NASH BARGAINING SOLUTION UNDER PREDONATION

A Master's Thesis

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July 2008

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The Institute of Economics and Social Sciences
of
Bilkent University

by

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In Partial Fulfillment of the Requirements For the Degree
of
MASTER OF ARTS

in

THE DEPARTMENT OF
ECONOMICS
BILKENT UNIVERSITY
ANKARA

July 2008

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

NASH BARGAINING SOLUTION UNDER PREDONATION

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We consider two person bargaining problems under predonation. Before the bargaining solution is applied we allow the alteration of the bargaining set by means of pre-donations of a share in one's would-be payoffs to one's opponent. Thus, a pre-bargain stage is instituted in which the bargainers may manipulate, via pre-donations, the (Nash) bargaining solution as applied in the next stage.We firstly concentrate on the simple bargaining problems with bargaining sets that have linear pareto frontier and show that the stronger bargainer (with greater ideal payoff) giving a pre-donation, her best pre-donation transforming the bargaining set into one on which the Nash bargaining solution distributes payoffs so that while other bargainer gets exactly the same payoff (as applied to the original simple bargaining problem), stronger bargainer makes strictly better off. Then, we look for Stackelberg and Nash equilibria of the so called "predonation game". Furthermore, we list our results for two by two normal form games.

Keywords: Bargaining, Nash Bargaining Solution,Predonation .

ÖZET

ÖNDEN BAĞIŞ ALTINDA NASH PAZARLIK ÇÖZÜMÜ

AKYOL, Ethem
Yüksek Lisans, Ekonomi Bölümü
Tez Yöneticisi: Prof. Semih Koray

Temmuz 2008

Önden bağış altında iki kişilik pazarlık problemlerini düşünüyoruz. Pazarlık çözümü uygulanmadan önce, pazarlık kümesinin birinin gelecekteki faydasının diğerine önden bağış yöntemiyle değişmesine izin veriyoruz. Dolayısıyla,pazarlıkçıların önden bağış yöntemiyle bir sonraki aşamada uygulanan (Nash) pazarlık çözümünü değiştirebileceği ön-pazarlık aşaması kuruluyor. Öncelikle doğrusal pareto cephesine sahip basit pazarlık kümeleri üzerinde yoğunlaşıyoruz ve güçlü oyuncu (daha yüksek ideal noktaya sahip) önden bağış verip en iyi önden bağışı pazarlık kümesini Nash pazarlık çözümü faydaları diğer pazarlıkçı aynı faydayı alırken (asıl basit pazarlık problemine uygulandığıyla) güçlü oyuncunun daha iyi duruma gelecek şekilde dağıtıyor. Daha sonra, "önden bağış oyunun" Stackelberg ve Nash dengelerini araştırıyoruz. Bununla beraber, ikiye iki normal form oyunlar için sonuçlarımızı listeliyoruz.

Anahtar Kelimeler: Pazarlık, Nash Pazarlık Çözümü, Önden Bağış.

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to Prof. Semih Koray for his guidance during this study.

I thank Prof. Tarık Kara and Ali Sinan Sertöz for participating in my thesis committee.

I am grateful to TÜBİTAK for their financial support during my graduate studies.

Of course, my greatest thanks go to my family: to my father, to my mother, to my brother Erhan and to my beloved (future) wife Pelin for their endless love and support.

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CHAPTER 1

INTRODUCTION

One of the simplest yet most fruitful paradigms in cooperative game theory is the "bargaining problem", in which a group of two or more participants is faced with a set of feasible outcomes, any of which will be the result if it is specified by the unanimous agreement of all the participants. If there is no unanimous agreement, a predetermined disagreement outcome (or, sometimes called status-quo outcome) will be the result. If there are feasible outcomes which all the participants prefer to the disagreement outcome, then there is an incentive to reach an agreement; however,so long as at least two of the participants differ over which outcome is the most preferable, there is a need for bargaining and negotiation over which outcome should be chosen.

We will look at history of bargaining problems. We will follow Roberto Serrano's study which can be found in The New Palgrave Dictionary of Economics (see Bibliography) here in brief. Before the adoption of game theoretic techniques, bargaining problems (also called bilateral monopolies at the time) were deemed indeterminate by economics. This was certainly the position stated by important economic theorists, including Edgeworth (1881) and Hicks (1932). More specifically, it was believed that the solution to a bargaining problem must satisfy both individual rationality and collective rationality properties: the former means that neither party should end up worse than at the status-quo and the latter refers to Pareto efficiency. Typically, the set of individually rational and Pareto efficient agreements is very large in a bargaining problem, and these theorists were inclined to think that theoretical arguments could not go further than this in getting a prediction. To be able to obtain such a prediction, one would have to rely on extra-economic variables, such as the bargaining power and abilities of either party, the psychological state of mind in negotiations, the religious beliefs of each party, the weather, and so on. A precursor to the game theoretic study of bargaining, at least in the attempt to provide a more determinate prediction, is the analysis of Zeuthen (1930). This Danish economist formulated a principle by which the solution to a bargaining problem be dictated by the two parties' risk attitudes (given the probability of breakdown of negotiations following the adoption of a tough position at the bargaining table).

Nash (1950, 1953) are two seminal papers that constitute the birth of the axiomatic theory of bargaining. Two assumptions are central in Nash's theory. First, bargainers are assumed to be fully rational individuals, and the theory is intended to yield predictions based exclusively on data relevant to them. Second, a bargaining problem is represented as a pair (S, d) where S is a compact and convex subset of $\mathbb{R}^2$, the feasible set of utilities and $d \in S$ is the disagreement utility point. Compactness follows from standard assumptions such as closed productions sets and bounded factor endowments, and convexity is obtained if one uses expected utility and lotteries over outcomes are allowed. Also, the set S must include points that dominate the disagreement point, i.e., there is a positive surplus to be enjoyed if agreement is reached and the question is how this surplus should be divided. As in most of game theory, by utility we mean von Neumann-Morgenstern expected utility; there may be underlying uncertainty, perhaps related to the probability of breakdown of negotiations.

With this second assumption, Nash is implying that all relevant information to the solution of the problem must be subsumed in the pair (S, d). In other words, two bargaining situations that may include distinct details ought to be solved the same way if both reduce to the same pair (S, d) in utility terms. In spite of this, it is sometimes convenient to distinguish between feasible utility pairs (points in S) from feasible outcomes in physical terms (such as the splits of a pie, to be created after agreement).

There, Nash introduced an idealized representation of the two-person bargaining problem and developed a methodology that gave hope to resolve undeterminateness of the terms of the bargaining that had been noted by Edgeworth (1881). After Nash's works there have been many applauds but his studies received also many criticsm. Many economists argue the fairness of Nash Bargaining solution and many other bargaining solutions have been proposed. This is due to the fact that no solution can be universally acceptable. The question which solution is better or fair is a question that can not be answered the same way by all people. This led many different bargaining solutions and studies on these solutions.

One of these studies is due to pioneering work of Murat Sertel on manipulability of well known bargaining solutions via predonations. He argued that no legal obstacle can stop agents from signing contracts under which they would be better off, thus far-sighted players, having the chance to change the bargaining set at hand by "predonating" share of their would-be payoffs, would reach to a better point. He studied on simple bargaining problems with disagreement point at the origin. His joint work with B.Zeki Orbay showed that Nash Bargaining solution is manipulable when applied to simple bargaining sets. Afterwards, Orbay(2000) showed that Kalai-Smorodinsky and Maschler-Perles Solutions give exactly the same solution under pre-donation when applied to simple bargaining problems. In addition to these studies, Akin(2001) showed that Kalai-Smorodinsky solution can be manipulated via predonation and the solution coincides with the concessionary division rule. The common point in these studies is that it is assumed that only one player makes predonation.

In this study we will examine predonation firstly on simple bargaining problems as in the previous works. Contrary to previous works, we will consider the cases where both player can make predonation. This will constitute Chapter 3. Then, in Chapter 4, we will investigate predonation going beyond simple bargaining problems. We will consider two by two normal form games with certain assumptions and our aim will be to determine whether players can make better off via predonation in these games. Final part will be the conclusion. Before these, in Chapter 2, we will introduce the formal treatment of bargaining and basic terminology that will be used throughout.

CHAPTER 2

PRELIMINARIES

2.1 Formal Treatment of Bargaining

In this section, we will introduce the formal characterization of bargaining problems and give basic terminology to the reader.

Definition. We define a two person bargaining problem by a pair (S, d) where S is a compact, convex subset of $\mathbb{R}^2$, and $d = (d_1, d_2) \in S$.

Remark 1. It is generally required that there is an element $s = (s_1, s_2) \in S$ such that $s_i > d_i$ to make sure that players have an incentive to bargain and we will assume this throughout.

Definition. Let B be the class of all bargaining problems. We mean by a bargaining solution a function that assigns a unique member of S to every bargaining problem $(S, d) \in B$.

2.1.1 Nash's Axiomatic Characterization

Nash, in his seminal work, listed the properties, or axioms, that he thought the solution should satisfy, and he established that there is a unique bargaining solution that satisfies these axioms. Now, we will list some axioms or properties that will be used throughout and state the Nash's theorem and its proof.

- **IAT (Independence of affine transformations):** The bargaining solution f is independent of affine transformations, if for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and for any $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$, where $\beta_i > 0$ for all $i \in \{1, 2\}$, we have

$$f_i(S^{'}, d^{'}) = \beta_i f(S, d) + \alpha_i$$

, where $S' = \{(\beta_1 s_1 + \alpha_1, \beta_2 s_2 + \alpha_2) : (s_1, s_2) \in S\}$

If we accept preferences, not utilities, as basic, then the two bargaining problems (S, d) and $(S^{'}, d^{'})$ represent the same situation. If the utility functions $u_i \in \{1, 2\}$ generate the set S when applied to agreement set, say A, then $u_i^{'} = \beta_i u_i + \alpha_i$ generate the set $S^{'}$ when applied to the same agreement set A.Thus, the outcome predicted by the bargaining solution should be same for (S, d) as for $(S^{'}, d^{'})$. Thus, the utility outcomes should be related in such a way that $f_i(S^{'}, d^{'}) = \beta_i f_i(S, d) + \alpha_i$ for $i \in \{1, 2\}$. In brief, the axiom requires that the utility outcome of bargaining co-vary with the representation of preferences, so that any physical outcome that corresponds to the solution of the problem (S, d) also corresponds to the solution of $(S^{'}, d^{'})$.

- **PAR (Pareto Optimality):** A bargaining solution f satisfies PAR, if for any S, there is no $s \in S$ such that $s_i > f_i(S, d)$ for every i.

This requires that the players never agree on an outcome s when there is an available outcome in which they are both better off. If they agreed on the inferior outcome s, then there would be room for "renegotiation": they could continue bargaining, the pair of utilities in the event of disagreement point being s.

-**IIA(Independence of Irrelevant Alternatives):** A bargaining solution f satisfies IIA if, whenever $S^{'} \subset S$ and $f(S, d) \in S^{'}$,then $f(S, d) = f(S^{'}, d)$.

In other words, suppose that when all the alternatives in S are available, the players agree on an outcome s in the smaller set $S^{'}$. Then we require that the players agree on the same outcome s when only the alternatives in $S^{'}$ are available. The idea is that in agreeing on s when they could have chosen any point in S, the players have discarded as "irrelevant" all the outcomes in S other than s. Consequently, when they are restricted to the smaller set $S^{'}$ they should also agree on s: the solution should not depend on "irrelevant" alternatives.

Note that this axiom makes the bargaining problems (S, d) and (S', d) where $S' = \{(s_1, s_2) \in S : (d_1, d_2) \leq (s_1, s_2)\}$ same.

- **SYM(Symmetry):** We say that a bargaining problem (S, d) is symmetric if the following two properties hold:

i) If $(s_1, s_2) \in S$,then we have $(s_2, s_1) \in S$.

ii) $d_1 = d_2$.

The bargaining solution f satisfies SYM if $f_1(S, d) = f_2(S, d)$ for any symmetric bargaining problem (S, d).

With Nash' s own words, this axiom states equality of bargaining skills of bargainers. If the positions of the two bargainers completely symmetric, then the solution should treat them symmetrically.

- **IR(Individual Rationality):** A bargaining solution f satisfies IR, if $f(S, d) \geqslant d$ for every bargaining problem (S, d).

This property is implicit in our treatment of bargaining. Many modern treatments of the subject explicitly include this property.

Now, we are ready to state Nash's theorem and its proof (We will follow the proof of this theorem from Osborne,M.J. and Rubinstein,A., A course in game theory (MIT Press, 1994)):

Theorem 1. *There is a unique bargaining solution $f : B \to S$ satisfying the axioms IAT,PAR,IIA and SYM, and it is given by*

$$f^N(S,d) = \underset{(d_1,d_2)\leq(s_1,s_2)\in S}{\arg\max} (s_1-d_1).(s_2-d_2)$$

Proof. Firstly, we will show that f^N is a well defined bargaining solution. The set $\{s \in S : s \geqslant d\}$ is a compact set and the function H defined by

$$H(s_1,s_2) = (s_1-d_1).(s_2-d_2)$$

is continuous, hence there is a solution to the maximization problem defining f^N. Furthermore, H is strictly quasi-concave on $\{s \in S : s > d\}$, and there exist $s \in S$ such that $s > d$, and S is convex, the maximizer is unique. Next, let's check that f^N satisfies the stated axioms:

IAT: If (S,d) and $(S^{'},d^{'})$ are as in the statement of the axiom, then $s^{'} \in S^{'}$ if and only if there is $s \in S$ such that $s_i^{'} = \alpha_i + \beta_i s_i$ for $i = 1,2. Now,$

$$(s_1^{'}-d_1^{'})(s_2^{'}-d_2^{'}) = \beta_1\beta_2(s_1-d_1)(s_2-d_2)$$

Thus, (s_1^*,s_2^*) maximizes $(s_1-d_1)(s_2-d_2)$ over S iff $(\alpha_1+\beta_1 s_1^*, \alpha_2+\beta_2 s_2^*)$ maximizes $(s_1^{'}-d_1^{'})(s_2^{'}-d_2^{'})$ over $S^{'}$.

