Spherical Wave Representation of the Dyadic Green's Function for a Spherical Impedance Boss at the Edge of a Perfectly Conducting Wedge

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Abstract— In this work, canonical problem of a scatterer at the edge of a wedge is considered and eigenfunction solution is developed. Initially, a dyadic Green's function for a spherical impedance boss at the edge of a perfect electrically conducting (PEC) wedge is obtained. Since scattering from objects at the edge is of interest, a three-dimensional Green's function is formulated in terms of spherical vector wave functions. First, an incomplete dyadic Green's function is expanded in terms of solenoidal vector wave functions with unknown coefficients, which is not valid in the source region. Unknown coefficients are calculated by utilizing the Green's second identity and orthogonality of the vector wave functions. Then, the solution is completed by adding general source correction term. Resulting Green's function is decomposed into two parts. First part is the dyadic Green's function of the wedge in the absence of the sphere and the second part represents the effects of the spherical boss and the interaction between the wedge and the scatterer. In contrast to cylindrical vector wave function expansions and asymptotic solutions which fail to converge in the paraxial region, proposed solution exhibits good convergence everywhere in space. Using the developed Green's function scattered field patterns are obtained for several impedance values and results are compared with those of a PEC spherical boss. Effects of the incident angle and surface impedance of the boss on the scattering pattern are also examined.

1. INTRODUCTION

In modeling radar targets and developing propagation models for wireless networks structures on wedges are often involved. Analytical treatment of such canonical structures will provide accurate and fast simulations and a physical insight of the problem. In this work, we present an eigenfunction solution for an impedance spherical boss placed at the edge of perfect electrically conducting (PEC) wedge which provides a basis to analyze impedance scatteres at the edge.

Configurations consisting of wedge structures are treated extensively in literature. A Geometrical Theory of Diffraction (GTD) based solution for scattering from cylinder-tipped wedge is first presented in [1], then numerical results in the shadow boundaries are improved by introducing higher order terms [2]. Also in [3], Green's function based on eigenfunction expansion is for cylinder-tipped wedge with a sectoral or annular groove. In addition, scattering from corrugated, grooved and cavity backed wedge are considered in [4] and [5] where hybrid methods using Uniform Theory of Diffraction, Method of Moments and Finite Element Method are employed. Nevertheless, published works mostly refer to configurations which conform to cylindrical coordinates and do not mention the scattering from structures at the edge.

Procedure followed here is an extension of that outlined in [6] where Dyadic Green's function is developed for spherical PEC boss at the edge. Due to three-dimensional geometry of the problem, spherical vector wave functions are utilized. This approach is convenient for accurate field calculation in the paraxial region where the scatterer is located.

2. FORMULATION

Geometry of the problem is shown in Fig. 1. A PEC wedge with exterior angle γ is considered which extends infinitely in the z direction. One side of the wedge lies on the xz plane. A spherical impedance boss with radius a is centered at the edge of the PEC wedge. $\bar{R} = r\hat{r}$ and $\bar{R}' = r'\hat{r}'$ are two position vectors denoting the observation and the source locations, respectively. S_w and S_B denotes the surface of the wedge and the boss, respectively. Σ is an imaginary spherical surface which extends to infinity. These surfaces enclose the volume V. $\hat{n}$ is the unit normal vector directed in the volume V. Throughout the paper $e^{j\omega t}$ time convention is assumed and suppressed.

The electric field due to the volume current density, $\bar{J}_v$, which is confined to volume V_j, can be written in the form of

$$\bar{E}\left(\bar{R}\right) = j\omega\mu_0 \int_{V_j} \bar{J}_v\left(\bar{R}'\right) \cdot \bar{\bar{\Gamma}}_{WB}\left(\bar{R}, \bar{R}'\right) dv' \tag{1}$$

