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Fatigue accumulation in dynamic contests

Kerim Keskin [a,*], Çağrı Sağlam [b]

[a] ADA University, Ahmadbay Aghaoglu Street 61, Baku AZ1008, Azerbaijan
[b] Bilkent University, Ankara 06800, Turkey

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ABSTRACT

We study a dynamic contest model where efforts exerted in previous periods accumulate as fatigue. As an individual's fatigue level increases, it becomes more costly to exert one unit of effort in the future. This creates a trade-off between exerting high efforts today to collect winning prizes sooner and exerting low efforts today to gain a cost advantage in the future. We characterize the steady state conditions for open-loop equilibrium and analyze equilibrium efforts in the presence of accumulated fatigue.

1. Introduction

It is well established that fatigue is an important problem for economic agents in various industries and that it mostly has a negative impact on productivity [see 2,3,7,9,18, among others], especially in competitive situations such as sport competitions or job promotions [see, for instance, 10,19,20]. Despite its significance, however, there are only a few numbers of studies that analytically investigate how fatigue affects behavior in strategic environments. Having a variety of application areas, such as sports, warfare, elections, labor market, and firm competition, contest games constitute a fruitful venue for the analysis of how fatigue affects performance in strategic interactions. The analysis would be particularly promising in dynamic contest models, where a player makes multiple decisions through time, so that it is natural to expect future decisions to be influenced by past decisions, due to an accumulated fatigue that may cause a decrease in productivity or an increase in cost. In this paper, we study how equilibrium behavior changes in a dynamic contest game in the presence of accumulated fatigue.

In a contest model, there are multiple players exerting costly and irreversible efforts in order to increase their chances of winning a set of valuable prizes [see 5,16, among others]. Here we consider a two-player infinite-horizon dynamic contest model where the two players compete in a component battle for a common winning prize awarded in each period. Each player exerts some effort, and a contest success function determines who wins the battle. Exerted efforts accumulate as fatigue in the following periods. Fatigue directly affects the marginal cost function in the future, which is modeled in a novel fashion (see Fig. 1 in the next section), so that an increase in fatigue level implies that it will be more costly to exert efforts in future periods. This creates an interesting trade-off. On the one hand, a player can choose to exert high efforts today, increasing his probability of collecting winning prizes in early periods, but decreasing his chances of being successful in later periods. On the other hand, a player can choose to exert low efforts today, saving his energy, so that although it is less likely that he collects prizes in early periods, it becomes more likely that he wins in later periods, since having a low fatigue level would give him a cost advantage in the future. In the current paper, following Grossmann et al. [12] and Keskin and Sağlam [14], we resort to open-loop strategies and analyze open-loop equilibrium. We characterize the steady state conditions and investigate how equilibrium efforts appear in the presence of accumulated fatigue. We conclude with a variety of comparative static results.

Our dynamic contest model has several interpretations. (i) Consider two individuals working in the same firm. In each month, they exert efforts such that one of them is entitled to a premium at the end of the month. (ii) Consider two firms competing in the same market. In each year, they exert efforts such that one of them improves its relative stature in the eyes of the consumers and obtains a larger market share in that year. (iii) Consider two countries fighting a war for the control of a territory that generates valuable resources. In each period, they exert efforts such that one of them earns the right to control the territory and to collect the resources in that period. Note that fatigue is a natural phenomenon for such interpretations in the sense that working overtime, overuse of pro-

* Corresponding author.
E-mail address: kkeskin@ada.edu.az (K. Keskin).

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ductive inputs, or fighting more fiercely may result in performance losses in the future in the respective examples. It is also worth noting that a discount factor regulates the future value of the winning prize in our model. The discount factor can also be interpreted as the probability that the interaction between the two players is terminated before the next period. As such, it can be argued that the current model potentially ends in finite time, but even then, the expected payoffs would be written over an infinite horizon.

The conventional assumption in most of the existing dynamic contest models is that efforts exerted in a period have a one-time effect, which is to influence the outcome of the component battle within the same period [see, for example, 8,15,17]. As opposed to this common way of modeling, there also are studies that consider additional direct and/or indirect effects of exerted efforts. For a selection of such examples, one can refer to the models with learning by doing [4], with talent investments [11], with resource constraints [13], or with fatigue [19]. In the current paper, since we analyze the effects of fatigue in a model where a player's fatigue level accumulates as he exerts efforts throughout the game and the cost from exerting one unit of effort is higher when a player is more tired, we also make a contribution to this strand of literature.

