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Residues of singular holomorphic foliations

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Introduction

In this article we give a geometric calculation of the residues of singular holomorphic foliations which are defined by Baum and Bott in [1]. Our objective is twofold; to show the geometric flavour of the Baum–Bott residues through the Nash and Grassmann graph constructions, and to reduce the rationality conjecture of [1, p. 287] from a calculation dealing with coherent sheaves to the relatively better understood domain of vector bundles under suitable conditions, namely when the singular holomorphic foliation is given as the image sheaf of a bundle morphism. The Baum–Bott residue is shown to be the sum of two terms. One of these terms is obtained through a Grassmann graph construction and the other is obtained by defining a basic connection for a suitable vector bundle. Our conclusion is that if the rationality conjecture holds for integrable vector bundles which give singular holomorphic foliations then it holds for integrable sheaves which are image sheaves of bundle morphisms. In Section IV we carry out this program with the further assumption that the associated Nash blow up (see Section I) is smooth. This has the advantage of exhibiting the geometric character of the problem more explicitly than the general case which is built on the techniques of the smooth case and is given in Section V.

In more general terms, we show that the Baum–Bott residue before and after the Nash blow up differ by a term coming from the Grassmann Graph construction. The main consequence is then a reduction of the rationality conjecture on algebraic manifolds to vector bundles.

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I. Nash construction

In this section we will define the Nash construction through which we will later define the Nash residue. We will be interested in the Nash blow-up N of a coherent subsheaf $\mathscr{F}$ of a locally free sheaf $\mathscr{G}$ on M, which we now define. We assume that the rank of $\mathscr{F}$ is k and the rank of $\mathscr{G}$ is m, with $0 < k < m$, and we define a map

$$F\colon M - S \to G(k, \mathscr{G})$$

$$x \mapsto [\mathscr{F}_x]$$

where S is the singular set of $\mathscr{F}$, $G(k, \mathscr{G})$ is the Grassmann bundle of k planes in $\mathscr{G}$ and $[\mathscr{F}_x]$ is the point in the Grassmannian that the k-plane $\mathscr{F}_x$ defines. With this preparation we can define N to be the topological closure of $F(M - S)$ in $G(k, \mathscr{G})$. This space N, if admissible, comes equipped with a short exact sequence of vector bundles

$$0 \to \tau \to \pi^* H \to W \to 0$$

where τ is the tautological bundle, W is the universal quotient bundle and H is the underlying vector bundle of $\mathscr{G}$. Here π is the natural projection from N to M. If we denote $\pi^{-1}(S)$ by S' then we can write the following relations

$$\underline{\tau} \,|\, N - S' = \pi^* \mathscr{F} \,|\, N - S' \quad \text{and}$$

$$\underline{W} \,|\, N - S' = \pi^* (\mathscr{G}/\mathscr{F}) \,|\, N - S'$$

where the underbar notation denotes the associated holomorphic sheaf as in [1]. These relations suggest that we can relate some sheaf theoretical problems to their akins in vector bundle theory and hope to get a more managable situation. There is one little(!) problem however; it is not clear a priori that N is an acceptable space in the complex category. Nobile, Rossi, and Riemenschneider have several analicity theorems that can be applied under similar set ups as above, see [5, 7, 6]. The following theorem combines and generalizes their results and is applicable in our setting. We present a sketch of the proof. For details and further discussions we refer to [8].

THEOREM I.1: *N is locally a monoidal transformation and is consequently a complex analytic space.*

Proof: Let U be an open neighbourhood on which $\mathscr{G} \mid U$ is trivial and $\mathscr{F} \mid U$ is generated by r sections $f_1, \ldots, f_r$. Through a trivialization of $\mathscr{G}$ each f_i can be considered as an n-tuple $(f_{i1}, \ldots, f_{in})$ of holomorphic functions on U. Define an $n \times r$-matrix A by letting the ij-th entry to be f_{ij}. For $x \in U - U \cap S$, the point $[\mathscr{F}_x]$ in the Grassmann variety $G(k, \mathscr{G}_x)$ can also be denoted by $[A(x)]$, where by this notation we suggest the linear span of the row vectors. The number of the rows r may be larger than k and for computational purposes we want to pick up a $k \times n$ submatrix of $A(x)$ to represent $[\mathscr{F}_x]$. For this in mind consider the indexing set

$$B_s = \{(N_1, \ldots, N_k) \in \mathbf{Z}^k \mid 1 \leqslant N_1 < \cdots < N_k \leqslant s\}, \text{ for any } s \in \mathbf{N}.$$

Choose μ from B_r and β from B_n, and define two ideals and a $k \times n$ matrix as follows;

(a) $\Delta_{\mu\beta} := det(f_{ij})$ where $i \in \mu$ and $j \in \beta$,
(b) $A_\mu := (f_{ij})$ where $i \in \mu$ and $j = 1, \ldots, n$.
(c) $I_\mu :=$ the ideal generated by the $k \times k$ minors of A_μ.