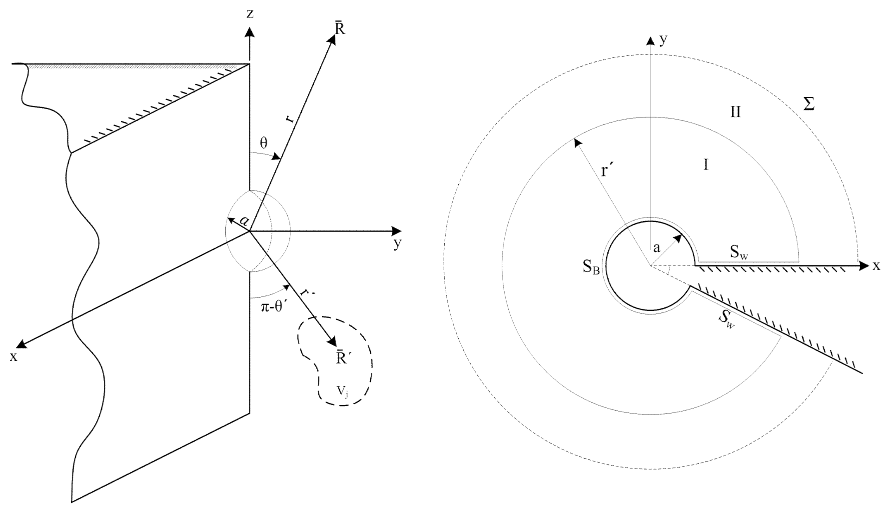

Figure 1: Geometry of the problem.

where $\bar{\bar{\Gamma}}_{WB}(\bar{R},\bar{R}')$ is the dyadic Green's function for the PEC wedge and boss. For the isotropic, homogeneous medium in V, k_0 and Z_0 are the wave number and characteristic impedance of free space, respectively.

Dyadic Green's function satisfies the following differential equation

$$\nabla \times \nabla \times \bar{\bar{\Gamma}}_{WB}\left(\bar{R},\bar{R}'\right) - k_0^2 \bar{\bar{\Gamma}}_{WB}\left(\bar{R},\bar{R}'\right) = \bar{\bar{I}}\delta\left(\bar{R}-\bar{R}'\right) \tag{2}$$

where $\bar{\bar{I}}$ is the unit dyad and $\delta(\bar{R}-\bar{R}')$ is the Dirac delta function. In addition to (2), $\bar{\bar{\Gamma}}_{WB}(\bar{R},\bar{R}')$ satisfies the following boundary condition at S_w

$$\hat{n} \times \bar{\bar{\Gamma}}_{WB}\left(\bar{R},\bar{R}'\right) = 0 \tag{3}$$

and also satisfies the impedance boundary condition at surface of the boss, S_B, $\hat{n} \times \hat{n} \times \bar{E}(\bar{R}) = -\eta \hat{n} \times \bar{H}(\bar{R})$ which can be written in terms of dyadic Green's function as follows:

$$\hat{n} \times \hat{n} \times \bar{\bar{\Gamma}}_{WB}\left(\bar{R},\bar{R}'\right) = \kappa \hat{n} \times \nabla \times \bar{\bar{\Gamma}}_{WB}\left(\bar{R},\bar{R}'\right) \tag{4}$$

where $\kappa = \frac{\eta}{jk_0 Z_0}$ and η is the surface impedance of the boss. $\bar{\bar{\Gamma}}_{WB}$ also satisfies the Meixner edge condition and the radiation condition.

To construct the dyadic Green's function, first an incomplete dyadic Green's function is expanded in terms of solenoidal spherical vector wave functions, which is valid for $\bar{R} \neq \bar{R}'$, then the solution is completed by adding a source correction term introduced in [7].

