



Cohomology of infinite groups realizing fusion systems

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Abstract

Given a fusion system \mathcal{F} defined on a p -group S , there exist infinite group models, constructed by Leary and Stancu, and Robinson, that realize \mathcal{F} . We study these models when \mathcal{F} is a fusion system of a finite group G and prove a theorem which relates the cohomology of an infinite group model π to the cohomology of the group G . We show that for the groups $GL(n, 2)$, where $n \geq 5$, the cohomology of the infinite group obtained using the Robinson model is different than the cohomology of the fusion system. We also discuss the signalizer functors $P \rightarrow \Theta(P)$ for infinite group models and obtain a long exact sequence for calculating the cohomology of a centric linking system with twisted coefficients.

Keywords Fusion systems · Graph of groups · Cohomology of groups · Signalizer functor

1 Introduction

Let Γ be a discrete group. If Γ has a finite p -subgroup S such that every p -subgroup of Γ is conjugate to a subgroup of S , then S is called a *Sylow p -subgroup* of Γ . The fusion system $\mathcal{F}_S(\Gamma)$ is defined as the category whose objects are subgroups of S , and whose morphisms are given by maps induced by conjugation by an element in Γ . In general, a fusion system \mathcal{F} is a category whose objects are the subgroups of a finite p -group S and whose morphisms are injective group homomorphisms satisfying certain properties. We say a fusion system \mathcal{F} is realized by a discrete group Γ , if Γ has a Sylow p -subgroup S such that $\mathcal{F} = \mathcal{F}_S(\Gamma)$.

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When a fusion system satisfies some further axioms that mimic Sylow theorems, it is called a *saturated fusion system*. Fusion systems realized by finite groups are the main examples of saturated fusion systems. There are exotic saturated fusion systems that are not realized by a finite group. However it has been shown independently by Leary and Stancu [13] and Robinson [18] that given a saturated fusion system \mathcal{F} , there is always a discrete group π that realizes \mathcal{F} . These infinite group models are constructed as fundamental groups of certain graphs of groups.

In this paper, we consider these infinite group constructions for a fusion system \mathcal{F} which is already realized by a finite group G . We find these infinite group models interesting from the point of view of group cohomology and cohomology of categories, even in the case where \mathcal{F} is realized by a finite group. The main aim of the paper is to prove a theorem which relates the mod- p cohomology of the fusion system \mathcal{F} of a group G to the mod- p cohomology of an infinite group model π that realizes \mathcal{F} and to provide an infinite family of examples where these two cohomology groups are not isomorphic. Throughout the paper when we say the cohomology of a group (or a fusion system), we always mean the mod- p cohomology for a fixed prime p unless otherwise is stated clearly.

Let G be a finite group and S be a Sylow p -subgroup of G . If there is subgroup $H \leq G$ which includes S as Sylow p -subgroup such that $\mathcal{F}_S(G) = \mathcal{F}_S(H)$, then we say H controls p -fusion in G . We say G is p -minimal if it has no proper subgroup that controls p -fusion in G . Assume that G is p -minimal and let π be an infinite group realizing the fusion system $\mathcal{F}_S(G)$ obtained by either the Leary–Stancu model or the Robinson model using the vertex groups as in Remark 2.11. We observe that in this case there is a surjective group homomorphism

$$\chi : \pi \rightarrow G$$

whose kernel is a free group F on which G acts by conjugation.

The homomorphism $\chi : \pi \rightarrow G$ satisfies some extra properties that makes it a *storing homomorphism* (see Definition 3.1). In Sect. 3 we study the cohomology of a discrete group π when there is a storing homomorphism $\chi : \pi \rightarrow G$. We consider the case where χ takes a Sylow p -subgroup of π to a Sylow p -subgroup of G . In this case we prove a theorem (Theorem 3.3) which relates the mod- p cohomology of π to the mod- p cohomology of G via a direct sum decomposition.

In Sect. 4 we apply the results of the previous section to an infinite group model π realizing a fusion system and prove the following theorem.

Theorem 1.1 *Let $\mathcal{F} = \mathcal{F}_S(G)$ be a fusion system of a finite group G . Assume that G is p -minimal, and let π denote the infinite group realizing \mathcal{F} obtained by either the Leary–Stancu model or the Robinson model (as in Remark 2.11). Then there is a group extension $1 \rightarrow F \rightarrow \pi \rightarrow G \rightarrow 1$ where F is a free group, and for every $n \geq 0$, there is an isomorphism of cohomology groups*

$$H^n(\pi; \mathbb{F}_p) \cong H^n(G; \mathbb{F}_p) \oplus H^{n-1}(G; \text{Hom}(F_{ab}, \mathbb{F}_p))$$

where $F_{ab} := F/[F, F]$ denotes the abelianization of F .

In [14, Thm 1.1], Libman and Seeliger consider the cohomology of an infinite model group π when \mathcal{F} is any saturated fusion system. In this case there is a map $f : B\pi \rightarrow |\mathcal{L}|_p^\wedge$ from the classifying space of π to the p -completion of the associated centric linking system \mathcal{L} . They showed that the \mathbb{F}_p -algebra homomorphism $f^* : H^*(|\mathcal{L}|; \mathbb{F}_p) \rightarrow H^*(\pi; \mathbb{F}_p)$ induced by the map f is a split monomorphism, and the splitting is given by the restriction map $\text{Res}_S^\pi : H^*(\pi; \mathbb{F}_p) \rightarrow H^*(S; \mathbb{F}_p)$. This gives an isomorphism

$$H^*(\pi; \mathbb{F}_p) \cong H^*(|\mathcal{L}|; \mathbb{F}_p) \oplus \ker(\text{Res}_S^\pi)$$

(see Sect. 2.4 for details). As a consequence of Theorem 1.1 we obtain that if \mathcal{F} is a fusion system realized by a finite group G that is p -minimal, then the kernel of the restriction map Res_S^π is isomorphic to $H^{*-1}(G; \text{Hom}(F_{ab}, \mathbb{F}_p))$.

At the end of Sect. 4, we also give some examples of storing homomorphisms and show that the mod- p cohomology of an infinite group constructed using the Leary–Stancu model is not in general isomorphic to the cohomology of the fusion system that it realizes.

In Sect. 5 we consider the mod-2 fusion systems for the groups $G = GL(n, 2)$ for $n \geq 5$ and show that they provide infinitely many examples where the Robinson model does not give discrete groups π with cohomology isomorphic to the cohomology of the corresponding fusion system.

Theorem 1.2 *Let $G = GL(n, 2)$ for $n \geq 5$, and let S be the Sylow-2 subgroup consisting of upper triangular matrices in G . Suppose that π is the infinite group realizing $\mathcal{F}_S(G)$ constructed using the Robinson model. Then $H^2(G, \mathbb{F}_2) \not\cong H^2(\pi, \mathbb{F}_2)$.*

In Sect. 6, we consider finite group actions on graphs and show that under certain conditions group actions on graphs can be used to obtain infinite group models realizing fusion systems. For a finite group G with p -rank equal to 2, we introduce a new infinite group model whose vertex groups are normalizers of elementary abelian p -subgroups (see Theorem 6.7). Note that for these infinite group models the cohomology of the infinite group π is isomorphic to the cohomology of the fusion system by a theorem of Webb (see Theorem 6.6).

In Sect. 7, we discuss signalizer functors $P \rightarrow \Theta(P)$ for infinite group models (see Definition 7.1 for a definition of signalizer functor). In the case where \mathcal{F} is the p -fusion system of a finite group G , we calculate the signalizer functors in terms of normalizers in the kernel of the storing homomorphism $\chi : \pi \rightarrow G$. For arbitrary fusion systems we show that for every \mathcal{F} -centric P , the mod- p homology of the group $\Theta(P)$ is zero in dimensions greater than 1 (see Proposition 7.5). In dimension 1 the homology group functor $P \rightarrow H_1(\Theta(P); \mathbb{F}_p)$ defines an $\mathbb{F}_p\mathcal{L}$ -module. We denote this module by $H_1\Theta$. As a consequence of the vanishing of homology groups of $\Theta(P)$ at dimensions greater than 1, we obtain the following theorem.

Theorem 1.3 *Let $\mathcal{T} := \mathcal{T}_S^c(\pi)$ denote the transporter category for an infinite group model π defined on the \mathcal{F} -centric subgroups of S , and let \mathcal{L} be the associated linking system defined by a signalizer functor $P \rightarrow \Theta(P)$. Then for every $\mathbb{F}_p\mathcal{L}$ -module M , there is a long exact sequence*

$$\begin{aligned} \cdots &\rightarrow H^{n-1}(\mathcal{T}; q^*M) \rightarrow \text{Ext}_{\mathcal{RL}}^{n-2}(H_1\Theta, M) \rightarrow H^n(\mathcal{L}; M) \rightarrow H^n(\mathcal{T}; q^*M) \\ &\rightarrow \text{Ext}_{\mathcal{RL}}^{n-1}(H_1\Theta, M) \rightarrow H^{n+1}(\mathcal{L}; M) \rightarrow \cdots \end{aligned}$$

where q^*M denote the $\mathbb{F}_p\mathcal{T}$ -module obtained from M via the quotient functor $q : \mathcal{T} \rightarrow \mathcal{L}$.

2 Definitions and preliminary results

In this section we introduce necessary definitions and preliminary results for the rest of the paper. The readers familiar with fusion systems and graphs of groups can skip most of this section. The standard reference for definitions on fusion systems is [9] and for graphs of groups is [21].

2.1 Fusion systems

Let S be a finite p -group. A fusion system \mathcal{F} on S is a category whose objects are the subgroups $P \leq S$, and for every $P, Q \leq S$ the morphism set $\text{Hom}_{\mathcal{F}}(P, Q)$ consists of injective group homomorphisms $P \rightarrow Q$ with following properties:

- (i) For all $P, Q \leq S$ we have $\text{Hom}_S(P, Q) \subseteq \text{Hom}_{\mathcal{F}}(P, Q)$, where $\text{Hom}_S(P, Q)$ is the set of all homomorphisms $c_s : P \rightarrow Q$ induced by conjugation with elements $s \in S$.
- (ii) For any morphism $\phi \in \text{Hom}_{\mathcal{F}}(P, Q)$, the isomorphism $\phi : P \rightarrow \phi(P)$, and its inverse $\phi^{-1} : \phi(P) \rightarrow P$, are morphisms in \mathcal{F} .

Let Γ be a discrete (possibly infinite) group, and let S be a finite p -subgroup of Γ . We say S is a *Sylow p -subgroup* of Γ if for every p -subgroup $P \leq \Gamma$ there is a $g \in \Gamma$ such that $gPg^{-1} \leq S$. When S is a Sylow p -subgroup of a discrete group Γ , the fusion system $\mathcal{F}_S(\Gamma)$ is defined as the fusion system on S whose morphisms $P \rightarrow Q$ are defined as the set of all maps $c_g : P \rightarrow Q$ induced by conjugations by elements in Γ . If a fusion system satisfies some additional axioms it is called a *saturated* fusion system (see [9, Def 1.37] for a definition). If G is finite group with a Sylow p -subgroup S , then the fusion system $\mathcal{F}_S(G)$ is saturated.

