is very ample if and only if for any two points P, $Q \in X$, (including the case Q infinitely near to P),*

$$\dim |D - P - Q| = \dim |D| - 2.$$

In other words, dimension drops by exactly one when we assign a base point which was not already an unassigned point of a linear system.

Remark 3.3. *If we apply Remark 3.1 to a blown up model of X, we observe that if $\delta = |D - P_1 - \cdots - P_r|$ is a linear system with assigned base points on X, then the associated linear system δ' on X' is very ample on X' if and only if* *(a)* δ *has no assigned base points and*

(b) for every $P \in X$*, (including the infinitely near points on* X'*)* $\delta - P$ *has no assigned base points.*

We will construct Del Pezzo surfaces by blowing up $\mathbb{P}^2$. Remark 3.3 provides us the necessary and sufficient conditions for a linear system on a given Del Pezzo surface to very ample by looking at the corresponding linear system with assigned base points at blow up points on $\mathbb{P}^2$.

3.2 Del Pezzo Surfaces

This section follows [10, pp 397-401]. Now we will focus on the particular situation of linear systems of plane curves of fixed degree with assigned base points. The first natural question that arises is whether they have unassigned base points or not. If not, we study the corresponding morphism of the blown up model to a projective space. We will use a linear system of cubic curves with six base points in order to get the cubic surface in $\mathbb{P}^3$. First we will consider linear system of conics with base points. In this context the word conic and cubic are used to mean any effective divisor in the plane of degree two and three, respectively.

Proposition 3.4 ([10], Proposition 4.1 p397). *Let* δ *be the linear system of conics in* $\mathbb{P}^2$ *with assigned base points* $P_1, \cdots, P_r$ *and assume that no three of the* P_i *are collinear. If* $r \leq 4$*, then* δ *has no unassigned base points. This result remains true if* P_2 *is infinitely near* P_1*.*

It is sufficient to prove the proposition for $r = 4$. The proposition states that if no three of the blow up points are collinear, then the linear system δ and also associated linear system δ' are very ample. Since δ is very ample, by Remark 3.2, for each blow up point, $\dim \delta$ drops one. The system of conics without base points has dimension 5, hence for $r \leq 5$, $\dim \delta = 5 - r$. So for $r = 5$, there is a unique conic passing through blow points $P_1, \cdots, P_5$.

Generalizing this result for linear system of cubic curves and using the results on conics, we get the following proposition.

Proposition 3.5 ([10], Proposition 4.3 p 399). *Let δ be the linear system of plane cubic curves with assigned base points $P_1, \cdots, P_r$ and assume that no 4 of the P_i are collinear, and no 7 of them lie on a conic. If $r \leq 7$, then δ has no unassigned base points. This result remains true if P_2 is infinitely near P_1.*

As in previous proposition, it is sufficient to consider maximal value of r. If no 4 of the blow up points are collinear and no 7 of them lie on a conic then δ is very ample. The system of cubics without base points has dimension 9. So with the same hypotheses in proposition, $\dim \delta = 9 - r$ for $r \leq 8$.

We will work with Del Pezzo surfaces of degree 3 to 6. So as a special case of Propositon 3.5, we will state the following theorem. Notice that the conditions of the theorem satisfy the hypotheses of the above proposition. So the linear system δ has no unassigned base points and the result remains true if P_2 is infinitely near P_1. By Remark 3.3,the associated linear system δ' is very ample.

Theorem 3.6 ([10], Theorem 4.6 p 400). *Let δ be the linear system of plane cubic curves with assigned (ordinary) base points $P_1, \cdots, P_r$ and assume that no 3 of the P_i are collinear, and no 6 of them lie on a conic. If $r \leq 6$, then the corresponding linear system δ' on the surface X' obtained from $\mathbb{P}^2$ by blowing up $P_1, \cdots, P_r$, is very ample.*

The following corollary of this theorem is the first step to construct non-singular cubic surface in $\mathbb{P}^3$. Moreover it gives the characterization of the canonical sheaf of a surface obtained by blowing up $\mathbb{P}^2$ at i points for $i \leq 6$. This characterization provides us the relation between Del Pezzo surfaces of degree i and surfaces obtained by blowing up $\mathbb{P}^2$ at $9 - i$ points.