PAR: Since H is increasing in each of its arguments, s does not maximize H over S if there exists $t \in S$ with $t_i > s_i$ for i = 1, 2.

IIA: If $S^{'} \subset S \ \ and\ s^* \in S^{'}$ maximizes H over S, then s^* also maximizes H over S$^{'}$.

SYM: If (S,d) is symmetric and (s_1^*,s_2^*) maximizes H over S, then, since H is a symmetric function, (s_2^*,s_1^*) also maximizes H over S. Since the maximizer is unique, we have $s_1^* = s_2^*$.

Final part will be the uniquess part. Suppose that f is a bargaining solution that satisfies the four axioms. We will show that $f = f^N$, that is $f(S,d) = f^N(S,d)$ for any bargaining problem (S,d). Let $f^N(S,d) = z$. Since there exist $s_i > d_i$ for $i = 1,2$, we have z$_i > d_i$ for $i = 1,2. Let\ (S^{'},d^{'})$ be a bargaining problem that is obtained from (S, d) by the transformation $s_i \rightarrow \beta_i s_i + \alpha_i$, where $\alpha_i = \frac{-d_i}{2(z_i - d_i)}$ and $\beta_i = \frac{1}{2(z_i - d_i)}$ which moves the disagreement point to the origin and the solution to $(\frac{1}{2}, \frac{1}{2})$. Since both f and f^N satisfies IAT, we have for any i=1,2

$$f_i(S', 0) = \beta_i f(S, d) + \alpha_i$$

and

$$f_i^N(S', 0) = \beta_i f^N(S, d) + \alpha_i$$

Hence, $f(S, d) = f^N(S, d)$ iff $f(S', 0) = f^N(S', 0)$. Since $f^N(S', 0) = (\frac{1}{2}, \frac{1}{2})$, it remains to show that $f(S', 0) = (\frac{1}{2}, \frac{1}{2})$.

We claim that S' contains no points (s_1', s_2') such that $s_1' + s_2' > 1$. Suppose not, i.e. let $(s_1', s_2') \in S'$ such that $s_1' + s_2' > 1$, then let $(t_1, t_2) = (\frac{1-\epsilon}{2} + \epsilon s_1', \frac{1-\epsilon}{2} + \epsilon s_2')$, where $0 < \epsilon < 1$. Since S' is convex, the point (t_1, t_2) is in S'; but for small enough ϵ, we have $t_1 t_2 > \frac{1}{4}$, contradicting the fact that $f^N(S', 0) = (\frac{1}{2}, \frac{1}{2})$.

Since S' is bounded, then we can find a rectangle T about the 45^o line that contains S', on the boundary of which is $(\frac{1}{2}, \frac{1}{2})$. Now, by PAR and SYM, we have $f(T, 0) = (\frac{1}{2}, \frac{1}{2})$. By IIA, $f(S', 0) = f(T, 0)$, so that $f(S', 0) = (\frac{1}{2}, \frac{1}{2})$, completing the proof. □

2.1.2 Other Bargaining Solutions

After Nash's work, many other bargaining solutions were proposed. We will present some examples here:

1. **The Kalai-Smorodinsky Solution:** $f^K(S, d)$ is the maximal point of S on the segment connecting d to $a(S, d)$ where

$$a_i(S, d) = \max\{x_i : x \in S, x \geqslant d\} \text{ for all } i \in \{1, 2\}$$

2. **Utilitarian Solution:** $f^U(S,d)$ is a maximizer in $s \in S$ of $\sum s_i$.

3. **Egalitarian Solution:** $f^E(S,d)$ is the maximal point of S of equal coordinates:

$$f_1^E(S,d) - d_1 = f_2^E(S,d) - d_2$$

4. **Asymmetric Nash Solution:**

$$f^{AN}(S,d) = \underset{(d_1,d_2) \leq (s_1,s_2) \in S}{\arg\max} (s_1 - d_1)^{\alpha}.(s_2 - d_2)^{\beta}$$

,where $\alpha, \beta > 0$

5. **Maschler-Perles Solution:** $f^{MP}(S,d)$ is the point p which satisfies

$$\int_k^p \sqrt{-ds_1 ds_2} = \int_p^t \sqrt{-ds_1 ds_2}$$

where the line integrals are taken on the pareto frontier of S and $k = (d_1, a_2(S,d))$ and $t = (a_1(S,d), d_2)$, where

$$a_i(S,d) = \max\{x_i : x \in S, x \geqslant d\} \text{ for all } i \in \{1,2\}$$

2.2 Normal Form Games

Definition. We call an ordered triple $g = (N, X, u)$ an N normal form game, where N is a nonempty set (set of players), $X = \prod_{i \in N} X_i$ where X_i is a nonempty set for each $i \in N$ and $u = (u_i)_{i \in N}$ with $u_i : X \to \mathbb{R}$, a function for each $i \in N$. We will denote a two person normal form game by $g = (X, Y, u_1, u_2)$ where $X = X_1$ and $Y = X_2$.

Definition. Given a game $g = (X, Y, u_1, u_2), \gamma \in X$ x Y is called a Nash

Equilibrium if

$$\text{For each } i \in \{1,2\} \text{ , for each } x_1 \in X, x_2 \in Y,\ u_i(\gamma) \geqslant u_i(x_i, \gamma_{-i})$$

Definition. Let $g = (X, Y, u_1, u_2)$ be a finite two person normal form game with $|X| = m, |Y| = n$, represent (u_1, u_2) by a bimatrix

$$(A,B) = \begin{bmatrix} (a_{11}, b_{11}) & . & . & (a_{1n}, b_{1n}) \\ . & . & . & . \\ . & . & . & . \\ (a_{m1}, b_{m1}) & . & . & (a_{mn}, b_{mn}) \end{bmatrix}$$

We define the mixed extension $g^{'} = (X^{'}, Y^{'}, u_1^{'}, u_2^{'})$ by

$$\begin{aligned} X^{'} &= \{x \in \mathbb{R}^m_+ : \sum_{i=1}^{m} x_i = 1\} \\ Y^{'} &= \{y \in \mathbb{R}^n_+ : \sum_{i=1}^{n} y_i = 1\} \end{aligned}$$

and for any $x \in X^{'}, y \in Y^{'}$

$$\begin{aligned} u_1^{'} &= xAy^t \\ u_2^{'} &= xBy^t \end{aligned}$$

Theorem 2. *Let $g = (X, Y, u_1, u_2)$ be a finite two person normal form game with a bimatrix (A, B).Let $g^{'} = (X^{'}, Y^{'}, u_1^{'}, u_2^{'})$ stand for the mixed extension of g. Now, $g^{'}$ has a Nash equilibrium.*

Proof. Any game theory book can be consulted for the proof. □

CHAPTER 3

SIMPLE BARGAINING PROBLEMS UNDER PREDONATION

3.1 Unilateral Predonation

Sertel (1991) formally defined a simple bargaining problem as (S_a, d) where

$$S_a = \{(u_1, u_2) \in \mathbb{R}^2_+ : u_1 \leqslant a(1 - u_2)\}$$

with $d = (0, 0)$ and $0 < a < 1$. Note that the ideal(maximal) payoff of Player 1 is a, and the ideal payoff of Player 2 is 1. We will sometimes call Player 2 richer or stronger player due to this fact.

Following Sertel(1991), we can interpret this problem as a property division problem as follows. We assume that there is a certain item property which can be monetarily valued. Claimants have different valuations on the property. The problem distributes 1, the highest claimed value, among the claimants. If the property is divisible, we can distribute it to agents. If not, we can give it to one of the claimants and require her to monetarily compensate the others.

In our model, it is assumed that there is no legal prohibition for monetary transfers but transfers are not considered as part of the bargaining stage but are confined to the pre-bargaining stage, where a player gives a share of her future payoffs to the other party before they bargain, which we call "predonation" .

Before going further, let us explain formally what we mean by predonation:

Definition. Given a bargaining problem (S, d), by a predonation from agent i to agent j we mean any function $\lambda_i : S \to \mathbb{R}^2$, parametrized by some $\lambda_i \in [0,1]$, which transforms each $(u_i, u_j) \in S$ into $\lambda_i(u_i, u_j) = ((1-\lambda_i)u_i, u_j + \lambda_i u_i)$.

After defining what we mean by predonation, let' s check whether predonation is beneficial for bargainers in simple bargaining problem:

Now, simple calculations show that

$$f^N(S_a, d) = (\frac{a}{2}, \frac{1}{2})$$

We will firstly determine whether players can make better off by unilateral predonations:

Consider firstly the case where Player 2 makes predonation, i.e. she donates a portion of her would-be payoff to player 1, say $\lambda \in [0,1]$ of her payoff. That means we make the following transformation:

$$(u_1, u_2) \to (u_1 + \lambda u_2, (1-\lambda)u_2)$$

Thus, given λ, our new bargaining set is:

$$S'_a(\lambda) = \{(u_1^{'}, u_2^{'}) \in \mathbb{R}^2 : (u_1^{'}, u_2^{'}) = (u_1 + \lambda u_2, (1-\lambda)u_2) \text{ for some } (u_1, u_2) \in S\}$$

Note that disagreement point $(0,0)$ is transformed to $(0,0)$.

Now, Player 2 will choose optimal λ that makes her payoff maximal when Nash Bargaining solution is applied to $(S'_a(\lambda), (0,0))$.Calculations show that (see Appendix A) $\lambda_2^* = \frac{a}{2}$, thus yielding a solution

$$(\frac{a}{2}, 1 - \frac{a}{2})$$

Note that while Player 1 gets exactly the same payoff as he gets before predonation, Player 2 becomes strictly better off.

Here, very obvious question to arise is whether this Nash Bargaining solution is a solution of some well known bargaining solutions of the original (before predonation) bargaining problem. The difficulty here arises since we can not guarantee that new solution stays in non-predonated bargaining set. However, it is obvious that the new solution will stay in $S' = \{(u_1, u_2) \in \mathbb{R}_+^2 : u_1 + u_2 \leqslant 1\}$. This bargaining set has another feature: It is the bargaining set when we assume existence of transferable utility. Now, it is very easy to see that asymmetric Nash Bargaining solution with weights $\frac{a}{2}$ and $1 - \frac{a}{2}$ gives the result $(\frac{a}{2}, 1 - \frac{a}{2})$ when applied to the bargaining problem $(S', 0)$. Furthermore, we have the following result :

Theorem 3. *Let* $A = \{(u_1, u_2) \in \mathbb{R}_+^2 : u_1 + u_2 \leqslant 1\}$. *For any* $\frac{a}{2} \in (0, 1)$, *there is a unique bargaining solution* $f^{\frac{a}{2}}$ *that satisfies IAT, IIA, IR, and* $f^{\frac{a}{2}}(A, 0) = (\frac{a}{2}, 1 - \frac{a}{2})$ *and it is:*

$$f^{\frac{a}{2}}(S, d) = \underset{(d_1,d_2) \leq (s_1,s_2) \in S}{\arg\max} (s_1 - d_1)^{\frac{a}{2}} . (s_2 - d_2)^{1-\frac{a}{2}}$$

Proof. The proof is very similar to proof of Theorem 1. It is easy to show that $f^{\frac{a}{2}}$ satisfies the given axioms and $f^{\frac{a}{2}}(A, 0) = (\frac{a}{2}, 1 - \frac{a}{2})$. Assume that f is a bargaining solution that satisfies the four axioms. We will show that $f = f^N$, that is $f(S, d) = f^N(S, d)$ for any bargaining problem (S, d). Let $f^N(S, d) = z$. Since there exist $s_i > d_i$ for $i = 1, 2$, we have $\mathrm{z}_i > d_i$ for $i = 1, 2$.*Let* (S', d') be a bargaining problem that is obtained from (S, d) by the transformation $s_i \rightarrow \beta_i s_i + \alpha_i$, where $\alpha_1 = \frac{-ad_1}{2(z_1 - d_1)}$, $\beta_1 = \frac{a}{2(z_1 - d_1)}$ and $\alpha_2 = \frac{-(1-\frac{a}{2})d_2}{2(z_2 - d_2)}$, $\beta_2 = \frac{(1-\frac{a}{2})}{(z_2 - d_2)}$which moves the disagreement point to the origin and the solution to $(\frac{a}{2}, 1-\frac{a}{2})$. Since both f and f^N satisfies IAT, we have

$$f_i(S^{'}, 0) = \beta_i f(S, d) + \alpha_i \text{ for } i = 1, 2$$

and

$$f_i^N(S^{'}, 0) = \beta_i f^N(S, d) + \alpha_i \text{ for } i = 1, 2$$

Hence, $f(S, d) = f^{\frac{a}{2}}(S, d)$ iff $f(S^{'}, 0) = f^{\frac{a}{2}}(S^{'}, 0)$. Since $f^{\frac{a}{2}}(S^{'}, 0) = f(S^{'}, 0) = (\frac{a}{2}, 1-\frac{a}{2})$, we have the result. □

After stating our result for the case where Player 2 makes pre-donation, we will now check whether Player 1 can benefit from unilateral predonation of himself. As shown again in Appendix A, Player 1, poorer or weaker player, can not become strictly better off via unilateral pre-donation. That is, $\lambda_1^* = 0$.