To elaborate, Harbaugh and Klumpp [13] study a four-player elimination tournament where each player is endowed with one unit of effort to allocate across two stages. The authors show that a weak player exerts more effort in the semi-final stage, but a strong player chooses to save his effort to have a better chance of winning the final stage. This can be interpreted as a rather obscure modeling of fatigue in the sense that increasing efforts in a stage decreases the efforts in the next stage. The first attempt for a more apparent modeling of fatigue appears in Ryvkin [19]. The author investigates the effect of fatigue in a two-player finite-horizon dynamic contest model. Each player makes a binary choice between exerting 'low' or 'high' effort in each period. Fatigue is introduced in such a way that a player's probability of winning the next component battle decreases in his own fatigue level and increases in the other player's fatigue level. As such, what is important is the difference between the players' fatigue levels, while the actual fatigue amounts do not have any direct effect on the nature of the strategic interaction. One major finding is that players are more likely to choose 'high' efforts in later periods. The paper also tests the model predictions in a laboratory experiment and reports that subjects behaved in line with the predictions. Later, Sela and Erez [23] also study resource constraints in a dynamic contest, but they regulate the effect of today's effort on tomorrow's resource budget using a fatigue parameter. For a lower value of this parameter, an effort level causes a smaller decrease in the next stage resources. The authors analyze a two-player n-stage contest and show that players' equilibrium efforts are weakly decreasing over the stages if the same prize is awarded in each stage. They also identify the prize distribution that balances players' equilibrium efforts over the stages. More recently, Sela [21,22] utilizes a similar idea in different contest models, namely in elimination tournaments and best-of-k contests. The author shows that the lowest amount of effort is always allocated in the last stage, but the comparison of equilibrium effort levels in earlier stages depends on the fatigue parameter and the contest type. In comparison to the papers summarized above, our paper takes a different approach in modeling the effect of fatigue: it is introduced as a state variable, such that it accumulates throughout the game and affects the marginal cost of exerting effort.

As a final note, upon completion of the current paper, we became aware of a recent working paper by Angelova et al. [1]. The authors study symmetric equilibria in an n-player finite-horizon dynamic contest model in the presence of fatigue accumulation. Although there appear similarities to our paper in how fatigue

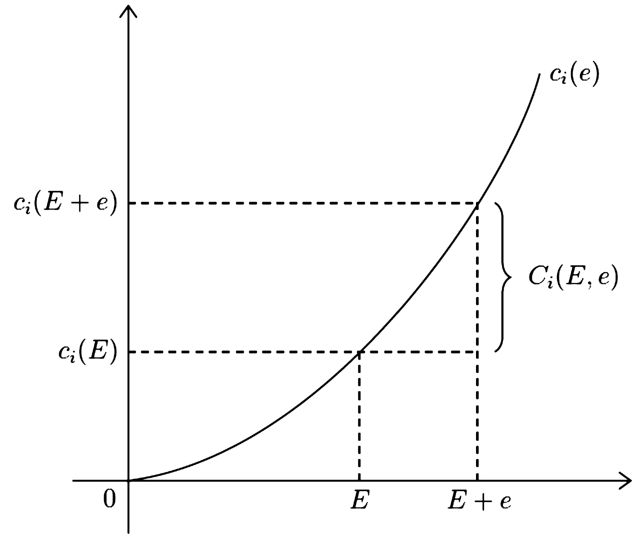


Fig. 1. The illustration of $C_i(E,e)$.

is modeled, there also are a number of differences between our model and their model. Those differences will be discussed in detail in the next section.

The remainder of the paper is organized as follows. In Section 2, we present the details of our model. In Section 3, we characterize the steady state conditions for an open-loop equilibrium and provide a comparative static analysis around the stable steady state. In Section 4, we conclude by further mentioning some numerical results on a finite-horizon version of the model and for a speed of convergence analysis in the original infinite-horizon model.