Corollary 3.7 ([10], Corollary 4.7 p 400). *With the same hypotheses, for each $r = 0, 1, \cdots, 6$, we obtain an embedding of X' in $\mathbb{P}^{9-r}$ as a surface of degree $9 - r$, whose canonical sheaf $\omega_{X'}$ is isomorphic to $\mathcal{O}_{X'}(-1)$. In particular, for $r = 6$, we obtain a nonsingular cubic surface in $\mathbb{P}^3$.*

We can give the definition of Del Pezzo surfaces after setting our definitions and developing the required results.

Definition 3.8. *A Del Pezzo surface is defined to be a surface X of degree d in $\mathbb{P}^d$ such that $\omega_X \cong \mathcal{O}_{X'}(-1)$.*

Corollary 3.7 shows a way to construct Del Pezzo surfaces of degree $d = 3, \cdots 9$. It states that for a cubic surface in $\mathbb{P}^3$, obtained by blowing up $\mathbb{P}^2$ at 6 points, the condition $\omega_X \cong \mathcal{O}_{X'}(-1)$ is satisfied. The family of all cubic surfaces in $\mathbb{P}^3$ has dimension $\dim H^0(\mathcal{O}_{\mathbb{P}^3}(3)) - 1 = 19$. In order to find the dimension of the family of cubic surfaces obtained by blowing up $\mathbb{P}^2$ at 6 points, one must count the choice of 6 points in the plane,and the automorphisms of $\mathbb{P}^2$ and $\mathbb{P}^3$. The choice of 6 points in the plane contributes 12 and automorphisms of $\mathbb{P}^3$ contributes 15 to this dimension. Automorphisms of $\mathbb{P}^2$ must not be counted so we subtract 8 from the sum. Hence dimension of the family of cubic surfaces obtained by blowing up $\mathbb{P}^2$ at 6 points is also 19. This proves that almost all non-singular cubic surfaces are obtained by blowing up $\mathbb{P}^2$ at 6 points.

3.3 Cubic surfaces in $\mathbb{P}^3$

This section basically follows [10, pp 401-405]. From now on, we will specialize to the case of cubic surfaces in $\mathbb{P}^3$ and study the properties of it. Let $P_1, \cdots, P_6$ be six points in the plane no 3 of them are collinear and not all 6 of them lie on a conic. Let δ be the linear system of plane cubic curves through $P_1, \cdots, P_6$ and let Q be the non-singular cubic surface in $\mathbb{P}^3$ obtained by the construction given in Corollary 3.7. Hence X is isomorphic to $\mathbb{P}^2$ with the points $P_1, \cdots, P_6$ blown up. Let $\pi : Q \to \mathbb{P}^2$ be the projection. Let $E_1, \cdots, E_6 \subseteq Q$ be the exceptional curves, and let $e_1, \cdots, e_6 \in \operatorname{Pic} Q$ be their linear equivalence classes. Let $l \in \operatorname{Pic} Q$ be the class of the pullback of a line in $\mathbb{P}^2$.

Following proposition summarizes the geometry on the nonsingular cubic surface. It gives formulas for the genus and degree of a divisor, intersection pairings of divisors, hyperplane sections on Q and canonical sheaf of Q, by using the isomorphism $\operatorname{Pic} Q \cong \mathbb{Z}^7$.

Proposition 3.9 ([10], Proposition 4.8 p 401). *Let X be the cubic surface in $\mathbb{P}^3$. Then*

(a) $\operatorname{Pic} Q \cong \mathbb{Z}^7$*, generated by* $l, e_1, \cdots, e_6$*;*

(b) the intersection pairing on Q is given by $l^2 = 1,\ e_i^2 = -1,\ l \cdot e_i = 0,\ e_i \cdot e_j = 0$ for $i \neq j$;

(c) the hyperplane section h is $3l - \sum e_i$;

(d) the canonical class is $K = -h = -3l + \sum e_i$;

(e) if D is any effective divisor on Q, $D \sim al - \sum b_i e_i$, then the degree of D, as a curve in $\mathbb{P}^3$, is

$$d = 3a - \sum b_i;$$

(f) the self-intersection of D is $D^2 = a^2 - \sum b_i^2$;

(g) the arithmetic genus of D is

$$p_a(D) = \frac{1}{2}(D^2 - d) + 1 = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum b_i(b_i - 1).$$