Thus, we have proved the following result:

Theorem 4. *Given a simple bargaining problem* (S_a, d)*, stronger player becomes strictly better off via unilateral predonation of herself without hurting the weaker player. On the other hand, weaker player has no incentive to predonate.*

3.2 Stackelberg and Nash Equilibria

In previous section, we showed that although weaker player can not gain from predonation, richer player becomes better off without hurting the other player. That is, unilateral predonation leads to more efficient result. At this point, we will find Stackelberg and Nash Equilibria of the predonation game where both players announces a predonation share. Let's firstly find the best responses of bargainers to the predonation of other bargainer and we will check whether players can benefit from responding to the predonation offer of the other player.

As shown in Appendix A,

$$BR_1(\lambda_2) = 0 \ \forall \lambda_2 \in [0,1]$$

and

$$BR_2(\lambda_1) = \frac{a(1-\lambda_1)}{2(1-a\lambda_1)} \ \forall \lambda_1 \in [0,1]$$

Given these, we have the following result:

Theorem 5. *Given a simple bargaining problem* (S_a, d)*, the Stackelberg Equilibria when either Player 1 or Player 2 is the leader and Nash Equilibrium yield payoff pair* $(\frac{a}{2}, 1-\frac{a}{2})$.

Proof. Firstly, let's check the equilibrium of the Stackelberg game where Player 1 is the leader. In that case, Player 1, knowing the best response of player 2, will announce λ_1 and then Player 2 will announce λ_2. As shown in Appendix A, $\lambda_1^* = 0$ and $\lambda_2^* = \frac{a}{2}$, thus yielding Nash Bargaining solution $(\frac{a}{2}, 1-\frac{a}{2})$.

Similarly, the Stackelberg game where Player 2 is the leader yields the same solution since player 2 will choose $\lambda_2^* = \frac{a}{2}$ since $\lambda_1^* = 0$. Furthermore, if players simultaneously choose a predonation share, we find that Nash Equilibrium is unique and it is $(\lambda_1^*, \lambda_2^*) = (0, \frac{a}{2})$, yielding the same solution.

It is easy to see that the unique Nash equilibrium is $(\lambda_1, \lambda_2) = (0, \frac{a}{2})$ □

CHAPTER 4

TWO BY TWO NORMAL FORM GAMES UNDER PREDONATION

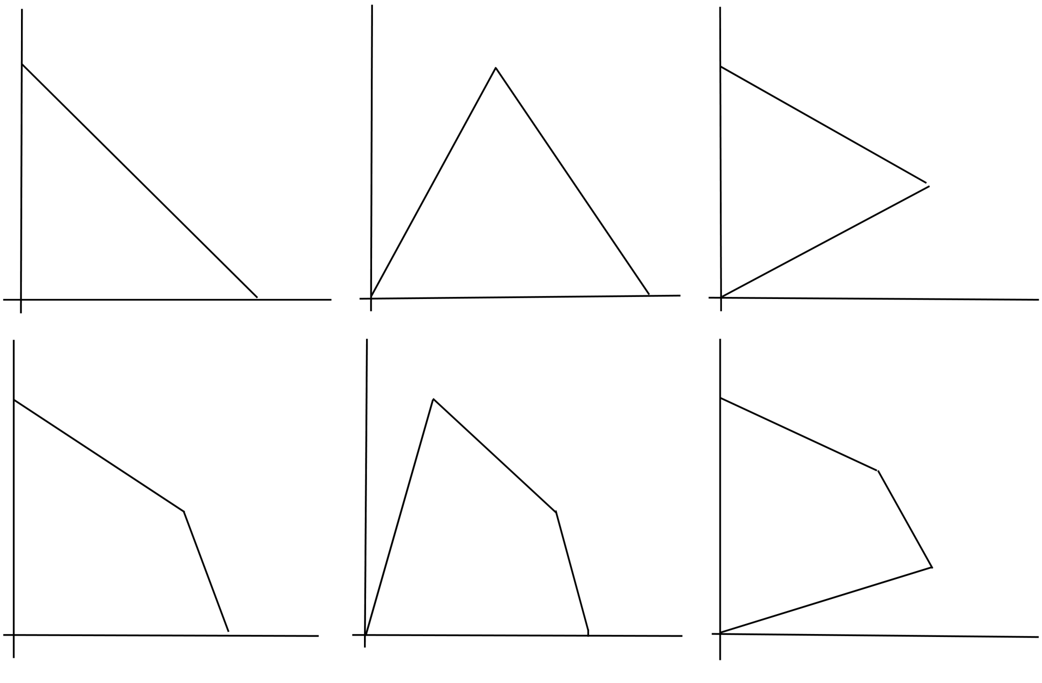
In Chapter 3, we considered simple bargaining problems. Next, we will consider two by two normal form games and will check whether players can become better off via predonation. Namely , we will consider two person normal form games with bimatrix

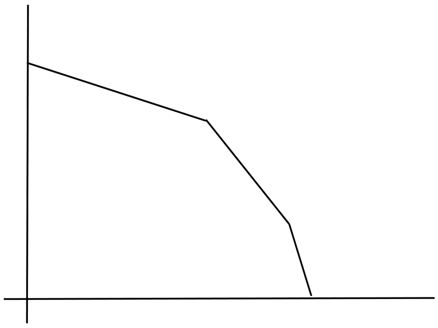
$$(A,B) = \begin{bmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) \end{bmatrix}$$

Given this two person normal form game, the set of all obtainable payoff pairs is just the convex hull of the four points. Call this set S. We will assume that $a_{ij} \geqslant 0$ to make sure that players can predonate share of their payoffs. That is, we do not allow players predonate more than they have. Furthermore, we will assume that mixed extension of this game has a unique Nash Equilibrium. (Note that we know existence of at least one Nash Equilibrium.) We will take the payoff pair that this Nash Equilibrium yields as the disagreement point. That is, they get the non-cooperative equilibrium result if they can not cooperate. This seems plausible considering that players can not commit themselves to any planned strategies in the event of disagreement. Now, we have a well defined bargaining problem (S, d). Now, since Nash Bargaining solution satisfies IIA, we can take the bargaining set to be

$$S' = \{(s_1, s_2) \in S : (d_1, d_2) \leq (s_1, s_2)\}$$

and since Nash Bargaining solution satisfies IAT, we can transfom disagreement point to the origin. Furthermore, noting that Nash Bargaining solution satisfies PAR we can represent all such bargaining problems with one of the seven bargaining problems with following bargaining sets. (Note also that we will disregard the case where there is a unique pareto optimal point in the bargaining set since it is trivially true that no player can benefit from predonation.)



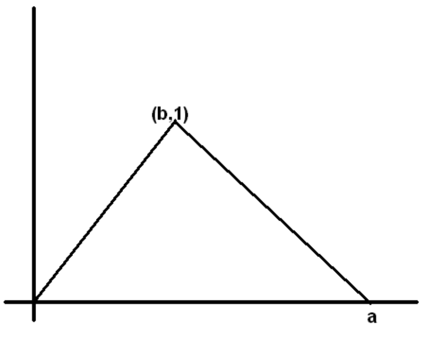


By investigating each case seperately, we have the following result:

Theorem 6. *Player 1 and Player 2 can become better off via unilateral predonation under some conditions on the slopes of the lines that determines the bargaining set.*

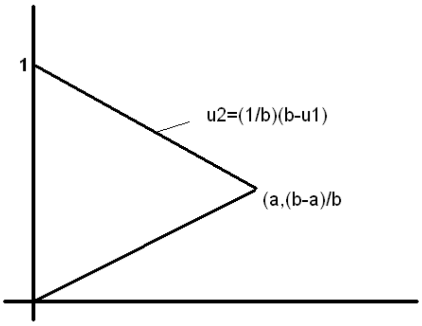
Proof. We can transform the bargaining set such that ideal point of Player 2 is 1 and ideal point of Player 1 is a and the disagreement point is at the origin since Nash Bargaining solution satisfies IAT and IIA.

Now, here we will state the results and leave calculations to Appendix B. The first bargaining set is the simple bargaining set when $a < 1$ which we discussed in previous chapter. The cases when $a = 1$ and $a > 1$ are considered in Appendix B and we showed that while stronger player can become better off via predonation, weaker player can not make better off. Other cases are as follows:



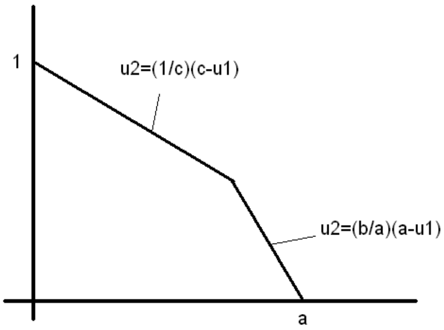
Player 2 becomes better off if $\frac{a}{2} > b$ and $a - b < 1$.

Player 1 becomes better off if $a > 2b$ and $(a - b) > 1$



Player 2 becomes better off via predonation if ($b < 1, b > 2a, \frac{a}{b} > \frac{b}{2}$) or ($b < 1, b \leq 2a$).

Player 1 becomes better off if $b \leq 2a$ and $b > 1$



Player 2 becomes better off if

$$(2a \leq c, c(2-b) \geqslant a, a < b)$$

or

$$(a > \frac{c}{2}, b(2a-c) < a, c(2-b) \geqslant a), b > a$$

or

$$(a > \frac{c}{2}, b \geqslant \frac{a}{2a-c}, c < 1)$$

Player 1 becomes better off if

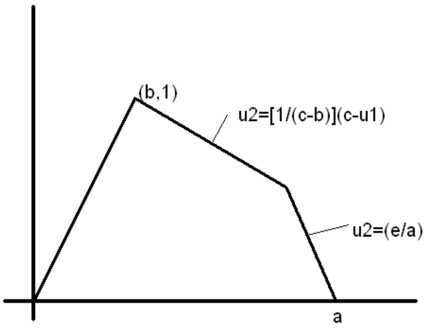
$$(2a \leq c, c(2-b) \geqslant a, a > b)$$

or

$$(a > \frac{c}{2}, b(2a-c) < a, c(2-b) \geqslant a, b < a)$$

or

$$(a > \frac{c}{2}, b \geqslant \frac{a}{2a-c}, c > 1)$$



Player 2 becomes better off if

$$(b < \frac{c}{2}, e(c-b) < 2c-a, \frac{c}{2} - b \geqslant 1 \text{ and } a < e)$$

or

$$(b < \frac{c}{2}, e(c-b) < 2c-a, \frac{c}{2} - b < 1 \text{ and } a < e \text{ and } a \leq e(a-b))$$

or

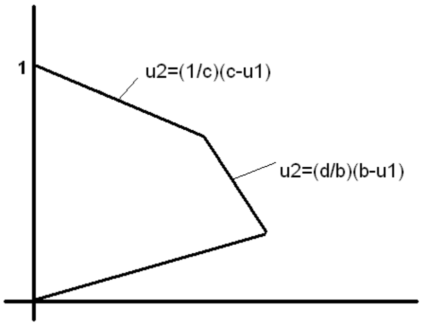
$$(b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac, c-b < 1)$$

Player 1 becomes better off if

$$b < \frac{c}{2}, e(c-b) < 2c-a, a > e$$

or

$$b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac, (c-b) > 1$$



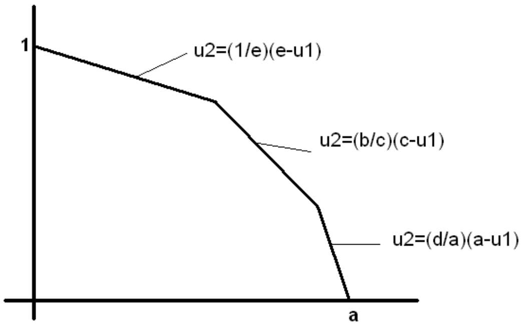
Player 2 becomes better off if Player 2 makes better off if $[\frac{b}{2} < a, (2-d)c \geqslant b, d > b]$, or $[\frac{b}{2} < a, b < (2b-c)d, c < 1]$.

Player 1 becomes better off if

$$\frac{b}{2} < a, (2-d)c \geqslant b, b > d$$

or

$$\frac{b}{2} < a, b < (2b-c)d, c > 1$$



Player 2 becomes better off via predonation if

$$\begin{aligned}
&[2c \leq e, b(2c-a) \geqslant cd, e(2-b) > c] \text{ or} \\
&[2c \leq e, (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab \leq d(2a-c)] \text{ or} \\
&[2c > e, (2c-e)b \geqslant c, e < 1] \text{ or} \\
&[2c > e, (2c-e)b < c, b(2c-a) \geqslant cd, e(2-b) > c] \text{ or} \\
&[2c > e, (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab \leq d(2a-c)]
\end{aligned}$$

Player 1 becomes better off via predonation if

$$\begin{aligned}
[2c &\le e, b(2c-a) \geqslant cd, e(2-b) > c, a > d] \text{ or}\\
[2c &\le e, (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab \le d(2a-c), b < c] \text{ or}\\
[2c &> e, (2c-e)b < c, b(2c-a) \geqslant cd, e(2-b) > c, a > d] \text{ or}\\
[2c &> e, (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab \le d(2a-c), b < c]
\end{aligned}$$

□

Note that conditions that Player 1 becomes better off and conditions that Player 2 becomes better off do not coincide.