2. The model

Consider an infinite-horizon dynamic contest model where two players compete in a component battle in each period $t \in \{0, 1, 2, \ldots\}$. Each player $i \in \{1, 2\}$ chooses an effort level $e_{i,t} \in [0, \infty)$, and a contest success function $P_i(e_{1,t}, e_{2,t})$ determines the probability that player i wins the component battle in period t. We assume that P_i is an increasing, twice-differentiable, and concave function of $e_{i,t}$.

The winner of each battle collects a common prize of $V > 0$. Exerting effort is costly. A player gets tired as he exerts efforts, which in turn changes the marginal cost of effort in the next period. More precisely, let $E_{i,0} = 0$ denote player i's initial fatigue level, and define the fatigue level in the beginning of period $t+1$ as

$$E_{i,t+1} = (1 - r_i)\left(E_{i,t} + e_{i,t}\right) \tag{1}$$

where $r_i \in [0, 1]$ denotes the recovery rate for player i. This indicates that the effort exerted by player i directly contributes to his fatigue level, but then a portion of the total fatigue "$E_{i,t} + e_{i,t}$" is recovered right before period $t+1$ starts.

The cost incurred by player i in period t depends on his current fatigue level and the effort exerted in this period: $C_i(E_{i,t}, e_{i,t})$. The intuition is as follows. Consider the function illustrated in Fig. 1. A marginal change in c_i represents the cost incurred from exerting one unit of effort at a given fatigue level. If player i starts with a fatigue of E, his fatigue increases to $E+e$ after exerting e units of effort, leading to a cost of $c_i(E+e)$, but $c_i(E)$ of that cost is not incurred within the current period. As such, we assume that for every $i \in \{1, 2\}$ and every $(E, e) \in [0, \infty)^2$,

$$C_i(E, e) = c_i(E + e) - c_i(E),$$

given an increasing, twice-differentiable, and strictly convex function, c_i.

Finally, everything is common knowledge.

Now, it can be argued that Angelova et al. [1] introduce fatigue into dynamic contests in a similar manner. To our understanding, however, there is one crucial difference in the model's interpretation. According to their model, fatigue is costly and the effort exerted in a period has an indirect cost. For example, having a positive fatigue level of $F > 0$ in the beginning of a period, a player incurs a cost of F^2 when he does not exert any effort in that period, while exerting a positive effort $e > 0$ increases his fatigue level to $F + e$, in which case he incurs a cost of $(F + e)^2$. Thus, the cost is indeed a function of the fatigue level at the end of a period. On the other hand, in the current paper, we assume that exerting effort is costly and that there are cost spillovers via fatigue. As such, fatigue has an indirect effect on costs in our model, or more precisely, the fatigue level only affects the marginal cost of exerting effort in a period. For the sake of comparison, we can further note that our model assigns a zero cost to a player who chooses not to exert any effort in a period independent of his fatigue level. On top of this major difference in the model's interpretation, there are other differences between the two papers: Angelova et al. [1] analyze symmetric equilibria in a finite-horizon game with n symmetric players, whereas we characterize the steady state conditions for an equilibrium in an infinite-horizon game with two asymmetric players.

3. The results

Assuming that a player's effort choices or his accumulated fatigue levels are not observable to his competitor, at least to some degree, we resort to open-loop strategies and analyze open-loop equilibrium. Accordingly, each player chooses an action path in the first period and commits to these strategies throughout the game. Taking a pre-committed action path $\mathbf{e}_j = (e_{j,t})_{t=0}^{\infty}$ for player j as given, player $i \in \{1, 2\}$ determines his best responses aiming to maximize his expected lifetime utility. The associated discrete-time optimal control problem for player i can be written as

$$\max \sum_{t=0}^{\infty} \beta^t \left[P_i(e_{1,t}, e_{2,t})V - C_i(E_{i,t}, e_{i,t})\right]$$

$$\text{subject to} \qquad E_{i,t+1} = (1 - r_i)\left(E_{i,t} + e_{i,t}\right)$$

for $E_{i,0} = 0$ and when $(e_{j,t})_{t=0}^{\infty}$ is given.