Let \mathcal{F} be a fusion system, and let P and Q be two subgroups in S . If there is an isomorphism $f : P \rightarrow Q$ in \mathcal{F} , then we say P and Q are *\mathcal{F} -conjugate* and denote this by $P \sim_{\mathcal{F}} Q$. A subgroup $Q \leq S$ is called *fully \mathcal{F} -normalized* if $|N_S(Q)| \geq |N_S(R)|$ for every $R \leq S$ with $Q \sim_{\mathcal{F}} R$. A subgroup $Q \leq S$ is called *fully \mathcal{F} -centralized* if $|C_S(Q)| \geq |C_S(R)|$ for every $R \leq S$ with $Q \sim_{\mathcal{F}} R$. We say Q is *\mathcal{F} -centric* if $C_S(R) \leq R$ for every $R \sim_{\mathcal{F}} Q$. A subgroup $Q \leq S$ is called *\mathcal{F} -radical* if $O_p(\text{Aut}_{\mathcal{F}}(Q)) = \text{Inn}(Q)$, where $O_p(G)$ denotes the largest normal p -subgroup in a group G .

If $\mathcal{F} = \mathcal{F}_S(G)$ for a finite group G with a Sylow p -subgroup S , then $P \leq S$ is fully normalized in \mathcal{F} if and only if $N_S(P)$ is a Sylow p -subgroup of $N_G(P)$ (see [9, Prop 1.38]). A subgroup $P \leq S$ is \mathcal{F} -centric if and only if $Z(P)$ is the Sylow p -subgroup of $C_G(P)$ (see [9, Prop 4.43]). In this case we have $C_G(P) = Z(P) \times C'_G(P)$ where $C'_G(P) := O_{p'}(C_G(P))$ denotes the largest normal subgroup of $C_G(P)$ whose order is coprime to p . A p -subgroup P in G is called *p -centric* if it satisfies this property.

We say a p -subgroup P of G is p -radical if $O_p(N_G(P)/P) = 1$. Note that a subgroup $P \leq S$ is $\mathcal{F}_S(G)$ -radical if and only if $O_p(N_G(P)/PC_G(P)) = 1$. Hence in general being p -radical and $\mathcal{F}_S(G)$ -radical are different conditions. However, the following holds.

Lemma 2.1 *Let G be finite group with a Sylow p -subgroup S , and let P be a subgroup of S . If P is $\mathcal{F}_S(G)$ -centric and $\mathcal{F}_S(G)$ -radical, then P is p -centric and p -radical. In general the converse does not hold.*

Proof We have seen above that P is $\mathcal{F}_S(G)$ -centric if and only if P is p -centric. Assume that P is not p -radical. Then there is a p -subgroup Q of $N_G(P)$ such that $P \triangleleft Q \triangleleft N_G(P)$. Since P is p -centric, $C_G(P) = Z(P) \times C'_G(P)$, hence $PC_G(P) = PC'_G(P)$. Since $C'_G(P)$ has order coprime to p , we have $Q \cap C'_G(P) = 1$. This gives that $QC'_G(P)/PC'_G(P)$ is a nontrivial normal p -subgroup in $N_G(P)/PC'_G(P)$. Hence P is not $\mathcal{F}_S(G)$ -radical.

To see that the converse that does not hold, let $G = D_{24}$ be the dihedral group of order 24 and let $P = O_2(G)$ be the normal cyclic subgroup of order 4. The centralizer of P in G is the cyclic subgroup of order 12, so P is 2-centric. Since $N_G(P)/P \cong S_3$, P is 2-radical. However P is not $\mathcal{F}_S(G)$ -radical since $N_G(P)/PC_G(P) \cong C_2$ (see [2, pg 11]). □

Given a p -group S , the largest fusion system on S is the system where morphisms from P to Q are all injective homomorphisms $f : P \rightarrow Q$. This fusion system is denoted by \mathcal{F}_S^{\max} . In general \mathcal{F}_S^{\max} is not a saturated fusion system. The fusion system generated by a collection of morphisms $\{f_i : P_i \rightarrow Q_i\}$ is defined as the smallest subfusion system of \mathcal{F}_S^{\max} that includes all the morphisms f_i . We denote this fusion system by $\langle f_i \mid i = 1, \dots, n \rangle$.

Alperin’s theorem for fusion systems states that if \mathcal{F} is a saturated fusion system, then \mathcal{F} is generated by \mathcal{F} -automorphisms of fully normalized, \mathcal{F} -radical, \mathcal{F} -centric subgroups of S (see [9, Thm 4.52]).

2.2 Graphs of groups

A graph Γ consists of two sets $E(\Gamma)$ and $V(\Gamma)$, called the edges and vertices of Γ , an involution on $E(\Gamma)$ which sends e to \bar{e} where $e \neq \bar{e}$, and two maps $o, t : E(\Gamma) \rightarrow V(\Gamma)$ which satisfy $t(e) = o(\bar{e})$. Each edge e is considered as an oriented edge, with origin $o(e)$ and terminus $t(e)$. The pair $\{e, \bar{e}\}$ is called an unoriented edge.

Definition 2.2 A graph of groups (\mathcal{G}, Y) consists of a connected nonempty graph Y together with a function \mathcal{G} assigning

- (i) to each vertex v of Y a group G_v and to each edge e of Y a group G_e , such that $G_{\bar{e}} = G_e$ for all e , and
- (ii) to each edge e , a monomorphism $\phi_e : G_e \rightarrow G_{t(e)}$.

The fundamental group of a graph of groups is a group that can be described by giving a presentation. Let E denote the free group with a basis given by the edges of Y . For this presentation we denote the edges of Y by y and write a^y for the image of

$a \in G_y$ under the monomorphism ϕ_y . Let $F(\mathcal{G}, Y)$ denote the quotient group of the free product

$$E * \left(\underset{v \in V(Y)}{*} G_v \right)$$

by the normal subgroup N , where N is the normal closure of the relations

$$ya^y y^{-1} = a^{\bar{y}} \quad \text{and} \quad \bar{y} = y^{-1}$$

for all $y \in E(Y)$ and $a \in G_y$. Let T be a maximal tree in Y , then we define the group $\pi(\mathcal{G}, Y, T)$ to be the quotient group of $F(\mathcal{G}, Y)$ subject to the relations $y = 1$ if $y \in E(T)$. It can be shown that the isomorphism class of $\pi(\mathcal{G}, Y, T)$ does not depend on the maximal tree T that is chosen (see [21, Proposition 20]). We call the group $\pi(\mathcal{G}, Y, T)$, the *fundamental group of (\mathcal{G}, Y)* , and denote it by $\pi(\mathcal{G}, Y)$.

There is also a topological description of the fundamental group of a graph of groups as the fundamental group of a topological space. For a discrete group G , let BG denote the classifying space of G . For each edge e , there is a continuous map $B\phi_e : BG_e \rightarrow BG_{t(e)}$ induced by the group homomorphism $\phi_e : G_e \rightarrow G_{t(e)}$.

Definition 2.3 The *total space* $X(\mathcal{G}, Y)$ of the graph of groups (\mathcal{G}, Y) is defined as the quotient space of

$$\left(\coprod_{v \in V(Y)} BG_v \right) \coprod \left(\coprod_{e \in E(Y)} (BG_e \times [0, 1]) \right)$$

by the identifications

$$BG_e \times [0, 1] \rightarrow BG_{\bar{e}} \times [0, 1] \quad \text{by } (x, t) \mapsto (x, 1 - t)$$

and

$$BG_e \times \{1\} \rightarrow BG_{t(e)} \quad \text{by } (x, 1) \mapsto (B\phi_e)(x).$$

Using van Kampen’s theorem and some other arguments, it is possible to show that the fundamental group of $X(\mathcal{G}, Y)$ is isomorphic to the group $\pi(\mathcal{G}, Y)$ defined above (see [8, Prop 23, pg 204]). The space $X(\mathcal{G}, Y)$ has a contractible universal covering, so it is a classifying space for the group $\pi(\mathcal{G}, Y)$ (see [13, Thm 22]).

Example 2.4 Two well-known examples of graph of groups are amalgamations and HNN-extensions. If Y is a graph with one unoriented edge and two distinct vertices, and if A and B are the vertex groups and C is the edge group with two monomorphism $A \hookrightarrow C \hookrightarrow B$, denoted by i_A and i_B , then the fundamental group π is the amalgamated product

$$A *_C B := \langle A * B \mid i_A(c)i_B(c)^{-1}, \text{ for all } c \in C \rangle.$$

For the HNN-extension, we take the graph Y as a graph with one unoriented edge and one vertex, i.e. the graph is just a loop. If the vertex group is A , the edge group is C , and its monomorphism is the identity embedding $C \hookrightarrow A$ and $\phi : C \hookrightarrow A$, we obtain the HNN-extension

$$A*_C := \langle A, t \mid tct^{-1} = \phi(c), \text{ for all } c \in C \rangle.$$

When Y is a finite graph as it is assumed to be throughout this paper, one can express the fundamental group π as a finite sequence of amalgamations and HNN-extensions. Because of this it is important to understand these two examples of graph of groups. □

One of the key properties of graphs of groups is that vertex groups G_v embed into the fundamental group of graphs of groups.

Lemma 2.5 (Lemma 19, pg. 200, [8]) *Let (\mathcal{G}, Y) be a graph of groups, let Z be a connected subgraph of Y . Then the natural homomorphism $\pi(\mathcal{G}|_Z, Z) \rightarrow \pi(\mathcal{G}, Y)$ is a monomorphism. In particular, for any vertex v of Y , the natural homomorphism $i_v : G_v \rightarrow \pi(\mathcal{G}, Y)$ is a monomorphism.*

Note that the natural map i_v is the map induced by inclusion of G_v into the free product $E*(\ast_{v \in V(Y)} G_v)$. Using the topological description of the fundamental group, it can also be described as the map induced by the inclusion of BG_v into the total space $X(\mathcal{G}, Y)$.

2.3 Groups acting on graphs

Let G be a group acting on a graph X . We say G acts *without inversion* if $ge \neq \bar{e}$ for every edge e in X and every $g \in G$. Sometimes this type of action is called a cellular action. Throughout the paper we will assume that all actions on graphs are without inversion. Assume that X is a connected graph. Then we can define a graph of groups (\mathcal{G}, Y) on the graph $Y = X/G$ using the G -action on X . The vertex groups of (\mathcal{G}, Y) are the stabilizers of vertices of X under G -action. The details of the construction of this graph of groups can be found in [21, Section 5.4]. The first structure theorem of the Bass-Serre theory is the following:

Theorem 2.6 (Theorem 12, pg. 52, [21]) *If π is the fundamental group of a graph of groups (\mathcal{G}, Y) , then there is a tree T on which π acts without inversions such that the graph of groups associated to the π action on T is isomorphic to (\mathcal{G}, Y) .*

The tree T is usually called the universal cover of the graph of groups (\mathcal{G}, Y) , and its construction is described in [21, pg. 51].