Choose a $\mu \in B_r$ for which the ideal I_μ is not trivial and define two maps

$$H\colon U - V(I_\mu) \to U \times G(k, n)$$

$$x \mapsto (x, [A_\mu(x)])$$

and

$$T\colon U - V(I_\mu) \to U \times \mathbf{P}^N$$

$$x \mapsto (x, [\Delta_{\mu\beta_0}; \cdots ; \Delta_{\mu\beta_N}]),$$

where $V(I_\mu)$ denotes as usual the associated variety and $\beta_0, \ldots, \beta_N$ is some fixed ordering of the elements of B_n. Of the three maps we have defined, F and H agree on $U - V(I_\mu)$. The domain of F contains the domain of H, each of which is dense in U. We then conclude that

$$\overline{H(U - V(I_\mu))} = \overline{F(U - U \cap S)}.$$

On the other hand $\overline{T(U - V(I_\mu))}$ is a monoidal transformation of U and is analytic. It remains to relate T to F and H. Using the map (id, pl): $U \times G(k, n) \to U \times \mathbf{P}^n$ where id is the identity on U and pl is the Plücker imbedding, it can be shown that $(id, pl) \circ H = T$ on their domains of definition. Since pl is a closed imbedding and (id, pl) is an isomorphism on the image of H, the closure of the images of H and T are isomorphic. That different neighbourhoods patch follows from the uniqueness of monoidal transformations. □

To investigate the smoothness of N let us agree to call a coherent sheaf $\mathscr{F}$ "nice" if it is of rank k on M with singularity set S and if for every $p \in S$ there exists an open neighbourhood U of p with sections $f_1, \ldots, f_k$ of $\mathscr{F} \mid U$ such that

(i) $f_1, \ldots, f_k$ generate $\mathscr{F} \mid U - U \cap S$

(ii) $f_1, \ldots, f_k$ are linearly dependent on $\mathscr{F} \mid U \cap S$.

Examples of "nice" sheaves are easy to find; complex actions of reductive groups generate a "nice" subsheaf of the tangent sheaf. Finitely generated subsheaves of locally free sheaves are "nice", and locally of finite type subsheaves of locally free sheaves are "nice".

COROLLARY I.2: *If $\mathscr{F}$ is "nice", then N is a monoidal transformation with centre S and furthermore if S is smooth then N is also smooth.*

II. Nash residue

We will define a residue class using the Nash construction for singular holomorphic foliations for which the Nash construction gives a smooth space. We take $\mathscr{F}$ to be an integrable full coherent subsheaf of the tangent sheaf $\underline{T}$ with rank $\mathscr{F} = k$. We assume that N, the Nash blow up of M with respect to $\mathscr{F}$ and $\underline{T}$, is smooth.

Recall that on N there is a short exact sequence of bundles

$$0 \to \tau \to \pi^*T \to W \to 0.$$

If we denote by Y the underlying vector bundle of $(\underline{T}/\mathscr{F}) \mid M - S$, then we observe that on $Y \mid M - S$, by Bott's vanishing theorem, there is a basic connection D_1, which pulls back to a connection D of W in the sense that D and $\pi^* D_1$ agree on $N - \Sigma$ where Σ is a compact subset of N which contains S', see [1, p. 300]. Let K_W be the corresponding curvature matrix. For a symmetric homogeneous polynomial $\Phi \in \mathbf{C}[X_1, \ldots, X_n]$, if $\deg \Phi > n - k$ then $\Phi(K_W)$ vanishes on $N - \Sigma$. To examine the local behaviour and hence to capture a residue class through this differential form let Z be a connected component of S, and let U be an open neighbourhood of Z which deformation retracts to Z through the map $\delta\colon U \to Z$. The differential form $\Phi(K_W)$ is of compact support on $\pi^{-1}(U)$. Its Poincare dual is a homology cycle which can be mapped to U and pushed into Z by the deformation retraction. This defines $N\ \mathrm{Res}_\Phi(\mathscr{F}, Z)$, the Nash residue of $\mathscr{F}$

at Z associated to the polynomial Φ. We can summarize this definition as

$$N \operatorname{Res}_\Phi (\mathscr{F}, Z) = \delta_* \pi_* (\Phi(K_W) \cap \pi^{-1}(U)).$$

If $\mathscr{F}$ is the image sheaf of a bundle morphism $\Psi: E \to T$, we will use $N \operatorname{Res}_\Phi(E, Z)$ instead of $N \operatorname{Res}_\Phi(\underline{\Psi(E)}, Z)$.

