The Paley graph conjecture and Diophantine $m$-tuples

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A Diophantine $m$-tuple with property $D(n)$, where $n$ is a nonzero integer, is a set of $m$ positive integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. It is known that $M_n = \sup\{|S| : S$ is a $D(n)$ $m$-tuple$|$ exists and is $O(\log |n|)$. In this paper, we show that the Paley graph conjecture implies that the upper bound can be improved to $\ll (\log |n|)^\epsilon$, for any $\epsilon > 0$.

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1. Introduction

A Diophantine $m$-tuple with property $D(n)$, where $n$ is a nonzero integer, is a set of $m$ positive integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$.

Diophantus found the quadruple $\{1, 13, 68, 105\}$ with property $D(256)$. The first $D(1)$-quadruple $\{1, 3, 8, 120\}$ was found by Fermat (cf. [9]). Baker and Davenport showed in 1969 (cf. [2]) that Fermat’s set is the only extension of $\{1, 3, 8\}$ to a $D(1)$-quadruple,
and thus it cannot be extended to a $D(1)$-quintuple. This result follows from a paper of Baker [1] published in 1968 on linear forms in logarithms of algebraic numbers, which is an effective version of Gelfond’s theorem used for solutions of Diophantine equations in two unknowns and a reduction method introduced in [2].

Baker and Davenport’s remarkable result was the first step towards the folklore conjecture which predicts that there are no $D(1)$-quintuples. In 2004, Dujella (cf. [13]) proved using similar methods to those of Baker and Davenport that there is no sextuple with property $D(1)$ and there are only finitely many effectively computable $D(1)$-quintuples. In 2018, in a very recent paper, He, Togbé and Ziegler (cf. [26]) have shown that there are no Diophantine quintuples, thereby settling this conjecture.

For $n \neq 1$, however, there are Diophantine quintuples and sextuples, such as the quintuple $\{1, 33, 105, 320, 18240\}$ with $n = 256$ (cf. [16]) and the sextuple $\{99, 315, 9920, 32768, 44460, 19534284\}$ with $n = 2985984$ (cf. [19]).

Thus, two related and important questions in the study of Diophantine $m$-tuples are (i) to determine, for a given $n$ and $m$, the number of possible $D(n)$-$m$-tuples; and (ii) to estimate the quantity

$$M_n = \sup \{|S| : S \text{ is a } D(n) \text{ } m\text{-tuple}\}.$$ 

The first observation is that there is no infinite Diophantine $m$-tuples, for any $n \neq 0$, since the number of integral points on the elliptic curve

$$y^2 = (a_1x + n)(a_2x + n)(a_3x + n)$$

is finite, which follows from a celebrated theorem of Siegel (see, for example, [31]). Unfortunately, known bounds (cf. [30]) for the number of integral solutions depend on $n$, $a_1$, $a_2$, and $a_3$. On the other hand, as a result of a conjecture of Caporaso, Harris, and Mazur [6], the hyperelliptic curve of genus 2 given by

$$y^2 = (a_1x + n)(a_2x + n)(a_3x + n)(a_4x + n)(a_5x + n)$$

has a bounded number of integral points, independent of $n$ and the coefficients $a_1, \ldots, a_5$. This would imply that $\sup_n M_n$ is bounded. This observation has been made by Dujella in [12] and the first result in this direction is due to Dujella and Luca [11], who proved that $M_n$ is bounded by an absolute constant whenever $|n|$ is prime, and that, for every $\varepsilon > 0$, the set of positive integers $n$ for which a $D(n)$ or $D(-n)$ Diophantine $m$-tuple exists with $m > (1 + \varepsilon) \log \log n$ is of asymptotic density zero.