We utilize the discrete-time maximum principle for open-loop strategies, as proposed by Corella and Hernández-Lerma [6]. By their Theorem 12, we know that there exists a sequence of multipliers $(\lambda_t)_{t=1}^{\infty} \in \mathbb{R}^{\infty}$ such that for every $t \geq 1$,

$$\frac{\partial H_{i,t}(\cdot)}{\partial E_{i,t}} = \lambda_t; \tag{2}$$

for every $t \geq 0$,

$$\frac{\partial H_{i,t}(\cdot)}{\partial e_{i,t}} = 0; \tag{3}$$

and for every $h \geq 1$,

$$\lim_{t\to\infty} \lambda_t \prod_{s=h}^{t-1} \frac{\partial E_{i,s+1}}{\partial E_{i,s}} = 0, \tag{4}$$

where the discrete Hamiltonian function for player i can be written as

$$H_{i,t}(e_{i,t}, E_{i,t}, \lambda_{t+1}) = \beta^t \left[P_i(e_{1,t}, e_{2,t})V - C_i(E_{i,t}, e_{i,t})\right] + \lambda_{t+1}(1 - r_i)\left(E_{i,t} + e_{i,t}\right).$$

Equations (2)–(4) constitute the set of necessary and sufficient conditions for an optimal solution under certain assumptions. Note from equations (1) and (2)–(3) that λ_t is finite and $\partial E_{i,t+1}/\partial E_{i,t} = 1 - r_i$ for every t, so that all the respective assumptions hold.

The first-order conditions for optimality are

$$\forall t \geq 1: \qquad \frac{\partial H_{i,t}(\cdot)}{\partial E_{i,t}} = -\beta^t \frac{\partial C_i(E_{i,t}, e_{i,t})}{\partial E_{i,t}} + \lambda_{t+1}(1 - r_i) = \lambda_t \tag{5}$$

and

$$\forall t \geq 0: \qquad \frac{\partial H_{i,t}(\cdot)}{\partial e_{i,t}} = \beta^t \left(\frac{\partial P_i(e_{1,t}, e_{2,t})}{\partial e_{i,t}} V - \frac{\partial C_i(E_{i,t}, e_{i,t})}{\partial e_{i,t}}\right) + \lambda_{t+1}(1 - r_i) = 0; \tag{6}$$

while the transversality conditions are given by

$$\forall h \geq 1: \qquad \lim_{t\to\infty} \lambda_t \prod_{s=h}^{t-1} \frac{\partial E_{i,s+1}}{\partial E_{i,s}} = \lim_{t\to\infty} \lambda_t \prod_{s=h}^{t-1} (1 - r_i) = \lim_{t\to\infty} \lambda_t (1 - r_i)^{t-h} = 0. \tag{7}$$

Substituting (6) into (5), we obtain the following Euler equation:

$$\frac{\partial P_i(e_{1,t}, e_{2,t})}{\partial e_{i,t}} V - \frac{\partial C_i(E_{i,t}, e_{i,t})}{\partial e_{i,t}} = \beta(1 - r_i)\left(\frac{\partial P_i(e_{1,t+1}, e_{2,t+1})}{\partial e_{i,t+1}} V - \frac{\partial C_i(E_{i,t+1}, e_{i,t+1})}{\partial e_{i,t+1}} + \frac{\partial C_i(E_{i,t+1}, e_{i,t+1})}{\partial E_{i,t+1}}\right). \tag{8}$$

The steady state is characterized by the requirement

$$E_{i,t+1} = E_{i,t} = E_i \qquad \text{for every} \quad i \in \{1, 2\}$$

imposed into equations (1), (7), and (8). This brings us to our first result.

Proposition 1. *In the dynamic contest game studied here, the steady state is implicitly characterized by*

$$E_i = \frac{1 - r_i}{r_i} e_i \tag{9}$$

and

$$(1 - \beta(1 - r_i))\left(\frac{\partial P_i(e_1, e_2)}{\partial e_i} V - \frac{\partial C_i(E_i, e_i)}{\partial e_i}\right) = \beta(1 - r_i)\frac{\partial C_i(E_i, e_i)}{\partial E_i} \tag{10}$$

for every $i \in \{1, 2\}$.