If C is an irreducible curve on X other than exceptional curves, then $\pi(C)$ is an irreducible plane curve C_0. Let C_0 have degree a and has multiplicity b_i at each point P_i. Note that $C_0 \sim a \cdot l$, where l is a line in $\mathbb{P}^2$, $\pi^*(C_0) = C + \sum b_i E_i$. Hence, we get $C \sim a \cdot l - \sum b_i e_i$. Therefore for any $a, b_1, \cdots, b_6$ we have an irreducible curve C on X in the class $a \cdot l - \sum b_i e_i$. This argument shows us that the study of certain plane curves will give us information about curves on X.

The following theorem characterizes the exceptional curves on Del Pezzo surfaces in general:

Theorem 3.10 ([12], Theorem 26.2 p 135). *Let Q be a Del Pezzo surface of degree $1 \leq d \leq 7$, and let $f : Q \to \mathbb{P}^2$ be its representation in the form of a monoidal transformation of the plane with as center the union of $r = 9 - d$ points $P_1 \cdots P_r$. Then the following assertion hold:*

(i) *The map* $D \to$ (*class of* $\mathcal{O}_Q(D)) \in \operatorname{Pic} Q$ *establishes a one to one and onto correspondence between exceptional curves on* Q *and exceptional classes in the Picard group. These classes generate the Picard group.*

(ii) *The image of* D *in* $\mathbb{P}^2$ *of an arbitrary exceptional curve* $D \subset Q$ *is of the following types:*

  (a) *one of the points* P_i;

  (b) *a line passing through two of the points* P_i;

  (c) *a conic passing through five of the points* P_i;

  (d) *a cubic passing through seven of the points* P_i *such that one them is a double point;*

  (e) *a quartic passing through eight of the points* P_i *such that three of them are double points;*

  (f) *a quintic passing through eight of the points* P_i *such that six of them are double points;*

  (g) *a sextic passing through eight of the points* P_i *such that seven of them are double points and one is a triple point.*

  (of course only for $r = 8$ *the whole list must be used; for* $r = 7$ *only (a)-(d);for* $r = 6, 5$ *only (a)-(c); for* $r = 4, 3$ *only (a)-(c))*

Although our aim is to analyze cubic surface, it is worthwhile to state Theorem 3.10 because most of the results we are stating are valid for all Del Pezzo surfaces and as a second step we aim to use the method illustrated in Example 2.7 to this wider setting. Now let us turn back to cubic surfaces and refine the previous theorem for our purposes.

Theorem 3.11 ([10], Theorem 4.9 p 402). *The cubic surface* Q *contains exactly 27 lines. Each one has self-intersection* -1*, and they are the only irreducible curves with negative self-intersection on* Q*. They are*

(a) *the exceptional curves* E_i, $i = 1, \cdots, 6$ *(six of these),*

(b) *the strict transform F_{ij} of the line in $\mathbb{P}^2$ containing P_i and P_j, $1 \leq i < j \leq 6$ (fifteen of these), and*

(c) *the strict transform G_j of the conic in $\mathbb{P}^2$ containing the five P_i for $i \neq j$, $j = 1, \cdots, 6$ (six of these).*

The 27 lines mentioned in the Theorem 3.11 have a high degree of symmetry and involves lots of geometry. We are concerned with intersection theory on cubic surface so our main concern will be the configuration of these 27 lines. The following proposition provides us the required information.

Proposition 3.12 ([10], Proposition 4.10, p 403). *Let Q be the cubic surface, and let $E'_1, \cdots, E'_6$ be any subset of six mutually skew lines chosen from among the 27 lines on Q. Then there is another morphism $\pi' : Q \to \mathbb{P}^3$, making Q isomorphic to that $\mathbb{P}^2$ with 6 points $P'_1, \cdots, P'_6$ blown up (no 3 collinear and not all 6 on a conic), such that $E'_1, \cdots, E'_6$ are the exceptional curves for π'.*

The proposition states that any 6 mutually skew lines among the 27 lines has the same properties with $E_1, \cdots, E_6$. In other words, the configuration of 27 lines are determined by 6 mutually skew lines among them. Naming the lines $E_1, \cdots, E_6$ determines the remaining 21 lines: F_{ij} is the unique line which meets E_i and E_j but not any other E_k's; G_i is the unique line which meets all E_j except E_i. Proposition states that, every ordered set of 6 mutually skew lines among the 27 lines there is a unique automorphism of the configuration sending $E_1, \cdots, E_6$ to those 6 mutually skew lines.