The second structure theorem of the Bass-Serre theory is in some sense a converse to Theorem 2.6. Let G be a group acting on a graph X without inversions, and let (\mathcal{G}, Y) be the associated graph of groups where $Y = X/G$. If $\pi = \pi(\mathcal{G}, Y)$, then there is a group homomorphism $\varphi : \pi \rightarrow G$ that takes the elements in $i(G_v)$ to the corresponding stabilizer subgroups in G and takes the HNN-extension generators t to

the corresponding group elements in G . There is also a map of graphs $\psi : T \rightarrow X$ from the universal cover T of (\mathcal{G}, Y) to the graph X .

Theorem 2.7 (Thm 13, pg. 55, [21]) *With the above notation and hypothesis, the following properties are equivalent:*

- (i) X is a tree.
- (ii) $\psi : T \rightarrow X$ is an isomorphism of graphs.
- (iii) $\varphi : \pi \rightarrow G$ is an isomorphism of groups.

One of the consequences of Theorem 2.7 is that if a group G acts freely on a tree then G is a free group. We can also conclude the following.

Corollary 2.8 (Cor 1, pg 212, [8]) *If H is a subgroup of the fundamental group $\pi(\mathcal{G}, Y)$ of a graph of groups such that the intersection of H with every conjugate of subgroups $i_v(G_v)$ is the trivial group, then H is free.*

In the situation described before Theorem 2.7, even for an arbitrary graph X , the homomorphism $\varphi : \pi \rightarrow G$ is surjective if X is connected (see [21, Lemma 4, pg 34]). From the way φ is defined it is easy to see that the kernel of φ meets every conjugate of a vertex group in the trivial group, hence by Corollary 2.8, the kernel of φ is a free group.

2.4 Graphs of groups realizing fusion systems

Given a saturated fusion system \mathcal{F} defined on a finite p -group S , there are two different constructions of a discrete group π with Sylow p -subgroup S , due to Leary and Stancu [13] and Robinson [18], such that $\mathcal{F}_S(\pi) = \mathcal{F}$. In both of these constructions the group π is the fundamental group of a graph of groups. We first state the result by Leary and Stancu, which defines π as an iterated HNN-extension of the group S and does not require the fusion system \mathcal{F} to be saturated; it works for any fusion system.

Theorem 2.9 (Leary and Stancu [13]) *Let \mathcal{F} be a fusion system on a p -group S generated by isomorphisms $f_i : P_i \rightarrow Q_i$ for $1 \leq i \leq r$. We define a graph of groups (\mathcal{G}, Y) where Y is the graph having only one vertex v and edges $e_1, \bar{e}_1, e_2, \bar{e}_2, \dots, e_r, \bar{e}_r$. We define the vertex group $G_v := S$ and edge groups $G_{e_i} = G_{\bar{e}_i} := P_i$. The morphisms $\phi_{e_i} : P_i \hookrightarrow S$ are the inclusions and the morphisms $\phi_{\bar{e}_i} : P_i \rightarrow S$ are f_i composed with inclusions of Q_i into S . Then the fundamental group $\pi := \pi(\mathcal{G}, Y)$ realizes the fusion system \mathcal{F} , that is $\mathcal{F} = \mathcal{F}_S(\pi)$.*

As an example of the Leary–Stancu model, consider the group

$$G = S_3 = \langle a, b \mid a^2 = b^3 = 1, aba = b^2 \rangle$$

at prime $p = 3$. The unique Sylow 3-subgroup of G is $S = \langle b \rangle \cong C_3$. The morphism $f : S \rightarrow S$ defined by $f(b) = b^2$ generates the fusion system $\mathcal{F}_S(G)$. In this case the Leary–Stancu model gives the infinite group

$$\pi = \langle b, t \mid b^3 = 1, tbt^{-1} = b^2 \rangle \cong C_3 \rtimes \mathbb{Z}.$$

We will come back to this example later in Example 4.3 when we discuss cohomology of infinite group models.

We now describe the Robinson model for realizing fusion systems.

Theorem 2.10 (Robinson [18]) *Let \mathcal{F} be a fusion system on a p -group S generated by the images $\mathcal{F}_{S_i}(G_i)$ under injective group homomorphisms $f_i : S_i \hookrightarrow S$ for $1 \leq i \leq r$. We define a graph of groups (\mathcal{G}, Y) , where Y has vertices $v_0, v_1, v_2, \dots, v_r$ and edges e_i, \bar{e}_i between v_0 and v_i for $1 \leq i \leq r$. The vertex groups are $G_{v_0} := S$ and $G_{v_i} = G_i$ for $1 \leq i \leq r$. The edge groups are $G_{e_i} = G_{\bar{e}_i} := S_i$ and monomorphisms $\phi_{e_i} : S_i \hookrightarrow S, \phi_{\bar{e}_i} : S_i \hookrightarrow G_i$ are inclusions. Then the fundamental group $\pi := \pi(\mathcal{G}, Y)$ realizes the fusion system \mathcal{F} .*

Robinson’s theorem is proved also in [13, Thm 3]. Note that to apply Robinson’s model to a particular fusion system, we need to start with a collection of subgroups G_i such that images of $\mathcal{F}_{S_i}(G_i)$ generate the fusion system \mathcal{F} . Such a collection always exists for a saturated fusion system, but does not exist for an arbitrary fusion system (see [13, Section 4]).

Given a saturated fusion system \mathcal{F} , a family of subgroups $\{P_i\}$ in S is called a *conjugation family* if \mathcal{F} is generated by morphisms in the normalizer fusions system $N_{\mathcal{F}}(P_i)$. By a theorem of Goldschmidt [11], the family of \mathcal{F} -centric and \mathcal{F} -radical subgroups in S form a conjugation family. For each i , the normalizer fusion system $N_{\mathcal{F}}(P_i)$ is realized by a finite group G_i with a Sylow p -subgroup isomorphic to $N_S(P_i)$ (see [9, Thm 3.70]). Hence if we take the groups G_i in Theorem 2.10 as these model groups and the subgroups S_i as their Sylow p -subgroups, then the infinite group π obtained using the Robinson construction will realize the fusion system \mathcal{F} .

Remark 2.11 If \mathcal{F} is a fusion system realized by a finite group G with a Sylow p -subgroup S , we take the subgroups G_i in the Robinson model as the normalizers $G_i = N_G(P_i)$ where $\{P_i\}$ is the family of all fully normalized, \mathcal{F} -radical, and \mathcal{F} -centric subgroups of S . For the edge groups we take the Sylow p -subgroups $S_i = N_S(P_i)$ for every i . When we refer to the Robinson model for a fusion system realized by a finite group G , we will always assume that the collection of groups $\{G_i\}$ and $\{S_i\}$ appearing in Theorem 2.10 are chosen as described here.

Associated to a saturated fusion system \mathcal{F} there is a *centric linking system* \mathcal{L} (see [9, Def 9.35]) and the triple $(S, \mathcal{F}, \mathcal{L})$ is called a p -local finite group. The cohomology of the p -local finite group $(S, \mathcal{F}, \mathcal{L})$ is defined to be the cohomology of the p -completion of the realization $|\mathcal{L}|$ of the linking system \mathcal{L} . It is shown in [6, Thm B] that the cohomology of a p -local finite group is isomorphic to the subalgebra of \mathcal{F} -stable elements in $H^*(S; \mathbb{F}_p)$, denoted by $H^*(\mathcal{F}; \mathbb{F}_p)$. We define the *cohomology of the fusion system \mathcal{F}* as the inverse limit

$$H^*(\mathcal{F}; \mathbb{F}_p) := \lim_{P \in \mathcal{F}} H^*(P; \mathbb{F}_p)$$

and it is easy to see that these two definitions for $H^*(\mathcal{F}; \mathbb{F}_p)$ coincide.

In general the cohomology of a fusion system may be different than the cohomology of an infinite group π that realizes it. The following theorem by Libman and Seeliger [14, Thm 1.1] explains the relation between these two cohomology groups.

Theorem 2.12 (Libman and Seeliger [14]) *Let \mathcal{F} be a saturated fusion system defined on a finite p -group S , and let π be an infinite group realizing \mathcal{F} , constructed using the Leary–Stancu model or the Robinson model. Then the map $\text{res}_S^\pi : H^*(\pi, \mathbb{F}_p) \rightarrow H^*(S, \mathbb{F}_p)$ splits as an \mathbb{F}_p -algebra map and has an image isomorphic to $H^*(\mathcal{F}; \mathbb{F}_p)$ that gives*

$$H^*(\pi; \mathbb{F}_p) \cong H^*(\mathcal{F}; \mathbb{F}_p) \oplus \ker(\text{Res}_S^\pi).$$

The proof of this theorem uses results from the homotopy theory of linking systems. We will give later another proof for this theorem for fusion systems realized by a finite group.

3 Graphs of groups with a storing homomorphism

In this section we use the definitions and notation introduced in the previous section.

Definition 3.1 Let (\mathcal{G}, Y) be a graph of groups and G be a finite group. A group homomorphism $\chi : \pi(\mathcal{G}, Y) \rightarrow G$ is called a *storing homomorphism* if it is surjective and for any vertex group G_v with inclusion map $i_v : G_v \rightarrow \pi(\mathcal{G}, Y)$, the composition $\chi \circ i_v : G_v \rightarrow G$ is injective.

The kernel of a storing homomorphism $\chi : \pi \rightarrow G$ has a trivial intersection with $i_v(G_v)$ for each vertex group G_v , hence by Corollary 2.8, the kernel of χ is a free group. Therefore, we have an exact sequence

$$1 \rightarrow F \rightarrow \pi(\mathcal{G}, Y) \xrightarrow{\chi} G \rightarrow 1$$

where F is a free group. This gives a G -action on the abelianization $F_{ab} = F/[F, F]$ induced by conjugation in $\pi(\mathcal{G}, Y)$.

By Theorem 2.6, the fundamental group $\pi := \pi(\mathcal{G}, Y)$ acts on a tree T without inversion in such a way that the isotropy subgroups of the vertices of T are conjugate to the vertex groups G_v of (\mathcal{G}, Y) . The π -action on T induces an action of $G \cong \pi(\mathcal{G}, Y)/F$ on the quotient graph $X = T/F$. From this we obtain a G -action on $H_1(X)$.

Lemma 3.2 *There is a $\mathbb{Z}G$ -module isomorphism between F_{ab} and $H_1(X)$.*

Proof Let $\pi : T \rightarrow X = T/F$ denote the quotient map which takes a point $t \in T$ to its F -orbit Ft . Fix a vertex $v \in T$, and let $\bar{v} = \pi(v)$. By covering space theory, there is an isomorphism $F \cong \pi_1(X, \bar{v})$ given by the map $\phi : F \rightarrow \pi_1(X, \bar{v})$ that takes an $f \in F$ to the path homotopy class $[\pi(\tau)]$, where $\tau = p(v, fv)$ is a path from v to fv .

Let $\hat{\phi}$ be the induced isomorphism between the abelianization groups $F_{ab} \rightarrow H_1(X) \cong (\pi_1(X, \bar{v}))_{ab}$. We have a commutative diagram

$$\begin{CD} F @>\phi>> \pi_1(X, \bar{v}) \\ @VjVV @VVkV \\ F_{ab} @>\hat{\phi}>> H_1(X) \end{CD}$$

where j and k are the abelianization maps. To show that $\hat{\phi}$ is a $\mathbb{Z}G$ -module isomorphism, it is enough to show that for every $f \in F$ and $g \in G$, the equality $k(\phi(\gamma f \gamma^{-1})) = g \cdot k(\phi(f))$ holds for every $\gamma \in \pi(\mathcal{G}, Y)$ such that $\chi(\gamma) = g$.