Theorem 7. *In each case, the Stackelberg Equilibria when either Player 1 or Player 2 is the leader and Nash Equilibrium yield the same payoff pair as the player that can become better off by unilateral predonation of himself unilaterally chooses optimal predonation share.*

Proof. All the calculations are shown in Appendix B, again. □

CHAPTER 5

CONCLUSION

In this study, we have dealt with bargaining problems under predonation. Firstly, as in previous studies on bargaining under predonation, we considered simple bargaining problems that can be viewed as a division problem. We extended results of previous studies on simple bargaining problems. We have stated a result that links Nash Bargaining solution with Asymmetric Nash Bargaining Solution. In addition, we not only considered the case of unilateral predonation of players but also considered the case where a player can respond to other player's predonation. We showed that weaker player always responds by " not predonating " to any predonation from the stronger player, and we showed that stronger player has a unique best response to any given predonation from weaker player. Using these results, we showed that Stackelberg equilibria where Player 1 or Player 2 leader and the Nash Equilibrium of the simultaneous game yield the same result.

Then, we considered two by two normal form games with unique Nash Equilibrium. We take these games as a bargaining problem with bargaining set which consists of all attainable utility pairs by cooperation of players and the disagreement point to be the utility pair that the unique Nash Equilibrium yields. We showed that, under some conditions both players can gain from unilateral predonation. However, conditions where Player 1 and Player 2 becomes better off are different. That is, under conditions when Player 1 can become better off via unilateral predonation of himself Player 2 can not become better off. Similarly, when Player 2 becomes better off via unilateral predonation of herself Player 1 can not become better off via predonation of himself. We, furthermore, checked the Stackelberg and Nash Equilibria and saw that all equilibria yield the same payoff pair to bargainers.

In this study, we have seen that although stronger player always become better off via predonation in simple bargaining problems and weaker player can not become better off via unilateral predonation of himself, we can not generalize this statement to any bargaining problem. That is, there are bargaining problems that stronger player can not become better off via pradonation and there are problems that weaker player can become better off. We have listed bargaining problems that arise from two by two normal form games where each player can become better off.

Given these, it is still an open question whether we can generalize these results to any bargaining problem. That is, to get general results about the cases in which stronger player can become better off via predonation. It would be next step to find conditions on any given bargaining problem that stronger and weaker players become better off.

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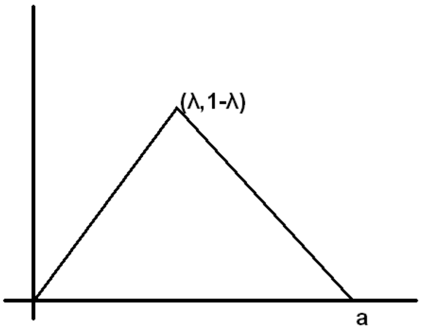
APPENDIX A

CALCULATIONS IN CHAPTER 3

A.1 Unilateral Pre-Donation

Consider firstly the case where Player 2 makes predonation, i.e. she donates a portion of her would-be payoff to player 1, say $\lambda \in [0, 1]$ of her payoff. Now, when $0 < \lambda < a$, we have the following bargaining set:



Note that when $\lambda \geqslant a$, there is a unique pareto optimal point $(\lambda, 1-\lambda)$.We know that Nash Bargaining solution will be a Pareto Optimal point.Thus, Nash Bargaining solution is the point where u_1u_2 is maximized on the line on the right.Thus, we will solve the following problem on the pareto optimal line:

$$\begin{aligned} & \max\ u_1u_2 \\ = & \max\ u_1^2(\frac{\lambda-1}{a-\lambda}) - u_1(\frac{\lambda-1}{a-\lambda})a \end{aligned}$$

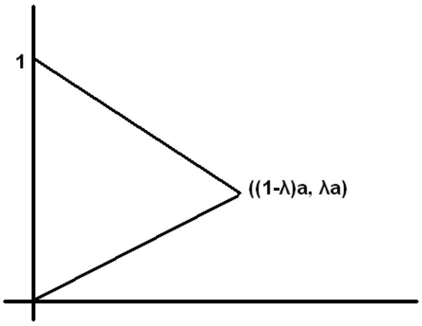
Now, $\frac{\partial(u_1u_2)}{\partial u_1} = \frac{2(\lambda-1)}{a-\lambda}u_1 - \frac{(\lambda-1)}{a-\lambda}a$. Thus,

$$\begin{aligned} \frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ iff } u_1 < \frac{a}{2} \\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ iff } u_1 > \frac{a}{2} \end{aligned}$$

Then, if player 2 chooses $\lambda < \frac{a}{2}$, NBS(Nash Bargaining Solution) will yield $\frac{a}{2}.\frac{1-\lambda}{a-\lambda}$ to Player 2, which is increasing in λ; if player 2 chooses $\lambda = \frac{a}{2}$, NBS will yield $1-\frac{a}{2}$ to Player 2 . If player 2 chooses $\frac{a}{2} < \lambda < a$, NBS will yield $1-\lambda < 1-\frac{a}{2}$. By choosing $a \leqslant \lambda < 1$, Player 2 will get $1-\lambda$, and 0 by choosing $\lambda = 1$. Thus, $\lambda^* = \frac{a}{2}$, yielding a solution

$$(\frac{a}{2}, 1-\frac{a}{2})$$

After stating our result for the case where Player 2 makes pre-donation, we will now check whether Player 1 can benefit from unilateral predonation of himself. Now, we have the following bargaining set if $\lambda \neq 1$



Now, it is trivially true that $\lambda^* \neq 1$. We know that Nash Bargaining solution will be a Pareto Optimal point.Thus, Nash Bargaining solution is the point where $u_1 u_2$ is maximized on the line on the right.Thus, we will solve the following problem on the pareto optimal line:

$$\begin{aligned} & \max \ u_1 u_2 \\ = \ & \max \ \text{-}u_1^2\left(\frac{1-\lambda a}{(1-\lambda)a}\right) + u_1 \end{aligned}$$

Now, $\frac{\partial(u_1 u_2)}{\partial u_1} = 1 - 2u_1(\frac{1-\lambda a}{(1-\lambda)a})$. Thus,

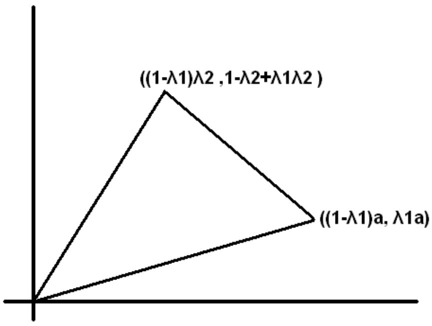
$$\begin{aligned} \frac{\partial(u_1 u_2)}{\partial u_1} &> 0 \text{ iff } u_1 < \left(\frac{(1-\lambda)a}{2(1-\lambda a)}\right) \\ \frac{\partial(u_1 u_2)}{\partial u_1} &< 0 \text{ iff } u_1 > \left(\frac{(1-\lambda)a}{2(1-\lambda a)}\right) \end{aligned}$$

Now, $\frac{(1-\lambda)a}{2(1-\lambda a)} \geqslant (1-\lambda)a$ iff $\lambda \geqslant \frac{1}{2a}$. Thus, if $a < \frac{1}{2}$, we have $\frac{(1-\lambda)a}{2(1-\lambda a)} < (1-\lambda)a$ and thus Nash Bargaining solution yields either $(\frac{(1-\lambda)a}{2(1-\lambda a)})$ to Player 1 which is decreasing in λ. Thus, $\lambda^* = 0$. If, on the other hand, $1 > a \geqslant \frac{1}{2}$, choosing $0 \leq \lambda \leq \frac{1}{2a}$, Nash Bargaining solution yields $(\frac{(1-\lambda)a}{2(1-\lambda a)})$ to Player 1 which is decreasing in λ and by choosing $\lambda > \frac{1}{2a}$, Player 1 gets $(1-\lambda)a$. Thus, $\lambda^* = 0$.

A.2 Best Responses

At this point, we will find the best responses of bargainers to the predonation of other bargainer.

Firstly, let's find the best response of Player 1 to a given predonation share λ_2 of Player 2. Now, when $0 < \lambda_2 < a$. we have the following bargaining set :



The pareto optimal line is $u_2 = \frac{\lambda_1(a-\lambda_2)-(1-\lambda_2)}{(1-\lambda_1)(a-\lambda_2)}u_1 + \frac{a(1-\lambda_2)}{a-\lambda_2}$

On the pareto optimal line:

$$u_1u_2 = \frac{\lambda_1(a-\lambda_2)-(1-\lambda_2)}{(1-\lambda_1)(a-\lambda_2)}u_1^2 + \frac{a(1-\lambda_2)}{a-\lambda_2}u_1$$

Then,

$$\frac{\partial(u_1u_2)}{\partial u_1} > 0 \text{ iff } \frac{a(1-\lambda_2)(1-\lambda_1)}{2[(1-\lambda_2)-\lambda_1(a-\lambda_2)]} > u_1$$

$$\frac{\partial(u_1u_2)}{\partial u_1} < 0 \text{ iff } \frac{a(1-\lambda_2)(1-\lambda_1)}{2[(1-\lambda_2)-\lambda_1(a-\lambda_2)]} < u_1$$

Now, if $\lambda_1 < \frac{(1-\lambda_2)(2-\frac{a}{\lambda_2})}{2(a-\lambda_2)}$, we have $\frac{a(1-\lambda_2)(1-\lambda_1)}{2[(1-\lambda_2)-\lambda_1(a-\lambda_2)]} < (1-\lambda_1)\lambda_2$ and if $\lambda_1 > \frac{1-\lambda_2}{2(a-\lambda_2)}$, we have $\frac{a(1-\lambda_2)(1-\lambda_1)}{2[(1-\lambda_2)-\lambda_1(a-\lambda_2)]} > (1-\lambda_1)a$. Furthermore, note that $\frac{(1-\lambda_2)(2-\frac{a}{\lambda_2})}{2(a-\lambda_2)} < \frac{1-\lambda_2}{2(a-\lambda_2)}$ since $\lambda_2 < a$.

Consider the case where $\lambda_2 > \frac{a}{2}$: In that case Player 1 by choosing $0 \leqslant \lambda_1 < \frac{(1-\lambda_2)(2-\frac{a}{\lambda_2})}{2(a-\lambda_2)}$ Player 1 will get $(1-\lambda_1)\lambda_2$, decreasing in λ_1.Thus, if $\frac{(1-\lambda_2)(2-\frac{a}{\lambda_2})}{2(a-\lambda_2)} \geqslant 1$, $\lambda_1^* = 0$. Suppose not. If $\frac{1-\lambda_2}{2(a-\lambda_2)} < 1$, by choosing $\frac{(1-\lambda_2)(2-\frac{a}{\lambda_2})}{2(a-\lambda_2)} \leqslant \lambda_1 < \frac{1-\lambda_2}{2(a-\lambda_2)} < 1$, will get $\frac{a(1-\lambda_2)(1-\lambda_1)}{2[(1-\lambda_2)-\lambda_1(a-\lambda_2)]}$,decreasing in λ_1. By choosing $\frac{1-\lambda_2}{2(a-\lambda_2)} \leqslant \lambda_1 \leqslant 1$, will get $(1-\lambda_1)a$. If, on the other hand, $\frac{1-\lambda_2}{2(a-\lambda_2)} \geqslant 1$, by choosing $\frac{(1-\lambda_2)(2-\frac{a}{\lambda_2})}{2(a-\lambda_2)} \leqslant \lambda_1 \leqslant 1$, he will get $(1-\lambda_1)a$. Thus, combining all the cases, $\lambda_1^* = 0$ if $a > \lambda_2 > \frac{a}{2}$.

Secondly, consider the case where $\lambda_2 \leqslant \frac{a}{2}$. In that case, we have

$$\frac{(1-\lambda_2)(2-\frac{a}{\lambda_2})}{2(a-\lambda_2)} \leqslant 0.$$

Then, If $\frac{1-\lambda_2}{2(a-\lambda_2)} < 1$, by choosing $0 \leqslant \lambda_1 < \frac{1-\lambda_2}{2(a-\lambda_2)} < 1$, will get $\frac{a(1-\lambda_2)(1-\lambda_1)}{2[(1-\lambda_2)-\lambda_1(a-\lambda_2)]}$,decreasing in λ_1. By choosing $\frac{1-\lambda_2}{2(a-\lambda_2)} \leqslant \lambda_1 \leqslant 1$, wii get $(1-\lambda_1)a$. If, on the other hand, $\frac{1-\lambda_2}{2(a-\lambda_2)} \geqslant 1$, by choosing $0 \leqslant \lambda_1 \leqslant 1$, he will get $(1-\lambda_1)a$. Thus, combining all the cases, $\lambda_1^* = 0$ if $\lambda_2 \leqslant \frac{a}{2}$. Thus, we have $\lambda_1^* = 0 \ \forall \ \lambda_2 \in [0, a)$

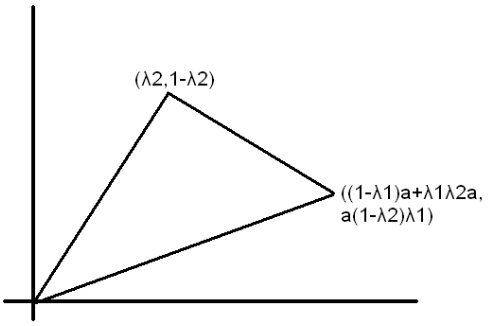
If $\lambda_2 = a$, it is trivially true that $\lambda_1^* = 0$.