Chapter 4

Cohomological Dimension and Cubic Surface

4.1 Introduction

In Chapter 2, we discussed the concept and basic properties of cohomological dimension and illustrated a method to compute the cohomological dimension of $\mathbb{P}^3$ minus an irreducible curve. We proved that $Q - C$ is affine for quadric surface Q and rational quartic curve C. We concluded that $\mathrm{cd}(\mathbb{P}^3 - C) = 1$ by proposition 2.7. In the previous chapter, we discussed the properties of cubic surfaces. We constructed the cubic surface by blowing up the projective plane at 6 points, and analyzed the properties of curves lying on this surface. Now, we will attack the problem: "Which curves satisfies the condition $\mathrm{cd}(\mathbb{P}^3 - C) = 1$ on the cubic surface Q?", in our new setting described by linear systems and cubic surfaces.

Remember that, in order $U = Q - C$ to be affine, we have used the following criteria:

(i) $\dim_k H^1(U, \mathcal{F}) < \infty$ for every coherent sheaf $\mathcal{F}$ on U,

(ii) U contains no complete curves.

The following criteria (Nakai-Moishezon Criterion)(see [10], p 365) which can also be taken as a definition of ampleness, provides us a new setting for the required curves:

> A divisor D on the surface X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curve C on X.

If we choose our curve C to correspond to an ample divisor then it will satisfy the condition (ii) since Nakai-Moishezon Criterion implies that C intersects all the irreducible curves on Q and then $Q - C$ contains no complete curves and $C^2 > 0$ so $q(Q - C) = 0$. Hence, we will search for the curves that correspond to ample divisors on a given cubic surface Q.

4.2 Ample divisors on cubic surface

After analyzing the properties of curves on cubic surface and stating our problem in our new setting, we can turn our attention to ample divisors which contain curves on a given cubic surface. We will first state a lemma which gives a combinatorial condition for a divisor class to be very ample on the cubic surface. The preceding theorem gives necessary conditions for a divisor D on the cubic surface to be ample in a more general setting.

Lemma 4.1 ([10], Lemma 4.12 p 405). *Let $D \sim al - \sum b_i e_i$ be divisor class on the cubic surface X, and suppose that $b_1 \geq b_2 \geq \cdots \geq b_6 > 0$ and $a \geq b_1 + b_2 + b_5$. Then D is very ample.*

For the proof, we choose a basis for $\operatorname{Pic} Q$ such that one of the divisors is very ample and others are in a linear system without base points. We write a given divisor $D \sim al - \Sigma b_i e_i$ in this basis.The conditions imposed on the coefficients of this bases give the required condition in terms of a and b_i's.

Theorem 4.2 ([10], Theorem 4.11 p 405). *The following conditions are equivalent for a divisor D on the cubic surface X:*

(i) D is very ample;

(ii) D is ample;

(iii) $D^2 > 0$*, and for every line* $L \subseteq X,\ D \cdot L > 0$*;*

(iv) for every line $L \subseteq X,\ D \cdot L > 0$.

The first three implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ follow from the Nakai-Moishezon Criterion. For the last implication $(iv) \Rightarrow (i)$ we do not have to consider all the lines on cubic surface. By proposition 3.12 we can choose 6 mutually skew lines $E'_1 \cdots E'_6$ among 27 lines. If we make our choice such that $D \cdot E'_i \geq D \cdot E'_{i+1}$, then F_{12}, the line passing through points P_1 and P_2 will be candidate in the third choice. We will have $D \cdot F_{12} \geq D \cdot E'_3$ which will satisfy the combinatorial condition of lemma 4.1. Hence D is a very ample divisor.

