IMA Journal of Mathematical Control and Information (2023) **40**, 789–803
https://doi.org/10.1093/imamci/dnad032
Advance Access publication on 1 December 2023

Existence of unattainable states for Schrödinger type flows on the half-line

Türker Özsari*
Department of Mathematics, Bilkent University, Çankaya, Ankara 06800, Turkey
*Corresponding author. Email: turker.ozsari@bilkent.edu.tr

and

Konstantinos Kalimeris
Mathematics Research Center, Academy of Athens, Soranou Efesiou 4, Athens 115 27, Greece

[Received on 12 April 2023; revised on 20 September 2023; accepted on 9 November 2023]

We prove that the solutions of the Schrödinger and biharmonic Schrödinger equations do not have the exact boundary controllability property on the half-line by showing that the associated adjoint models lack observability. We consider the framework of L^2 boundary controls with data spaces $H^{-1}(\mathbb{R}_+)$ and $H^{-2}(\mathbb{R}_+)$ for the classical and biharmonic Schrödinger equations, respectively. The lack of controllability on the half-line contrasts with the corresponding dynamics on a finite interval for a similar regularity setting. Our proof is based on an argument that uses the sharp fractional time trace estimates for solutions of the adjoint models. We also make several remarks on the connection of controllability and temporal regularity of spatial traces.

Keywords: Fokas method; Schrödinger equation; biharmonic Schrödinger equation; controllability.

1. Introduction

In this paper, we consider the exact boundary controllability problem for the linear Schrödinger and biharmonic Schrödinger equations:

$$u_t + Pu = 0, \quad x \in \Omega = \mathbb{R}_+, \, t \in (0, T),$$

where $P = -i\partial_x^2$ (Schrödinger) or $P = -i\partial_x^4$ (biharmonic Schrödinger). For both equations, we consider an L^2 control acting through the boundary. As canonical examples, for the Schrödinger equation, we consider an L^2 control g acting through the Dirichlet actuation: $u(0,t) = g(t)$ and for the biharmonic Schrödinger equation, we consider an L^2 control $g = g(t)$ acting through the Neumann actuation while Dirichlet boundary datum is set to be zero, namely for the half-line problem we take boundary conditions as $u(0,t) = 0, u_x(0,t) = g(t)$. Of course, these choices of boundary conditions are not restrictive and the method of this paper can be extended to also some other boundary conditions.

We say that the system has exact boundary controllability property if for any initial state $u_0 = u(0)$ and a desirable final state u_T, one can find an L^2 control such that $u(T) = u_T$. Here, we will consider the case that the initial and final states belong to a Sobolev space of negative order, which is $H^{-1}(\Omega)$ for the Schrödinger and $H^{-2}(\Omega)$ for the biharmonic Schrödinger equation. Since both the Schrödinger and biharmonic Schrödinger equations are time-reversible, without loss of generality one can simply take $u_T \equiv 0$.

In control theory of partial differential equations (PDEs), it is common to consider systems in which the control acts from the interior of domain. This contrasts with the framework of this paper where we consider a control that acts from the boundary. Regarding a review of interior controllability properties for the Schrödinger equation, we refer the reader to Zuazua (2003) and Phung (2001). For exact boundary controllability on bounded domains, some classical references are Lebeau (1992); Lasiecka & Triggiani (1992, 2000); Machtyngier (1994); Triggiani (2007, 2008); and Tataru (1996). There are also some controllability results for the nonlinear Schrödinger equation such as Beauchard (2005) and Rosier & Zhang (2009b); Rosier & Zhang (2009a) and Laurent (2010). In the case of the biharmonic Schrödinger equation, see for instance Wen *et al.* (2014) and Capistrano-Filho & Cavalcante (2021). However, the controllability results in the literature for the nonlinear Schrödinger equation are of local character in the sense that the initial and final states are generally assumed to be close to each other. We believe that the nonlinear control problem without such assumptions remains open to date.

Regarding the lack of controllability, Illner *et al.* (2006) proved some results on non-controllability for the linear Schrödinger equation considered with interior controls on the whole space, see also Lampart (2021). There are non-controllability results for parabolic equations (see e.g. Micu & Zuazua, 2001; Kalimeris & Özsarı, 2020) and some dispersive equations such as the Korteweg de-Vries equation, see Rosier (2000).

To the best of our knowledge, there is no result on the lack of exact boundary controllability for the Schrödinger equation when the domain is half-line. The goal of this paper is to explore this problem as well as extend the relevant result to the case of the biharmonic Schrödinger equation.

The authors of this work proved the lack of null-controllability for the heat equation in Kalimeris & Özsarı (2020), using the Fokas method, which is also known as the unified transform, and was introduced in Fokas (1997) for the analysis of initial-boundary value problems (IBVPs) of integrable nonlinear PDEs. Later, the Fokas method has led to the emergence of a novel approach for studying, both analytically and numerically, IBVPs for linear PDEs, see Fokas (2008); Fokas & Kaxiras (2023). By exploiting the numerical efficiency of this method, the Neumann controller of the heat equation on the finite interval was obtained in Kalimeris *et al.* (2023). We find worth mentioning that in this work we are able to provide a proof for the lack of exact controllability for the Schrödinger equation on the half-line without the usage of the Fokas method, but this is not the case for the biharmonic Schrödinger equation. Furthermore, the Fokas method provides a unified approach for obtaining both results.

2. Noncontrollability for the Schrödinger equation

We consider the following IBVP with inhomogeneous Dirichlet boundary input that acts as the control:

$$iu_t + u_{xx} = 0; \quad u(x,0) = u_0(x); \quad u(0,t) = g(t), \quad x \in \mathbb{R}_+, \, t \in (0,T). \tag{2.1}$$

We are interested in studying the exact boundary controllability problem for (2.1): 'Given any target state u_T, does there exist a suitable boundary input (control) g so that $u(T) = u_T$?'

In order to place this problem in a concrete functional analytical framework, we will consider L^2-controls (i.e. $g \in L^2_{\text{loc}}$) and also assume $u_0, u_T \in H^{-1}(\mathbb{R}_+)$. It can be shown that (2.1) has a solution (in the transposition sense) which belongs to $C([0,T]; H^{-1}(\mathbb{R}_+))$ under these assumptions on g and u_0, see e.g. for instance Lions & Magenes (1968).