Proof. See the arguments above. □

To continue our analysis, for every $i \in \{1, 2\}$, we now consider specific functional forms for P_i and C_i functions. Let

$$P_i(e_{1,t}, e_{2,t}) = \frac{e_{i,t}}{e_{1,t} + e_{2,t}}. \tag{11}$$

Moreover, letting $c_i(e) = e^{\alpha}$ for some $\alpha > 1$, we assume that

$$C_i(E_{i,t}, e_{i,t}) = \left(E_{i,t} + e_{i,t}\right)^{\alpha} - {E_{i,t}}^{\alpha}. \tag{12}$$

The next result presents an explicit characterization of steady-state efforts.

Proposition 2. *In the dynamic contest game studied here, if P_i and C_i are as assumed above, the steady state is characterized by*

$$E_i^* = \frac{1-r_i}{r_i} e_i^*$$

and

$$e_i^* = \left(\frac{\left(\frac{\phi_i}{\phi_j}\right)^{\frac{1}{\alpha}}}{\phi_i \left(1 + \left(\frac{\phi_i}{\phi_j}\right)^{\frac{1}{\alpha}}\right)^2} \right)^{\frac{1}{\alpha}},$$

where

$$\phi_i = \frac{\alpha r_i^{1-\alpha}\left(1-\beta(1-r_i)^\alpha\right)}{(1-\beta(1-r_i))V} \tag{13}$$

for every $i \in \{1,2\}$.

Proof. Plugging (11) and (12) into (10), and then using (9), we obtain

$$\begin{aligned}(1-\beta(1-r_i))\left(\frac{e_j}{(e_i+e_j)^2}V - \alpha\left(\frac{e_i}{r_i}\right)^{\alpha-1}\right) \\ = \beta(1-r_i)\alpha\left(\left(\frac{e_i}{r_i}\right)^{\alpha-1} - \left(\frac{1-r_i}{r_i}e_i\right)^{\alpha-1}\right)\end{aligned} \tag{14}$$

for $j \neq i$. A series of algebraic operations return

$$\frac{e_i e_j}{(e_i+e_j)^2} = \frac{\alpha r_i^{1-\alpha}\left(1-\beta(1-r_i)^\alpha\right)}{(1-\beta(1-r_i))V} e_i{}^\alpha.$$

Using the symmetric version of the equation above for player $j \neq i$, and dividing the two equations side-by-side, we obtain

$$e_i^* = \left(\frac{\phi_j}{\phi_i}\right)^{\frac{1}{\alpha}} e_j^* \tag{15}$$

where ϕ_i is given by (13). Substituting (15) into (14), we find

$$e_i^* = \left(\frac{\left(\frac{\phi_i}{\phi_j}\right)^{\frac{1}{\alpha}}}{\phi_i \left(1 + \left(\frac{\phi_i}{\phi_j}\right)^{\frac{1}{\alpha}}\right)^2} \right)^{\frac{1}{\alpha}}. \quad \square$$

In the symmetric case, i.e., when $r_1 = r_2$, the steady-state effort equation reduces to

$$e_1^* = e_2^* = \left(\frac{1}{4\phi}\right)^{\frac{1}{\alpha}} = \left(\frac{(1-\beta(1-r))V}{4\alpha r^{1-\alpha}\left(1-\beta(1-r)^\alpha\right)}\right)^{\frac{1}{\alpha}}. \tag{16}$$

The symmetric efforts indicate that players have the same winning probability in each battle. Further note that if $r_i = r_j = 1$, the fatigue never accumulates for either player, so that the model reduces to a version where the same one-shot contest game is repeated infinitely many times. In the corresponding equilibrium, the equilibrium efforts are

$$\left(\frac{V}{4\alpha}\right)^{\frac{1}{\alpha}},$$

which is greater than e_i^* reported in (16). This implies that the accumulating nature of fatigue discourages players, leading them to exert lower efforts in the steady state.

Under asymmetry, however, one player ends up in an advantaged position in the steady state, exerting a higher steady-state effort, which results in a higher winning probability in each battle after the steady state is reached. Without loss of generality, assume that $r_i > r_j$. Then, using (15), we have $e_i^* > e_j^*$ if $\phi_j > \phi_i$, i.e., if

$$\left(\frac{r_i}{r_j}\right)^{\alpha-1} > \frac{\left(1-\beta(1-r_j)\right)}{(1-\beta(1-r_j)^\alpha)} \frac{\left(1-\beta(1-r_i)^\alpha\right)}{(1-\beta(1-r_i))}.$$

This inequality is always true, since starting from the same value when $r_i = r_j$, the left-hand-side rises faster as r_i increases. Hence, having a higher recovery rate than the other player is an advantage in the steady state.