Volume V is divided into two regions which is separated by the sphere of radius r'. To simplify the derivation, vector Green's function is defined as follows

$$\bar{G}\left(\bar{R},\bar{R}'\right) = \bar{\bar{\Gamma}}\left(\bar{R},\bar{R}'\right) \cdot \hat{u} \tag{5}$$

where $\hat{u}$ is a unit vector in an arbitrary direction. Vector Green's function can be expanded in terms of spherical vector wave functions which are presented in [8].

$$\bar{G}^{I} = \sum_q a_q\left(\bar{R}'\right) \bar{M}_{eq}^{(IM)}\left(k_0\bar{R}\right) + b_q\left(\bar{R}'\right) \bar{N}_{oq}^{(IN)}\left(k_0\bar{R}\right) \tag{6}$$

$$\bar{G}^{II} = \sum_q c_q\left(\bar{R}'\right) \bar{M}_{eq}^{(4)}\left(k_0\bar{R}\right) + d_q\left(\bar{R}'\right) \bar{N}_{oq}^{(4)}\left(k_0\bar{R}\right) \tag{7}$$

where $\bar{G}^{I}$ and $\bar{G}^{II}$ are the Green's function in the region I and region II, respectively. q is the compact summation index representing m, n. Since $\bar{G}^{I}$ satisfies the impedance boundary condition, vector wave functions are modified as follows:

$$\bar{M}_{e\mu n}^{(IM),(4)}\left(k_0\bar{R}\right) = k_0 z_{\mu+n}^{(IM),(4)}\left(k_0 r\right) \bar{m}_{e\mu n}\left(\theta,\phi\right) \tag{8}$$

$$\bar{N}_{o\mu n}^{(IN),(4)}\left(k_0\bar{R}\right) = \frac{1}{r} z_{\mu+n}^{(IN),(4)}\left(k_0 r\right) \bar{l}_{o\mu n}\left(\theta,\phi\right) + \frac{1}{r}\frac{d}{dr}\left[r z_{\mu+n}^{(IN),(4)}\left(k_0 r\right)\right] \bar{n}_{o\mu n}\left(\theta,\phi\right) \tag{9}$$

where

$$z_{\mu+n}^{(4)}\left(k_0 r\right) = h_{\mu+n}^{(2)}\left(k_0 r\right) \tag{10}$$

$$z_{\mu+n}^{(IM)} = j_{\mu+n}\left(k_0 r\right) - \frac{k_0\kappa j'_{\mu+n}\left(k_0 a\right) + \left(\frac{\kappa}{a}-1\right) j_{\mu+n}\left(k_0 a\right)}{k_0\kappa h_{\mu+n}^{(2)\prime}\left(k_0 a\right) + \left(\frac{\kappa}{a}-1\right) h_{\mu+n}^{(2)}\left(k_0 a\right)} h_{\mu+n}^{(2)}\left(k_0 r\right) \tag{11}$$

$$z_{\mu+n}^{(IN)} = j_{\mu+n}\left(k_0 r\right) - \frac{k_0 j'_{\mu+n}\left(k_0 a\right) + \left(\frac{1}{a}+k_0^2\kappa\right) j_{\mu+n}\left(k_0 a\right)}{k_0 h_{\mu+n}^{(2)\prime}\left(k_0 a\right) + \left(\frac{1}{a}+k_0^2\kappa\right) h_{\mu+n}^{(2)}\left(k_0 a\right)} h_{\mu+n}^{(2)}\left(k_0 r\right) \tag{12}$$

$\bar{m}_{e\mu n}$ and $\bar{n}_{o\mu n}$ are auxiliary vector wave functions which are defined in [8]. To satisfy the boundary conditions on S_w, $\bar{M}$ and $\bar{N}$ functions are chosen as even and odd, respectively. Additionally, $\mu = \frac{m\pi}{\gamma}$ are the eigenvalues defined by the exterior angle of the PEC wedge. The primes in the Equation (12) denote derivatives of the functions with respect to their argument.

To solve for the unknown coefficients, Green's second identity is applied in the volume V_1 and V_2 and orthoganality of the vector wave functions over spherical surfaces is used. Consequently, the coefficients are calculated as,

$$a_q\left(\bar{R}'\right) = \frac{j\pi}{2k_0}\frac{1}{Q_{\mu n}\left(\mu+n\right)\left(\mu+n+1\right)}\bar{M}_{eq}^{(4)}\left(k_0\bar{R}'\right)\cdot\hat{u} \tag{13}$$