2.1. *Main result*

In this section, we prove the lack of exact boundary controllability for (2.1) in $H^{-1}(\mathbb{R}_+)$. This is achieved by showing that there are solutions of the adjoint problem that violate the observability inequality (2.2).

The latter inequality provides a characterization of the exact boundary controllability property based on the well-known Hilbert Uniqueness Method (HUM) regarding (2.1), which is stated as follows (whose proof uses the same arguments in Machtyngier (1994), therefore omitted):

LEMMA 2.1. (2.1) has the exact boundary controllability property iff there is $c = c(T) > 0$ such that

$$|\phi_0|^2_{H^1_0(\mathbb{R}_+)} \le c \int_0^T |\phi_x(0,t)|^2 dt \tag{2.2}$$

for all $\phi_0 \in H^1_0(\mathbb{R}_+)$, where ϕ solves the homogeneous IBVP

$$i\phi_t + \phi_{xx} = 0; \quad \phi(x,0) = \phi_0(x); \quad \phi(0,t) = 0; \quad x \in \mathbb{R}_+,\ t \in (0,T). \tag{2.3}$$

Next, we recall the following fractional trace estimate.

LEMMA 2.2. Let $s \in (1/2, 1]$ and $\phi_0 \in H^s_0(\mathbb{R}_+)$, then

$$\sup_{x\in[0,\infty)} |\phi_x(x,t)|_{H_t^{\frac{2s-1}{4}}(0,T)} \lesssim |\tilde{\phi}_0|_{H^s(\mathbb{R})}, \tag{2.4}$$

where $\tilde{\phi}_0$ is the odd extension of ϕ_0.

Proof. Proof of Lemma 2.2 is given in Appendix A. It relies on similar Fourier analysis arguments which were previously used for establishing temporal regularity of spatial traces for the solutions of the Cauchy problem associated with the linear Schrödinger equation, see for instance Fokas *et al.* (2017); Batal *et al.* (2020). □

THEOREM 2.3. (2.1) does not have the exact boundary controllability property in $H^{-1}(\mathbb{R}_+)$.

Proof. Let $s = 1/2 + 2\alpha$ for some fixed $\alpha \in (0, 1/4)$; Lemma 2.2 yields the following estimate:

$$|\phi_x(0,t)|_{L^2_t(0,T)} \le |\phi_x(0,t)|_{H^\alpha_t(0,T)} \le c|\tilde{\phi}_0|_{H^{1/2+2\alpha}(\mathbb{R})} \le c|\phi_0|_{H^{1/2+2\alpha}(\mathbb{R}_+)}. \tag{2.5}$$

The first inequality is due to $H^\alpha(0,T) \hookrightarrow L^2(0,T)$ and the last inequality is a property of the odd extension.

The proof is completed by an argument of contradiction assuming exact controllability, namely (2.2) is true. To this end, we construct data that violate observability: Let $f \in H_0^{1/2+2\alpha}(\mathbb{R}_+)$ be such that $f \notin H^1_0(\mathbb{R}_+)$. By density there exists a sequence of functions $\phi_0^m \in H^1_0(\mathbb{R}_+)$ such that $\phi_0^m \to f$ in $H_0^{1/2+2\alpha}(\mathbb{R}_+)$ as $m \to \infty$. Then, ϕ_0^m has a subsequence that satisfies $|\phi_0^{m_k}|_{H^1_0(\mathbb{R}_+)} \to \infty$ as $k \to \infty$. Then, for such a sequence, in view of the observability inequality (2.2) and the trace inequality (2.5), we get

$$\infty > M > c|\phi_0^{m_k}|_{H^{1/2+2\alpha}(\mathbb{R}_+)} \ge |\phi_x^{m_k}(0,t)|_{L^2_t(0,T)} \ge c|\phi_0^{m_k}|_{H^1_0(\mathbb{R}_+)}$$

for some $M > 0$ independent of k. Therefore, letting $k \to \infty$, we get a contradiction. □

REMARK 2.4. (Solution representation for the half-line problem). The proof of Lemma 2.2 requires an explicit integral representation formula of the solution. Such a formula, obtained via the Laplace

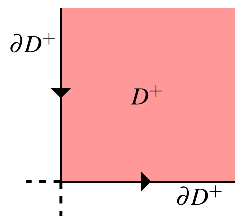

FIG. 1. Contour of integration for the Schrödinger equation.

transform, was used in Batal & Özsarı (2016). A different integral representation of the solution of (2.3) was obtained via Fokas method in Fokas *et al.* (2017):

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ik^2t}\hat{\phi}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-ik^2t}\hat{\phi}_0(-k)dk, \tag{2.6}$$

where D^+ is the quarter complex plane depicted in Fig. 1 and $\hat{\phi}_0(k)$ denotes the half-line Fourier transform of ϕ_0. The boundary of D^+ is traversed with the orientation that D^+ stays to the left of ∂D^+. The integral on the boundary of ∂D^+ can be deformed back onto the real axis. This is because for sufficiently smooth ϕ_0 which decays sufficiently fast as $x \to \infty$, we have that $e^{ikx-ik^2t}\hat{\phi}_0(-k)$ is analytic and decays as $k \to \infty$ in $\mathbb{C}_+ - D^+$so that we can use the Cauchy's theorem and Jordan's lemma. Therefore, one can simply write the solution as

$$\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ik^2t}\hat{\tilde{\phi}}_0(k)dk, \tag{2.7}$$

where $\tilde{\phi}_0(x) := \mathscr{F}^{-1}\{\hat{\phi}_0(\cdot) - \hat{\phi}_0(-\cdot)\}(x)$ is the odd extension of ϕ_0 from $\mathbb{R}_+$ to $\mathbb{R}$.