III. Grassmann graph construction

The Grassmann graph construction for the algebraic case is defined in [2] and the compact Kaehler case is shown in [9]. Here we recollect the notation that will be used in the rest of the article. For E, F vector bundles on M of ranks k and n, let $G(k, E \oplus F) \to M$ be the Grassmann bundle of k-planes in $E \oplus F$, and let $\operatorname{Hom}(E, F) \to M$ be the bundle of morphisms and denote the natural imbedding of $\operatorname{Hom}(E, F)$ into $G(k, E \oplus F)$ by j.

For every $\varphi \in \operatorname{Hom}(E, F)$ and $\lambda \in \mathbf{C}$ we define the graph of $\lambda\varphi$ as

$$s(\varphi): M \times \mathbf{C} \to G(k, E \oplus F) \times \mathbf{P}^1$$

$$(x, \lambda) \mapsto ([j_x(\lambda\varphi_x)], [1 : \lambda]).$$

The cycle at infinity $\mathbf{Z}_\infty$ is defined as the limit of $s(\varphi)(M, \lambda)$ as $\lambda \to \infty$. In the complex category it is not clear that $\mathbf{Z}_\infty$ is always an admissible cycle, but for the compact Kaehler case we have the following result.

THEOREM III.1 [9]: *If M is a compact Kaehler manifold then for any $\varphi \in \operatorname{Hom}(E, F)$ the corresponding cycle at infinity, $\mathbf{Z}_\infty$, is an analytic cycle.*

For the proof and further discussions we refer to [8, 9]. Here we report the main line of proof for completeness. Let ϱ: $\mathbf{C}^* \times G(k, E \oplus F) \to G(k, E \oplus F)$ be a $\mathbf{C}^*$ action defined as follows; any point $q \in G(k, E \oplus F)_x$ is defined by a k-plane H in $E_x \oplus F_x$, and any point $p \in H$ can be decomposed as $p = p_1 \oplus p_2$ where $p_1 \subset E_x$ and $p_2 \in F_x$. Construct a new plane $\lambda \cdot H$ by

$$\lambda \cdot H = \{p_1 \oplus \lambda p_2 \in E \oplus F | p_1 \oplus p_2 \in H\}.$$

Then define the $\mathbf{C}^*$ action ϱ as $\varrho(\lambda, [H]) = [\lambda \cdot H]$. Corresponding to the $\varphi \in \operatorname{Hom}(E, F)$ with respect to which we want to calculate $\mathbf{Z}_\infty$, we first define a holomorphic map A: $M \times \mathbf{C}^* \to G(k, E \oplus F)$ by $A(m, t) = s(\varphi)(m, t)$, which is equivariant with respect to ϱ on $G(k, E \oplus F)$ and the trivial action of $\mathbf{C}^*$ on $M \times \mathbf{C}^*$. Since ϱ has fixed points, by a lemma of Sommese A extends meromorphically to $M \times \mathbf{P}^1$, [10, p. 111, II-B]. Hence the closure C of the graph of A is an analytic space and $\mathbf{Z}_\infty = C \cap (M \times \{\infty\} \times G(k, E \oplus F))$ being the intersection of two analytic spaces is analytic. $\mathbf{Z}_\infty$ is called the cycle at infinity corresponding to the map φ.

We now define the cycle at infinity corresponding to a complex of vector bundles. Let

$$0 \to E_m \to E_{m-1} \to \cdots \to E_0 \to E_{-1} = 0 \qquad (E.)$$

be a complex of bundles with maps φ_i: $E_i \to E_{i-1}$. Assume that there is a subvariety S of M such that $(E.)$ is exact on $M - S$. Let $G_i = G(\text{rank } E_i, E_i \oplus E_{i-1})$ and $G = G_0 \times_M \cdots \times_M G_m$ be the total space of the bundle π: $G \to M$. Let τ_i denote the tautological bundle on G_i and $\tau = \tau_0 - \tau_1 + \cdots + (-1)^m \tau_m$ be the virtual tautological bundle on G. Define an imbedding Γ_λ: $M \to G$ by $\Gamma_\lambda(x) = (\Gamma_\lambda^0(x), \ldots, \Gamma_\lambda^m(x))$ where $\Gamma_\lambda^i(x) = s(\varphi_i)(x, \lambda)$, $\lambda \in \mathbf{C}$. Finally define $\mathbf{Z}_\infty$ as

$$\mathbf{Z}_\infty = \lim_{\lambda \to \infty} \Gamma_\lambda(M).$$

This is the cycle at infinity corresponding to the complex $(E.)$. Note that if we denote $\Gamma_\lambda(M)$ by Z_λ then $\{Z_\lambda\}_{\lambda \in \mathbf{P}^1}$ is a set of rationally equivalent cycles. The cycle at infinity decomposes as $\mathbf{Z}_\infty = Z_* + M_*$ where $M_* - \pi^{-1}(S)$ is biholomorphic to $M - S$, π: $M_* \to M$ is a meromorphic map, $\pi(Z_*)$ is in S and $\tau | M_*$ is the zero bundle. Letting E denote the virtual bundle $E_0 - E_1 + \cdots + (-1)^m E_m$ on M and $c(E)$ the total chern class we have

$$\begin{aligned} c(E) \cap [M] &= c(\tau) \cap Z_0 \\ &= c(\tau) \cap Z_*. \end{aligned}$$