$$b_q\left(\bar{R}'\right) = \frac{j\pi}{2k_0}\frac{1}{Q_{\mu n}\left(\mu+n\right)\left(\mu+n+1\right)}\bar{N}_{oq}^{(4)}\left(k_0\bar{R}'\right)\cdot\hat{u} \tag{14}$$

$$c_q\left(\bar{R}'\right) = \frac{j\pi}{2k_0}\frac{1}{Q_{\mu n}\left(\mu+n\right)\left(\mu+n+1\right)}\bar{M}_{eq}^{(IM)}\left(k_0\bar{R}'\right)\cdot\hat{u} \tag{15}$$

$$d_q\left(\bar{R}'\right) = \frac{j\pi}{2k_0}\frac{1}{Q_{\mu n}\left(\mu+n\right)\left(\mu+n+1\right)}\bar{N}_{oq}^{(IN)}\left(k_0\bar{R}'\right)\cdot\hat{u} \tag{16}$$

where

$$Q_{\mu n} = \frac{\epsilon_m\pi\gamma n!}{2\left(2\mu+2n+1\right)\Gamma\left(2\mu+n+1\right)} \tag{17}$$

$$\epsilon_m = \begin{cases} 2 & m=0 \\ 1 & m\neq 0. \end{cases} \tag{18}$$

By comparing the Equations (6) and (7) with Equation (5) and adding the source correction term, one could obtain the dyadic Green's function as

$$\bar{\bar{\Gamma}}_{WB}\left(\bar{R},\bar{R}'\right) = \frac{\hat{r}\hat{r}}{k_0^2}\delta\left(\bar{R}-\bar{R}'\right) + \frac{j\pi}{2k_0}\begin{cases} \sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty}\frac{\bar{M}_{e\mu n}^{(4)}\left(k_0\bar{R}\right)\bar{M}_{e\mu n}^{(IM)}\left(k_0\bar{R}'\right)+\bar{N}_{o\mu n}^{(4)}\left(k_0\bar{R}\right)\bar{N}_{o\mu n}^{(IN)}\left(k_0\bar{R}'\right)}{Q_{\mu n}(\mu+n)(\mu+n+1)} & r\geqslant r' \\ \sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty}\frac{\bar{M}_{e\mu n}^{(IM)}\left(k_0\bar{R}\right)\bar{M}_{e\mu n}^{(4)}\left(k_0\bar{R}'\right)+\bar{N}_{o\mu n}^{(IN)}\left(k_0\bar{R}\right)\bar{N}_{o\mu n}^{(4)}\left(k_0\bar{R}'\right)}{Q_{\mu n}(\mu+n)(\mu+n+1)} & r\leqslant r'. \end{cases} \tag{19}$$

Using the explicit equations for $\bar{M}^{(IM)}$ and $\bar{N}^{(IN)}$, one could obtain

$$\bar{\bar{\Gamma}}_{WB}\left(\bar{R},\bar{R}'\right) = \bar{\bar{\Gamma}}_{B}\left(\bar{R},\bar{R}'\right) + \bar{\bar{\Gamma}}_{W}\left(\bar{R},\bar{R}'\right) \tag{20}$$

where $\bar{\bar{\Gamma}}_W(\bar{R},\bar{R}')$ is the dyadic Green's function for the PEC wedge defined in the [8]. $\bar{\bar{\Gamma}}_B(\bar{R},\bar{R}')$ includes the terms added due to the presence of the spherical scatterer which represents the scattering from the sphere and its interaction with the wedge.

$$\bar{\bar{\Gamma}}_{B}\left(\bar{R},\bar{R}'\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha_{\mu n}\left(k_0 a\right)\bar{M}_{e\mu n}^{(4)}\left(k_0\bar{R}'\right)\bar{M}_{e\mu n}^{(4)}\left(k_0\bar{R}\right)+\beta_{\mu n}\left(k_0 a\right)\bar{N}_{o\mu n}^{(4)}\left(k_0\bar{R}\right)\bar{N}_{o\mu n}^{(4)}\left(k_0\bar{R}'\right)}{Q_{\mu n}\left(\mu+n\right)\left(\mu+n+1\right)} \quad r \gtreqless r' \tag{21}$$