Clearly, the solution formula in (2.7) can be obtained by first taking an odd extension of initial datum and then solving the corresponding Cauchy problem on the whole line. This makes the solution of the extended problem also odd so that its restriction to boundary a fortiori satisfies the desired Dirichlet boundary condition. However, this method does not work with higher order PDEs for which more than one boundary condition is necessary. The biharmonic Schrödinger equation which is treated in the next section is an example. Therefore, an integral representation formula as in (2.6) allows treatment of more general and multiple boundary conditions.

3. Further discussion of controllability for the Schrödinger equation

3.1. *Classical hidden trace regularity versus fractional trace estimates*

A trace estimate similar to (2.4) for $x = 0$ can be obtained using the classical multiplier $\bar{\phi}_x$. We note that the latter approach yields a weaker version of (2.2), which is not sufficient for the proof of the main result of this section.

Indeed, assuming $\phi_0 \in H_0^1(\mathbb{R}_+)$, this estimate has the form

$$\int_0^T |\phi_x(0,t)|^2 dt = -Im(\phi(T), \phi_x(T)) + Im(\phi_0, \dot{\phi}_0) \leq c|\phi_0|^2_{H_0^1(\mathbb{R}_+)}, \tag{3.1}$$

where $\dot{\Phi}_0$ represents the derivative of Φ_0. This can be obtained first for smooth data and then extend to $H_0^1(\mathbb{R}_+)$ by density. Let $\phi_0 \in C_0^\infty(\mathbb{R}_+)$ and ϕ be the corresponding smooth solution of (2.3) that vanishes at infinity together with its derivatives. Then, multiplying the equation by $\bar{\phi}_x$ and integrating over $\mathbb{R}_+ \times (0,T)$, we obtain (3.1) where the last estimate follows from Cauchy–Schwarz and Poincare inequalities as well as the conservation of H^1- norm. Note that the above estimate does not follow from Sobolev trace theory and intrinsic to the Schrödinger operator. Therefore, it is generally referred to as the hidden regularity property in the control community.

We emphasize that the sharp fractional temporal estimates of Lemma 2.2 are essential for the method of the paper. More precisely, the classical trace estimate (3.1) which can easily be obtained through energy method is not sufficient. Namely, the contradiction argument in the proof of Theorem 2.3 is based on the combination of the inverse inequality (2.2) with a trace estimate that involves a norm of the Neumann derivative stronger than L^2; hence, we chose $s > 1/2 \Rightarrow \frac{2s-1}{4} > 0$. This creates a motivation also for the effort given to non-energy method based trace estimates for the biharmonic Schrödinger equation in Section 4.

3.2. *Regularity gap between half-line and finite interval settings*

Lemma 2.2 shows that there is a gain of regularity of the boundary term in the half-line setting, compared with the finite interval case. In the finite interval setting, the Ingham inequalities imply that the trace $\phi_x(0,\cdot) \in L^2(0,T)$ if and only if $\phi_0 \in H_0^1(0,L)$. This is because the Schrödinger equation has the interesting property that the temporal regularity of spatial traces of solutions depends on the geometry. There is some loss when the equation is posed on a finite interval compared with the case of half-line. One can infer this by comparing the integral formulas of boundary traces in these two settings. When the domain is the half-line, differentiating formula (2.6) with respect to x and plugging $x = 0$, one gets

$$\begin{aligned}\phi_x(0,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik^2t} ik\hat{\phi}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{-ik^2t} ik\hat{\phi}_0(-k)dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik^2t} ik\hat{\tilde{\phi}}_0(k)dk.\end{aligned} \tag{3.2}$$

The last integral in (3.2) can be interpreted as an inverse Fourier transform of a certain function. Indeed, changing variable via $k = \sqrt{-\tau}$ for $k \geq 0$ and $k = -\sqrt{-\tau}$ for $k < 0$, we can rewrite the above formula as

$$\phi_x(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}\hat{\Phi}(\tau)d\tau, \tag{3.3}$$

where $\hat{\Phi}(\tau) = 1_{\{\tau<0\}} i\hat{\tilde{\phi}}_0(\sqrt{-\tau})$. This implies the estimate

$$\begin{aligned}|\phi_x(0,\cdot)|^2_{L_t^2(0,T)} &\leq |\phi_x(0,\cdot)|^2_{L_t^2(\mathbb{R})} \leq \int_{-\infty}^{0} |\hat{\tilde{\phi}}_0(\sqrt{-\tau})|^2 d\tau \\ &= 2\int_0^\infty k|\hat{\tilde{\phi}}_0(k)|^2 dk \leq 2|\tilde{\phi}_0|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R})} \lesssim |\phi_0|^2_{H_0^{\frac{1}{2}+\epsilon}(\mathbb{R}_+)},\end{aligned}$$

for any $\epsilon > 0$. Hence, $\phi_x(\cdot,0)$ is in $L^2(0,T)$ for the half-line setting even when we take ϕ_0 from less regular spaces than $H_0^1(\mathbb{R}_+)$, namely the spaces in the form $H_0^{\frac{1}{2}+\epsilon}(\mathbb{R}_+)$ with ϵ being arbitrarily small.

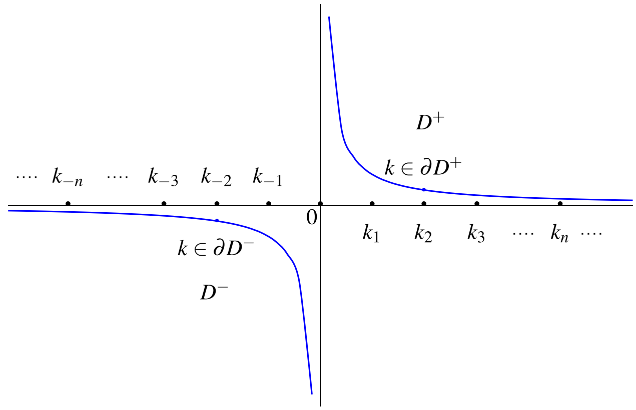

FIG. 2. The blue line is the contour of integration for the last two integrals in the formula (3.4). The countably many black dots along the real axis denote the zeros $k_n = n\pi/L$, $n \in \mathbb{Z}$, of the mapping $k \mapsto e^{ikL} - e^{-ikL}$.