We have $\phi(f) = [\pi(\tau)]$ where $\tau = p(v, f v)$ is a path from v to $f v$. For $\gamma \in \pi(\mathcal{G}, Y)$, we have $\phi(\gamma f \gamma^{-1}) = [\pi p(v, \gamma f \gamma^{-1} v)]$. Note that

$$p(v, \gamma f \gamma^{-1} v) \simeq p(v, \gamma v) \cdot p(\gamma v, \gamma f v) \cdot p(\gamma f v, \gamma f \gamma^{-1} v)$$

in T . Since π annihilates the F -action, we have

$$\pi(p(\gamma f v, \gamma f \gamma^{-1} v)) = \pi(\gamma f \gamma^{-1} p(\gamma v, v)) = \pi(p(\gamma v, v)) = \pi(p(v, \gamma v))^{-1}.$$

This gives

$$\begin{aligned} k(\phi(\gamma f \gamma^{-1})) &= k[\pi p(v, \gamma v)] + k[\pi p(\gamma v, \gamma f v)] - k[\pi p(\gamma v, v)] \\ &= k[\pi p(\gamma v, \gamma f v)] \end{aligned}$$

in $H_1(X)$. Note that $\pi p(\gamma v, \gamma f v)$ is a loop at $g\bar{v}$ whose homology class is equal to $gk[\pi p(v, f v)]$. Hence we have $k(\phi(\gamma f \gamma^{-1})) = gk(\phi(f))$ as desired. We conclude that $\hat{\phi}$ is a $\mathbb{Z}G$ -module isomorphism between F_{ab} and $H_1(X)$. □

Theorem 3.3 *Let (\mathcal{G}, Y) be a graph of groups and let $\pi := \pi(\mathcal{G}, Y)$. Suppose that π has a Sylow p -subgroup and that there is a surjective homomorphism $\chi : \pi \rightarrow G$ that takes a Sylow p -subgroup of π to a Sylow p -subgroup of G . Then, there is an isomorphism*

$$H^n(\pi; \mathbb{F}_p) \cong H^n(G; \mathbb{F}_p) \oplus H^{n-1}(G; \text{Hom}(F_{ab}, \mathbb{F}_p))$$

for every $n \geq 0$, where F is the kernel of χ .

Proof To simplify the notation we will denote the images of vertex groups G_v and edge groups G_e under $\chi : \pi \rightarrow G$ also by G_v and G_e . Let $F = \ker \chi$ and T be the tree on which π acts with isotropy given by (\mathcal{G}, Y) . Consider the G -action on the graph $X = T/F$. Since T is connected, X is also a connected graph.

The cellular cochain complex for X with coefficients in $R := \mathbb{F}_p$ gives an exact sequence of RG -modules

$$0 \rightarrow R \rightarrow C^0(X, R) \xrightarrow{\delta^0} C^1(X, R) \rightarrow H^1(X; R) \rightarrow 0. \tag{3.1}$$

The G -action on X permutes the cells in X , hence we have

$$C^0(X, R) = \bigoplus_{v \in OV} R[G/G_v] \text{ and } C^1(X, R) = \bigoplus_{e \in OE} R[G/G_e]$$

where OE and OV are orbit representative sets for edges and vertices in X , respectively. Since π has a Sylow p -subgroup, there exists a vertex group G_v containing a Sylow p -subgroup S . This means that in G the subgroup G_v also includes a Sylow p -subgroup of G . From this we conclude that the map $\mu : R \rightarrow C^0(X, R)$ splits since $|G : G_v|$ is not divisible by p . We can divide the exact sequence in (3.1) into two sequences

$$\begin{aligned} 0 \rightarrow R \rightarrow C^0(X, R) \rightarrow K \rightarrow 0 \\ 0 \rightarrow K \rightarrow C^1(X, R) \rightarrow H^1(X; R) \rightarrow 0 \end{aligned}$$

where the first sequence splits. By Shapiro’s lemma, the first sequence gives an isomorphism

$$\bigoplus_{v \in OV} H^*(G_v, R) \cong H^*(G; K) \oplus H^*(G; R). \tag{3.2}$$

From the second short exact sequence, we also obtain a long exact sequence. By adding $H^*(G; R)$ to two consecutive terms in this sequence and by using the isomorphism in (3.2), we get a long exact sequence

$$\begin{aligned} \dots \rightarrow H^n(G; R) \oplus H^{n-1}(G, H^1(X; R)) \rightarrow \bigoplus_{v \in OV} H^n(G_v, R) \xrightarrow{(\delta^0)^*} \bigoplus_{e \in OE} H^n(G_e, R) \\ \rightarrow H^n(G, H^1(X; R)) \oplus H^{n+1}(G; R) \rightarrow \dots \end{aligned}$$

The group $\pi = \pi(\mathcal{G}, Y)$ acts on a tree T with the same isotropy subgroups as the G -action on X . This gives a similar long exact sequence from the π -action on T :

$$\dots \rightarrow H^n(\pi; R) \rightarrow \bigoplus_{v \in OV} H^n(G_v; R) \rightarrow \bigoplus_{e \in OE} H^n(G_e, R) \rightarrow H^{n+1}(\pi; R) \rightarrow \dots$$

Since the maps in the middle coincide, by the five-lemma we obtain an isomorphism

$$H^n(\pi, R) \cong H^n(G, R) \oplus H^{n-1}(G, H^1(X; R))$$

for every $n \geq 0$. By Lemma 3.2 we have

$$H^1(X; R) = \text{Hom}(H_1(X); R) = \text{Hom}(F_{ab}, R).$$

Hence the proof is complete. □

Remark 3.4 Theorem 3.3 is a generalization of [4, Lemma 3.1]. The proof we give here is very similar to the proof in [4]. There is an alternative approach to proving Theorem 3.3 using the Lyndon–Hochschild–Serre spectral sequence [7, Thm 6.3] for the extension

$$1 \rightarrow F \rightarrow \pi \rightarrow G \rightarrow 1.$$

We use this approach later in the proof of Theorem 1.3.

In Theorem 3.3, the assumption that the fundamental group π has a Sylow p -subgroup is necessary, as the following example illustrates.

Example 3.5 Let $\pi = C_2 * C_2 = \langle a_1, a_2 \mid a_i^2, \text{ for } i = 1, 2 \rangle$. Then G has no Sylow 2-subgroup since the subgroups $\langle a_1 \rangle$ and $\langle a_2 \rangle$ are not conjugate to each other in π . Note that if we take $G = C_2 \times C_2$ and define the storing homomorphism $\chi : \pi \rightarrow G$ by taking a_1 and a_2 to the generators of G , then the kernel of χ is the subgroup $F = \langle (a_1 a_2)^2 \rangle \cong \mathbb{Z}$. We have $H^*(G; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2]$ where $\deg x_i = 1$, and $H^i(\pi; \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ for all $i \geq 1$. Hence, the isomorphism in Theorem 3.3 does not hold in this case.

4 Proof of Theorem 1.1

In this section we prove the following theorem which was stated as Theorem 1.1 in the introduction. For a discrete group F , $F_{ab} := F/[F, F]$ denotes the abelianization of F .

Theorem 4.1 *Let $\mathcal{F} = \mathcal{F}_S(G)$ be a fusion system of a finite group G . Assume that G is p -minimal, and let π denote the infinite group realizing \mathcal{F} obtained by either the Leary–Stancu model or the Robinson model (as in Remark 2.11). Then there is a group extension $1 \rightarrow F \rightarrow \pi \rightarrow G \rightarrow 1$ where F is a free group, and there is an isomorphism of cohomology groups*

$$H^n(\pi; \mathbb{F}_p) \cong H^n(G; \mathbb{F}_p) \oplus H^{n-1}(G; \text{Hom}(F_{ab}, \mathbb{F}_p))$$

for every $n \geq 0$.

Proof Let G be a finite group with Sylow p -subgroup S . Suppose that G has no proper subgroups that control p -fusion in G . Let $\pi_{LS} = \pi(\mathcal{G}, Y)$ denote the infinite group realizing the fusion system $\mathcal{F} := \mathcal{F}_S(G)$ constructed according to the Leary–Stancu model, as explained in Theorem 2.9. Let $\chi : \pi_{LS} \rightarrow G$ denote the group homomorphism that takes $S \leq \pi_{LS}$ to $S \leq G$ and the generators t_i to the group elements $g_i \in G$ where g_i is an element in G such that $c_{g_i} = f_i : P_i \rightarrow Q_i$ for $i = 1, \dots, r$. Note that the image of χ controls p -fusion in G , hence by our assumption above χ is surjective. The only vertex group of \mathcal{G} is S , and the restriction of χ to S is injective, hence χ is a storing homomorphism.

Now let π_R denote the infinite group π_R obtained using the Robinson model with vertex groups $N_G(P_i)$ for a collection of p -centric subgroups $\{P_i\}$, where Y is a

star-shaped graph with the center having vertex group S . In this case the group π is generated by the subgroups $i_v(N_G(P_i))$ in π , so χ is defined as a map that takes the subgroups $i_v(N_G(P_i))$ injectively to the subgroups $N_G(P_i)$ in G . It is easy to see that in this case too, χ is a storing homomorphism.

In both cases there is a storing homomorphism $\chi : \pi \rightarrow G$. Since the kernel of a storing homomorphism is a free group, this gives a group extension $1 \rightarrow F \rightarrow \pi \rightarrow G \rightarrow 1$, where F is a free group. Applying Theorem 3.3 to the storing homomorphism $\chi : \pi \rightarrow G$ gives the isomorphism in the statement of the theorem. \square

As a corollary of Theorem 4.1, we obtain the following.

Corollary 4.2 *Let G and π be as in Theorem 1.1 Then the kernel of the restriction map $\text{Res}_S^\pi : H^n(\pi; \mathbb{F}_p) \rightarrow H^n(S; \mathbb{F}_p)$ is isomorphic to $H^{n-1}(G; \text{Hom}(F_{ab}, \mathbb{F}_p))$ for all $n \geq 0$.*

Proof The image of the restriction map Res_S^π is isomorphic $H^*(\mathcal{F}; \mathbb{F}_p) \cong H^*(G; \mathbb{F}_p)$. Hence Theorem 1.1 gives the desired isomorphism. \square

We now give an example to illustrate that infinite groups obtained using the Leary–Stancu model may have cohomology groups that are not isomorphic to the cohomology of the fusion systems that they realize.