The localized chern classes are then defined as

$$C_S^i(E.) = \pi_*(c_i(\tau) \cap Z_*) \in H_*(S; \mathbf{C}),$$

where c_i denotes the i-th chern class. Similarly we define $C_Z^i(E.)$ if Z is a connected component of S. For details we refer to [2, 3, 9].

IV. Main result

In this section we calculate the difference between the Baum–Bott residue $\text{Res}_\Phi(\mathscr{F}, Z)$ and the Nash residue $N\,\text{Res}_\Phi(\mathscr{F}, Z)$. Our set up is as follows:

- M is a compact Kaehler manifold. We assume that the singular holomorphic foliation $\mathscr{F}$ is given as the image sheaf of a maximal rank bundle morphism, i.e. there is a vector bundle E of rank $k < n$, and there is a maximal rank morphism $\Psi: E \to T$, and $\mathscr{F}$ is the image sheaf $\underline{\Psi(E)}$ in the tangent sheaf $\underline{T}$. The singular set of the foliation is S. We denote the associated Nash construction by $\pi: N \to M$, and we assume that N is smooth.

We first prove a lemma about Ψ;

LEMMA IV.1 *If Ψ is of maximal rank then $\underline{\Psi}$, the map induced at the sheaf level is injective.*

Proof: Let U be an open subset of M such that E and T are trivial on U. Let $e_1, \ldots, e_k$ and $t_1, \ldots, t_n$ be local generators for E and T respectively. Then Ψ can be defined in terms of these basis elements as:

$$\Psi(e_i) = f_{i1}t_1 + \cdots + f_{in}t_n, \quad i = 1, \ldots, k,$$

where f_{ij} are holomorphic functions on U. If h is a section of $\underline{E}|U$ it can be expressed as $h = h_1e_1 + \cdots + h_ke_k$ where h_i are holomorphic functions on U. Then

$$\Psi(h) = h_1\Psi(e_1) + \cdots + h_k\Psi(e_k).$$

If $\Psi(h) = 0$ then in particular $\Psi_x(h(x)) = 0$ for $x \in U$. Since $k < n$ and Ψ is of maximal rank, Ψ is injective on $U - U \cap S$. Then $h(x) = 0$ for $x \in U - U \cap S$, i.e. each $h_i(x) = 0$ for $x \in U - U \cap S$, $i = 1, \ldots, k$. Since h_i's are holomorphic on U, and vanish on an open subset, they are identically zero on U, hence $h = 0$. This proves that $\underline{\Psi}$ is injective. □

The Baum–Bott residue is calculated on M. To calculate the Nash residue we construct the Nash blow up N. Finally to measure how much Nash residue deviates from Baum–Bott residue we carry a Grassmann graph construction on N for a particular complex of vector bundles on N. The resulting localized class, properly mapped to M, is precisely the looked for difference.

We start with the construction of the Grassmann graph. On M there is the exact sequence of sheaves

$$0 \to \underline{E} \to \underline{T} \to \underline{Q} \to 0 \tag{1}$$

where $\underline{E} \to \underline{T}$ is induced by Ψ and Q is the quotient sheaf. This sequence pulls back to a sequence of sheaves on N

$$0 \to \pi^*\underline{E} \to \pi^*\underline{T} \to \pi^*Q \to 0. \tag{2}$$

On N there is also the natural sequence of vector bundles

$$0 \to \tau \to \pi^*T \to W \to 0 \tag{3}$$

where τ is the tautological bundle and W is the universal quotient bundle. Denote $\pi^{-1}(S)$ by X. On $N - X$ the sheaves π^*Q and $\underline{W}$ agree, hence the sequence of locally free sheaves on N

$$0 \to \pi^*\underline{E} \to \pi^*\underline{T} \to \underline{W} \to 0 \tag{4}$$

is exact on $N - X$. The underlying vector bundles of this sequence gives a complex of vector bundles on N

$$0 \to \pi^*E \to \pi^*T \to W \to 0 \tag{5}$$

which is exact on $N - X$. It is this particular sequence of vector bundles which we use for the Grassmann graph construction

$$p\colon G \to N.$$

Let ξ be the virtual tautological bundle on G and let $\{Z_\lambda\}_{\lambda \in \mathbf{P}^1}$ be the family of rationally equivalent cycles on G obtained by the graph construction. We then have

$$\xi \mid Z_0 \;=\; p^*(\gamma) \tag{6}$$

where γ is the virtual bundle $\pi^*T - \pi^*E - W$ on N. The cycle at infinity $\mathbf{Z}_\infty$ decomposes as

$$\mathbf{Z}_\infty \;=\; \mathbf{N}_* + \mathbf{Z}_* \tag{7}$$

where $\mathbf{N}_*$ is bimeromorphic to N, and $\mathbf{Z}_*$ is in the fibre above X. It is known that

$$\xi \mid \mathbf{N}_* \;=\; 0 \tag{8}$$

see [2, p. 122], [3, pp. 340–341].