Namely, one needs almost $\frac{1}{2}$ units less spatial regularity for Neumann trace to be in $L^2(0,T)$ compared with the case where domain is a finite interval.

On the other hand, the integral formula for the Neumann trace on a finite interval $(0,L)$ (obtained through Fokas method Fokas, 1997) has the following more complicated form:

$$\begin{aligned} \phi_x(0,t) := \frac{1}{2\pi} \int_{k\in\mathbb{R}} e^{-ik^2 t}\, ik\widehat{\phi}_0(k)\, dk \\ - \frac{1}{2\pi} \int_{k\in\partial D^+} \frac{e^{-ik^2 t}}{e^{ikL} - e^{-ikL}} ik \left[e^{ikL}\, \widehat{\phi}_0(k) - e^{-ikL}\, \widehat{\phi}_0(-k) \right] dk \\ + \frac{1}{2\pi} \int_{k\in\partial D^-} \frac{e^{-ikL - ik^2 t}}{e^{ikL} - e^{-ikL}} ik \left[\widehat{\phi}_0(k) - \widehat{\phi}_0(-k) \right] dk, \end{aligned} \tag{3.4}$$

where $\widehat{\phi}_0(k)$ is the finite-line Fourier transform of $\phi_0(x)$:

$$\widehat{\phi}(k) = \int_0^L e^{-ikx} \phi(x) dx, \quad k \in \mathbb{C}, \quad \phi(x) = \frac{1}{2\pi} \int_{k\in\mathbb{R}} e^{ikx}\, \widehat{\phi}(k) dk, \quad x \in (0,L),$$

and the complex contours $\partial D^{\pm}$ are the positively oriented boundaries of the regions $D^{\pm}$ given by

$$D^{\pm} := \left\{ k \in \mathbb{C} : \mathrm{Im}(k) \gtrless 0 \text{ and } \mathrm{Re}(ik^2 + 1) < 0 \right\} \quad \text{(See Fig. 2).}$$

Now comparing the formulas for the Neumann trace in the half-line and finite-interval settings, we observe that although the first term in (3.4) is comparable with the half-line solution formula in (2.7), the last two integrals in (3.4) have rather different characteristics due to the abrupt behaviour of the mapping $f(k) := \frac{1}{e^{ikL} - e^{-ikL}}$ when $k \in \partial D^+ \cup \partial D^-$ (the points on the blue line) is close to the points $k_n = \frac{n\pi}{L}$ (black dots on the real axis), see Figs 2 and 3. The situation gets worse as n gets larger. We strongly believe these countably many impulses with growing peaks (which grow like $|k|$ when k is near k_n) cause loss of some smoothness in the finite interval setting. Such rough oscillatory distortion of the integrand is completely

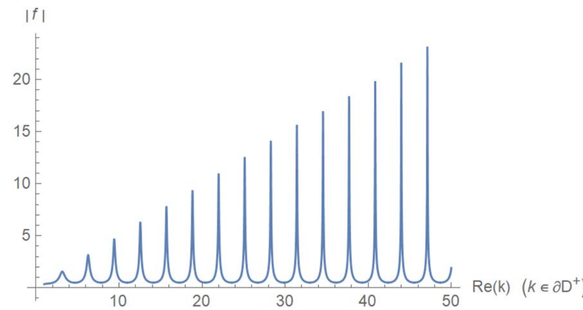

FIG. 3. The graph shows $|f(k)| = \left|\frac{1}{e^{ikL}-e^{-ikL}}\right|$ as k varies over ∂D^+.

absent in the solution formula for the half-line problem. This affects the sought-after regularity gain in the half-line setting compared with the finite interval problem.

3.3. *Other regularity settings for data and control spaces*

The estimate in Lemma 2.2 shows that there are initial data in $H_0^1(\mathbb{R}_+)$ which are not observable by the $L^2(0,T)$ norm of the normal derivative. Therefore, one may wonder whether controllability can be achieved with smoother data (i.e. $u_0, u_T \in H^{-\theta}(\mathbb{R}_+)$ with $\theta < 1$, $g \in L^2(0,T)$) or alternatively with rougher controls (e.g. $g \in H^{-\alpha}(0,T)$ with some $\alpha < 0$, $u_0, u_T \in H^{-1}(\mathbb{R}_+)$).

The proof of noncontrollability for the Schrödinger equation in this paper will remain valid if $H_0^1(\mathbb{R}_+)$ norm at the left-hand side of the observability inequality is replaced with $H_0^\theta(\mathbb{R}_+)$ norm for $\frac{1}{2} < \theta < 1$. Therefore, the exact boundary controllability with boundary controls in $L^2(0,T)$ should fail also with data at least in the slightly smoother spaces $H^{-\theta}(\mathbb{R}_+)$ with $\frac{1}{2} < \theta < 1$.

However, our approach does not rule out the possibility that a sufficiently high regularity setting for the data space or a sufficiently low regularity setting for the control space may allow for exact controllability. There are examples in the literature which show that there is a tight relation between control and regularity. For instance, as pointed out in Illner *et al.* (2006), compare the noncontrollability result of Turinici (2000) for the linear Schrödinger equation posed on the Euclidean space with the contrasting controllability result of Beauchard & Coron (2006) for the same model with relatively much higher data space.

Regarding the use of rougher controls with data space $H^{-1}(\mathbb{R}_+)$, we remark that the regularity theory for the linear Schrödinger equation suggests that if $\alpha < 0$, then boundary controls belonging to the space $H^{-\alpha}(0,T)$ and initial data in $H^{-1}(\mathbb{R}_+)$ will lead to solutions satisfying $u(\cdot,t) \in H^{-2\alpha-\frac{1}{2}}(\mathbb{R}_+)$. Therefore, with the choice of $\alpha = \frac{1}{4}$, the trajectories will remain in the data space $H^{-1}(\mathbb{R}_+)$. Hence, one may still hope to achieve controllability using controls in say $H^{-\frac{1}{4}}(0,T)$. The method of this paper does not prohibit such feature.