Example 4.3 Let $G = S_3 = \langle a, b \mid b^3 = a^2 = 1, aba = b^2 \rangle$ and $R = \mathbb{F}_3$. The Sylow 3-subgroup of G is $S = \langle b \rangle \cong C_3$. The Leary–Stancu model is the infinite group

$$\pi = \langle b, t \mid b^3 = 1, tbt^{-1} = b^2 \rangle \cong C_3 \rtimes \mathbb{Z}.$$

The storing homomorphism $\chi : \pi \rightarrow G$ takes $t \in \pi$ to $a \in G$, so the kernel of χ is $F = \langle t^2 \rangle \cong \mathbb{Z}$. The G -action on F is trivial, hence Theorem 3.3 gives that

$$H^n(\pi; \mathbb{F}_3) \cong H^n(S_3; \mathbb{F}_3) \oplus H^{n-1}(S_3; \mathbb{F}_3). \tag{4.1}$$

for all $n \geq 0$. The cohomology ring of C_3 is $H^*(C_3; \mathbb{F}_3) = \bigwedge_{\mathbb{F}_3} (x) \otimes \mathbb{F}_3[y]$, where $\deg x = 1$ and $\deg y = 2$, and the cohomology ring of S_3 is the subalgebra

$$H^*(S_3; \mathbb{F}_3) = \bigwedge_{\mathbb{F}_3} (xy) \otimes \mathbb{F}_3[y^2].$$

We can calculate the cohomology of π using the sequence $1 \rightarrow C_3 \rightarrow \pi \rightarrow \mathbb{Z} \rightarrow 1$. The LHS-spectral sequence has only two nonzero vertical lines and $d_1 : H^n(C_3; \mathbb{F}_3) \rightarrow H^n(C_3, \mathbb{F}_3)$ is identity only at dimensions n where $n \equiv 1, 2 \pmod 4$. From this calculation, we can easily see that $H^n(\pi; \mathbb{F}_3) \not\cong H^n(S_3; \mathbb{F}_3)$ and $\ker(\text{Res}_S^\pi) \cong H^{n-1}(S_3; \mathbb{F}_3)$ for all $n \geq 0$. \square

We end this section with an example of storing homomorphism $\pi \rightarrow S_3$ where π is an amalgamation of two finite groups and S_3 acts nontrivially on F .

Example 4.4 Let $\pi = S_3 *_{C_3} S_3$, and $R = \mathbb{F}_3$. We can give a presentation for π as follows:

$$\pi = \langle b, a_1, a_2 \mid a_i^2 = 1, a_i b a_i = b^2 \text{ for } i = 1, 2 \rangle.$$

The group $G = S_3$ is a store of π with storing homomorphism which takes both a_1 and a_2 to $a \in G$. The kernel of χ is $F = \langle a_1 a_2 \rangle \cong \mathbb{Z}$. In this case $G = S_3$ acts nontrivially on F since $a_1(a_1 a_2)a_1 = a_2 a_1 = (a_1 a_2)^{-1}$. We usually denote this one-dimensional $\mathbb{Z}S_3$ -module by $\tilde{\mathbb{Z}}$. By Theorem 3.3, we have

$$H^n(\pi; \mathbb{F}_3) \cong H^n(S_3; \mathbb{F}_3) \oplus H^{n-1}(S_3; \tilde{\mathbb{F}}_3)$$

for all $n \geq 0$, where $\tilde{\mathbb{F}}_3 = \mathbb{F}_3 \otimes \tilde{\mathbb{Z}}$ is the one dimensional $\mathbb{F}_3 S_3$ -module where the generators $b \in S_3$ acts trivially and a acts by multiplication with (-1) . We can calculate the cohomology groups $H^n(S_3; \tilde{\mathbb{F}}_3)$ using the sequence of $\mathbb{F}_3 G$ -modules $0 \rightarrow \mathbb{F}_3 \rightarrow \mathbb{F}_3[G/S] \rightarrow \mathbb{F}_3 \rightarrow 0$. We obtain that $H^n(S_3; \tilde{\mathbb{F}}_3) \cong \mathbb{F}_3$ for $n \equiv 1, 2 \pmod 4$, and 0 otherwise. The cohomology of π can be calculated using the long exact sequence for groups acting on a tree. From these we can verify that the isomorphism above holds. In this case the kernel of the restriction map to the Sylow 3-subgroup is $H^{n-1}(S_3; \tilde{\mathbb{F}}_3)$. □

5 An infinite family of examples

In this section we consider the 2-fusion system of the group $GL(n, 2)$ and show that the cohomology of the infinite group π_R constructed using the Robinson model is not isomorphic to the cohomology of the 2-fusion system for $n \geq 5$. This gives an infinite family of examples with this property. Examples of groups with this property were already known. In [20, Prop 6.8], it is shown that the mod-2 cohomology of $G = C_2^3 \rtimes GL(3, 2)$ is not isomorphic to the cohomology of π_R .

To construct an infinite group using the Robinson model for the 2-fusion system of $GL(n, 2)$, we must understand all fully normalized, \mathcal{F} -radical, \mathcal{F} -centric subgroups for the fusion system $\mathcal{F} = \mathcal{F}_S(G)$, where $G = GL(n, 2)$ and S is a Sylow 2-subgroup of G . Since $GL(n, p)$ is an algebraic group we will quote some standard results from [3, Sec 6.8] and [16] to describe its p -radical and p -centric subgroups. We also refer to [15, Appendix B] for some of the results below.

Let S be the subgroup of $G = GL(n, 2)$ consisting of the upper triangular matrices. Since the order of S is $2^{(n-1)(n-2)/2}$ and the order of G is $(2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1})$, the index $|G : S|$ is odd. Hence S is a Sylow 2-subgroup of G . The Borel subgroups of G are the conjugates of S and $N_G(S) = S$ (see [16, Thm 6.12]). Parabolic subgroups of G are stabilizers of flags $0 = V_0 < \cdots < V_k = \mathbb{F}_p^n$, so every parabolic subgroup is conjugate to a subgroup N consisting of matrices of the form

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

The unipotent radical of N is the subgroup U of matrices of the form

$$\begin{bmatrix} I & * & * \\ 0 & I & * \\ 0 & 0 & I \end{bmatrix}$$

We now state a special case of the Borel–Tits theorem (see [3, Thm 6.8.4]) to identify the p -radical subgroups of G .

Theorem 5.1 (Borel–Tits) *If $G = GL(n, p)$ then a p -subgroup U is p -radical if and only if $N_G(U)$ is parabolic and U is its unipotent radical.*

We also have the following observation.

Lemma 5.2 *Let S be the group of upper triangular matrices in $G = GL(n, 2)$ and $\mathcal{F} = \mathcal{F}_S(G)$. Then any unipotent radical U of a parabolic group P containing S is \mathcal{F} -centric.*

Proof If U is \mathcal{F} -centric and $V \geq U$, then V is also \mathcal{F} -centric. We know that the maximal parabolic subgroup corresponds to the minimal unipotent radicals. Then, it is enough to prove that the statement holds for all maximal parabolic subgroups containing S .

Take any maximal parabolic subgroup $N_G(U)$ containing S , where U is a subgroup of the form

$$U_m = \begin{bmatrix} I_m & M_{m, n-m}(\mathbb{F}_2) \\ 0 & I_{n-m} \end{bmatrix}$$

for some m . Then $U \triangleleft S$, hence U is fully \mathcal{F} -normalized. This implies that fully \mathcal{F} -centralized, and hence it is enough to show that $C_S(U) = Z(U)$. Take any $s \in S$ centralizing U_m . If we write

$$s = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A and C are upper triangular matrices with diagonal entries equal to 1, then the equation $su = us$ gives that we must have $AM = MC$ for any $M \in M_{m, n-m}(\mathbb{F}_2)$. Fix any $1 \leq i \leq m$ and $1 \leq j \leq m - n$. Choosing M to have all entries 0 except the (i, j) -th entry, which is equal to 1, the equality $AM = MC$ gives that $c_{j, k} = 0$ for $k \neq j$ and $a_{l, i} = 0$ for $l \neq i$. This gives $A = I_m$ and $C = I_{n-m}$. Hence s lies in U_m . We conclude that U is \mathcal{F} -centric. \square

The argument above can be extended to show that $C_G(U) = Z(U)$ for every unipotent radical U normal in S . This gives that $C'_G(U) = 1$ and $N_G(U)/U = N_G(U)/C_G(U)U$ for these subgroups. In particular, U is \mathcal{F} -radical since U is p -radical in G . We conclude the following.

Theorem 5.3 *Let $G = GL(n, 2)$. The subgroup of upper triangular matrices S in G is a Sylow 2-subgroup of G . Let $\mathcal{F} = \mathcal{F}_S(G)$. Then U is fully normalized, \mathcal{F} -radical, \mathcal{F} -centric subgroup of S if and only if $N_G(U)$ is parabolic containing S and U is its unipotent radical.*

Proof The first sentence is explained above. Let U be a fully normalized, \mathcal{F} -centric, and \mathcal{F} -radical subgroup in S . By Lemma 2.1, an \mathcal{F} -centric, \mathcal{F} -radical subgroup of S is p -radical, p -centric in G . Hence by Theorem 5.1, $N_G(U)$ is parabolic and U is its unipotent radical. Since $N_G(U)$ is parabolic, $N_G(U) \supset B$ for some Borel subgroup B . Since Borel subgroups are conjugate, there exists $g \in G$ such that $S = gBg^{-1}$. Let $P = gUg^{-1}$. Then $N_G(P) = gN_G(U)g^{-1} \supset gBg^{-1} = S$. Since U is fully normalized, we have $|N_S(U)| \geq |N_S(P)|$. So $N_S(P) = S$ gives that $N_S(U) = S$, which means $N_G(U)$ contains S as desired.

For the other direction, assume that U is a subgroup of S such that $N_G(U)$ is parabolic containing S and U is its unipotent radical. By Theorem 5.1, U is p -radical. By Lemma 5.2, U is \mathcal{F} -centric. By the remark after the proof of Lemma 5.2, U is \mathcal{F} -radical. Since $N_S(U) = S$, we can also say that U is fully normalized. \square

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 Let $G = GL(n, 2)$ with $n \geq 5$. By [12, Table 6.1.3], we have $H^2(\mathcal{F}; \mathbb{F}_2) = H^2(G; \mathbb{F}_2) = 0$. We will show that $H^2(\pi; \mathbb{F}_2) \neq 0$. The vertex groups of π are the subgroup S and the normalizers $N_G(P_i)$ of fully normalized, \mathcal{F} -radical, \mathcal{F} -centric subgroups P_i of S . From Theorem 5.3, the vertex groups of π are $N_0 = S$ and the parabolic subgroups N_1, N_2, \dots, N_k containing S . This gives a long exact sequence

$$\dots \rightarrow \bigoplus_{0 \leq i \leq k} H^1(N_i; \mathbb{F}_2) \xrightarrow{f} \bigoplus_1^k H^1(S; \mathbb{F}_2) \rightarrow H^2(\pi; \mathbb{F}_2) \rightarrow \dots \quad (5.1)$$

Note that for any i , we have $|H^1(N_i; \mathbb{F}_2)| \leq |H^1(S; \mathbb{F}_2)|$ because S is a Sylow 2-subgroup of N_i . Without loss of generality, assume that N_1, N_2, \dots, N_{n-1} are maximal parabolic subgroups such that, for $1 \leq m \leq n - 1$, we have

$$P_m = \begin{bmatrix} GL(m, 2) & M_{m,n-m}(\mathbb{F}_2) \\ 0 & GL(m - n, 2) \end{bmatrix}.$$

Then we have that $N_1 \cong N_{n-1} \cong C_2^{n-1} \rtimes GL(n - 1, 2)$. Note that $H^1(N_1; \mathbb{F}_2) \cong \text{Hom}(N_1, C_2)$. Take any $\phi \in \text{Hom}(N_1, C_2)$. The restriction of ϕ to $GL(n - 1, 2)$ is the zero homomorphism because $GL(n - 1, 2)$ is a simple group. If ϕ is non-zero, then we have $\phi(a) = 1$ for some $a \in C_2^{n-1}$. Take any nonzero $b \in C_2^{n-1}$ such that $b \neq a$. Since $GL(n - 1, 2)$ acts on C_2^{n-1} by conjugation and it sends any nonzero element to a nonzero element, we have $\phi(a) = \phi(b) = \phi(a + b) = 1$ which is a contradiction. We conclude that

$$H^1(N_1; \mathbb{F}_2) = H^1(N_{n-1}; \mathbb{F}_2) = 0.$$

From this we obtain that

$$\sum_{0 \leq i \leq k} \dim H^1(N_i; \mathbb{F}_2) < k \dim H^1(S; \mathbb{F}_2)$$

because in the left hand side two terms are zero as shown above, and for all the other summands we have $|H^1(N_i; \mathbb{F}_2)| \leq |H^1(S; \mathbb{F}_2)|$. This gives that the map f in the long exact sequence (5.1) is not surjective. Hence $H^2(\pi; \mathbb{F}_2) \neq 0$. This completes the proof.