Since $\mathbf{Z}_0$ is rationally equivalent to $\mathbf{Z}_\infty$ we have the following equalities

$$\begin{aligned} c_i(\xi) \cap \mathbf{Z}_0 &= c_i(\xi) \cap \mathbf{Z}_\infty \\ &= c_i(\xi) \cap \mathbf{N}_* + c_i(\xi) \cap \mathbf{Z}_* \\ &= c_i(\xi) \cap \mathbf{Z}_*, \quad i > 0 \end{aligned} \tag{9}$$

where the last equation follows from 8. Note that $c_i(\xi) \cap \mathbf{Z}_*$ is in the homology group of $\mathbf{Z}_*$. Combining 6 and 9 will give

$$\begin{aligned} c_i(\gamma) \cap [N] &= p_*(c_i(\xi) \cap \mathbf{Z}_0) \\ &= p_*(c_i(\xi) \cap \mathbf{Z}_*) \in H_*(X; \mathbf{C}). \end{aligned} \tag{10}$$

We use the last line as the definition of a localized chern class

$$C_X^i := p_*(c_i(\xi) \cap \mathbf{Z}_*), \quad i > 0. \tag{11}$$

With this notation 10 can be rewritten as

$$c_i(\gamma) \cap [N] = C_X^i, \quad i > 0. \tag{12}$$

At this point we pause to make an observation regarding what we are aiming at. One of the quantities we want to capture is $\mathrm{Res}_\Phi(\mathscr{F}, Z)$, which basically has to do with the chern classes of Q, and $N\ \mathrm{Res}_\Phi(\mathscr{F}, Z)$ which is related to the chern classes of W. We now relate these chern classes in a basic lemma:

LEMMA IV.2:

$$c(\pi^* Q) \cap [N] = c(W) \cap [N] + \sum_{i=1}^{n} c(W) \cap C_X^i.$$

Proof: The chern class of the virtual bundle γ is given by

$$\begin{aligned} c(\gamma) &= c(\pi^* T - \pi^* E - W) \\ &= c(\pi^*(T - E) - W) \\ &= c(\pi^*(T - E))/c(W) \\ &= c(\pi^* Q)/c(W). \end{aligned} \tag{13}$$

On the other hand using 12

$$c(\gamma) \cap [N] = (1 + c_1(\gamma) + \cdots + c_n(\gamma)) \cap [N]$$
$$= [N] + C_X^1 + \cdots + C_X^n. \qquad (14)$$

Putting 13 and 14 together

$$\{c(\pi^*Q)/c(W)\} \cap [N] = [N] + \sum_{i=1}^{n} C_X^i. \qquad (15)$$

Cap both sides of this by the cohomology element $c(W)$ to obtain

$$c(W) \cap (\{c(\pi^*Q)/c(W)\} \cap [N]) = c(W) \cap [N] + \sum_{i=1}^{n} c(W) \cap C_X^i.$$

The left hand side of this can be simplified as

$$(c(W) \cup \{c(\pi^*Q)/c(W)\}) \cap [N],$$

see [11, p. 254, (18)]. In this expression the $c(W)$'s cancel each other since the total chern class of a vector bundle is invertible. Finally putting the last two equations together with the mentioned cancellation gives the required equation. □

To simplify the notation define $\beta_i \in H_{2n-2i}(X; \mathbf{C})$ as

$$\beta_1 + \cdots + \beta_n = \sum_{i=1}^{n} c(W) \cap C_X^i. \qquad (16)$$

Then the above lemma can be restated in the form

$$c_i(\pi^*Q) \cap [N] = c_i(W) \cap [N] + \beta_i, \quad i = 1, \ldots, n. \qquad (17)$$

It is time to introduce the effect of the symmetric, homogeneous polynomial $\Phi \in \mathbf{C}[X_1, \ldots, X_n]$. Recall that $\Phi(\pi^*Q)$ is defined as

$$\Phi(\pi^*Q) = \tilde{\Phi}(c_1(\pi^*Q), \ldots, c_n(\pi^*Q))$$

where $\tilde{\Phi}(\sigma_1(X_1, \ldots, X_n), \ldots, \sigma_n(X_1, \ldots, X_n))$ is the unique polynomial, on the elementary symmetric polynomials, whose polarization is Φ. To calculate $\Phi(\pi^*Q)$ we first assume that Φ is of the form

$$\Phi(\cdot) = c_i(\cdot)c_j(\cdot) \tag{18}$$

for some i, j, with $0 \leqslant i, j \leqslant n$. The result for the general form of Φ can be deduced by using induction and forcing linearity. The following lemma summarizes the effect of Φ on the chern classes calculated in the previous lemma.