If $1 \geqslant \lambda_2 > a$, then Player 1 will get $(1-\lambda_1)\lambda_2$, thus $\lambda_1^* = 0$.

Thus, we have

$$\lambda_1^* = 0 \ \forall \ \lambda_2 \in [0,1]$$

Let's now find the best response of Player 2 to a given λ_1. When $\lambda_1 \neq 1$ and $\lambda_2 < \frac{a(1-\lambda_1)}{(1-a\lambda_1)}$:(Note that when $\lambda_2 \geqslant \frac{a(1-\lambda_1)}{(1-a\lambda_1)}$ we have unique pareto optimal point.)



On the pareto optimal line:

$$u_1 u_2 = \frac{1-\lambda_2}{a(1-\lambda_1) + a\lambda_1\lambda_2 - \lambda_2}((a\lambda_1 - 1)u_1^2 + a(1-\lambda_1)u_1)$$

Thus, we have

$$\frac{\partial(u_1u_2)}{\partial u_1} > 0 \text{ iff } \frac{a(1-\lambda_1)}{2(1-a\lambda_1)} > u_1$$

$$\frac{\partial(u_1u_2)}{\partial u_1} < 0 \text{ iff } \frac{a(1-\lambda_1)}{2(1-a\lambda_1)} < u_1$$

Now,

$$\frac{a(1-\lambda_1)}{2(1-a\lambda_1)} > (1-\lambda_1)a + \lambda_1\lambda_2 a \text{ iff } \lambda_2 < \frac{a(1-\lambda_1)(2a\lambda_1 - 1)}{2a\lambda_1(1-a\lambda_1)}$$

Note that if $2a\lambda_1 - 1 \leq 0$, i.e. $\lambda_1 \leq \frac{1}{2a}$ we have $\frac{a(1-\lambda_1)}{2(1-a\lambda_1)} \leq (1-\lambda_1)a + \lambda_1\lambda_2 a$. In that case: By choosing $0 \leqslant \lambda_2 \leqslant \frac{a(1-\lambda_1)}{2(1-a\lambda_1)}$, will get $\frac{a}{2} \cdot \frac{(1-\lambda_1)(1-\lambda_2)}{a(1-\lambda_1)+\lambda_2(a\lambda_1-1)}$, increasing in λ_2. By choosing $\frac{a(1-\lambda_1)}{2(1-a\lambda_1)} < \lambda_2 < \frac{a(1-\lambda_1)}{(1-a\lambda_1)}$, she will get $1-\lambda_2$. By choosing $\frac{a(1-\lambda_1)}{(1-a\lambda_1)} \leqslant \lambda_2 < 1$, will get $1-\lambda_2$.

If $'\lambda_1 > \frac{1}{2a}$, choosing $0 \leq \lambda_2 \leq \frac{a(1-\lambda_1)(2a\lambda_1-1)}{2a\lambda_1(1-a\lambda_1)}$ Player 2 gets $(1-\lambda_2)\lambda_1 a$ which is decreasing in λ_2, choosing By choosing $\frac{a(1-\lambda_1)(2a\lambda_1-1)}{2a\lambda_1(1-a\lambda_1)} < \lambda_2 \leq \frac{a(1-\lambda_1)}{2(1-a\lambda_1)}$, she will get $\frac{a}{2}.\frac{(1-\lambda_1)(1-\lambda_2)}{a(1-\lambda_1)+\lambda_2(a\lambda_1-1)}$. By choosing $\frac{a(1-\lambda_1)}{2(1-a\lambda_1)} \leqslant \lambda_2 < 1$, will get $1-\lambda_2$. Thus, we should check $\lambda_2 = 0$ and $\lambda_2 = \frac{a(1-\lambda_1)}{2(1-a\lambda_1)}$. Simple calculation shows that $\lambda_2^* = \frac{a(1-\lambda_1)}{2(1-a\lambda_1)}$.

Thus, we have

$$\lambda_2^* = \frac{a(1-\lambda_1)}{2(1-a\lambda_1)} \ \forall \lambda_1 \in [0,1)$$

When $\lambda_1 = 1$, it is trivial that $\lambda_2^* = 0 = \frac{a(1-\lambda_1)}{2(1-a\lambda_1)}$.

A.3 Stackelberg and Nash Equilibria

Firstly, let's check the equilibrium of the Stackelberg game where Player 1 is the leader. In that case, Player 1, knowing the best response of player 2, will announce λ_1 and then Player 2 will announce λ_2. Now,

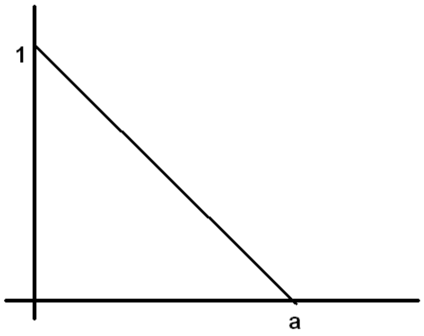
$$u_1 u_2 = \frac{2-a-a\lambda_1}{a(1-\lambda_1)(1-a\lambda_1)}[2(a\lambda_1-1)u_1 + a(1-\lambda_1)]$$

,hence $u_1 u_2$ is maximized at point $(\frac{a(1-\lambda_1)}{2(1-a\lambda_1)}\ ,1-\frac{a(1-\lambda_1)}{2(1-a\lambda_1)}\ )$. Thus, $\lambda_1^* = 0$. Then, $\lambda_2^* = \frac{a}{2}$, yielding a solution $(\frac{a}{2}, 1-\frac{a}{2})$.

APPENDIX B

CALCULATIONS IN CHAPTER 4

Now, as we stated before we can transform the bargaining set such that ideal point of Player 2 is 1 and ideal point of Player 1 is a and the disagreement point is the origin since Nash Bargaining solution satisfies IAT and IIA. We will look at each case seperately. First case:



Now, $f^N(S,d) = (\frac{a}{2}, \frac{1}{2})$

This is the so called simple bargaining problem. We investigated this case in previous chapter when $0 < a < 1$. Let's look at the case when $a = 1$:

When $a = 1$, Player 2 gets $\frac{1}{2}$ if she chooses $\lambda \leq \frac{1}{2}$. If she chooses $\lambda > \frac{1}{2}$, she gets $1 - \lambda$.Thus, she can get $\frac{1}{2}$ at most, thus she can not gain from predonation.Let' s find responses of each player:Firstly, let's find the best response of Player 2 to a given λ_1 : Now, if $\lambda_1 > \frac{1}{2}$: Player 2 gets λ_1 by choosing $0 \leq \lambda_2 \leq 1-\lambda_1$. If player 2 chooses λ_2 bigger than 1-λ_1, Player 2 gets $\frac{1}{2}$ if $\lambda_2 \geqslant \frac{1}{2}$ and $1-\lambda_2$ if $\lambda_2 < \frac{1}{2}$. If $\lambda_1 = \frac{1}{2}$, player 2 gets $\frac{1}{2}$. If $\lambda_1 < \frac{1}{2}$, Player 2 gets $\frac{1}{2}$ by choosing $0 \leq \lambda_2 \leq \frac{1}{2} < 1-\lambda_1$. By choosing $\frac{1}{2} < \lambda_2 \leq 1-\lambda_1$, player 2 gets $1-\lambda_2$ and choosing $\frac{1}{2} < 1-\lambda_1 < \lambda_2 \leq 1$, gets λ_1. Thus,

$$\begin{aligned} BR_2(\lambda_1) &= [0, 1-\lambda_1] \text{ if } \lambda_1 > \frac{1}{2} \\ &= [0,1] \qquad \text{if } \lambda_1 = \frac{1}{2} \\ &= [0, \frac{1}{2}] \qquad \text{if } \lambda_1 < \frac{1}{2} \end{aligned}$$

Similarly,

$$\begin{aligned} BR_1(\lambda_2) &= [0, 1-\lambda_2] \text{ if } \lambda_2 > \frac{1}{2} \\ &= [0,1] \qquad \text{if } \lambda_2 = \frac{1}{2} \\ &= [0, \frac{1}{2}] \qquad \text{if } \lambda_2 < \frac{1}{2} \end{aligned}$$

Let's find Stackelberg equilibrium where player 1 is the leader. If Player 1 chooses $0 \leq \lambda_1 \leq \frac{1}{2}$, Player 2 will choose $\lambda_2 \in [0, \frac{1}{2}]$, yielding $(\frac{1}{2}, \frac{1}{2})$, if chooses $\lambda_1 \leq \frac{1}{2}$, player 2 will choose $\lambda_2 \in [0,1]$ yielding $(\frac{1}{2}, \frac{1}{2})$,if chooses $\lambda_1 > \frac{1}{2}$, player 2 will choose $\lambda_2 \in [0, 1-\lambda_1]$, player 1 will get $1-\lambda_1$. Thus, Stackelberg equlibria where player 1 is the leader will yield $(\frac{1}{2}, \frac{1}{2})$. Similarly, Stackelberg equlibria where player 2 is the leader will yield $(\frac{1}{2}, \frac{1}{2})$. Furthermore, it is easy to see that all Nash equilibria yield $(\frac{1}{2}, \frac{1}{2})$.

Let's look at the case where $a > 1$. In that case, if Player 2 makes predonation:

Now, note that if $\frac{a}{2} < \lambda b$, then NBS will yield $(\lambda b, (1-\lambda)b)$.

If $a \geqslant 2b$, we have $\frac{a}{2} \geqslant \lambda b$ for each $\lambda \in [0,1]$. In that case, $f^N(S,d) = (\frac{a}{2}, \frac{(1-\lambda)b}{1-\lambda b}.\frac{a}{2})$ whose second component is decreasing in λ.

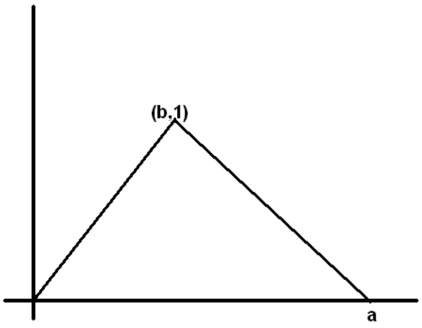
If $a < 2b$, choosing $0 \leq \lambda \leq \frac{a}{2b}$,NBS will yield $(\frac{a}{2}, \frac{(1-\lambda)b}{1-\lambda b}.\frac{a}{2})$ and by choosing $\frac{a}{2b} \leq \lambda \leq 1$, NBS will yield $(\lambda b, (1-\lambda)b)$. Thus, $\lambda^* = 0$.

If Player 1 makes predonation: By choosing choosing $0 \leq \lambda \leq \frac{b}{2a}$,NBS will yield $(\frac{(1-\lambda)ab}{2(b-\lambda a)}, \frac{b}{2})$ whose first component is increasing since $b < a$. Choosing $\frac{b}{2a} \leq \lambda \leq 1$, NBS will yield $a(1-\lambda)$ to Player 1. Thus, Player 1 becomes better off in that case by choosing $\lambda = \frac{b}{2a}$.

Easy calculations show that Stackelberg and Nash Equilibria yield the same payoff pair as the payoff paie we get when Player 1 chooses his optimal λ.

To sum up, in this case the stronger player(i.e. player with higher ideal point) becomes strictly better off by predonating. If the ideal points of players are same then they get at most what they get when they do not predonate.

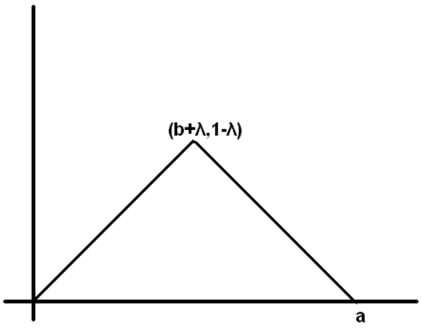
Next case:



$f^N(S,d) = (b,1) \qquad$ if $b > \frac{a}{2}$

$f^N(S,d) = (\frac{a}{2}, \frac{a}{2(a-b)})$ if $b \leq \frac{a}{2}$

If Player 2 makes predonation, we have the following bargaining set when $0 < \lambda < a - b$:



Thus, on the pareto optimal line:

$$\begin{aligned}\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{a}{2} > u_1\\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{a}{2} < u_1\end{aligned}$$

Firstly assume that $a-b>1$.Then, $\lambda \geqslant a-b$, hence there is a unique pareto optimal point $(b+\lambda, 1-\lambda)$. Thus, $\lambda^* = 0$.