A related elementary observation made independently by Brown [4], Gupta and Singh [21], and Mohanty and Ramasamy [28] is that $M_n \leq 3$ if $n \equiv 2 \pmod{4}$. Indeed, being a square, $a_ia_j + n \equiv 0, 1 \pmod{4}$. It follows that $a_i \not\equiv a_j \pmod{4}$ and that at most one $a_i$ can be even. On the other hand, if $n \not\equiv 2 \pmod{4}$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then $M_n \geq 4$ (cf. [15]).
More generally, Dujella proved in [14] that $M_n \leq 31$ for $|n| \leq 400$ and

$$M_n < C \log |n|,$$

where $C = 15.476$ if $|n| > 400$ and $C = 9.078$ if $n > 10^{100}$. This is done by estimating separately the quantities

$$A_n = \sup\{|S \cap [|n|^3, \infty)| : S \text{ has } D(n)\}$$

$$B_n = \sup\{|S \cap [n^2, |n|^3)| : S \text{ has } D(n)\}$$

$$C_n = \sup\{|S \cap [1, n^2]| : S \text{ has } D(n)\}.$$  

He proved that $A_n \leq 21$, $B_n < 0.6071 \log |n| + 2.152$, and $C_n < 11.006 \log |n|$ for $|n| > 400$. If $|n| > 10^{100}$, he showed that $C_n < 8.37 \log |n|$ and the final result is derived by combining all of these estimates. The most significant contribution comes from $C_n$ and is obtained by using Gallagher’s sieve inequality together with an estimate on double Dirichlet character sums due to Vinogradov. Improving these results, Murty and Becker [3] have recently shown that for any $n$,

$$M_n \leq 2.6071 \log |n| + O\left(\frac{\log |n|}{(\log \log |n|)^2}\right).$$

Our purpose in this manuscript is to relate the estimate of $M_n$ for all sufficiently large $n$ to the Paley graph conjecture (stated below) and show how one can improve the known estimates. The basic idea is to use this conjecture together with Gallagher’s sieve inequality to handle $B_n$ and $C_n$ simultaneously which will lead to Theorem 1. We first recall the Paley Graph Conjecture.

**Conjecture 1** (Paley graph conjecture). Let $\varepsilon > 0$ be a real number, $S, T \subseteq \mathbb{F}_p$ for an odd prime $p$ with $|S|, |T| > p^\varepsilon$, and $\chi$ any nontrivial multiplicative character modulo $p$. Then, there is some number $\delta = \delta(\varepsilon)$ for which the inequality

$$\left| \sum_{a \in S, b \in T} \chi(a + b) \right| \leq p^{-\delta}|S||T|$$

holds for primes larger than some constant $C(\varepsilon)$.

The related estimate from Murty and Becker [3] yields

$$\left| \sum_{a \in S, b \in T} \chi(ab + n) \right| \leq \sqrt{p|S||T|},$$

where $p \nmid n$ is an odd prime and not both sets $S$ and $T$ contain 0.
In general, a positive answer for this conjecture is known only in the case $|S| > p^{1/2+\varepsilon}$, and $|T| > p^\varepsilon$. However, if the sets $S$ and $T$ have a certain structure, there are nontrivial estimates that can be obtained under weaker constraints on their size (see [7,18,27,32,33]) using recent advances in additive combinatorics. We quote from [32] why this conjecture is called the Paley graph conjecture below.

A Paley graph is a graph $G(V,E)$ with vertex set $V = \mathbb{F}_p$ and edge set $E$ such that $(a, b) \in E$ if and only if $a - b$ is a quadratic residue modulo $p$. For this graph to be undirected, it is also necessary that $p \equiv 1 \mod 4$. Under this assumption, setting $S = -T$ in the conjecture and taking the Legendre symbol for the multiplicative character $\chi$, we obtain the following remarkable statement: the clique number of the Paley graph and its independence number increase slower than $p^\varepsilon$ for any positive $\varepsilon$.

On the other hand, Graham and Ringrose [20] proved that for infinitely many primes $p$, the least quadratic non-residue $q$ is at least $c(\log p)(\log \log \log p)$ for some constant $c > 0$. Obviously, for these primes, (1) does not hold for $S = T = \{1, 2, \ldots, q/2\}$. See also [29] for a recent result of Mrazović relating Paley graphs to the result of Graham and Ringrose.

Our main theorem is:

**Theorem 1.** If the Paley graph conjecture holds for some $\varepsilon \in (0,1)$, then

$$M_n \ll \varepsilon (\log |n|)^{\varepsilon} \left(1 + O\left(\frac{1}{(\log \log |n|)^2}\right)\right).$$

**Remark 1.** In particular, if for any $\varepsilon > 0$, the Paley graph conjecture holds, then

$$M_n \ll \varepsilon (\log |n|)^{\varepsilon}.$$

We refer the reader to several other related papers that apply results from graph theory to certain problems concerning Diophantine $m$-tuples such as [23,24,5,10,25].