LEMMA IV.3:

$$(\Phi(\pi^*Q) \cap [N]) = (\Phi(W) \cap [N]) + \beta$$

where β *is in* $H_*(X; \mathbf{C})$.

Proof: First assume 18 and cap it by $[N]$,

$$\begin{aligned}\Phi(\pi^*Q) \cap [N] &= \{c_i(\pi^*Q)c_j(\pi^*Q)\} \cap [N] \\ &= (c_i(\pi^*Q) \cap [N]) \cdot (c_j(\pi^*Q) \cap [N])\end{aligned} \tag{19}$$

where $\cdot$ is the cycle intersection, [4, p. 59]. Substituting 17 into this equation

$$\begin{aligned}\Phi(\pi^*Q) \cap [N] &= (c_i(W) \cap [N] + \beta_i) \cdot (c_j(W) \cap [N] + \beta_j) \\ &= (c_i(W) \cap [N]) \cdot (c_j(W) \cap [N]) + (c_i(W) \cap [N]) \cdot \beta_j \\ &\quad + \beta_i \cdot (c_j(W) \cap [N]) + \beta_i \cdot \beta_j .\end{aligned} \tag{20}$$

To shorten the notation define $\beta_{ij} \in H_*(X; \mathbf{C})$ as

$$\beta_{ij} = (c_i(W) \cap [N]) \cdot \beta_j + \beta_i \cdot (c_j(W) \cap [N]) + \beta_i \cdot \beta_j . \tag{21}$$

That β_{ij} is a cycle in X follows from the fact that each β_i is a cycle in X; see the definition of β_i in 16. With this short notation 20 becomes

$$\begin{aligned}\Phi(\pi^*Q) \cap [N] &= (c_i(W) \cap [N]) \cdot (c_j(W) \cap [N]) + \beta_{ij} \\ &= (c_i(W)c_j(W)) \cap [N] + \beta_{ij} \\ &= \Phi(W) \cap [N] + \beta_{ij}\end{aligned}$$

where the second line follows from the same reasoning as 19 and the last line follows from the current assumption 18. By induction on the size of Φ and forcing linearity we obtain for general Φ

$$\Phi(\pi^* Q) \cap [N] \;=\; \Phi(W) \cap [N] + \beta \tag{22}$$

where $\beta \in H_*(X; \mathbf{C})$. To have some idea about the form of β assume in general that

$$\Phi(\cdot) \;=\; c_{i_1}(\cdot) \cdots c_{i_r}(\cdot). \tag{23}$$

Then

$$\Phi(\pi^* Q) \;=\; \Pi A_1 \ldots A_r \tag{24}$$

where $A_m \in \{c_{i_m}(W) \cap [N], \beta_{i_m}\}$, $m = \{i_1, \ldots, i_r\}$ and the product is over all possible choices. We decompose the RHS of 24

$$\begin{aligned} \Phi(\pi^* Q) \;&=\; c_{i_1} \ldots c_{i_r} + (\Pi A_1 \ldots A_r - c_{i_1} \ldots c_{i_r}) \\ &=\; \Phi(W) + (\Pi A_1 \ldots A_r - c_{i_1} \ldots c_{i_r}) \end{aligned} \tag{25}$$

where c_{i_t} stands for $c_{i_t}(W) \cap [N]$, $t = 1, \ldots, r$. In the second line we use 23 to write $\Phi(W)$ and in this case

$$\beta \;=\; \Pi A_1 \ldots A_r - c_{i_1} \ldots c_{i_r}.$$

For a general polynomial Φ we impose linearity on this to obtain β's general form. □

Notice that so far the degree of Φ did not enter into any discussion. But to obtain the residues of singular holomorphic foliations we need to consider the degree too.

THEOREM IV.4: *Let Z be a connected component of S and* $\deg \Phi > n - k$, *then*

$$\mathrm{Res}_\Phi(E, Z) \;=\; N\, \mathrm{Res}_\Phi(E, Z) + \kappa$$

where κ is a homology cycle in Z and is calculated by a Grassmann graph construction.