6 Realizing fusion systems via group actions on graphs

Let G be a finite group acting on a connected graph X without inversion. As we discussed in Sect. 2.3, using the isotropy subgroups of the vertices and edges of X , we can define a graph of groups \mathcal{G} on the graph $Y = X/G$. Let $\pi := \pi(\mathcal{G}, Y)$ denote the fundamental group of this graph of groups. The map $\chi : \pi \rightarrow G$ defined by sending the vertex groups G_v of π to the corresponding isotropy subgroups in G is a storing homomorphism. The surjectivity of χ follows from the fact that X is connected. We also know that the kernel of χ is a free group by Corollary 2.8. This gives an extension of groups

$$1 \rightarrow F \rightarrow \pi \rightarrow G \rightarrow 1.$$

There is an alternative description of the group $\pi = \pi(\mathcal{G}, Y)$ that is associated to a G -action on a graph X . Consider the Borel construction $EG \times_G X$. From the description of the total space of (\mathcal{G}, Y) it is easy to see that the total space $X(\mathcal{G}, X)$ is homotopy equivalent to the Borel construction $EG \times_G X$ (see [19, pg. 167]). Hence $\pi = \pi_1(EG \times_G X)$. The Borel construction gives a fibration

$$X \rightarrow EG \times_G X \rightarrow BG$$

that induces a long exact sequence in homotopy groups

$$\cdots \rightarrow \pi_2(BG) \rightarrow \pi_1(X) \rightarrow \pi_1(EG \times_G X) \rightarrow \pi_1(BG) \rightarrow \pi_0(X) \rightarrow \cdots$$

Since X is connected and BG is a classifying space of a finite group, we obtain a short exact sequence

$$1 \rightarrow \pi_1 X \rightarrow \pi_1(EG \times_G X) \rightarrow \pi_1(BG) \rightarrow 1.$$

This shows that the map $\chi : \pi \rightarrow G$ is surjective and its kernel is isomorphic to $\pi_1(X)$, which is a free group.

Note that infinite groups obtained in this way may not have a Sylow p -subgroup in general.

Example 6.1 Consider the $G = C_2 = \langle a \rangle$ action on a circle X with the action $g(x, y) = (x, -y)$. We can view X as the realization of the graph with two vertices and two edges. Then the quotient graph Y is a graph with single edge and two vertices, where vertex groups are $G_v = C_2$ at both vertices, and the edge group is 1. The fundamental group $\pi := \pi(EG \times_G X)$ is the free product $C_2 * C_2$ which is isomorphic to the infinite Dihedral group D_∞ . In this case π does not have a Sylow 2-subgroup. \square

We can give a list of conditions on the G -action on X to guarantee the existence of a Sylow p -subgroup in π .

Proposition 6.2 *Let G be a finite group with a Sylow p -subgroup S , and let X be a connected G -graph where G acts without inversion. If there is a vertex v_0 of X such that*

- (1) S fixes v_0 , and
- (2) for any vertex $v \in X$ there exists a path y_1, \dots, y_n from v_0 to gv for some $g \in G$ such that for each i , a Sylow p -subgroup of the stabilizer group $\text{Stab}_G(t(y_i))$ lies in the stabilizer of the edge y_i ,

then the fundamental group $\pi = \pi(\mathcal{G}, Y)$ associated to the G -action on X has a Sylow p -subgroup that maps isomorphically to S under the storing homomorphism $\chi : \pi \rightarrow G$.

Proof The vertex groups of \mathcal{G} are the stabilizers of the G -action on X . Conditions (1) and (2) in the theorem implies the condition (ii) of [14, Prop 3.3]. Condition (i) of this proposition already holds since all vertex groups are finite, hence by [14, Prop 3.3], π has a Sylow p -subgroup isomorphic to S .

Since S fixes the point v_0 , S maps into π via the inclusion $i_v : G_v \rightarrow \pi$ where v is the image of v_0 under the quotient map $X \rightarrow Y$. It is clear that the storing homomorphism $\chi : \pi \rightarrow G$ takes a Sylow p -subgroup of π isomorphically onto $S \leq G$. \square

The storing homomorphism $\chi : \pi \rightarrow G$ can also be used to compare the corresponding fusion systems.

Theorem 6.3 *Let G be a finite group with a Sylow p -subgroup S , and let X be a connected graph on which G acts without inversion. Assume that X has a vertex v_0 satisfying conditions (1) and (2) of Proposition 6.2. Then after identifying the Sylow p -subgroups, we have $\mathcal{F}_S(\pi) \subseteq \mathcal{F}_S(G)$. Furthermore, if for every p -centric, p -radical subgroup P in G , the normalizer $N_G(P)$ fixes a vertex on X , then $\mathcal{F}_S(\pi) = \mathcal{F}_S(G)$.*

Proof Let $c_\gamma : Q \rightarrow R$ be a morphism in $\mathcal{F}_S(\pi)$, where $\gamma \in \pi$. Let $g = \chi(\gamma)$. After identifications, c_γ is equal to the conjugation map $c_g : Q \rightarrow R$ which is a morphism in $\mathcal{F}_S(G)$. Hence $\mathcal{F}_S(\pi) \subseteq \mathcal{F}_S(G)$.

To prove the second statement, let $c_g : P \rightarrow P$ be a morphism in $\mathcal{F}_S(G)$ where $P \leq S$ is a p -centric, p -radical subgroup in G , and $g \in N_G(P)$. By the given condition, $N_G(P) \leq \text{Stab}_G(v)$ for some v in X . Since χ is a storing homomorphism, it induces an isomorphism between $i_v(G_v) \leq \pi$ and $\text{Stab}_G(v) \leq G$. This means that there is a $\gamma \in i_v(G_v) \leq \pi$ such that $c_\gamma : P \rightarrow P$ is equal to the morphism $c_g : P \rightarrow P$, hence

$c_g \in \mathcal{F}_S(\pi)$. Since, by Alperin’s theorem $\mathcal{F} := \mathcal{F}_S(G)$ is generated by morphisms $c_g : P \rightarrow P$ where P is fully normalized, \mathcal{F} -centric, and \mathcal{F} -radical. By Lemma 2.1, \mathcal{F} -centric, \mathcal{F} -radical subgroups in $\mathcal{F}_S(G)$ are p -centric and p -radical in G , hence we can conclude that $\mathcal{F}_S(\pi) = \mathcal{F}_S(G)$. \square

Remark 6.4 If X is a G -graph which is not connected but such that the quotient graph $Y = X/G$ is connected, then there is a subgroup $H \leq G$ formed by elements $g \in G$ such that $gX_1 = X_1$ for some connected component X_1 of X . It is easy to see that there is an isomorphism $X \cong G \times_H X_1$ of G -spaces. This gives

$$EG \times_G X \simeq EG \times_G (G \times_H X_1) \simeq EH \times_H X_1.$$

If the connected H -graph X_1 satisfies the conditions in Theorem 6.3, then

$$\pi := \pi_1(EG \times_G X) = \pi_1(EH \times_H X_1)$$

realizes the fusion $\mathcal{F}_S(H)$. In general the fusion system $\mathcal{F}_S(H)$ will not be equal to $\mathcal{F}_S(G)$, but in some of the cases we consider below this will be true, and we will have applications to Theorem 6.3 even when the G -graph X is not connected. \square

The main example that motivated this section is the action of G on the graph of p -centric, p -radical subgroups of G .

Example 6.5 Let G be a finite group and X be the graph whose vertices are the p -centric, p -radical subgroups of G . There is an edge between P and Q in X if $Q < P$. The group G acts on X by conjugation, and this action is without inversion of edges. The graph X is not connected in general but the subgroup H that stabilizes a connected component is generated by the stabilizers of vertices of that component. Let X_1 be the connected component that includes S , then H is generated by normalizers of the p -centric, p -radical subgroups in S . The stabilizers are normalizer subgroups $N_G(P)$, and by Alperin’s fusion theorem they generate the fusion system $\mathcal{F}_S(G)$, so in this case we have $\mathcal{F}_S(H) = \mathcal{F}_S(G)$.

Let $\pi := \pi_1(EG \times_G X) \simeq \pi_1(EH \times_H X_1)$. Consider the H -action on the connected graph X_1 . If we take v_0 as S , then the stabilizer of v_0 in H contains a Sylow p -subgroup of G . Condition (ii) of Proposition 6.2 can be checked easily. Let P be any vertex in X_1 . There is a vertex R that is conjugate to P such that R is fully normalized and $R \leq S$. Then the path between R and S formed by a single edge satisfies the required condition because the stabilizer of the edge $R < S$ includes the Sylow p -subgroup of $N_G(R)$, which is $N_S(R)$. So π has a Sylow p -subgroup that maps isomorphically to the Sylow p -subgroup of G . The condition of Theorem 6.3 also holds because for every p -centric, p -radical subgroup of G , the subgroup $N_G(P)$ is the stabilizer of the vertex P . Hence π realizes $\mathcal{F}_S(G)$.