The stability analysis confirms that the steady state cannot be a "source" and that it is mostly saddle-path stable for a wide range of parameter values. The details of the stability analysis are omitted due to space limitations and are available upon request from the corresponding author.

Given that the steady state is stable, the following comparative static results apply. For the following, we focus on the ratio of steady-state efforts:

$$\epsilon = \frac{e_i^*}{e_j^*} = \left(\frac{\phi_j}{\phi_i}\right)^{\frac{1}{\alpha}}.$$

Taking its derivative with respect to r_i, we find that

$$\beta < \frac{(\alpha-1)(1-r_i)^{1-\alpha}}{\alpha + r_i - 1}$$

is a sufficient condition for ϵ to increase as r_i increases. This condition is always satisfied when $\alpha \geq 2$. This implies that a higher recovery rate has an encouragement effect, so that the player becomes more advantaged in the steady state.

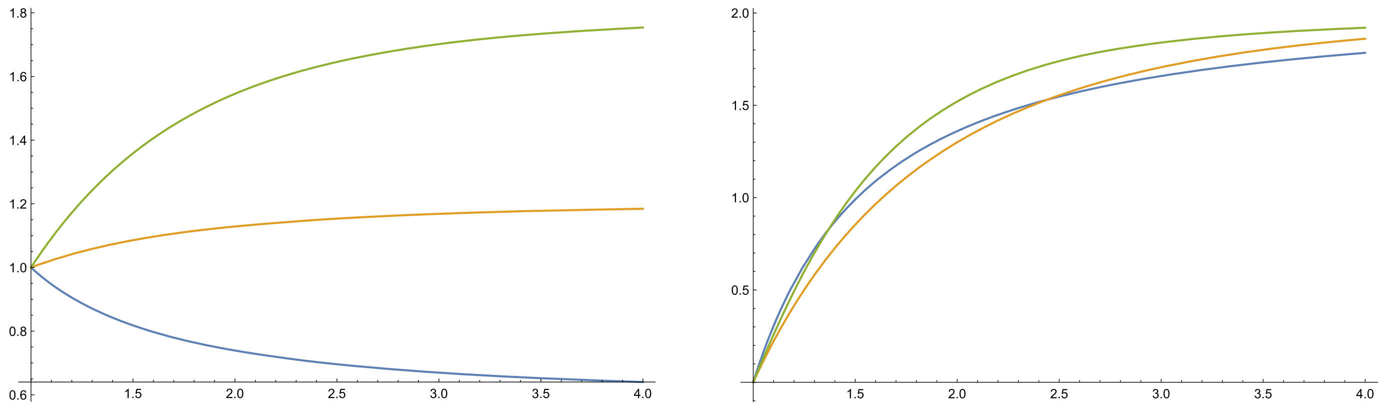
When α increases, a graphical analysis reveals that ϵ increases if $r_i > r_j$ and decreases if $r_i < r_j$ (see Fig. 2(a)). Furthermore, we see that $\partial\epsilon/\partial r_i$ increases as α does (see Fig. 2(b)). The former observation indicates that the cost function's curvature gives an advantage to the faster-recovering player. The latter observation shows that as the cost function's curvature increases, the advantage gained from having a higher recovery rate increases. Finally, as $\alpha \to 1$, the importance of r_i vanishes, since the fatigue's impact on the cost function disappears, as such the model reduces to the version with $r_i = r_j = 1$ in equilibrium.

4. Further remarks

We studied a dynamic contest model where efforts exerted in previous periods accumulate as fatigue. We were interested in the analysis of the trade-off between exerting high efforts today to collect winning prizes sooner and exerting low efforts today to have a potential to be more successful in the future. We characterized the steady state conditions for open-loop equilibrium in an infinite-horizon model with two asymmetric players. We presented the extent to which having a higher fatigue recovery rate turns out to be an advantage in the steady state and how that advantage is influenced by the curvature of the effort cost function.