Proof: We start by mapping the result of lemma IV.3 by π_* into M,

$$\pi_*(\Phi(\pi^*Q) \cap [N]) = \pi_*(\Phi(W) \cap [N]) + \pi_*\beta. \tag{26}$$

Let $Z_1, \ldots, Z_m$ be the connected components of S. For each i choose an open neighbourhood U_i of Z_i such that U_i deformation retracts to Z_i through the map

$$\delta_i \colon U_i \to Z_i \tag{27}$$

and $U_i \cap U_j = \emptyset$ if $i \neq j$. This induces an isomorphism

$$\delta_{i*} \colon H_*(U_i; \mathbf{C}) \to H_*(Z_i; \mathbf{C}). \tag{28}$$

The statement of the theorem will be derived from 26 using the map δ_{i*}. In particular we will show that

$$\mathrm{Res}_\Phi(E, Z_i) = \delta_{i*}\{\pi_*(\Phi(\pi^*Q) \cap [N]) \mid U_i\} \tag{29}$$

$$N\, \mathrm{Res}_\Phi(E_i, Z_i) = \delta_{i*}\{\pi_*(\Phi(W) \cap [N]) \mid U_i\} \tag{30}$$

$$\kappa = \delta_{i*}\{\pi_*\beta \mid U_i\} \tag{31}$$

Proof of 29: We first simplify the LHS of 26

$$\begin{aligned} \pi_*(\Phi(\pi^*Q) \cap [N]) &= \pi_*(\pi^*\Phi(Q) \cap [N]) \\ &= \Phi(Q) \cap \pi_*[N] \\ &= \Phi(Q) \cap (\deg \pi)\, [M] \\ &= \Phi(Q) \cap [M] \end{aligned} \tag{32}$$

where the first equation follows from the functoriality of chern classes, the second equation is a property of cap products, see [11, p. 254, (16)], the third equation holds by definition since $[N]$ and $[M]$ are fundamental cycles and finally the last equation holds since $\deg \pi = 1$.

We want to represent $\Phi(Q)$ by a differential form. With this in mind choose in each U_i a compact set Σ_i such that Z_i is contained in the interior of Σ_i, and let Σ be the union of them all;

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_m. \tag{33}$$

There exists a closed differential form ω on M with support on Σ such that

$$[\omega] = \Phi(Q) \tag{34}$$

where $[\cdot]$ denotes the cohomology element defined by that form, see [1, pp. 312–313]. Let ω_i be a differential form on M defined as

$$\omega_i | U_i = \omega | U_i \quad \text{and} \quad \omega_i | M - U_i = 0, \quad i = 1, \ldots, m. \tag{35}$$

We then have

$$\begin{aligned} \Phi(Q) \cap [M] &= [\omega] \cap [M] \\ &= [\omega_1] \cap [M] + \cdots + [\omega_m] \cap [M] \\ &= [\omega_1] \cap [U_1] + \cdots + [\omega_m] \cap [U_m] \end{aligned} \tag{36}$$

since each ω_i has support in U_i. But notice that

$$\delta_{i*}([\omega_i] \cap [U_i]) = \mathrm{Res}_\Phi(E, Z_i), \quad i = 1, \ldots, m \tag{37}$$

see [1, p. 313, (7.14)]. Putting 32, 36 and 37 together proves 29.

Proof of 30: The main observation here is that $T/\Psi(E)$ is a vector bundle on $M - S$ and since $\Psi(E) | M - S$ is integrable, $T/\Psi(E)$ has a basic connection D. Also since the restriction of W to $N - X$ is the same as the restriction of $\pi^*(T/\Psi(E))$ to $N - X$, this basic connection can be pulled to a connection of W on $N - X$. There then exists a connection $\tilde{D}$ for W on N such that $\tilde{D}$ agrees with $\pi^* D$ on $N - \pi^{-1}(\Sigma)$, see [1, p. 330, (4.41)]. If $\tilde{K}$ is the curvature matrix for $\tilde{D}$ then

$$\Phi(\tilde{K}) = 0 \text{ on } N - \pi^{-1}(\Sigma) \tag{38}$$

since D and $\tilde{D}$ agree on $\pi^{-1}(\Sigma)$. For simplicity of notation use V_i to denote $\pi^{-1}(U_i)$. Since $\deg \Phi > n - k$ we can define closed forms Φ_i on N by

$$\Phi_i | V_i = \imath\Phi(\tilde{K}) | U_i, \quad \text{and} \quad \Phi_i | N - V_i = 0 \tag{39}$$

where

$$\iota = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\deg\Phi}.$$

It follows that

$$\iota\Phi(\tilde{K}) = \Phi_1 + \cdots + \Phi_m. \tag{40}$$

By definition we have

$$\iota[\Phi(\tilde{K})] = \Phi(W) \tag{41}$$

Capping both sides by $[N]$ we will get

$$\begin{aligned}\Phi(W) \cap [N] &= [\Phi_1] \cap [N] + \cdots + [\Phi_m] \cap [N] \\ &= [\Phi_1] \cap [V_1] + \cdots + [\Phi_m] \cap [V_m] \end{aligned}\tag{42}$$

and

$$\pi_*(\Phi(W) \cap [N]) = \pi_*([\Phi_1] \cap [V_1]) + \cdots + \pi_*([\Phi_m] \cap [V_m]). \tag{43}$$

Finally 30 follows from 43 and the definition of Nash residue, see the end of section II.