If $a-b<1$: if $b+\lambda > \frac{a}{2}$, i.e. $\lambda > \frac{a}{2} - b$, $f^N(S,d) = (b+\lambda, 1-\lambda)$. Thus, if $\frac{a}{2} \leq b$, then $\lambda^* = 0$. Assume $\frac{a}{2} > b$. If $\lambda \leq \frac{a}{2} - b$, $f^N(S,d) = (\frac{a}{2}, \frac{1-\lambda}{(a-b)-\lambda} \cdot \frac{a}{2})$ whose second component is increasing in λ when $a-b>\lambda$.Thus,

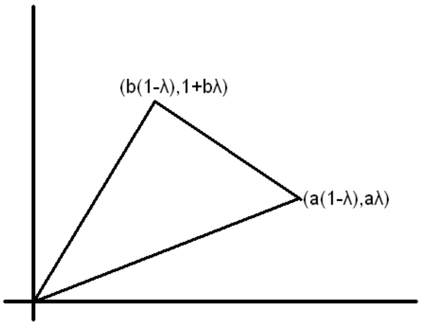
$$\text{if } \frac{a}{2} > b \text{ and } a-b<1, \text{ then } \lambda^* = \frac{a}{2} - b, \text{ yielding } f^N(S,d) = (\frac{a}{2}, 1-(\frac{a}{2}-b))$$

Note that if $a-b=1$, Player 2, by predonating, gets at most what she got before predonation.

To sum up, Player 2 becomes better off via predonation if

$$\frac{a}{2} > b \text{ and } a-b<1.$$

If player 1 makes predonation, we have the following bargaining set:



On the pareto optimal line:

$$\begin{aligned}\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{a(1-\lambda)}{2(1-\lambda(a-b))} > u_1\\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{a(1-\lambda)}{2(1-\lambda(a-b))} < u_1\end{aligned}$$

Now, if $b(1-\lambda) \geqq \frac{a(1-\lambda)}{2(1-\lambda(a-b))}$, $f^N(S,d) = (b(1-\lambda), 1+b\lambda)$ and if $b(1-\lambda) < \frac{a(1-\lambda)}{2(1-\lambda(a-b))}$, Nash Bargaining solution yields $\frac{a(1-\lambda)}{2(1-\lambda(a-b))}$ to player 1 which is decreasing in λ. Furthermore, $\frac{a(1-\lambda)}{2(1-\lambda(a-b))} \leq a(1-\lambda)$ iff $\lambda \leq \frac{1}{2(a-b)}$ and $\frac{a(1-\lambda)}{2(1-\lambda(a-b))} < b(1-\lambda)$ if $\lambda < \frac{2b-a}{2b(a-b)}$.

Now, if $a > 2b$: If $1 \geqslant 2(a-b)$ we have $b(1-\lambda) < \frac{a(1-\lambda)}{2(1-\lambda(a-b))} \leq a(1-\lambda)$. In that case $f^N(S,d) = (\frac{a(1-\lambda)}{2(1-\lambda(a-b))}, \frac{a}{2(a-b)})$ whose first componenent is increasing in λ if $(a-b) > 1$ and Thus, $\lambda^* = 0$.

If $1 < 2(a-b)$, choosing $0 \leq \lambda \leq \frac{1}{2(a-b)}$, $f^N(S,d) = (\frac{a(1-\lambda)}{2(1-\lambda(a-b))}, \frac{a}{2(a-b)})$, and choosing $\lambda \geqslant \frac{1}{2(a-b)}$, $f^N(S,d) = (a(1-\lambda), a\lambda)$. Thus, if $(a-b) > 1$, Player 1 becomes better off.

Secondly, consider the case where $a \leq 2b$. If $\frac{2b-a}{2b(a-b)} \geqslant 1$, $f^N(S,d) = (b(1-\lambda), 1+b\lambda)$. Thus, $\lambda^* = 0$. Assume that $\frac{2b-a}{2b(a-b)} < 1 \leq \frac{1}{2(a-b)}$. By choosing $\frac{2b-a}{2b(a-b)} < \lambda \leq 1$, NBS yields $\frac{a(1-\lambda)}{2(1-\lambda(a-b))}$ to Player 1, which is decreasing in λ since $\frac{1}{2} \geqslant (a-b)$. If $\frac{2b-a}{2b(a-b)} < \frac{1}{2(a-b)} \leq 1$, choosing $\frac{2b-a}{2b(a-b)} < \lambda \leq \frac{1}{2(a-b)}$, NBS yields $\frac{a(1-\lambda)}{2(1-\lambda(a-b))}$ to player 1. Choosing $\frac{1}{2(a-b)} \leq \lambda \leq 1$, Player 1 gets $a(1-\lambda)$. Thus, if $(a-b) \leq 1$, $\lambda^* = 0$ and when $(a-b) > 1$, we should check $\lambda = 0$ and $\lambda = \frac{1}{2(a-b)}$. Simple calculations show that $\lambda^* = 0$.

To sum up, Player 1 becomes better off if

$$a > 2b \text{ and } (a-b) > 1$$

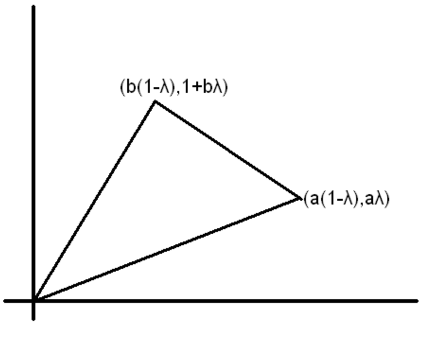
Now, let' s find the best responses of players. By calculations we get,

$$\begin{aligned} BR_1(\lambda_2) &= \frac{1-\lambda_2}{2((a-b)-\lambda_2)} \text{ if } a > 2b \\ &= 0 \text{ otherwise} \end{aligned}$$

and

$$\begin{aligned} BR_2(\lambda_1) &= \frac{a(1-\lambda_2)(1-\lambda_1)}{2[(a-b)(1-\lambda_1)+\lambda_2((a-b)\lambda_1-1)]} \text{ if } \frac{a}{2}-b>0,\ a-b<1 \\ &= 0 \text{ otherwise} \end{aligned}$$

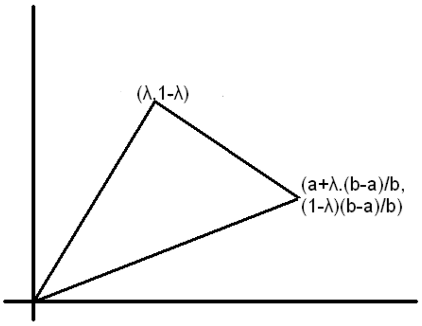
Thus, by similar calculations as above case all Stackelberg equilibria and Nash equilibria yield the same result where Player 2 chooses her optimal predonation share when $\frac{a}{2}-b > 0$, $a-b<1$ and yield the same result as Player 1 chooses his optimal predonation when $a > 2b$ and $(a-b) > 1$. Next case:



Now, if $b > 2a$, then $f^N(S,d) = (a, \frac{b-a}{b})$. If $b \leqslant 2a$, $f^N(S,d) = (\frac{b}{2}, \frac{1}{2})$.

If player 2 predonates:We will consider this case in two cases:

If $b < 1$: Then, if $\lambda < b$, we have the following bargaining set:

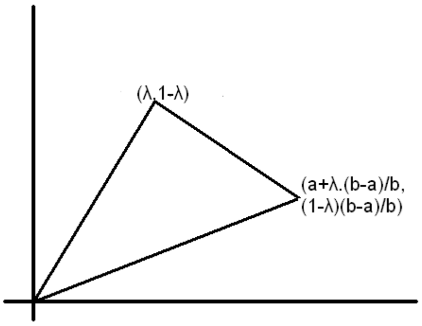


If $\lambda \geqslant b$, then there is a unique pareto optimal point $(\lambda, 1-\lambda)$. If $b > 2a$, $\frac{b}{2} > a + \frac{\lambda(b-a)}{b}$ iff $\frac{b(b-2a)}{2(b-a)} > \lambda$.

Choosing $0 \leqslant \lambda < \frac{b(b-2a)}{2(b-a)}$, Nash Bargaining solution (NBS) yields $(1-\lambda)\frac{b-a}{a}$; choosing $\frac{b(b-2a)}{2(b-a)} \leq \lambda \leq \frac{b}{2}$, $f^N(S,d) = (\frac{b}{2}, \frac{1-\lambda}{b-\lambda}.\frac{b}{2})$ whose second component is increasing in λ. Choosing $\frac{b}{2} \leq \lambda \leq 1$, $f^N(S,d) = (\lambda, 1-\lambda)$. Thus, we should check $\lambda = 0$ and $\lambda = \frac{b}{2}$. Simple calculations show that $\lambda^* = \frac{b}{2}$ if $\frac{a}{b} > \frac{b}{2}$.

If $b \leq 2a$, $\lambda^* = \frac{b}{2}$.

Now, we will consider the case $b \geqslant 1$:

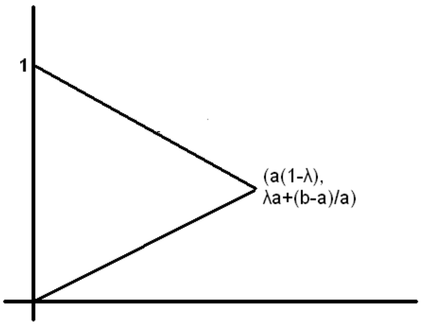


If $b > 2a$, $\frac{b}{2} > a + \frac{\lambda(b-a)}{b}$ iff $\frac{b(b-2a)}{2(b-a)} > \lambda$. Now, if $1 \leq b < 2$, $\frac{b(b-2a)}{2(b-a)} \leq 1$. Then, by choosing $0 \leq \lambda \leq \frac{b(b-2a)}{2(b-a)}$, NBS yields $(1-\lambda)\frac{b-a}{a}$ to Player 2. Choosing $\frac{b(b-2a)}{2(b-a)} \leq \lambda \leq \frac{b}{2}$, NBS yields $\frac{b}{2}.\frac{1-\lambda}{b-\lambda}$ which is decreasing in λ since $b > 1$. Thus, $\lambda^* = 0$.

If $b \geqslant 2$, if $\frac{b(b-2a)}{2(b-a)} \leq 1$, we have the above case. If not, then NBS yields $(1-\lambda)\frac{b-a}{a}$ to player 2. Hence, $\lambda^* = 0$.

To sum up, Player 2 becomes better off via predonation if $(b < 1, b > 2a, \frac{a}{b} > \frac{b}{2})$ or $(b < 1, b \leq 2a)$.

If player 1 makes predonation, we have the following bargaining set if $b \leq 1$, or $b > 1$ and $0 \leq \lambda < \frac{1}{b}$:



If $b > 1$, $\lambda \geqslant \frac{1}{b}$, there is a unique pareto optimal point whose first componet is $a(1-\lambda)$.

Now, on the pareto optimal line

$$\begin{aligned} \frac{\partial(u_1 u_2)}{\partial u_1} &> 0 \text{ if } \frac{b(1-\lambda)}{2(1-\lambda b)} > u_1 \\ \frac{\partial(u_1 u_2)}{\partial u_1} &< 0 \text{ if } \frac{b(1-\lambda)}{2(1-\lambda b)} < u_1 \end{aligned}$$

Now,

$$\begin{aligned} \frac{b(1-\lambda)}{2(1-\lambda b)} &\leq a(1-\lambda) \text{ iff } \frac{2a-b}{2ab} \geqslant \lambda \\ \frac{2a-b}{2ab} &< 1 \text{ if } b \leq 1 \end{aligned}$$

. Thus, if $b > 2a$, then $\frac{b(1-\lambda)}{2(1-\lambda b)} > a(1-\lambda)$, thus NBS yields $a(1-\lambda)$. Thus, $\lambda^* = 0$.

If $b \leq 2a$, by choosing $0 \leq \lambda \leq \frac{2a-b}{2ab}$, NBS yields $\frac{b(1-\lambda)}{2(1-\lambda b)}$, which is increasing in λ iff $b > 1$. Choosing $\frac{2a-b}{2ab} \leq \lambda \leq 1$ NBS yields $a(1-\lambda)$. Thus, $\lambda^* = 0$.

If $b > 1$, and $b \leq 2a$, $\lambda^* = \frac{2a-b}{2ab}$, yielding $f^N(S, d) = (\frac{2ab-2a+b}{2b}, \frac{1}{2})$.

Thus, Player 1 becomes better off if

$$b \leq 2a \text{ and } b > 1$$

Now, let' s find the best responses of players. By calculations we get,

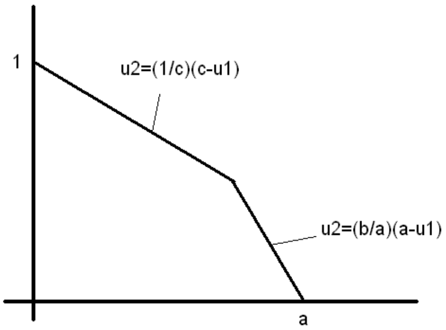
$$\begin{aligned} BR_1(\lambda_2) &= \frac{b(1-\lambda_1)}{2(1-\lambda_1 b)} \text{ if } b \leq 2a \text{ and } b < 1 \\ &= \frac{(1-\lambda_1)(b-2a+2ab\lambda_1)}{2(1-\lambda_1 b)(a\lambda_1 + \frac{b-a}{b})} \text{ if } b < 1, b > 2a, \frac{a}{b} > \frac{b}{2} \\ &= 0 \text{ otherwise} \end{aligned}$$

and

$$\begin{aligned} BR_2(\lambda_1) &= \frac{b(1-\lambda_2)(1-\lambda_1)}{2[(1-\lambda_2+\lambda_2\lambda_1)-\lambda_1 b)]} \text{ if } b \leq 2a \text{ and } b > 1 \\ &= 0 \text{ otherwise} \end{aligned}$$

Thus, by similar calculations as above case all Stackelberg equilibria and Nash equilibria yield the same result where Player 2 chooses her optimal predonation share when ($b \leq 2a$ and $b < 1$) or ($b < 1, b > 2a, \frac{a}{b} > \frac{b}{2}$)and yield the same result as Player 1 chooses his optimal predonation when $b \leq 2a$ and $b > 1$.