Theorem 4.2 gives us the necessary and sufficient conditions for a divisor D to be ample on a cubic surface. Now we are left with the question whether D contains an irreducible curve or not. Following corollary provides the answer:

Corollary 4.3 ([10], Corollary 4.13 p 406). *Let* $D \sim al - \sum b_i e_i$ *be a divisor class on* X*. Then:*

(a) D is ample $\Leftrightarrow$ *very ample* $\Leftrightarrow$ $b_i > 0$ *for each* i*, and* $a > b_i + b_j$ *for each* $i,\ j$*, and* $2a > \sum_{i \neq j} b_i$ *for each* j*;*

(b) in any divisor class satisfying the conditions of (a), there is an irreducible non-singular curve.

Proof. (a) The first implication is just the Nakai-Moishezon Criterion, and the second one is obtained from (i)$\Leftrightarrow$ (iv) in Theorem 4.2. Remember that $F_{ij} \sim l - e_i - e_j$, hence $D \cdot F_{ij} > 0$ means $a > b_i + b_j$, and $G_j \sim 2l - \sum_{i \neq j} e_i$, so $D \cdot G_j > 0$ means $2a > \sum_{i \neq j} b_i$.

(b) Proof is a consequence of Bertini's Theorem:

Theorem 4.4 ([10], Theorem 8.18 p 179). *Let X be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where k is an algebraically closed field. Then there exists a hyperplane $H \subseteq \mathbb{P}^n_k$, not containing X, and such that the scheme $H \cap X$ is regular at every point, (in addition, if $\dim X \geq 2$, then $H \cap X$ is connected, hence irreducible, and so $H \cap X$ is a non-singular variety). Furthermore, the set of hyperplanes with the property forms an open dense subset of the complete linear system $|H|$, considered as a projective space.*

□

Although we proved Corollary 4.3 for nonsingular cubic surface, analogous of this corollary can be proved for all Del Pezzo surfaces of degree 3 to 9. We know all the lines on Del Pezzo surface of degree 1 to 7 by Theorem 3.10. So we can prove the validity of Theorem 4.3 for Del Pezzo surfaces degree 3 to 7. Characterization of very ample divisors on the surface obtained by blowing up $\mathbb{P}^2$ at a single point, which is the rational ruled surface and also Del Pezzo surface of degree 8, is different(see [10], Corollary 2.18 p 380).

Up to now, we proved that a divisor $D \sim al - \sum b_i e_i$ satisfying the conditions $a > b_i + b_j$ for each i, j, and $2a > \sum_{i \neq j} b_i$ for each j contains an irreducible smooth curve C. Moreover self intersection of C is positive and since D is very ample C intersects with all curves on cubic surface Q. We are ready to state our main theorem:

Theorem 4.5. *Let Q be a cubic surface in $\mathbb{P}^3$ and $D \sim al - \sum b_i e_i$ be a divisor satisfying the conditions $a > b_i + b_j$ for each i, j, and $2a > \sum_{i \neq j} b_i$ for each j. Let C denote a curve in divisor class D, then $cd(\mathbb{P}^3 - C) = 1$.*

Proof. We will apply the same technique we illustrated in Example 1. Let $X = \mathbb{P}^3 - Q$ and $Y = Q - C$. For any curve C on cubic surfaces, $H^i(Q - C, \mathcal{F})$ is finite dimensional for $i > 0$. In fact the combinatorial conditions $a > b_i + b_j$ for each i, j and $2a > \sum_{i \neq j} b_i$ for each j implies that the self intersection of the curve C is strictly positive; $C^2 = a^2 - \sum b_i > 0$. Moreover D is very ample, hence C intersects all curves on Q, which means that $Q - C$ contains no complete curves.

Therefore $Q - C$ is affine and $\mathrm{cd}(Q - C) = 0$. Since $Y = Q - C$ is locally defined by a single equation, $H^i_Y(\mathcal{F}) = 0$ for all $i > 1$ and $X - Y = \mathbb{P}^3 - Q$ is affine then $\mathrm{cd}(X - Y) = 0$. By Proposition 2.7, $\mathrm{cd}(\mathbb{P}^3 - C) \leq 1$. $\mathbb{P}^3 - C$ is not affine so $\mathrm{cd}(\mathbb{P}^3 - C) \neq 0$, hence $\mathrm{cd}(\mathbb{P}^3 - C) = 1$. □

In order to understand the combinatorial conditions ; $a > b_i + b_j$ for each i, j, and $2a > \sum_{i \neq j} b_i$, consider the tuples in the following table. They satisfy the combinatorial conditions of theorem, so they correspond to curves with given degree and genus. By Theorem 4.5, cohomological dimension of $\mathbb{P}^3$ minus one of these curves is one.

