equal to the number of parameters under test. If $\mathscr{I}_{\gamma\theta}$ has rank q, the generalized inverses* which occur in the theorem are the inverses of the matrices.

Although the derivation of Hausman's test from the maximum likelihood approach has been obtained under a sequence of local alternative hypotheses, the null hypothesis actually tested by this procedure is $H_0^*: \mathscr{I}_{\gamma\gamma}^{-1}\mathscr{I}_{\gamma\theta}\boldsymbol{\beta} = \mathbf{0}$ against $H_1^*: \mathscr{I}_{\gamma\gamma}^{-1}\mathscr{I}_{\gamma\theta}\boldsymbol{\beta} \neq \mathbf{0}$. Notice that H_0^* reduces to H_0 whenever we have simultaneously $q \geqslant p$ and rank $(\mathscr{I}_{\gamma\theta}) = p$.

Now, suppose that H_0 and H_0^* are not equivalent, and that Hausman's procedure is used for the problem of testing the null hypothesis H_0. Two main results can be derived. First, if n is sufficiently large and $\boldsymbol{\theta}_n^0$ not near $\boldsymbol{\theta}^0$, the power of Hausman's test will not be near 1 in all the directions of the parameter space. Thus there is a strong possibility that the test might not be consistent. Second, there exist directions for which the test has a better power than conventional procedures such as the likelihood ratio test*.

Finally, Hausman [2] suggested an alternative procedure for the specification error testing problem. He pointed out that in many situations the null hypothesis of no specification error may be tested in an expanded regression framework.

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See also ECONOMETRICS.

ALBERTO HOLLY

HAZARD CHANGE POINT ESTIMATION

In reliability, survival, or warranty studies, the immediate risk of failure of items or individuals becomes an important quantity for inference and decision-making purposes. Hence, the hazard- or failure-rate function that describes the instantaneous risk of failure of items at a time point that has not failed before, plays an essential role. In this respect, there are two basic issues, one of them being the characterization of the hazard function that captures the underlying hazard dynamics of the quantities under study and the second one is the estimation of the hazard function. Most of the existing models in the literature consider continuous monotone increasing or decreasing, or bathtub-shaped hazard-rate functions that are commonly observed in real applications. Another important class of hazard-rate functions is described by monotone functions with a single or a finite number of jump (change) points. In some medical or reliability applications, sudden changes in the hazard function may occur due to treatment effects or maintenance activities. The problem of statistical interest is then the estimation of the times of the changes, as well as their sizes. For the change-point models, testing a constant hazard hypothesis against a change-point alternative also becomes an important issue. Also, in survival models, it is common to observe censored or truncated data, which further complicates the estimation and inference problems. In this short review, we confine ourselves with the basic results in the literature regarding the estimation of monotone-hazard functions with a single change point. Some references will be provided, however, for the estimation with incomplete data and testing issues.

Let X be a nonnegative random variable denoting a survival time, such as the time to failure of an item or the time until the first recurrence of a disease after a treatment or surgery, with probability density function (p.d.f) f, distribution function F, and the hazard rate function λ. We focus on the simple model introduced by Matthews and Farewell [7]:

$$\lambda(x) = \beta + \theta I(x \geqslant \tau), \qquad (1)$$

where β, θ, and τ are unknown constants and $I(\cdot)$ is the indicator function. In this model, the risk of immediate failure is described by a step function, where a single jump occurs at the point τ, which may correspond to the time epoch where a sharp change occurs due to the effect of a treatment in medical studies or an overhaul action in maintenance. The corresponding pdf is given by

$$f(x) = \lambda(x)\exp\left\{-\int_0^x \lambda(t)dt\right\} = \begin{cases} \beta\exp(-\beta x) & \text{if } x < \tau \\ (\beta+\theta)\exp\{-\beta x - \theta(x-\tau)\} & \text{if } x \geqslant \tau \end{cases} \quad (2)$$

In an application to the data of time to relapse after remission induction for leukemia patients, Mathews and Farewell [7] consider the likelihood ratio test for testing $\tau = 0$. They use numerical techniques to obtain the maximum likelihood estimators and simulation results for assessing the performance of the proposed test.

Let $X_1, X_2, \ldots, X_n$ be a set of i.i.d. observations from the density given in (2). Also, let $X(\tau) = \#\{i : X_i \leqslant \tau\}$ be the number of observations not exceeding τ. Then, the log likelihood can be written as

$$L(\beta,\theta,\tau) = X(\tau)\log\beta - \beta\sum_{i=1}^{n} X_i I(X_i \leqslant \tau) - \theta\sum_{i=1}^{n} X_i I(X_i > \tau) + (n - X(\tau))\{\log\theta - (\beta-\theta)\tau\} \quad (3)$$