In our concluding remarks, we aim to give further insights on the equilibrium dynamics inherent in our model. In that regard, we first provide the analysis of the speed of convergence to the steady state.

Following Stokey and Lucas [24, p. 147–153], the speed of convergence is defined as $1-|\bar{\lambda}|$ where $\bar{\lambda}$ is the largest characteristic root of the respective Jacobian matrix that is less than one



(a) Changes in ϵ (b) Changes in $\partial\epsilon/\partial r_i$

Fig. 2. The illustrations of how ϵ and $\partial\epsilon/\partial r_i$ change in response to a change in α [$r_j = 0.5$, $\beta = 0.95$, and $r_i = 0.3$ (Blue), $r_i = 0.6$ (Orange), $r_i = 0.9$ (Green)]. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

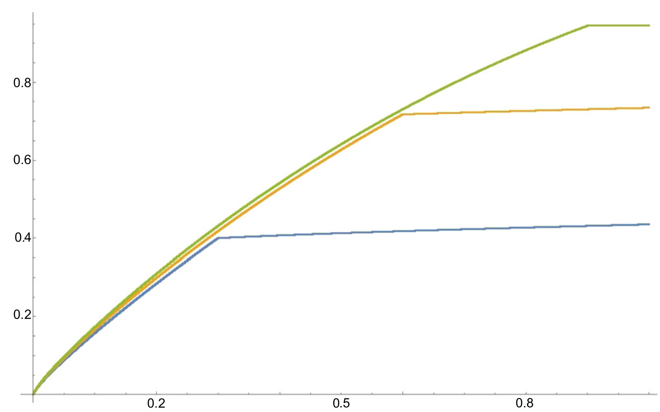


Fig. 3. The speed of convergence to the steady state as a function of r_i [$\alpha = 2$, $\beta = 0.95$, and $r_j = 0.3$ (Blue), $r_j = 0.6$ (Orange), $r_j = 0.9$ (Green)].

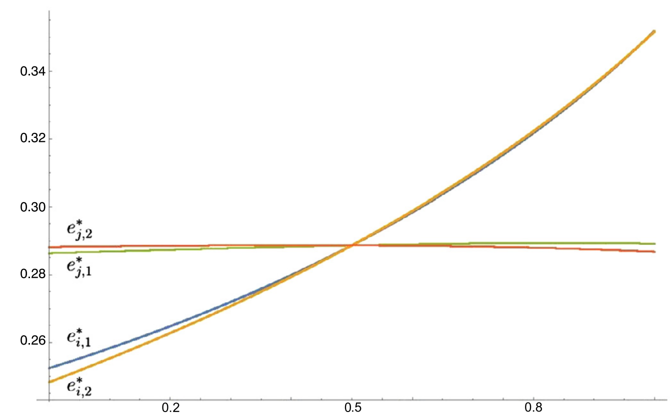


Fig. 4. The equilibrium efforts $\left(e^*_{i,1}, e^*_{j,1}\right)$ and $\left(e^*_{i,2}, e^*_{j,2}\right)$ as a function of r_i.

in absolute value. This indicates that the closer the largest stable eigenvalue of the linearized system is to zero, the faster a player's equilibrium effort and fatigue levels converge to their steady state values. Under the assumption that $\alpha = 2$, $\beta = 0.95$, and $V = 1$, Fig. 3 considers three different values for r_j and illustrates the speed of convergence as a function of r_i. It can be observed that as r_i or r_j increases, i.e., as either player has a higher recovery rate, which means that there is less fatigue effect for the player, the equilibrium strategies tend to converge to their steady state values faster. The figure also reveals a point where there is a substantial change in the evolution of the speed of convergence with respect to r_i. This happens when $r_i = r_j$. The respective observation is that when $r_i < r_j$, i.e., when player i is in a disadvantaged position in terms of recovery rate, an increase in his recovery rate increases the speed of convergence to a great extent; whereas when $r_i > r_j$, i.e., when it is player i who has the recovery rate advantage, an increase in his recovery rate slightly increases the speed of convergence.