where

$$\alpha_{\mu n}\left(k_0 a\right) = -\frac{k_0 \kappa j'_{\mu+n}\left(k_0 a\right) + \left(\frac{\kappa}{a} - 1\right) j_{\mu+n}\left(k_0 a\right)}{k_0 \kappa h^{(2)\prime}_{\mu+n}\left(k_0 a\right) + \left(\frac{\kappa}{a} - 1\right) h^{(2)}_{\mu+n}\left(k_0 a\right)} \tag{22}$$

$$\beta_{\mu n}\left(k_0 a\right) = -\frac{k_0 j'_{\mu+n}\left(k_0 a\right) + \left(\frac{1}{a} + k_0^2 \kappa\right) j_{\mu+n}\left(k_0 a\right)}{k_0 h^{(2)\prime}_{\mu+n}\left(k_0 a\right) + \left(\frac{1}{a} + k_0^2 \kappa\right) h^{(2)}_{\mu+n}\left(k_0 a\right)}. \tag{23}$$

For the limiting case of PEC scatterer, $\eta = 0$, $\bar{\bar{\Gamma}}_B$ reduces to the dyadic Green's function for a PEC boss which is presented in [6].

3. NUMERICAL RESULTS

In this section, dyadic Green's function given by (21) alongside with (22) and (23) is used to investigate the effect of the spherical boss on the scattering of the PEC wedge. The field scattered from the impedance boss is defined as follows:

$$\bar{E}^s\left(\bar{R}\right) = jk_0 Z_0 \int_{V_j} \bar{\bar{\Gamma}}_B\left(\bar{R}, \bar{R}'\right) \cdot \bar{J}_V\left(\bar{R}'\right) dv'. \tag{24}$$

A point source is assumed at $\bar{R}_0 = r_0 \hat{r}_0$ which is in the far zone of the sphere.

$$\bar{J}_v\left(\bar{R}'\right) = \delta\left(\bar{R}' - \bar{R}_0\right) \bar{p}_e \tag{25}$$

where $\bar{p}_e$ is an electric dipole moment which is determined by the angular unit vectors $\hat{\theta}$ and $\hat{\phi}$. In the following examples, a sphere of radius 0.25λ is centered at the edge of a half plane. Origin of the coordinate system is chosen as the center of the sphere.

In Figs. 2 and 3, monostatic scattered field is plotted for normalized surface impedance values of $\eta_s = \eta/Z_0 = 0, 1.5, 3$. Scattered field is calculated with elevation angle fixed at a value θ_0, and ϕ is varied from 0 to 360°. $E_{\theta\theta}$ represents the scattered electric field in $\hat{\theta}$ direction when the incident field is in $\hat{\theta}$ direction and $E_{\phi\phi}$ represents the scattered electric field in $\hat{\phi}$ direction when incident field is in $\hat{\phi}$ direction.

In Fig. 2, incident and scattered fields are in the paraxial region, hence the scattered field is high due to the edge excited guided waves. In this region field pattern varies as $\cos^2(\frac{\phi}{2})$ and $\sin^2(\frac{\phi}{2})$ for $E_{\theta\theta}$ and $E_{\phi\phi}$, respectively which indicates that $n = 0$, $m = 1$ mode is dominant.

As the incident field moves away from the paraxial region (Fig. 3), higher order modes are excited. In addition, field intensity drops significantly due to the weakened edge guidance. The amount of decrease in the field amplitude is observed to be more for the PEC scatterer.