2. Preliminaries and proof of Theorem 1

We collect here some results needed to prove our main theorem.

**Lemma 1 (cf. [17, Thm 4.2]).** For $x \geq 2$,

$$|\theta(x) - x| < 3.965 \frac{x}{\log^2 x},$$

where $\theta(x) = \sum_{p \leq x} \log p$. Also, for $x \geq 1$, $\theta(x) < 2x$. 
Lemma 2. For any $\alpha \in (0, 1)$,
\[
\sum_{p\leq Q} \frac{\log p}{p^\alpha} = \frac{Q^{1-\alpha}}{1-\alpha} \left(1 + O\left(\frac{1}{\log^2 Q}\right)\right)
\]
with an effectively computable implied constant. Furthermore,
\[
\sum_{p\leq Q} \frac{\log p}{p^\epsilon} < \frac{2Q^{1-\epsilon}}{1-\epsilon}.
\tag{2}
\]

Proof. Both results follow from Lemma 1 and partial integration. \qed

Lemma 3 (cf. [17, Thm 5.1, Lemma 5.10]). For $x > 1$,
\[
\pi(x) \leq \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2}{\log^2 x} + 7.59\right).
\tag{3}
\]
Furthermore, for $k \geq 4$,
\[
p_k \leq k(\log k + \log \log k + 1) < 2k \log k,
\tag{4}
\]
where $p_k$ denotes the $k$th prime.

Lemma 4 (cf. [22, Thm 11]). For $n \geq 3$,
\[
\omega(n) \leq 1.38402 \frac{\log n}{\log \log n}
\tag{5}
\]
with equality for $n = p_1 p_2 \cdots p_9$, where $\omega(n)$ is the number of distinct prime divisors of $n$.

The following is an inequality which is a result of Gallagher’s Sieve Inequality.

Lemma 5. Let $S$ be a subset of $\{1, 2, \ldots, N\}$ for some positive integer $N$. For any $1 < Q \leq N$,
\[
|S| \leq \frac{\sum_{p\leq Q} \log p - \log N}{\sum_{p\leq Q} \frac{\log p}{|S_p|} - \log N},
\]
where $S_p = S \mod p$, provided the denominator is positive.

Proof of Theorem 1. As mentioned in the introduction, $A_n \leq 21$. Thus, it is enough to take a $D(n)-m$-tuple $S$ lying inside $[1, N]$, where $N = |n|^3$. 


Assume Conjecture 1 holds for some $\varepsilon > 0$ at least for the quadratic character given by the Legendre’s symbol $(\frac{\cdot}{p})$. Then, there exists some $\delta = \delta(\varepsilon) > 0$ such that (1) holds for $p > C(\varepsilon)$ for some constant $C(\varepsilon) > 0$. Increasing $C(\varepsilon)$ if necessary we can make sure that the inequality

$$p^\varepsilon(1 - p^{-\delta}) \geq 3$$

also holds for $p > C(\varepsilon)$. Note that $p^\varepsilon > 3$ for these primes. If $N \leq C(\varepsilon)$, we get $|S| \leq C(\varepsilon)$. Otherwise, assume that $N$ is large enough so that we can take a prime $p \nmid n$ with $C(\varepsilon) < p \leq Q < N$, where $Q$ is to be chosen later.

For $i = \pm 1$, let $S_i$ denote the elements $a$ of $S_p$ for which $(\frac{a}{p}) = i$. Thus, we have

$$|S_p| \leq |S_1| + |S_{-1}| + 1,$$

with equality when $0 \in S_p$. Since $p \nmid n$, for each $a \in S_i$, there is at most one $b_0 \in S_i$ such that $p \mid ab_0 + n$, and for $b \in S_i \setminus \{b_0, a\}$, $(\frac{ab + 1}{p}) = 1$. Finally, it may happen that $(\frac{a^2 + 1}{p}) = -1$. Thus, assuming $|S_i| > p^\varepsilon$ (so that $|S_i| > 3$ as well) would result in

$$0 < |S_i|(|S_i| - 3) \leq \sum_{a,b \in S_i} \left( \frac{ab + n}{p} \right) = \left| \sum_{a,b \in S_i} \left( \frac{b + na^{-1}}{p} \right) \right| = \left| \sum_{b \in S_i, a \in nS_i^{-1}} \left( \frac{b + a}{p} \right) \right| \leq p^{-\delta(\varepsilon)}|S_i|^2,$$

implying

$$p^\varepsilon < |S_i| \leq \frac{3}{1 - p^{-\delta}},$$

which contradicts (6). Thus, we must have $|S_i| \leq p^\varepsilon$ for $C(\varepsilon) < p \leq Q$ with $p \nmid n$.