3.4. *Interior and approximate controllability*

It is interesting to check whether other well-known controllability properties such as interior exact controllability and the boundary approximate controllability hold in the half-line setting. To this end, let us first consider the interior exact controllability problem for the Schrödinger equation:

$$iu_t + u_{xx} = 1_\omega(\omega)h(x,t); \quad u(x,0) = u_0(x); \quad u(0,t) = 0, \quad x \in \mathbb{R}_+,\ t \in (0,T), \tag{3.5}$$

where 1_ω is the characteristic function supported on an open bounded set $\omega \subset \mathbb{R}_+$ and h is a control function with $h(\cdot,t) \in L^2_x(\mathbb{R}_+)$. The proof of the fact that the interior controllability fails for the above problem in a certain setting is very similar to the treatment of the associated problem on the whole real line and can be done by adapting the arguments in the paper of Illner *et al.* (2006) to the half-line setting. Indeed, it is well known (see Zabczyk, 2020) that the problem (3.5) has the interior exact controllability property in $L^2(\mathbb{R}_+)$ iff there is some constant $c_T > 0$ for which

$$|\phi_0|^2_{L^2(\mathbb{R}_+)} \le c_T \int_0^T |1_\omega \mathscr{T}(-t)\phi_0|^2_{L^2(\mathbb{R}_+)}dt, \quad \forall \phi_0 \in L^2(\mathbb{R}_+), \tag{3.6}$$

where $\mathscr{T}(t)\phi_0 = \phi(t)$ denotes the solution of the associated homogeneous IBVP

$$i\phi_t + \phi_{xx} = 0; \quad \phi(x,0) = \phi_0(x); \quad \phi(0,t) = 0, \quad x \in \mathbb{R}_+, t \in \mathbb{R}. \tag{3.7}$$

Recall that the solution of above problem is given by $\phi = \tilde{\phi}|_{\mathbb{R}_+}$, where

$$\tilde{\phi}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ik^2t}\hat{\tilde{\phi}}_0(k)dk,$$

$\tilde{\phi}_0(x)$ being the odd extension of ϕ_0 from $\mathbb{R}_+$ to $\mathbb{R}$. Namely, $\tilde{\phi}$ satisfies the IBVP

$$i\tilde{\phi}_t + \tilde{\phi}_{xx} = 0; \quad \tilde{\phi}(x,0) = \tilde{\phi}_0(x), \quad x \in \mathbb{R}, t \in \mathbb{R}. \tag{3.8}$$

Therefore, we have

$$|\mathscr{T}(t)\phi_0|_{L^\infty_x(\mathbb{R}_+)} = |\phi(\cdot,t)|_{L^\infty_x(\mathbb{R}_+)} \le |\tilde{\phi}(\cdot,t)|_{L^\infty_x(\mathbb{R})} \lesssim \frac{1}{|t|^{\frac{1}{2}}}|\tilde{\phi}_0|_{L^1(\mathbb{R})} = \frac{2}{|t|^{\frac{1}{2}}}|\phi_0|_{L^1(\mathbb{R}_+)}, \; t \ne 0, \tag{3.9}$$

where the second inequality follows from the dispersion estimate for the free Schrödinger operator and the last equality is due to the definition of the odd extension. Now, let η be the standard mollifier function and set

$$\eta_j(x) := j \cdot \eta(j(x-1)), \; x \ge 0 \text{ and } \phi_0^j = \mathscr{T}(2T)\eta_j, \quad j \in \mathbb{Z}_+.$$

Using the conservation of $L^2(\mathbb{R}_+)$ norm for the IBVP (3.7), assuming (3.6) and utilizing (3.9) with t replaced by $2T - t$, we get

$$\begin{aligned} j \cong |\eta_j|^2_{L^2(\mathbb{R}_+)} &= |\phi_0^j|^2_{L^2(\mathbb{R}_+)} \\ &\le c_T \int_0^T |1_\omega \mathscr{T}(-t)\phi_0^j|^2_{L^2(\mathbb{R}_+)}dt = c_T \int_0^T |1_\omega \mathscr{T}(2T-t)\eta_j|^2_{L^2(\mathbb{R}_+)}dt \\ &\le c_T \ell(\omega)|\eta_j|^2_{L^1(\mathbb{R}_+)} \int_0^T (2T-t)^{-1}dt = \ln(2)c_T\ell(\omega) < \infty, \end{aligned}$$

where $\ell(\omega)$ is the size of the finite region ω. Letting $j \to \infty$ in the above estimate, we get a contradiction. Hence, linear Schrödinger equation posed on the half-line lacks interior exact boundary controllability feature in the L^2 framework when the control acts through a bounded region $\omega \subset \mathbb{R}_+$. Of course, the above proof for noncontrollability is not valid if ω is unbounded as $\ell(\omega) = \infty$ in that case.

Regarding the approximate boundary controllability, it is known that this property is equivalent to the statement 'if $\phi_x(0,t) = 0$ for a.e. $t \in (0,T)$, then $\phi_0 \equiv 0$, where ϕ is the solution of (3.7)', see Zabczyk (2020). So, let us assume $\phi_x(0,t) = 0$ for a.e. $t \in (0,T)$. Then, it follows from (3.3) that

$$\hat{\tilde{\phi}}_0(\sqrt{-\tau}) = 0 \text{ for} \tau < 0,$$

where $\tilde{\phi}_0$ is the odd extension of ϕ_0. Therefore, $\hat{\tilde{\phi}}_0(k) = 0$ for $k > 0$. Since $\tilde{\phi}_0$ is odd, we know that its Fourier transform $\hat{\tilde{\phi}}_0$ is also odd. It follows from this fact that we also have $\hat{\tilde{\phi}}_0(k) = 0$ for $k \le 0$. We just showed that $\hat{\phi}_0(k) = 0$ for $k \in \mathbb{R}$. This gives $\tilde{\phi}_0 \equiv 0$ on $\mathbb{R}$ (and therefore $\phi_0 \equiv 0$ on $\mathbb{R}_+$) by the Fourier inversion. Therefore, the Schrödinger equation seems to have the approximate boundary controllability property in the half-line setting.