As for 31, we can take it as the definition of κ. Note however that since $\pi_*\beta \mid U_i$ is already in $H_*(Z_i; \mathbf{C})$, the isomorphism δ_{i*} does not change it. Hence in fact

$$\kappa = \pi_*\beta \mid U_i$$

where Z is taken as Z_i. For a closed expression of κ see the last part in the proof of the previous lemma. This completes the proof of the theorem. □

When Φ has rational coefficients κ will always be a rational homology cycle by construction. Hence the rationality of Res will be determined from the rationality of N Res in our particular set up, see the beginning of this section.

COROLLARY IV.5: *If the rationality conjecture holds for vector bundles, then it holds for sheaves which are image sheaves of bundle morphisms.*

For a discussion of the rationality conjecture see [1].

V. Removing the smoothness requirement on N

Our present set up is as given in the beginning of section IV except that N is no longer assumed to be smooth. Let f: $L \to N$ be a desingularization of N with the exceptional divisor D_e. The rest of the notations is as in section IV.

Equation IV.5 pulls back to L as

$$0 \to f^*\pi^*E \to f^*\pi^*T \to f^*W \to 0. \qquad (E^*)$$

Each of the terms in this sequence denotes a vector bundle on a smooth space L. Note that N may not be a smooth space but it is always an admissible space in the complex category, see theorem I.1. The complex (E^*) is exact off D_e, and can be used for the Grassmann graph construction as before, to give a localized class on D_e, which can further be pushed down to S. Recall the basic connection D of $T/\Psi(E)$ as given in "the proof of 30" in the previous section. Pull this connection to a connection of f^*W on $L - D_e$, in a similar way as it was pulled back to a connection of W. Extending this connection for f^*W on all of L we can calculate a 'Nash type' residue in $H_*(S; \mathbf{C})$. We may call this residue class a Nash-Hironaka residue and denote it by $NH\ \mathrm{Res}_\Phi(E, S)$. This is basically the residue of a vector bundle. We then have the analogue of theorem IV.4;

THEOREM V.1: *If the singular holomorphic foliation $\mathscr{F}$ is given as the image sheaf of a maximal rank bundle morphism and Z is a connected component of the singular set S, then*

$$\mathrm{Res}_\Phi(E, Z) = NH\ \mathrm{Res}_\Phi(E, Z) + \kappa$$

where κ is a homology cycle in Z and is calculated by a Grassmann graph construction.

The proof of theorem V.1 carried out in detail will duplicate most of the proof of theorem IV.4, therefore we leave it out. The insight of the problem and the interplay between geometry and topology is best exhibited in the case when N is assumed to be smooth and everything happens right on the Nash blow up. The techniques developed in the smooth case can then be applied to the general case. Also note that corollary IV.5 now holds verbatim without the further assumption that the Nash blow up is smooth.

REMARKS (1) If $\mathscr{F}$ is generated by a bundle morphism Ψ: $E \to T$ but the rank of E is larger than the rank of $\mathscr{F}$, then $\underline{\Psi}$ is not injective. To make the theorem work in this case some restrictions must be imposed on the kernel of Ψ. For example we may require that for every connected component Z of S there exist an open neighbourhood U of Z and a vector bundle H on U with a bundle morphism η: $H|U - Z \to E|U - Z$ such that η is injective, and $\Psi \circ \eta | U - Z = 0$. Then the theorem holds for $\mathscr{F}$. Note that η need not be defined on all of U but H should, to ensure the construction of the Grassmann graph. In general if there is a locally free resolution of $\mathscr{F}$ on U, then this can be used for the Grassmann graph construction and the theorem can be derived.

(2) Let M be a compact complex manifold with a positive line bundle. Then M is algebraic by the Kodaira imbedding theorem, [4, p. 181]. Hence N being a subvariety of $M \times G(k, \underline{T})$ is also algebraic together with its desingularization f: $L \to N$. On algebraic manifolds coherent sheaves have global syzygies, [4, p. 701]. Then $f^*\pi^*\mathscr{F}$ will have global syzygies. This syzygy can be used together with the previous remark to establish the result of theorem V.1 without requiring that $\mathscr{F}$ be the image sheaf of a bundle morphism. Thus we conclude that the rationality conjecture on algebraic manifolds can be deduced from a similar statement for vector bundles.

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