We conclude that $|S_p| \leq 1 + 2p^\varepsilon$. Take $\gamma = 2 + C(\varepsilon)^{-\varepsilon}$. Then, $|S_p| < \gamma p^\varepsilon$ for the primes in question. Therefore,

$$\sum_{p \leq Q} \frac{\gamma \log p}{|S_p|} > \sum_{C(\varepsilon) < p \leq Q} \frac{\log p}{p^\varepsilon},$$

$$\geq \sum_{p \leq Q} \frac{\log p}{p^\varepsilon} - \sum_{p \mid \gamma} \frac{\log p}{p^\varepsilon} - \sum_{p \mid n} \frac{\log p}{p^\varepsilon}.$$
\[
\sum_{\substack{p \mid n}} \frac{\log p}{p^\varepsilon} \leq \frac{\pi(1/\varepsilon)}{e\varepsilon} \leq (1 + 11\varepsilon)\varepsilon^{1/\varepsilon-1}.
\]

Otherwise, using (2) we obtain

\[
\sum_{\substack{p \mid n}} \frac{\log p}{p^\varepsilon} \leq \sum_{p \leq p_{\omega(n)}} \frac{\log p}{p^\varepsilon} < \frac{2p^{1-\varepsilon}}{1-\varepsilon}.
\]

Using the inequalities (4) and (5) to estimate the last term, and combining this with the previous estimate above we derive that

\[
\sum_{\substack{p \mid n}} \frac{\log p}{p^\varepsilon} \ll \varepsilon (\log |n|)^{1-\varepsilon}.
\]

Using Lemma 2 again yields

\[
\sum_{p \leq Q} \frac{\gamma \log p}{|S_p|} > Q^{1-\varepsilon} \left(1 - \frac{c_1}{\log^2 Q}\right) - c_2 (\log N)^{1-\varepsilon}
\]

for some positive constants \(c_1\) and \(c_2\) depending on \(\varepsilon\). Since we need the sum on the left larger than \(\gamma \log N\) to be able to use Lemma 5, we choose \(Q = (\lambda^{-1} \log N)^{1/(1-\varepsilon)}\) for some \(\lambda < 1\). Combining the estimates above and using Lemmas 2 and 5 we obtain

\[
|S| \leq \gamma \frac{Q(1 + O(1/\log^2 Q)) - \log N}{1 - \frac{c_1}{\log^2 Q} - c_2 (\log N)^{1-\varepsilon} - \log N} \leq \frac{\gamma (\log N)^{\frac{1}{1-\varepsilon}} (1 + O(1/(\log \log N)^2))}{Q^{1-\varepsilon} (1 - \frac{c_4}{(\log \log N)^2}}.
\]

Choosing \(\lambda = \varepsilon\) minimizes the coefficient above and we obtain

\[
|S| \leq \frac{2C(\varepsilon)\varepsilon + 1}{C(\varepsilon)^{\epsilon}(1 - \varepsilon)} \frac{3^{\frac{1}{1-\varepsilon}}}{\varepsilon} (\log |n|)^{\frac{1}{1-\varepsilon}} \left(1 + O\left(\frac{1}{(\log \log |n|)^2}\right)\right). \quad \square
\]

3. Concluding remarks

The Paley graph conjecture is not only important in graph theory but has important consequences in computer science as explained in [8]. There is some progress towards this conjecture in the emerging field of additive combinatorics as evident in the paper of [7]. However, here, one needs to have some information on the additive structure of our sets \(S\) and \(T\). Our main idea in the paper is to demonstrate the intimate connection between this important conjecture and the problem of Diophantine \(m\)-tuples, which was hitherto unknown.
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