4. Noncontrollability for the biharmonic Schrödinger equation

In this section, we extend the noncontrollability result to the biharmonic Schrödinger equation:

$$iu_t + u_{xxxx} = 0; u(x,0) = u_0(x); u(0,t) = 0; u_x(0,t) = h(t), x \in \mathbb{R}_+, t \in (0,T). \tag{4.1}$$

Here, the control acts through the Neumann actuation at $x = 0$. It can be shown that given $h \in L^2_{\text{loc}}$, $u_0 \in H^{-2}(\mathbb{R}_+)$, the above problem has a solution in the space $C([0,T]; H^{-2}(\mathbb{R}_+))$.

4.1. *Main result*

The characterization of the exact controllability of (4.1) is given by the observability property (4.2) in the following lemma (whose proof can be adapted from arguments similar to those in Wen *et al.* (2014), therefore omitted).

LEMMA 4.1. (4.1) has the exact boundary controllability property iff there is $c = c(T) > 0$ such that

$$|\phi_0|^2_{H^2_0(\mathbb{R}_+)} \le c \int_0^T |\phi_{xx}(0,t)|^2 dt \tag{4.2}$$

for all $\phi_0 \in H^2_0(\mathbb{R}_+)$, where ϕ solves the homogeneous IBVP

$$i\phi_t + \phi_{xxxx} = 0; \phi(x,0) = \phi_0(x); \phi(0,t) = 0; \quad \phi_x(0,t) = 0; x \in \mathbb{R}_+, \ t \in (0,T). \tag{4.3}$$

Thus, we follow the same approach with Section 2, where we now use the following second-order fractional trace estimate:

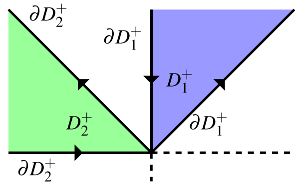

FIG. 4. Contour of integration for the biharmonic Schrödinger equation.

LEMMA 4.2. Let $s \geq 0$, $s \neq 1/2 + n$ for $n \in \mathbb{N}_0$, $\phi_0 \in H_0^s(\mathbb{R}_+)$ and ϕ be the solution of (4.3). Then,

$$|\phi_{xx}(0,\cdot)|_{H_t^{\frac{2s-1}{8}}(0,T)} \lesssim |\phi_0|_{H_0^s(\mathbb{R}_+)}. \tag{4.4}$$

Proof. The proof of this lemma is given in Section 4.2. □

Our main result for the biharmonic Schrödinger equation reads as follows:

THEOREM 4.3. (4.1) does not have the exact boundary controllability property in $H^{-2}(\mathbb{R}_+)$.

Proof. Letting $s = 1/2 + 4\alpha$ for fixed $\alpha \in (0, 3/8)$, we deduce the following estimate from (4.4):

$$|\phi_{xx}(0,t)|_{L_t^2(0,T)} \leq |\phi_{xx}(0,t)|_{H_t^\alpha(0,T)} \leq c|\phi_0|_{H_0^{1/2+4\alpha}(\mathbb{R}_+)}. \tag{4.5}$$

Now, we pick a sequence of initial data ϕ_0^k which satisfies $|\phi_0^k|_{H_0^2(\mathbb{R}_+)} \to \infty$, as $k \to \infty$, while $|\phi_0^k|_{H_0^{1/2+4\alpha}(\mathbb{R}_+)}$ is uniformly bounded with respect to k. Assuming that (4.2) is true for such initial data and employing (4.5), the corresponding solutions will satisfy

$$\infty > c|\phi_0^k|_{H^{1/2+2\alpha}(\mathbb{R}_+)} \geq |\phi_{xx}^k(0,t)|_{L_t^2(0,T)} \geq c|\phi_0^k|_{H_0^2(\mathbb{R}_+)} \to \infty$$

as $k \to \infty$. Thus, (4.2) does not hold. □

4.2. *A second-order fractional trace estimate (Proof of Lemma 4.2)*

We will first derive an explicit formula of the second-order trace of the solution of (4.3). This solution can be derived using Fokas's method (see Özsarı & Yolcu, 2019 for details) and reads as follows:

$$\begin{aligned}\phi(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+ik^4t}\hat{\phi}_0(k)dk - \frac{1}{2\pi}\int_{\partial D_1^+} e^{ikx+ik^4t}\left[(1+i)\hat{\phi}_0(-ik) - i\hat{\phi}_0(-k)\right]dk \\ &- \frac{1}{2\pi}\int_{\partial D_2^+} e^{ikx+ik^4t}\left[(1-i)\hat{\phi}_0(ik) + i\hat{\phi}_0(-k)\right]dk,\end{aligned} \tag{4.6}$$

where D_1^+ and D_2^+ are depicted in Fig. 4. Their boundaries are formed as follows:

$$\partial D_1^+ = (i\infty, 0] \cup [0, (1+i)\infty) \qquad \text{and} \qquad \partial D_2^+ = (-\infty, 0] \cup [0, (1-i)\infty).$$

Using (4.6) we obtain a formula for the second-order trace

$$\phi_{xx}(0,t) = -\frac{1+i}{\pi}\int_0^\infty e^{ik^4 t}k^2\left[\hat{\phi}_0(k)+\hat{\phi}_0(-k)\right]dk + \frac{2}{\pi}\int_0^\infty e^{ik^4 t}k^2\Phi_0(k)dk, \tag{4.7}$$

where $\Phi_0 \equiv \mathscr{L}\{\phi_0\}$ is the Laplace transform of ϕ_0.

Indeed, this is achieved in three steps. We first deform ∂D_1^+ and ∂D_2^+ to the boundaries of the first and second quadrant, respectively; this is possible because the second integrand of (4.6) is bounded and analytic in the domain contained by the rays $((1+i)\infty, 0] \cup [0, +\infty)$ and the third integrand of (4.6) is bounded and analytic in the domain contained by the rays $((1-i)\infty, 0] \cup [0, i\infty)$ Second, we differentiate twice with respect to x and we set $x = 0$. Third, we employ change of variables $k \to -k$ and $k \to ik$ to the integrals which are integrated along the half-lines $(-\infty, 0]$ and $[0, +i\infty)$, respectively.