Tuple	degree	genus
$(8, 4, 3, 3, 2, 2, 2)$	8	6
$(10, 4, 4, 4, 4, 3, 3)$	8	6
$(10, 5, 4, 4, 3, 3, 2)$	9	7
$(11, 5, 4, 4, 4, 4, 3)$	9	8
$(9, 4, 4, 4, 4, 1, 1)$	9	10

Chapter 5

Conclusion

In this thesis we have proved that the method of Budach can be applied to cubic surface for curves satisfying some conditions. We have tried to present the required results for cubic surface in accordance with Del Pezzo surfaces of other degrees. The same arguments we did for cubic surface Q in order to prove the necessary conditions for a curve C on Q so that $Q - C$ is affine, also works for Del Pezzo surfaces of other degrees. Remember that a Del Pezzo surface of degree d lies in $\mathbb{P}^d$. Hartshorne generalizes Budach's method for higher dimensions hence we must use this generalization Proposition 2.7. The problem is that we do not know cohomological dimension of $\mathbb{P}^d$ minus a Del Pezzo surface of degree d. This needs more local analysis. We think that Del Pezzo surfaces of other degrees are good candidates for a second step towards.

In the appendix, we gave a complete classification of varieties in $\mathbb{P}^3$ by using the invariants $\mathrm{cd}(X)$ and $q(X)$. We do not have the complete classification in $\mathbb{P}^4$. Generalization of this approach may also be used on specific 3-folds for this classification.

Appendix A

Classification in $\mathbb{P}^3$

In Chapter 2, we have started the classification of varieties of the form $\mathbb{P}^3 - X$ and gave some examples depending on the invariants q and cd. Then, we have focused on the method of Budach and paid attention to the problem whether does there exists a surface Q containing the irreducible non-singular curve C such that $Q - C$ is affine. In fact, the situation in $\mathbb{P}^3$ is completely done by using topological and algebraic arguments. Here is the complete classification ([9], p 445).

Description of X	Invariants of $\mathbb{P}^3 - X$	
	q	cd
X of pure dim $= 2$	0	0
X connected, and has some components of dim $= 1$	1	1
X disconnected, of pure dim $= 1$	1	2
X has an isolated point	2	2
X is empty	-1	3

Bibliography

[1] Arnaud Beauville, *Complex algebraic surfaces*, second ed., London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996, Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid.

[2] L. Budach, *Quotientenfunktoren und Erweiterungstheorie*, Mathematische Forschungsberichte, Band XXII, VEB Deutscher Verlag der Wissenschaften, Berlin, 1967.

[3] Michel Demazure and Henry Charles Pinkham (eds.), *Séminaire sur les Singularités des Surfaces*, Lecture Notes in Mathematics, vol. 777, Springer, Berlin, 1980, Held at the Centre de Mathématiques de l'École Polytechnique, Palaiseau, 1976–1977.

[4] Jacob Goodman and Robin Hartshorne, *Schemes with finite-dimensional cohomology groups*, Amer. J. Math. **91** (1969), 258–266.

[5] Jacob Eli Goodman, *Affine open subsets of algebraic varieties and ample divisors*, Ann. of Math. (2) **89** (1969), 160–183.

[6] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221.

[7] Robin Hartshorne, *Complete intersections and connectedness*, Amer. J. Math. **84** (1962), 497–508.

[8] ———, *Local cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin, 1967.

[9] ———, *Cohomological dimension of algebraic varieties*, Ann. of Math. (2) **88** (1968), 403–450.

[10] ———, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[11] Steven L. Kleiman, *On the vanishing of $H^n(X, \mathcal{F})$ for an n-dimensional variety*, Proc. Amer. Math. Soc. **18** (1967), 940–944.

[12] Yu. I. Manin, *Cubic forms*, second ed., North-Holland Mathematical Library, vol. 4, North-Holland Publishing Co., Amsterdam, 1986, Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

[13] Meltem Önal, *On complete intersections and connectedness*, Master's thesis, Bilkent University, 2002.

[14] Jean-Pierre Serre, *Sur la cohomologie des variétés algébriques*, J. Math. Pures Appl. (9) **36** (1957), 1–16.