Nyugen, Rogers, and Walker [12] (NRW) point out that the likelihood above is unbounded unless $\beta < \theta$. In particular, if τ is chosen to satisfy $X_{(n-1)} \leqslant \tau \leqslant X_{(n)}$, the likelihood in (3) is proportional to

$$(n-1)\log\beta - \beta\sum_{i=1}^{n-1} X_i - \beta\tau + \log\theta - \theta(X_{(n)} - \tau)$$

and letting $\theta = 1/(X_{(n)} - \tau)$, $\tau \to X_{(n)}$, the likelihood becomes unbounded. When $\beta > \theta$, however, it is bounded but since the parameter space is not bounded, it is not clear whether the supremum of the likelihood can be achieved. For fixed τ, differentiating (3) w.r.t. β and θ yields the critical points:

$$\hat{\beta}(\tau) = \frac{X(\tau)}{\sum_{i=1}^{n}\min(X_i,\tau)}; \qquad \hat{\theta}(\tau) = \frac{n - X(\tau)}{\sum_{i=1}^{n}(X_i-\tau)I(X_i > \tau)} \quad (4)$$

Nyugen, Rogers, and Walker suggest the following estimators for $1/\beta$ and $1/\theta$

$$\hat{B}_n(\tau) = \frac{\sum_{i=1}^{n}\min(X_i,\tau)}{X(\tau)+1}; \qquad \hat{T}_n(\tau) = \frac{\sum_{i=1}^{n}(X_i-\tau)I(X_i>\tau)}{n - X(\tau)+1} \quad (5)$$

and prove the existance of a strongly consistent estimator for τ. To this end, they define a stochastic process $X_n(t)$, which converges to zero, so that a candidate for an estimator of τ is the value $\hat{t}$ such that $X_n(\hat{t})$ is close to zero. However, an explicit expression for this estimator is not provided. As their final result, they prove that if $\hat{\tau}$ is a strongly consistent estimator of τ, then, substituting $\hat{\tau}$ into (5) with probability one, it holds that

$$\hat{B}_n(\hat{\tau}) \to 1/\beta; \quad \hat{T}_n(\hat{\tau}) \to 1/\theta.$$

Yao [16] considers the same model for the estimation of (β,θ,τ) and suggest the maximum likelihood estimators with the restriction that $\tau \leqslant X_{(n-1)}$. In particular, substituting (4) into (3), the following function is maximized w.r.t. τ

$$l(\tau) = -[\sup_{\alpha,\beta} L(\beta,\theta,\tau) + n]/n = \begin{cases} \log[n^{-1}\sum_{i=1}^{n}(X_i-\tau)] & \text{if } \tau < X_{(1)} \\ (X(\tau)/n)\log\left(\frac{\sum_{i=1}^{n}\min(X_i,\tau)}{X(\tau)}\right) + (n - X(\tau)/n)\log\left(\frac{\sum_{i=1}^{n}(X_i-\tau)I(X_i>\tau)}{n-X(\tau)}\right) & \text{if } \tau \geqslant X_{(1)} \end{cases}$$

Yao [16] proposes an estimator $\hat{\tau}$ which minimizes $l(\tau)$ subject to $\tau \leqslant X_{(n-1)}$ and shows that it is unique with probability one. Yao also proves that $\hat{\tau} \to \tau$ in probability. Estimators $\hat{\beta}$ and $\hat{\theta}$ for β and θ are then obtained by substituting $\hat{\tau}$ into (4). As to the asymptotic distribution of the estimators, it is shown that the scaled random vector below converges in distribution:

$$\{\sqrt{n}(\hat{\beta}-\beta), \sqrt{n}(\hat{\theta}-\theta), n(\hat{\tau}-\tau)\} \to (X, Y, Z)$$

where X, Y, and Z are independent, X and Y are normal variables with mean zero and variances $\beta^2/(1-\exp(-\beta\tau))$, $\theta^2/\exp(\beta\tau)$ respectively, and Z is defined as a function of independent unit exponential variables (see Ref. 16 for details). Note here that the rate of convergence for the estimator of τ is n as opposed to the usual $\sqrt{n}$ rate encountered in most of the ”regular” models and the limiting distribution is not normal. Pham and Nguyen [14] employ a similar maximum-likelihood estimation method for the parameters of the model in (2), where the maximization over τ values is restricted to the random interval $[T_n^1, T_n^2]$ with $0 \leqslant T_n^1 \leqslant T_n^2$. Under minor conditions on T_n^1, T_n^2, Pham and Nguyen [14] show that the proposed estimator is strongly consistent. Choice of these quantities that satisfy the requirements include a fixed interval, provided that τ lies within that interval, or, $[X_{1:n}, X_{n:n}-\delta]$ where δ is a fixed positive number and the natural choice $[X_{1:n}, X_{n-1:n}]$, where $X_{k:n}$ is the kth order statistic of a sample of size n. Pham and Nguyen [14] also provide the asymptotic distributions of the estimators as in Reference 16. In particular, it is shown that $n(\hat{\tau}-\tau)$ converges in distribution to a random variable R_I, where I is the index which maximizes