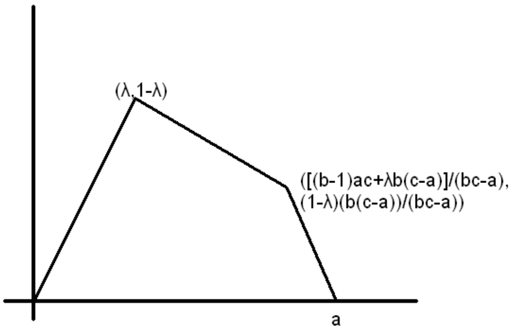
Another case:



Now, calculations show that

$$\begin{aligned}
f^N(S,d) &= (\frac{a}{2},\frac{b}{2}) \qquad\qquad \text{if } 2a \leq c, c(2-b) \geqslant a \\
&= (\frac{(b-1)ac}{bc-a},\frac{b(c-a)}{bc-a}) \text{ if } 2a \leq c, c(2-b) < a \\
&= (\frac{c}{2},\frac{1}{2}) \qquad\qquad \text{if } 2a > c, b \geqslant \frac{a}{2a-c} \\
&= (\frac{a}{2},\frac{b}{2}) \qquad\qquad \text{if } 2a > c, b < \frac{a}{2a-c}, c(2-b) \geqslant a \\
&= (\frac{(b-1)ac}{bc-a},\frac{b(c-a)}{bc-a})\text{if } \ 2a > c, b < \frac{a}{2a-c}, c(2-b) < a
\end{aligned}$$

Let's check whether Player 2 can make better off by predonation.



Now, the upper line transforms to $u_2 = (\frac{1-\lambda}{c-\lambda})(c-u_1)$. Thus, on this line

$$\begin{aligned}
\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{c}{2} > u_1 \\
\frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{c}{2} < u_1
\end{aligned}$$

The lower line trasnforms to $u_2 = \frac{b(1-\lambda)}{a-\lambda b}(a-u_1)$. Thus,

$$\begin{aligned}
\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{a}{2} > u_1 \\
\frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{a}{2} < u_1
\end{aligned}$$

Some facts before we continue:

$$\begin{aligned}\frac{c}{2} &\leq \frac{(b-1)ac+\lambda b(c-a)}{bc-a} \text{ iff } \lambda \geqslant \frac{c}{2b(c-a)}(bc+a(1-2b))\\ \frac{a}{2} &\leq \frac{(b-1)ac+\lambda b(c-a)}{bc-a} \text{ iff } \lambda \geqslant \frac{a}{2b(c-a)}((2-b)c-a)\end{aligned}$$

We will continue case by case:

If $2a \leq c, c(2-b) \geqslant a$: In that case, we have:

$$0 \leq \frac{a}{2b(c-a)}((2-b)c-a) < \frac{a}{b} \leq \frac{c}{2b(c-a)}(bc+a(1-2b)) < \frac{c}{2}$$

Consider firstly the case where $a < b$: By choosing $0 \leq \lambda \leq \frac{a}{2b(c-a)}((2-b)c-a) < \frac{a}{b}$, $f^N(S,d) = (\frac{a}{2}, \frac{b(1-\lambda)}{a-\lambda b} \cdot \frac{a}{2})$ whose second component is increasing in λ. By choosing $\frac{a}{2b(c-a)}((2-b)c-a) \leq \lambda \leq \frac{c}{2b(c-a)}(bc+a(1-2b))$, NBS yields $(1-\lambda)\frac{b(c-a)}{bc-a}$. If $\frac{c}{2b(c-a)}(bc+a(1-2b)) \geqslant 1$, then we are done. consider the case $\frac{c}{2b(c-a)}(bc+a(1-2b)) < 1$: If $\frac{c}{2b(c-a)}(bc+a(1-2b)) \leq 1 < \frac{c}{2}$, by choosing $\frac{c}{2b(c-a)}(bc+a(1-2b)) \leq \lambda \leq 1$, NBS yields $(1-\lambda)$ to Player 2. If $\frac{c}{2} < 1$, then by choosing $\frac{c}{2b(c-a)}(bc+a(1-2b)) \leq \lambda \leq \frac{c}{2}$, $f^N(S,d) = (\frac{c}{2}, \frac{1-\lambda}{c-\lambda} \cdot \frac{c}{2})$ whose second component is increasing in λ if $c < 1$. Thus, if $c > 1$, $\lambda^* = \frac{a}{2b(c-a)}((2-b)c-a)$, yielding a payoff $\frac{a}{2}$ to Player 1. If $c \leq 1$, one needs to check $\lambda^* = \frac{a}{2b(c-a)}((2-b)c-a)$ and $\lambda^* = \frac{c}{2}$. Note that in both cases Player 1 gets at least his previous(without predonation) payoff. Thus, player 2 makes better off in that case. If $a = b$, choosing $0 \leq \lambda \leq \frac{a}{2b(c-a)}((2-b)c-a)$, $f^N(S,d) = (\frac{a}{2}, \frac{a}{2})$. Thus, no gaim from predonation. If $b < a$, choosing $0 \leq \lambda \leq \frac{a}{2b(c-a)}((2-b)c-a)$ NBS yields $\frac{b(1-\lambda)}{a-\lambda b} \cdot \frac{a}{2}$ to Player 2 which is decreasing since $b < a$. Choosing $\frac{a}{2b(c-a)}((2-b)c-a) \leq \lambda \leq 1$, Player 2 gets $a(1-\lambda)$. Thus, $\lambda^* = 0$.

If $2a \leq c, c(2-b) < a$: In that case, we have $\frac{c}{2b(c-a)}(bc+a(1-2b)) \geqslant \frac{a}{b} > 0 > \frac{a}{2b(c-a)}((2-b)c-a)$. By choosing $0 \leq \lambda < \frac{a}{b}$, NBS yields $(1-\lambda)\frac{b(c-a)}{bc-a}$ to Player 2. Thus, if $a \geqslant b$, $\lambda^* = 0$. If $a < b$: If $c < 1$, by choosing

$\frac{a}{b} \leq \lambda < \frac{c}{2b(c-a)}(bc + a(1-2b))$ Player 2 gets $(1-\lambda)\frac{b(c-a)}{bc-a}$, by choosing $\frac{c}{2b(c-a)}(bc+a(1-2b)) \leq \lambda \leq \frac{c}{2}$, Player 2 gets $\frac{1-\lambda}{c-\lambda}.\frac{c}{2}$. Thus, one needs to check $\lambda = 0$ and $\lambda = \frac{c}{2}$. Easily shown that $\lambda^* = 0$. If $c \geqslant 1, \lambda^* = 0$ by similar calculations.

If $a > \frac{c}{2}, b \geqslant \frac{a}{2a-c}$: In that case $\frac{a}{b} > 0 > \frac{c}{2b(c-a)}(bc + a(1-2b)) > \frac{a}{2b(c-a)}((2-b)c - a)$. Thus, $\frac{a}{2}, \frac{c}{2} \leqslant \frac{(b-1)ac+\lambda b(c-a)}{bc-a}$. Now, if $\frac{c}{2} < \frac{a}{b} < 1$, by choosing $0 \leq \lambda \leq \frac{c}{2}$, $f^N(S,d) = (\frac{c}{2}, \frac{1-\lambda}{c-\lambda}.\frac{c}{2})$ whose second component is increasing if $c < 1$. Choosing $\frac{c}{2} \leq \lambda \leq 1$, NBS yields $1-\lambda$ to Player 2. If $\frac{a}{b} < \frac{c}{2}$; by choosing $0 \leq \lambda \leq \frac{c}{2}$, $f^N(S,d) = (\frac{c}{2}, \frac{1-\lambda}{c-\lambda}.\frac{c}{2})$ whose second component is increasing if $c < 1$. In remaining cases Player 2 gets $1-\lambda$.

Thus, Player 2 becomes better off if $c < 1$.

If $a > \frac{c}{2}, b(2a-c) < a, c(2-b) \geqslant a$: This case reduces to case 1 where $2a \leq c, c(2-b) \geqslant a$. That is, Player 2 makes better off if $b > a$.

If $2a > c, b < \frac{a}{2a-c}, c(2-b) < a$: Similarly, this case reduces to case where $2a \leq c, c(2-b) < a$.

To sum up, Player 2 becomes better off if

$$(2a \leq c, c(2-b) \geqslant a, a < b)$$

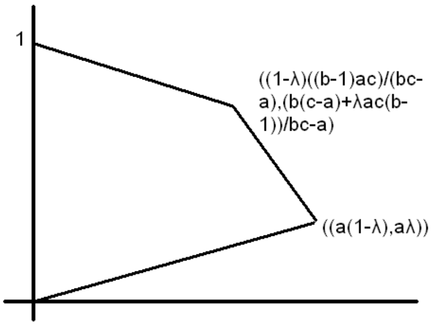
or

$$(a > \frac{c}{2}, b(2a-c) < a, c(2-b) \geqslant a), b > a$$

or

$$(a > \frac{c}{2}, b \geqslant \frac{a}{2a-c}, c < 1)$$

Secondly, consider the case where Player1 predonates:



Now, the upper line is transformed so that $u_2 = 1 + \frac{c\lambda - 1}{c(1-\lambda)}u_1$, thus unless $\lambda > \frac{1}{c}$, it is downward sloped. Thus, on this line

$$\begin{aligned}\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{c(1-\lambda)}{2(1-c\lambda)} > u_1 \\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{c(1-\lambda)}{2(1-c\lambda)} < u_1\end{aligned}$$

Lower line is transformed so that $u_2 = b + \frac{a\lambda - b}{a(1-\lambda)}u_1$. Thus,

$$\begin{aligned}\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{ab(1-\lambda)}{2(b-\lambda a)} > u_1 \\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{ab(1-\lambda)}{2(b-\lambda a)} < u_1\end{aligned}$$

Lower line is transformed so that $u_2 = b + \frac{a\lambda - b}{a(1-\lambda)}u_1$. Again, let's start with some facts:

$$\begin{aligned}\frac{c(1-\lambda)}{2(1-c\lambda)} &\leq (1-\lambda)\frac{(b-1)ac}{bc-a} \text{ iff } \lambda \leq \frac{b(2a-c)-a}{2ac(b-1)} \\ \frac{ab(1-\lambda)}{2(b-\lambda a)} &\geqslant (1-\lambda)\frac{(b-1)ac}{bc-a} \text{ iff } \lambda \geqslant \frac{b(a+(b-2)c)}{2ac(b-1)} \\ \frac{b(2a-c)-a}{2ac(b-1)} &< \frac{b(a+(b-2)c)}{2ac(b-1)} < \frac{b}{2a}\end{aligned}$$

Now, if $2a \leq c, c(2-b) \geqslant a$: In that case, $\frac{c(1-\lambda)}{2(1-c\lambda)} > (1-\lambda)\frac{(b-1)ac}{bc-a}$ and $\frac{ab(1-\lambda)}{2(b-\lambda a)} \geqslant (1-\lambda)\frac{(b-1)ac}{bc-a}$. Consider firstly when $b > a$:Now, if $\frac{b}{2a} < \lambda$, then $\frac{ab(1-\lambda)}{2(b-\lambda a)} \geqslant (1-\lambda)a$. In that case NBS yields $(1-\lambda)a$ to Player 1, decreasing in λ. By choosing $0 \leq \lambda \leq \frac{b}{2a}$, NBS yields $\frac{ab(1-\lambda)}{2(b-\lambda a)}$ to player 1 which is decreasing in λ. Thus, if $b \geqslant 2a$, then $\lambda^* = 0$. If $b < 2a$, by choosing $\frac{b}{2a} < \lambda < 1$, NBS yields $1 - \lambda$ to player 1, hence $\lambda^* = 0$. If $b < a$, choosing $0 \leq \lambda \leq \frac{b}{2a}$, NBS yields $\frac{ab(1-\lambda)}{2(b-\lambda a)}$ to player 1 which is increasing in λ in that case.