In order to prove (4.4), in the remaining of this section we will estimate the RHS of (4.7).

Lemma 4.4. Let $\phi_0 \in H_0^s(\mathbb{R}_+)$, $\Phi_0 \equiv \mathscr{L}\{\phi_0\}$, $s \geq 0$ and $s \neq n + 1/2$, $n \in \mathbb{N}_0$. Then,

$$\int_0^\infty (1+k^2)^s|\Phi_0(k)|^2 dk \lesssim |\phi_0|^2_{H_0^s(\mathbb{R}_+)}. \tag{4.8}$$

Proof. We first introduce the function Ψ_0 to be the inverse Fourier transform of

$$\widehat{\Psi}_0(k) := \begin{cases} \Phi_0(k), & k \geq 0, \\ 0, & k < 0. \end{cases}$$

Then,

$$|\Psi_0|^2_{H^s(\mathbb{R})} = \int_0^\infty (1+k^2)^s|\Phi_0(k)|^2 dk.$$

It is well known that Laplace transform is a linear bounded operator from $L_x^2(\mathbb{R}_+)$ into $L_k^2(\mathbb{R}_+)$, which gives (4.8) for $s = 0$. Now, we will consider the case $s = 1$ for which we have $\phi_0(0) = 0$. Then, using the first derivative formula for the transform, we get $[\mathscr{L}\phi_0'](k) = k\Phi_0(k) - \phi_0(0) = k\Phi_0(k)$. Therefore, we have $|[\mathscr{L}(\phi_0)](k)|^2 + |[\mathscr{L}(\phi_0')](k)|^2 = (1+k^2)|\Phi_0(k)|^2$ so that

$$\begin{aligned} |\Psi_0|^2_{H^1(\mathbb{R})} &= \int_0^\infty (1+k^2)|\Phi_0(k)|^2 dk = \int_0^\infty \left(|[\mathscr{L}(\phi_0)](k)|^2 + |[\mathscr{L}(\phi_0')](k)|^2\right) dk \\ &\lesssim |\phi_0|^2_{L^2(\mathbb{R}_+)} + |\phi_0'|^2_{L^2(\mathbb{R}_+)} \equiv |\phi_0|^2_{H_0^1(\mathbb{R}_+)}. \end{aligned}$$

This settles the case $s = 1$. Regarding the case $s = 2$ for which we have $\phi_0(0) = \phi_0'(0) = 0$, using the transform formula for second derivative $[\mathscr{L}\phi_0''](k) = k^2\Phi_0(k) - k\phi_0'(0) - \phi_0(0) = k^2\Phi_0(k)$. Therefore, we have $|[\mathscr{L}(\phi_0)](k)|^2 + |[\mathscr{L}(\phi_0')](k)|^2 + |[\mathscr{L}(\phi_0'')](k)|^2 = (1+k^2+k^4)|\Phi_0(k)|^2$ so that

$$\begin{aligned}|\Psi_0|^2_{H^2(\mathbb{R})} &= \int_0^\infty (1+k^2)^2|\Phi_0(k)|^2 dk \\ &\le 2\int_0^\infty \left(|[\mathscr{L}(\phi_0)](k)|^2 + |[\mathscr{L}(\phi_0')](k)|^2 + |[\mathscr{L}(\phi_0'')](k)|^2\right) dk \\ &\lesssim |\phi_0|^2_{L^2(\mathbb{R}_+)} + |\phi_0'|^2_{L^2(\mathbb{R}_+)} + |\phi_0''|^2_{L^2(\mathbb{R}_+)} \equiv |\phi_0|^2_{H_0^2(\mathbb{R}_+)}.\end{aligned}$$

This settles the case $s = 2$. Using the inequality $(1+k^2)^n \le \frac{2^n}{n+1}\sum_{j=0}^n k^{2j}$ we are able to obtain (4.8) for all $s \in \mathbb{N}_0$. We just showed that the operator $T : \phi_0 \in H_0^m(\mathbb{R}_+) \to \Psi_0 \in H^m(\mathbb{R})$ is linear bounded for every $m \in \mathbb{N}_0$. Therefore, by interpolation T is also linear bounded from $H_0^s(\mathbb{R}_+)$ into $H^s(\mathbb{R})$ for $s \ge 0$ (except for $s = 1/2 + n$, $n \in \mathbb{N}_0$, in which case one needs to replace $H_0^s(\mathbb{R}_+)$ with the space $H_{00}^s(\mathbb{R}_+)$). □

Next, we recall the following well-known property of zero extension.

LEMMA 4.5. [Lions & Magenes, 1972] Let $s \ge 0$, $s \ne 1/2 + n$ for $n \in \mathbb{N}_0$, and $\phi_0 \in H_0^s(\mathbb{R}_+)$. Then, the zero extension of ϕ_0, denoted by $\tilde{\phi}_0$, belongs to $H^s(\mathbb{R})$. Moreover, $|\tilde{\phi}_0|_{H^s(\mathbb{R})} \lesssim |\phi_0|_{H_0^s(\mathbb{R}_+)}$.

Now, we are ready to prove the second-order trace estimate.

LEMMA 4.6. Let $A(t)$ and $B(t)$ denote the first and second integral of (4.7), respectively. Then,

$$|A|_{H_t^{\frac{2s-1}{8}}(0,T)} \lesssim |\phi_0|_{H_0^s(\mathbb{R}_+)}, \tag{4.9}$$

$$|B|_{H_t^{\frac{2s-1}{8}}(0,T)} \lesssim |\phi_0|_{H_0^s(\mathbb{R}_+)}. \tag{4.10}$$