Given the tractability problems encountered in the identification of the equilibrium efforts before the steady state is reached, we now turn our attention to a finite-horizon model, aiming to give additional insights on how equilibrium efforts evolve from one period to another. Accordingly, we consider a two-period version of our model. All model assumptions are preserved except that we now have $t \in \{1, 2\}$ and $\beta = 1$. For simplicity, we also assume that $V = 1$ and $\alpha = 2$. Analyzing subgame perfect Nash equilibrium, in the second period, player $i \in \{1, 2\}$ maximizes

$$\frac{e_{i,2}}{e_{1,2}+e_{2,2}} - \left(E_{i,2}+e_{i,2}\right)^2 + E_{i,2}{}^2, \tag{17}$$

where $E_{i,2} = (1-r_i)\left(e_{i,1}\right)$ is player i's fatigue level at the beginning of period 2. It can be calculated that the equilibrium effort level is

$$e^*_{i,2} = \frac{1+E_{i,2}(E_{1,2}+E_{2,2})}{2\sqrt{2+\left(E_{1,2}+E_{2,2}\right)^2}} - \frac{E_{i,2}}{2}$$

for each player. Writing $e^*_{1,2}$ and $e^*_{2,2}$ into (17) returns player i's equilibrium expected payoff in period 2:

$$EU^*_{i,2} = \frac{1}{4}\left(2+4E_{i,2}{}^2 + \frac{2\left(E_{j,2}-E_{i,2}\right)}{\sqrt{2+\left(E_{1,2}+E_{2,2}\right)^2}} - \left(E_{i,2} + \frac{1+E_{i,2}\left(E_{1,2}+E_{2,2}\right)}{\sqrt{2+\left(E_{1,2}+E_{2,2}\right)^2}}\right)^2\right).$$

Then, in the first period, player $i \in \{1, 2\}$ maximizes

$$\frac{e_{i,1}}{e_{1,1}+e_{2,1}} - e_{i,1}{}^2 + EU^*_{i,2}.$$

Given that $E_{i,2}$ is a function of $e_{i,1}$, the rest of the equilibrium analysis turns out to be intractable, mostly due to the asymmetry in players' recovery rates. At this point, we continue with a numerical analysis. Assuming that $r_j = 0.5$, we let r_i change between 0 and 1. The respective equilibrium efforts are illustrated in Fig. 4.

It can be seen that when $r_i > r_j$, we have $e^*_{i,1} > e^*_{j,1}$ and $e^*_{i,2} > e^*_{j,2}$, so that a player with a higher recovery rate has an advantage in the equilibrium, similar to what is observed in the steady state of the infinite-horizon version. An increase in r_i motivates both players, but it motivates player i more, which gives him an edge in each battle. Further note that for sufficiently high values of r_i, a decrease in $e^*_{j,1}$ is observed. This signals that an increase in r_i may have a discouragement effect for the disadvantaged player j, if there is a sufficiently large difference between their recovery rates.

As for the equilibrium efforts across periods, we see that a player's effort in the second period is not further apart from his effort in the first period. However, the direction of change is different for the two players. For instance, when $r_i < r_j$, player i has a relatively lower recovery rate, and his second period effort is lower than his first period effort, whereas player j exerts a higher effort in the second period compared to his own first period effort. This can be interpreted as follows: Knowing the recovery rates for both players, player i is aware that he would be in a disadvantaged position in the second period. This is because his fatigue level will be higher than his opponent's in most cases, which is even possible when he exerts less effort than what player j exerts in the first period. Anticipating this, player i is more motivated to exert effort in the first period, where both players have the same marginal costs since the fatigue effect is not present. Conversely, it can be argued that player j chooses to save his energy in the first period, so that he will have a better chance of winning the second period battle.

In earlier works, we are not aware of any paper reporting results on asymmetric players except for [13]. These authors find that a weak player exerts more effort in the first stage, but a strong player chooses to save his effort to have a better chance of winning the second stage game. This is similar to what we observe in equilibrium efforts across periods, although in the first period of our model, the equilibrium effort exerted by the advantaged player is greater than what is exerted by the disadvantaged player. It seems that this difference is caused by how efforts are modeled: we assume costly efforts with no boundary conditions, whereas they assume effort budgets such that if a player increases his effort in a stage, his effort in the other stage should decrease.

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