Note that the infinite group π obtained from this group action is the same as the model given by Libman and Seeliger in [14, Sec. 4.1], which is different than the Robinson model given in Theorem 2.10. This version of Robinson model is interesting from the point of view of the normalizer decomposition of classifying spaces. Let \mathcal{C} denote the collection of all p -centric, p -radical subgroups in G . The graph X is the

1-skeleton of the poset of subgroups in \mathcal{C} . Since the collection of p -centric, p -radical subgroups of G is a p -ample collection, the projection map $EG \times_G |\mathcal{C}| \rightarrow BG$ induces a mod- p isomorphism (see Definition 7.7 and 8.10 in [10]). In addition, we have $B\pi \cong EG \times_G X$, hence the map $f : B\pi \rightarrow BG$ induced by $\chi : \pi \rightarrow G$ gives a mod- p cohomology isomorphism if and only if the inclusion map $i : X \rightarrow |\mathcal{C}|$ induces a mod- p cohomology isomorphism

$$i^* : H^*(EG \times_G |\mathcal{C}|; \mathbb{F}_p) \rightarrow H^*(EG \times_G X; \mathbb{F}_p).$$

It is easy to see that for many groups these two cohomology rings will not be isomorphic, in particular, when the permutation modules for the higher dimensional cells in $|\mathcal{C}|$ are not free. □

In the rest of the section we consider the group action on the graph of elementary abelian p -subgroups of G . Through out this discussion, we assume G is a finite group with p -rank equal to 2, meaning that G has a subgroup isomorphic to $(\mathbb{Z}/p)^2$ but it has no subgroups isomorphic to $(\mathbb{Z}/p)^3$. Let $X = \mathcal{A}_p(G)$ be the poset of all nontrivial elementary abelian p -subgroups in G . Since G has p -rank equal to 2 this is a one-dimensional poset, hence we may consider it as a graph whose vertices are the nontrivial elementary abelian p -subgroups of G and where there is an edge between E_1 and E_2 if and only if $E_1 < E_2$. The group G acts on X by conjugation.

Let $\pi := \pi(\mathcal{G}, Y)$ denote the fundamental group of the graph of groups associated to the G -action on X . Note that the vertex groups of \mathcal{G} are the normalizers $N_G(E)$. By the discussion above, the group π can also be described as the fundamental group $\pi := \pi_1(EG \times_G |\mathcal{A}_p(G)|)$. In this case the mod p cohomology of π is known to be isomorphic to the cohomology of G . This is a theorem due to P. Webb.

Theorem 6.6 (Webb [22, Thm E]) *Assume that G is a finite group with $rk_p(G) = 2$. Let $\mathcal{A}_p(G)$ be the poset of nontrivial elementary abelian p -subgroups in G , and let $\pi := \pi_1(EG \times_G |\mathcal{A}_p(G)|)$. Then $\tilde{H}_i(|\mathcal{A}_p(G)|; \mathbb{F}_p)$ is a projective $\mathbb{F}_p G$ -module for $i = 0, 1$, and there is an isomorphism*

$$H^*(\pi, \mathbb{F}_p) \cong H^*(G, \mathbb{F}_p).$$

Proof The first part is proved in [22, Thm E]. The isomorphism of cohomology groups is given in [22, pg. 153]. □

Using Theorem 6.3 we can also conclude the following.

Theorem 6.7 *Let G be a finite group with $rk_p(G) = 2$, and let S be a Sylow p -subgroup of G . Then $\pi := \pi_1(EG \times_G |\mathcal{A}_p(G)|)$ has a Sylow p -subgroup isomorphic to S , and $\mathcal{F}_S(\pi) = \mathcal{F}_S(G)$.*

Proof Since the center $Z(S)$ of S is not trivial, we can take a subgroup C_1 of order p which lies in $Z(S)$. Let X_1, X_2, \dots, X_k be the connected components of $X := |\mathcal{A}_p(G)|$, assume that $C_1 \in X_1$. Define

$$H := \{g \in G \mid gX_1 = X_1\}.$$

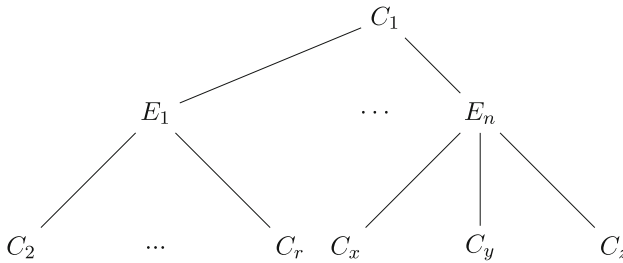


Fig. 1 Poset $\mathcal{A}_p(G)$

Since S fixes C_1 , it also fixes the component X_1 . Hence, we have $S \leq H$. Any elementary abelian $E \leq S$ is connected to C_1 by a path in X_1 because the elementary abelian subgroup EC_1 of S is connected to both E and C_1 . For any subgroup $Q \leq S$, the normalizer $N_G(Q)$ normalizes the largest central elementary abelian subgroup Z_Q of Q because Z_Q is a characteristic subgroup of Q . This means that $N_G(Q)$ fixes the vertex Z_Q in X , which lies in X_1 . Hence, $N_G(Q) \leq H$ for all $Q \leq S$. By Alperin’s fusion theorem, we obtain that

$$\mathcal{F}_S(H) = \mathcal{F}_S(G).$$

To complete the proof, it is enough to prove that $\pi \cong \pi_1(EG \times_H X_1)$ realizes the fusion system $\mathcal{F} := \mathcal{F}_S(H)$. We argue that in this situation conditions (1) and (2) of Theorem 6.3 are satisfied. For condition (1), we choose the vertex v_0 as the subgroup C_1 . It is clear that S fixes C_1 since $C_1 \leq Z(S)$.

For condition (2), take any E in the poset X_1 . Since the group E fixes the vertex E , it also fixes the component X_1 , hence $E \leq H$. By replacing E with an H -conjugate subgroup we can assume E is a fully \mathcal{F} -normalized subgroup of S . Since $\text{rk}_p(G) = 2$, the subgroup E has order p or p^2 . We will analyze these two cases.

We denote the elementary abelian p -subgroups of S order p and p^2 by C_i ’s and E_i ’s, respectively. Since $C_1 \leq Z(S)$, we have $C_1 \leq E_i$ for all i because otherwise the group C_1E_i is an elementary abelian group of order p^3 , giving a contradiction with $\text{rk}_p(G) = 2$ (Fig. 1).

If $E = E_i$ for some i , then there is an edge between C_1 and E . The stabilizer of E has Sylow p -subgroup $N_S(E)$ because E is fully \mathcal{F} -normalized. The subgroup $N_S(E)$ is contained in $N_H(C_1) \cap N_H(E)$, which is the stabilizer of the edge between C_1 and E . Hence, condition (2) is satisfied in this case.

Now assume that $E = C_i$ for some i . Then C_i is contained in $E_j = C_1C_i$. Now we consider the path C_1, E_j, C_i for the condition (2). Since C_i is fully \mathcal{F} -normalized, $N_H(C_i)$ has Sylow p -subgroup $N_S(C_i)$ contained in $N_H(E_j)$ because if $s \in S$ fixes C_i then it fixes $E_j = C_1C_i$. This shows that the edge from E_j to C_i satisfies the Sylow p -subgroup condition.

For the first edge from C_1 to E_j , if E_j is also fully \mathcal{F} -normalized then we are done in a similar way. Assume to the contrary that E_j is not fully \mathcal{F} -normalized. Now, we need to change the path. There is an $h \in H$ such that $E_{j'} = hE_jh^{-1}$ is fully \mathcal{F} -normalized. $N_S(E_{j'})$ is a Sylow p -subgroup of $N_H(E_{j'})$, which lies in the

stabilizer of the edge C_1 to $E_{j'}$. By taking the h -conjugate of the conclusion from the previous paragraph we see that the edge from $E_{j'}$ to hEh^{-1} also satisfies the Sylow p -subgroup condition. Thus each of the edges in the path $C_1, E_{j'}, hEh^{-1}$ satisfies the Sylow p -subgroup condition, hence by Theorem 6.3, we conclude that π has a Sylow p -subgroup isomorphic to S .

For every p -centric subgroup $P \leq S$, let E be the maximal central elementary abelian subgroup of P . Then E is characteristic in P , so $N_G(P) \leq N_G(E)$. This means $N_G(P)$ stabilizes E . Hence, by Theorem 6.3, we can conclude that $\mathcal{F}_S(\pi) = \mathcal{F}_S(G)$. \square

7 Signalizer functors for infinite group models

Let \mathcal{F} be a saturated fusion system on a finite p -group S . The *centric linking system* \mathcal{L} associated to \mathcal{F} is a category whose objects are the \mathcal{F} -centric subgroups of S , together with a functor $\pi : \mathcal{L} \rightarrow \mathcal{F}^c$ and a monomorphism $\delta_P : P \rightarrow \text{Aut}_{\mathcal{L}}(P)$ for each \mathcal{F} -centric subgroup $P \leq S$ satisfying certain properties (see [9, Def 9.35] for details). The triple $(S, \mathcal{F}, \mathcal{L})$ is called a *p -local finite group* and its *classifying space* is the space $|\mathcal{L}|_p^\wedge$. When \mathcal{F} is realized by a finite group G , then \mathcal{F} -centric subgroups of S are p -centric in G . Hence for every $P \in \mathcal{F}^c$, we have $C_G(P) = Z(P) \times C'_G(P)$ where $C'_G(P)$ has order coprime to p . In this case the morphisms of the category \mathcal{L} are given by

$$\text{Mor}_{\mathcal{L}}(P, Q) = \{gC'_G(P) \mid g \in G, gPg^{-1} \leq Q\} = \{g \in G \mid gPg^{-1} \leq Q\} / C'_G(P).$$

For a discrete group π with a Sylow p -subgroup S , the transporter category $\mathcal{T}_S^c(\pi)$ is defined as the category whose objects are \mathcal{F} -centric subgroups of S and whose morphisms are given by

$$\text{Mor}_{\mathcal{T}_S^c(\pi)}(P, Q) = \{g \in \pi \mid gPg^{-1} \leq Q\}.$$

Definition 7.1 A *signalizer functor* Θ on a discrete group π is an assignment $P \rightarrow \Theta(P)$ for every \mathcal{F} -centric $P \leq S$ such that $\Theta(P)$ is a complement of $Z(P)$ in $C_\pi(P)$ and such that if $gPg^{-1} \leq Q$ then $\Theta(Q) \leq g\Theta(P)g^{-1}$.

A signalizer functor Θ is a functor from the transporter category $\mathcal{T}_S^c(\pi)$ to the category of groups. Given a signalizer functor Θ on a discrete group π , we can define a quotient category \mathcal{L}_Θ whose morphisms from P to Q are

$$\text{Mor}_{\mathcal{L}}(P, Q) = \text{Mor}_{\mathcal{T}_S^c(\pi)}(P, Q) / \Theta(P).$$

It is proved in [1, Lemma 2.6] that the category \mathcal{L}_Θ is a linking system for the fusion system $\mathcal{F}_S(\pi)$. For infinite group models π_R and π_{LS} realizing fusion systems, we have the following theorem.

Theorem 7.2 (Libman and Seeliger [14]) *Fix a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ and let π be an infinite group model realizing \mathcal{F} obtained by either the Leary–Stancu model*

or the Robinson model. Then there is a signalizer function Θ on π such that \mathcal{L} is a quotient of the transporter system $T_S^c(\pi)$.