$$S_i = i\log(\beta/\theta) + e^{-\beta\tau}(\theta-\beta)R_i, \quad -\infty < i < \infty$$

where

$$R_i = \begin{cases} -\sum_{j=i}^{0}(e^{\beta\tau}/\beta)Z_j & \text{if } i \leqslant 0 \\ \sum_{j=1}^{i}(e^{\beta\tau}/\theta)Z_j & \text{if } i > 0 \end{cases}$$

and Z_j, $-\infty < j < \infty$ are independent exponential variables with unit mean. The result above provides the limiting distribution of $n(\hat{\tau}-\tau)$, however, it is seen that it is a highly complicated one, which requires heavy computation. Pham and Nguyen [13] propose to overcome this difficulty by considering the parametric bootstrap distribution. Writing $n(\hat{\tau}-\tau) = U_n(X_1, X_2, \ldots, X_n, \tau)$, let $\alpha_n = (\beta_n, \theta_n, \tau_n)$ be the estimators based on an i.i.d. sample $(X_1, X_2, \ldots, X_n)$ and let the bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$ be independent random variables sampled conditionally from the model (2) where the parameters are replaced with α_n, and τ_n^* be the estimator of τ obtained similar to τ_n from the bootstrap sample. Then, Pham and Nguyen [13] show that the distribution of $n(\hat{\tau_n^*}-\tau_n) = U_n(X_1^*, X_2^*, \ldots, X_n^*, \tau_n)$ converges weakly to that of R_I defined above. They also note that, if nonparametric bootstrap, where the estimation is based on the random sample obtained from $(X_1, X_2, \ldots, X_n)$, is used, then the support of the limiting distribution is discrete and cannot converge in law to the distribution of R_I, which has continuous support, and hence nonparametric bootstrap is inconsistent.

Mi [9] proposed a consistent estimator for τ based on the total time on test (TTT) transformation which is strongly consistent and is not subject to any constraints. The scaled TTT transform of a survival distribution function F with mean $\mu = \int \overline{F}(t)dt$ is defined as

$$\phi(u) = \frac{1}{\mu}\int_0^{F^{-1}}(u)\overline{F}(t)dt, \quad \forall\, 0 \leqslant u \leqslant 1$$

where $F^{-1}(u) \equiv \inf\{s : F(s) \geqslant u\}$ and $F^{-1}(1) \equiv 1$.

Consider a unit square with vertices (0,0), (0,1), (1,0), and (0,0) and let $\mathbf{D}$ be the diagonal, connecting (0,0) and (1,1). Also, let $\rho(u,v) = (1/\sqrt{2})|u-v|$ be the distance from the point (u,v) to $\mathbf{D}$. Mi (1996) shows that if the hazard function of F is given by (1), the maximum value of $\rho(u,\phi(u))$ is attained at $u_0 = F(\tau)$. Based on this observation, the empirical TTT is obtained as

$$\phi_n\left(\frac{i}{n}\right) = \frac{\sum_{k=1}^{i} X_{k:n} + (n-i)X_{i:n}}{\sum_{k=1}^{n} X_k}$$

Define the integer $i(n)$ by

$$\rho\left(\frac{i(n)}{n}, \phi_n(\frac{i(n)}{n})\right) = \max_{1\leqslant j\leqslant n-1} \rho\left(\frac{j(n)}{n}, \phi_n(\frac{j(n)}{n})\right)$$

and set

$$\hat{\tau} \equiv X_{i(n):n}$$

$$\hat{\beta} \equiv \frac{i(n)}{\sum_{k=1}^{i(n)} X_{k:n} + (n-i(n))X_{i(n):n}}$$

$$\hat{\theta} \equiv \frac{n-i(n)}{\sum_{k=i(n)}^{n} X_{k:n} - (n-i(n))X_{i(n):n}}$$

Mi [9] shows that the estimators above are strongly consistent. These estimators also provide strongly consistent estimators for the hazard, distribution, and the density functions of the survival variable. As seen from the above expressions, the advantage of these estimators is that they are explicitly defined and are easy to compute.

As mentioned before, censored/truncated observations are common in survival applications and extensions of some of the methods discussed above are available for such data. Readers can refer to References 1, 2, 3, 5, and 10 for estimation with censored data, to Reference 11 for a review of hazard change point models, and Reference 17 for estimation with randomly truncated data. Results on testing for constant hazard against a change point alternative can be found in References 4, 5, 6, 7, 8, 15, and 16.

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Ülkü Gürler

HAZARD PLOTTING

The old Chinese proverb "one picture is worth a thousand words" is exemplified by the practitioners of graphical* methods in statistics. One particular graphical method that is used in the analysis of reliability and survival data is *hazard plotting*, first introduced by Nelson [5]. The principal purpose of hazard plotting is to determine graphically how well a particular probability distribution, when characterized by its cumulative hazard function, fits a given set of failure data. The procedure allows for censoring* of the data.

It is first necessary to review some basic concepts of survival/reliability theory*. The *survival function* $S_T(t)$ is defined as the