Proof. By making the change of variables $\tau = k^4$ in $A(t)$, we obtain

$$A(t) = -\frac{1+i}{4\pi}\left[\int_0^\infty e^{i\tau t}\tau^{-\frac{1}{4}}\hat{\phi}_0(\tau^{\frac{1}{4}})d\tau + \int_0^\infty e^{i\tau t}\tau^{-\frac{1}{4}}\hat{\phi}_0(-\tau^{\frac{1}{4}})d\tau\right] = I(t) + II(t),$$

where

$$\widehat{I}^{(t)}(\tau) := \begin{cases} \frac{(i-1)}{8\pi}\tau^{-\frac{1}{4}}\hat{\phi}_0(\tau^{\frac{1}{4}}), & \tau \ge 0, \\ 0, & \tau < 0\end{cases}$$

and

$$\widehat{II}^{(t)}(\tau) := \begin{cases} \frac{(i-1)}{8\pi}\tau^{-\frac{1}{4}}\hat{\phi}_0(-\tau^{\frac{1}{4}}), & \tau \ge 0, \\ 0, & \tau < 0.\end{cases}$$

Let $\tilde{\phi}_0$ be the zero extension of ϕ_0, then by direct calculation, we get

$$|I|^2_{H_t^{\frac{2s-1}{8}}} = c\int_0^\infty (1+\tau^2)^{\frac{2s-1}{8}}\tau^{-1/2}|\hat{\phi}_0(\tau^{\frac{1}{4}})|^2 d\tau \lesssim \int_{-\infty}^\infty (1+k^2)^s|\hat{\phi}_0(k)|^2 dk = |\tilde{\phi}_0|^2_{H^s}$$

for $s\in\mathbb{R}$. Similarly, we have $|II|^2_{H_t^{\frac{2s-1}{8}}} \lesssim \int_{-\infty}^\infty (1+k^2)^s|\hat{\phi}_0(k)|^2 dk$. Using Lemma 4.5 and the estimates of I and II, we deduce

$$\begin{aligned}|A|^2_{H_t^{\frac{2s-1}{8}}(0,T)} &\le |A|^2_{H_t^{\frac{2s-1}{8}}} \le |I|^2_{H_t^{\frac{2s-1}{8}}} + |II|^2_{H_t^{\frac{2s-1}{8}}} \\ &\le \int_{-\infty}^\infty (1+k^2)^s|\hat{\phi}_0(k)|^2 dk \lesssim |\phi_0|^2_{H_0^s(\mathbb{R}_+)}.\end{aligned} \tag{4.11}$$

By similar arguments, we also find

$$|B|^2_{H_t^{\frac{2s-1}{8}}(0,T)} \lesssim \int_0^\infty (1+k^2)^s|\Phi_0(k)|^2 dk \lesssim |\phi_0|^2_{H_0^s(\mathbb{R}_+)},$$

where the second inequality follows from Lemma 4.4. □

Applying the estimates (4.9) and (4.10) to (4.7) we obtain (4.4).

5. Concluding remarks

It is important to notice that the proofs of our main results must fail when the spatial domain $\Omega = \mathbb{R}_+$ is replaced with a finite interval $\Omega = (0,L)$, since it is well known that in the latter case controllability holds true. This can be seen by comparing the associated fractional trace estimates on the half-line and the finite interval. Indeed, for the classical Schrödinger equation, it is known that there are $\frac{1}{4}$ units loss of temporal trace regularity in the finite interval case Bona *et al.* (2018). Namely, $H^s(\mathbb{R}_+)$ norm at the right-hand side of (2.4) must be replaced with $H^{s+1/2}(0,L)$ norm. This would lead to the occurrence of $H^{1+2\alpha}(0,L)$ norm in an estimate similar to (2.5) in place of the $H^{1/2+2\alpha}(\mathbb{R}_+)$ norm, where $\alpha > 0$. However, $1+2\alpha > 1$, therefore construction of an observability violating sequence with the properties (i) ϕ_0^m uniformly bounded in $H^{1+2\alpha}(0,L)$, (ii) $|\phi_0^m|_{H_0^1(0,L)} \to \infty$ as $m\to\infty$ would be impossible. A similar remark also applies to biharmonic Schrödinger equation and based on same arguments one should expect a loss of $\frac{3}{8}$ units of temporal trace regularity in the finite-interval case. Namely, $H^s(\mathbb{R}_+)$ norm at the right-hand side of (4.4) must be replaced with $H^{s+3/2}(0,L)$ norm, preventing to make the conclusion of non-controllability.

Acknowledgments

The authors thank the anonymous reviewers for their valuable suggestions. This work was accomplished during a scientific visit of Konstantinos Kalimeris at Bilkent University, Ankara in January 2023.

Funding

This work was supported by the Scientific and Technological Research Council of Turkey [TÜBİTAK 1001 grant # 117F449 to T.Ö.]; and the Research Committee of the Academy of Athens [Research grant # 200/984 to K.K.].

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A. Proof of Lemma 2.2

The trace estimate (2.4) can be easily obtained from (2.7) via standard Fourier analysis. Indeed, let $s \in (1/2, 1]$ and $\phi_0 \in H_0^s(\mathbb{R}_+)$. Then, using the inverse Fourier transform formula deduced in (3.3) above, we have

$$\begin{aligned}
|\phi_x(0,\cdot)|^2_{H_t^{\frac{2s-1}{4}}(0,T)} &\le |\phi_x(0,\cdot)|^2_{H_t^{\frac{2s-1}{4}}(\mathbb{R})} \\
&\le \int_{-\infty}^0 (1+\tau^2)^{\frac{2s-1}{4}} |\hat{\tilde{\phi}}_0(\sqrt{-\tau})|^2 d\tau \\
&= 2\int_0^\infty (1+k^4)^{\frac{2s-1}{4}} k |\hat{\tilde{\phi}}_0(k)|^2 dk \\
&\le 2\int_0^\infty (1+k^2)^{2(\frac{2s-1}{4})}(1+k^2)^{\frac{1}{2}} |\hat{\tilde{\phi}}_0(k)|^2 dk \\
&= 2\int_0^\infty (1+k^2)^s |\hat{\tilde{\phi}}_0(k)|^2 dk \\
&= 2|\tilde{\phi}_0|^2_{H^s(\mathbb{R})} \lesssim |\phi_0|^2_{H_0^s(\mathbb{R}_+)},
\end{aligned} \tag{A.1}$$

where the third inequality follows from $1+k^4 \le (1+k^2)^2$ and $k \le (1+k^2)^{\frac{1}{2}}$ and the last inequality is a property of the odd extension for the given range of s.