Theorem 7.2 is proved as part of [14, Thm 1.1] using topological arguments, in particular, using the definition of linking system associated to a map $f : BS \rightarrow X$. In the proof it is shown that for each $P \in \mathcal{F}^c$, there is a group extension

$$1 \rightarrow \Theta(P) \rightarrow N_\pi(P) \xrightarrow{\chi_P} \text{Aut}_{\mathcal{L}}(P) \rightarrow 1.$$

The group $\text{Aut}_{\mathcal{L}}(P)$ has $N_S(P)$ as a Sylow p -subgroup, and it is known that the induced fusion system is isomorphic to the normalizer fusion system $N_{\mathcal{F}}(P)$. If π is an infinite group model constructed by the Leary–Stancu model or the Robinson model, then π has a Sylow p -subgroup isomorphic to S , but in general this property is not inherited by subgroups. So, it is not clear that the normalizer group $N_\pi(P)$ has a Sylow p -subgroup.

Question 7.3 Does the normalizer group $N_\pi(P)$ in general have a Sylow p -subgroup isomorphic to $N_S(P)$? Is it possible to see $N_\pi(P)$ as an infinite model for the fusion system $N_{\mathcal{F}}(P)$ which is realizable by the finite group $\text{Aut}_{\mathcal{L}}(P)$ and view the above sequence as a sequence coming from a storing homomorphism of a realizable fusion system?

Example 7.4 Let \mathcal{F} be a fusion system of a finite group G with Sylow p -subgroup S . Assume that G is p -minimal, and that π is an infinite group realizing \mathcal{F} obtained by either the Leary–Stancu model or by the Robinson model. There is a storing homomorphism $\chi : \pi \rightarrow G$ whose kernel is a free group F . For each $P \in \mathcal{F}^c$, reducing the storing homomorphism to $N_\pi(P)$, we obtain a short exact sequence

$$1 \rightarrow N_F(P) \rightarrow N_\pi(P) \rightarrow \overline{N}_\pi(P) \rightarrow 1,$$

where $\overline{N}_G(P) = \chi(N_\pi(P))$ is a subgroup of $N_G(P)$. By Theorem 7.2, the homomorphism

$$\chi_P : N_\pi(P) \rightarrow \text{Aut}_{\mathcal{L}}(P) = N_G(P)/C'_G(P)$$

is surjective, hence $\overline{N}_\pi(P)C'_G(P) = N_G(P)$. This gives that Θ fits into an extension of the form

$$1 \rightarrow N_F(P) \rightarrow \Theta(P) \rightarrow \overline{\Theta}(P) \rightarrow 1$$

where $\overline{\Theta}(P) = \overline{N}_\pi(P) \cap C'_G(P)$. Note that $\overline{\Theta}(P)$ is a finite group whose order is coprime to p and $N_F(P) \leq F$ is a free group.

One of the consequences of the calculation given in Example 7.4 is that mod- p cohomology of the group $\Theta(P)$ is zero at dimensions greater than 1. It turns out that this holds more generally for any saturated fusion system.

Proposition 7.5 *Let \mathcal{F} be a saturated fusion system on a finite p -group S . Suppose π is an infinite group realizing \mathcal{F} obtained by either the Leary–Stancu model or the Robinson model. Let $P \rightarrow \Theta(P)$ be the signalizer functor for π such that \mathcal{L} is a quotient of the transporter system $\mathcal{T}_S^c(\pi)$. Then for every $P \in \mathcal{F}^c$ and for every $i \geq 2$, we have $H_i(\Theta(P); \mathbb{F}_p) = 0$.*

Proof Note that π has a Sylow p -subgroup S and $\mathcal{F} = \mathcal{F}_S(\pi)$. Let P be an \mathcal{F} -centric subgroup of S . Since $\Theta(P)$ is a subgroup of the fundamental group of a graph of groups with finite vertex groups, it is itself a graph of groups with finite vertex groups. The result follows from the cohomology sequence for a graph of groups once we show that all finite vertex groups of $\Theta(P)$ have order coprime to p . Let $Q \leq \Theta(P)$ be a finite p -group. Then Q centralizes P , hence PQ is a finite p -subgroup of π . Let $g \in \pi$ be such that $g(PQ)g^{-1} \leq S$. Then $P' = gPg^{-1}$ is a subgroup of S and $gQg^{-1} \leq S$ centralizes P' . Since P is \mathcal{F} -centric, and P' is \mathcal{F} -isomorphic to P , we have $C_S(P') = Z(P')$. This gives that $gQg^{-1} \leq Z(P')$, which implies $Q \leq Z(P)$. Since $\Theta(P) \cap Z(P) = 1$, we get $Q = 1$. We conclude that all finite subgroups of $\Theta(P)$ have order coprime to p . \square

The proposition we proved above has consequences for cohomology groups of the category \mathcal{L} with twisted coefficients. For a commutative ring R , an $R\mathcal{L}$ -module M is defined as a contravariant functor from \mathcal{L} to the category of R -modules. Let $\mathcal{T} := \mathcal{T}_S^c(\pi)$ denote the transporter category for π defined on \mathcal{F} -centric subgroups of S . The quotient functor $q : \mathcal{T} \rightarrow \mathcal{L}$ gives an extension of small categories

$$1 \rightarrow \{\Theta(P)\} \rightarrow \mathcal{T} \rightarrow \mathcal{L} \rightarrow 1.$$

Theorem 1.2 is proved using a spectral sequence for extensions of small categories. We first introduce necessary definitions. We refer the reader to [17, Appendix A] for details.

Definition 7.6 (Def A.5, [17]) Let \mathcal{C} and \mathcal{D} be two small categories such that $\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{D})$. Let $\varphi : \mathcal{C} \rightarrow \mathcal{D}$ be a functor that is the identity on objects and surjective on morphism sets. For each $x \in \text{Ob}(\mathcal{C})$, set

$$K(x) := \ker\{\varphi_{x,x} : \text{Aut}_{\mathcal{C}}(x) \rightarrow \text{Aut}_{\mathcal{D}}(x)\}.$$

We say φ is *source regular* if for every $x, y \in \text{Ob}(\mathcal{C})$, the group $K(x)$ acts freely on $\text{Mor}_{\mathcal{C}}(x, y)$ and $\varphi_{x,y} : \text{Mor}_{\mathcal{C}}(x, y) \rightarrow \text{Mor}_{\mathcal{D}}(x, y)$ is the orbit map of this action.

When $\varphi : \mathcal{C} \rightarrow \mathcal{D}$ is a source regular functor, we say

$$1 \rightarrow \{K(x)\} \rightarrow \mathcal{C} \xrightarrow{\varphi} \mathcal{D} \rightarrow 1$$

is a source regular extension, and call the family $\{K(x)\}$ the kernel of the functor φ . For a small category \mathcal{C} , an $R\mathcal{C}$ -module is defined as a functor $M : \mathcal{C}^{op} \rightarrow R\text{-mod}$. Note that the assignment $x \rightarrow H_i(K(x); R)$ defines an $R\mathcal{D}$ -module since $K(x)$ acts trivially on homology groups $H_i(K(x); R)$ when R denotes a trivial $RK(x)$ -module.

Given a functor $\varphi : \mathcal{C} \rightarrow \mathcal{D}$ and an $R\mathcal{D}$ -module M , the RC -module defined as the composition

$$\mathcal{C} \xrightarrow{\varphi} \mathcal{D} \xrightarrow{M} R\text{-mod}$$

is denoted by φ^*M .

Proposition 7.7 *Let $\varphi : \mathcal{C} \rightarrow \mathcal{D}$ be a source regular functor with kernel $\{K(x)\}$. Then for every $R\mathcal{D}$ -module M , there is a spectral sequence*

$$E_2^{i,j} = \text{Ext}_{R\mathcal{D}}^j(H_i(K(-); R), M) \Rightarrow H^{i+j}(\mathcal{C}; \varphi^*M).$$

Proof There is a standard resolution for calculating the cohomology groups of a small category \mathcal{C} with coefficients in an RC -module M . For each n , let $C_n(-)$ be defined as the RC -module

$$C_n(-) = \bigoplus_{x_0 \rightarrow \dots \rightarrow x_n} \text{RMor}_{\mathcal{C}}(-, x_0)$$

where the sum is over all sequences of n morphisms $x_0 \rightarrow \dots \rightarrow x_n$ in \mathcal{C} (see [17, Lemma A.1] for details). The complex $C_*(-)$ gives a projective resolution of the constant functor \underline{R} as an RC -module. The cohomology of \mathcal{C} with coefficients in φ^*M is defined as the cohomology of the cochain complex $\text{Hom}_{RC}(C_*(-); \varphi^*M)$. It is easy to see that

$$\text{Hom}_{RC}(\text{Mor}_{\mathcal{C}}(-, x_0), \varphi^*M) \cong M(x_0) \cong \text{Hom}_{R\mathcal{D}}(\text{Mor}_{\mathcal{D}}(-, x_0), M).$$

Hence, we can write

$$H^*(\mathcal{C}; \varphi^*M) = H^*(\text{Hom}_{RC}(C_*(-); M)) \cong H^*(\text{Hom}_{R\mathcal{D}}(C'_*(-); M))$$

where $C'_*(-)$ is the complex of projective $R\mathcal{D}$ -modules such that

$$C'_n(x) = \bigoplus_{x_0 \rightarrow \dots \rightarrow x_n} R[\text{Mor}_{\mathcal{C}}(x, x_0)/K(x)]$$

for every $x \in \text{Ob}(\mathcal{C})$. Alternatively we can consider $C'_n(-)$ as the chain complex obtained from $C_n(-)$ by applying the functor $(-)_K$ introduced in the proof of [5, Lem 1.3].

For the calculation of the cohomology group $H^*(\text{Hom}_{R\mathcal{D}}(C'_*(-); M))$, there is an hyper-cohomology spectral sequence (see [3, Prop 3.4.3]) with E_2 -term

$$E_2^{i,j} = \text{Ext}_{R\mathcal{D}}^j(H_i(C'_*(-); R), M) \Rightarrow H^{i+j}(\mathcal{C}; \varphi^*M).$$

Since $K(x)$ acts freely on $\text{Mor}_{\mathcal{C}}(x, y)$ for every $x, y \in \text{Ob}(\mathcal{C})$, we have

$$H_i(C'_*(x); R) \cong H_i(K(x); R)$$

for every $x \in \text{Ob}(\mathcal{C})$, and for every i . This completes the proof. \square

Proof of Theorem 1.3 Let $R = \mathbb{F}_p$. The quotient functor $q : \mathcal{T} \rightarrow \mathcal{L}$ is a source regular functor with kernel $\{\Theta(P)\}$. Hence by Proposition 7.7, for every $R\mathcal{L}$ -module M , there is a spectral sequence

$$E_2^{i,j} = \text{Ext}_{R\mathcal{L}}^j(H_i(\Theta(-); R), M) \Rightarrow H^{i+j}(\mathcal{T}; q^*M).$$

By Proposition 7.5, $H_i(\Theta(P); R) = 0$ for every $i \geq 2$, hence the E_2 -term of the spectral sequence has only two nonzero horizontal lines at $i = 0, 1$ with a single differential

$$d_2 : \text{Ext}_{R\mathcal{L}}^j(H_1(\Theta(-); R), M) \rightarrow \text{Ext}_{R\mathcal{L}}^{j+2}(R, M)$$

for $j \geq 0$. The long exact sequence given in the statement of the theorem follows from the fact that E_2 -page is a two line spectral sequence. \square

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