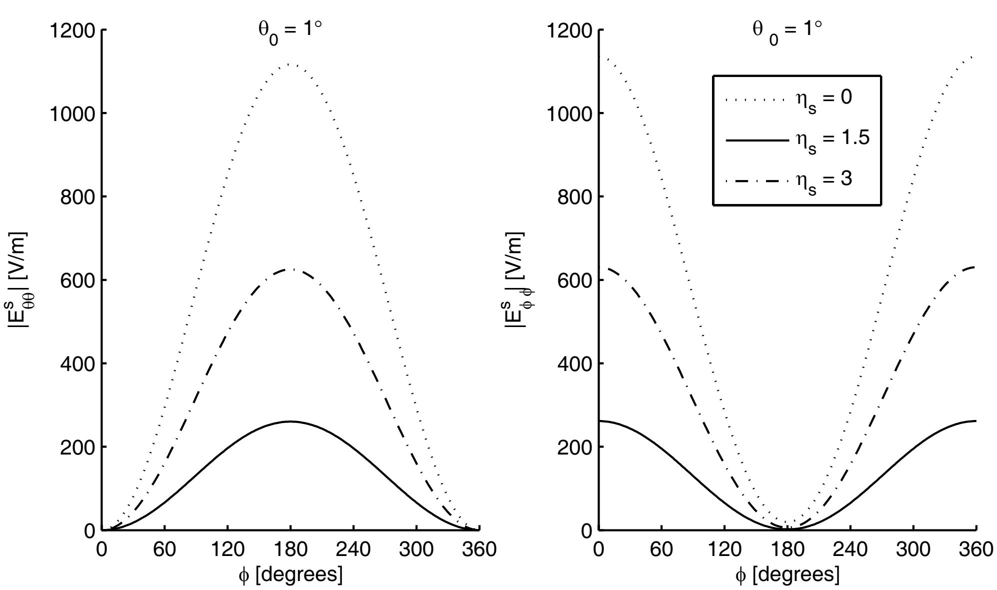

Figure 2: Monostatic scattered field for the spherical boss at the edge. The incident angle $\theta_0 = 1°$, $a = 0.25\lambda$.

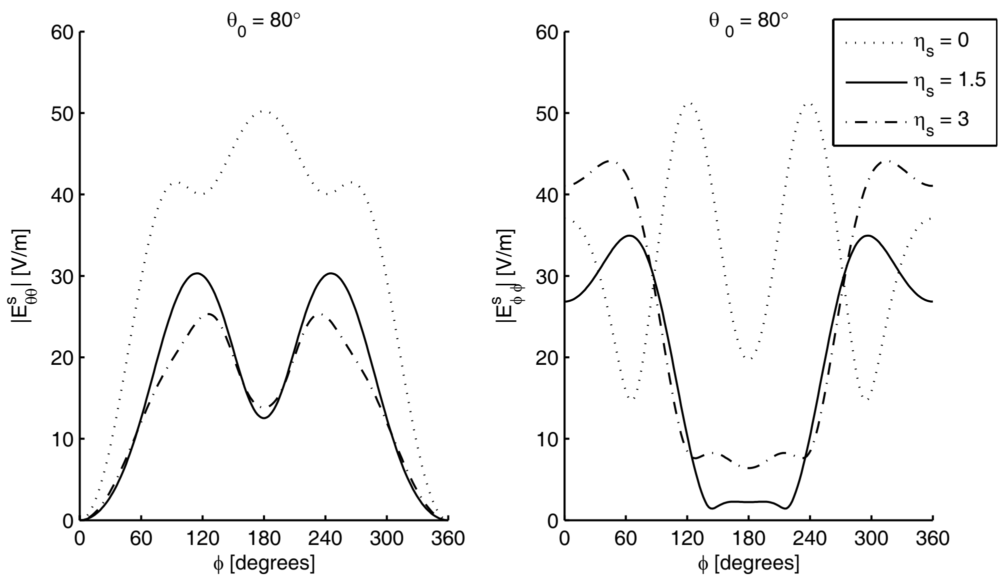

Figure 3: Monostatic scattered field for the spherical boss at the edge. The incident angle $\theta_0 = 80°$, $a = 0.25\lambda$.

4. CONCLUSION

We have developed a dyadic Green's function for an impedance boss at the edge based on the spherical vector wave function expansion that gives accurate results everywhere in space. In the limiting case of $\eta = 0$, dyadic Green's function expression reduces to the one for the PEC boss. Numerical results are obtained for impedance scatterer and compared with the PEC scatterer. This study could set a benchmark for numerical solvers and lead to an extension for the current high-frequency approaches.

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