If $2a \leq c, c(2-b) < a$: In that case, $\frac{c(1-\lambda)}{2(1-c\lambda)} > (1-\lambda)\frac{(b-1)ac}{bc-a}$ and $\frac{b(a+(b-2)c)}{2ac(b-1)} > 0$. Thus, by choosing $0 \leq \lambda \leq \frac{b(a+(b-2)c)}{2ac(b-1)}$, Player 1 gets $(1-\lambda)\frac{(b-1)ac}{bc-a}$. Hence,if $\frac{b(a+(b-2)c)}{2ac(b-1)} > 1$, $\lambda^* = 0$. Suppose not. Then, assume $\frac{b}{2a} < 1$: Choosing $\frac{b(a+(b-2)c)}{2ac(b-1)} \leq \lambda \leq \frac{b}{2a}$, NBS yields $\frac{ab(1-\lambda)}{2(b-\lambda a)}$ decreasing in λ. Choosing $\frac{b}{2a} \leq \lambda \leq 1$, NBS yields $a(1-\lambda)$ to player 1. Thus, $\lambda^* = 0$. If $\frac{b}{2a} \geqslant 1$:Choosing $\frac{b(a+(b-2)c)}{2ac(b-1)} \leq \lambda \leq 1$, NBS yields $\frac{ab(1-\lambda)}{2(b-\lambda a)}$ decreasing in λ.Thus, $\lambda^* = 0$.

If $a > \frac{c}{2}, b \geqslant \frac{a}{2a-c}$: In that case, by choosing $0 \leq \lambda \leq \frac{b(a+(b-2)c)}{2ac(b-1)} (< \frac{1}{c})$, Player 1 gets $(\frac{c}{2}.\frac{1-\lambda}{1-c\lambda})$ which is increasing in λ iff $c > 1$. Thus, Player 1 makes better off if $c > 1$, easily shown $\lambda^* = 0$ if $c \leq 1$.

If $a > \frac{c}{2}, b(2a-c) < a, c(2-b) \geqslant a$: This case redeces to the case where $2a \leq c, c(2-b) \geqslant a$.

If $2a > c, b < \frac{a}{2a-c}, c(2-b) < a$: Similarly, this case reduces to case where $2a \leq c, c(2-b) < a$.

Thus, to sum up, Player 1 becomes better off if

$$(2a \leq c, c(2-b) \geqslant a, a > b)$$

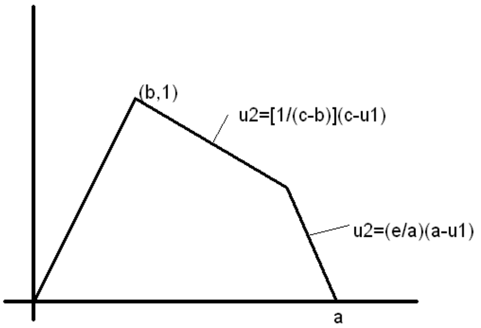
or

$$(a > \frac{c}{2}, b(2a-c) < a, c(2-b) \geqslant a, b < a)$$

or

$$(a > \frac{c}{2}, b \geqslant \frac{a}{2a-c}, c > 1)$$

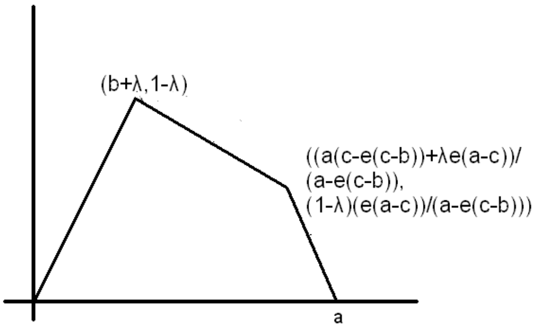
For this case and for the remaining cases, similar to previous cases Steckelberg and Nash equilibria yields the same solution when either player 1 or player 2 chooses optimal predonation share noting that cases when player 1 becomes better off and cases when player 2 becomes better off do not coincide.Next case:



$$\begin{aligned} f^N(S,d) &= (b,1) && \text{if } b \geqslant \frac{c}{2} \\ &= (\frac{a}{2},\frac{e}{2}) && \text{if } b < \frac{c}{2}, e(c-b) < 2c-a \\ &= (\frac{c}{2},\frac{c}{2(c-b)}) && \text{if } b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac \end{aligned}$$

and $f^N(S,d) = (\frac{a(c-e(c-b))}{a-e(c-b)}, \frac{e(a-c)}{a-e(c-b)})$ if $b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac, e(c-b) \geqslant 2c-a$

If Player 2 makes predonation:



Then the upper line transforms to $u_2 = \frac{1-\lambda}{(c-b)-\lambda}(c-u_1)$ which is downward sloped if $\lambda \leq (c-b)$. On this line:

$$\begin{aligned} \frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{c}{2} > u_1 \\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{c}{2} < u_1 \end{aligned}$$

The lower line transforms to $u_2 = \frac{(1-\lambda)e}{a-\lambda e}(a-u_1)$. On this line:

$$\begin{aligned} \frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{a}{2} > u_1 \\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{a}{2} < u_1 \end{aligned}$$

Again we will state some facts before continuing:

$$\begin{aligned} \frac{c}{2} &\geqslant \frac{a(c-e(c-b))+\lambda e(a-c)}{a-e(c-b)} \text{ if } \lambda \leq \frac{ac+(c-2a)e(c-b)}{2e(c-a)} \\ \frac{a}{2} &\geqslant \frac{a(c-e(c-b))+\lambda e(a-c)}{a-e(c-b)} \text{ if } \lambda \leq \frac{a((2c-a)-e(c-b))}{2e(c-a)} \end{aligned}$$

$$\frac{a((2c-a)-e(c-b))}{2e(c-a)} < \frac{a}{e}$$

Again we will continue case by case:

If $b \geqslant \frac{c}{2}$, we have $b+\lambda \geqslant \frac{c}{2}$, yielding $1-\lambda$ to player 2 ,thus $\lambda^* = 0$.

If $b < \frac{c}{2}, e(c-b) < 2c-a$: In that case if $\frac{c}{2}-b \geqslant 1$ we have $\frac{c}{2} \geqslant b+\lambda$. In that case, we have $c-b \geqslant 1$. Assume firstly that $a \geqslant e$. By choosing $0 \leq \lambda \leq \frac{a((2c-a)-e(c-b))}{2e(c-a)}$, NBS yields $(\frac{a}{2}, \frac{a}{2}.\frac{(1-\lambda)e}{a-\lambda e})$ whose second component is increasing in λ if $e > a$. Thus, if $\frac{a((2c-a)-e(c-b))}{2e(c-a)} \geqslant 1$, $\lambda^* = 0$. Suppose $\frac{a((2c-a)-e(c-b))}{2e(c-a)} < 1 \leq \frac{ac+(c-2a)e(c-b)}{2e(c-a)}$. Choosing $\frac{a((2c-a)-e(c-b))}{2e(c-a)} < \lambda \leq 1$,Player 2 gets $(1-\lambda)\frac{e(a-c)}{a-e(c-b)}$,thus $\lambda^* = 0$. If $\frac{ac+(c-2a)e(c-b)}{2e(c-a)} < 1$, choosing $\frac{ac+(c-2a)e(c-b)}{2e(c-a)} < \lambda \leq 1$ NBS yields $(\frac{c}{2}, \frac{1-\lambda}{(c-b)-\lambda}.\frac{c}{2})$ whose second component is increasing if $c-b < 1$ Thus, if $\frac{c}{2}-b \geqslant 1$ and $a \geqslant e$ Player 2 can not become better off. Assume $a < e$. In that case choosing $0 \leq \lambda \leq \frac{a((2c-a)-e(c-b))}{2e(c-a)}$, NBS yields $(\frac{a}{2}, \frac{a}{2}.\frac{(1-\lambda)e}{a-\lambda e})$ whose second component is increasing in λ. Thus, Player 2 makes better off. If $\frac{c}{2}-b < 1$, then we have $\frac{c}{2} \geqslant b+\lambda$ if $\frac{c}{2}-b \geqslant \lambda$. Thus, when $\frac{ac+(c-2a)e(c-b)}{2e(c-a)} \leq (\frac{c}{2}-b)$,that is when $a \leq e(a-b)$ and $a < e$ Player 2 becomes better off. Thus, Player 2 becomes better off if

$$\frac{c}{2}-b \geqslant 1 \text{ and } a < e$$

or

$$\frac{c}{2} - b < 1 \text{ and } a < e \text{ and } a \leq e(a-b)$$

If $b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac,$ we have $\frac{c}{2} < \frac{a(c-e(c-b))+\lambda e(a-c)}{a-e(c-b)}$: If $\frac{c}{2} - b \geqslant \lambda$, $f^N(S,d) = (\frac{c}{2}, \frac{c}{2} \cdot \frac{(1-\lambda)}{(c-b)-\lambda})$ whose second component is increasing if $(c-b) < 1$. If $\frac{c}{2} - b < \lambda$, NBS yields $1-\lambda$ to player 2. Thus, if $c - b \geqslant 1$, $\lambda^* = 0$. If $c - b < 1$, by choosing $\lambda = \frac{c}{2} - b$, player 2 makes better off if $c - b < 1$.

If $b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac, e(c-b) \geqslant 2c - a$, we have $\frac{a}{2} < \frac{a(c-e(c-b))+\lambda e(a-c)}{a-e(c-b)}$. By choosing $0 \leq \lambda \leq \frac{ac+(c-2a)e(c-b)}{2e(c-a)}$, NBS yields $(1-\lambda)\frac{e(a-c)}{a-e(c-b)}$ to player 2, hence $\lambda^* = 0$.

Thus,Player 2 becomes better off if

$$(b < \frac{c}{2}, e(c-b) < 2c - a, \frac{c}{2} - b \geqslant 1 \text{ and } a < e)$$

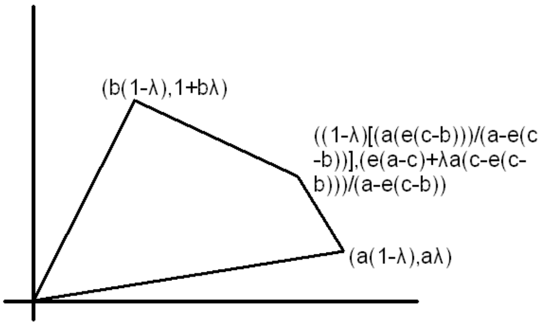
or

$$(b < \frac{c}{2}, e(c-b) < 2c - a, \frac{c}{2} - b < 1 \text{ and } a < e \text{ and } a \leq e(a-b))$$

or

$$(b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac, c - b < 1)$$

Now, check whether Player 1 becomes better off:



Then the upper line transforms to $u_2 = \frac{c}{c-b} + \frac{\lambda(c-b)-1}{(c-b)(1-\lambda)}u_1$. On this line:

$$\begin{aligned}
\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{(1-\lambda)c}{2(1-\lambda(c-b))} > u_1 \\
\frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{(1-\lambda)c}{2(1-\lambda(c-b))} < u_1
\end{aligned}$$

The lower line transforms to $u_2 = e + \frac{(\lambda a-e)}{a(1-\lambda)}u_1$. On this line:

$$\begin{aligned}
\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{ae(1-\lambda)}{2(e-\lambda a)} > u_1 \\
\frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{ae(1-\lambda)}{2(e-\lambda a)} < u_1
\end{aligned}$$

Again, we will consider each case seperately:

If $b \geqslant \frac{c}{2}$, Player 1 gets $b(1-\lambda)$.Thus, $\lambda^* = 0$.

If $b < \frac{c}{2}, e(c-b) < 2c-a$: In that case we have $\frac{ae(1-\lambda)}{2(e-\lambda a)} \geqslant (1-\lambda)\frac{a(c-e(c-b))}{a-e(c-b)}$. If $\lambda < \frac{e}{2a}$, we have $\frac{ae(1-\lambda)}{2(e-\lambda a)} < a(1-\lambda)$. Thus when $e \geqslant 2a$, $\frac{ae(1-\lambda)}{2(e-\lambda a)} < a(1-\lambda)$.In that case NBS yields $\frac{ae(1-\lambda)}{2(e-\lambda a)}$ to player 1 which is increasing if $a < e$. Thus, Player 1 cannot become better off. On the other hand, if $a > e$, Player 1 can become better off by choosing $\lambda = \frac{e}{2a}$.

Thus, Player 1 becomes better off if

$$b < \frac{c}{2}, e(c-b) < 2c-a, a > e$$

If $b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac$, we have $b(1-\lambda) < \frac{(1-\lambda)c}{2(1-\lambda(c-b))}$. Choosing $0 \leq \lambda \leq \frac{e(c-b)(2a-c)-ac}{2a(c-b)(e(c-b)-c)}$, Player 1 gets $\frac{c(1-\lambda)}{2(1-\lambda(c-b))}$ which is increasing if $(c-b) > 1$.

Thus, Player 1 becomes better off if

$$b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac, (c-b) > 1$$

If $b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac, e(c-b) \geqslant 2c-a$, again NBS yields $\frac{ae}{2}.\frac{1-\lambda}{e-\lambda a}$ to player 1 decreasing in λ.

Thus, $\lambda^* = 0$.

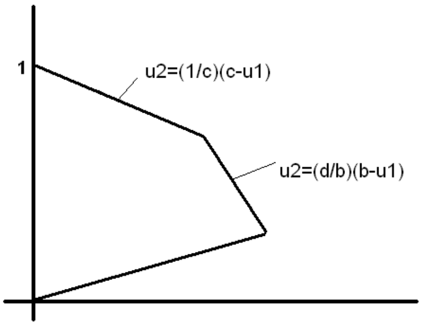
Thus, Player 1 becomes better off if

$$b < \frac{c}{2}, e(c-b) < 2c-a, a > e$$

or

$$b < \frac{c}{2}, (2a-c)e(c-b) \geqslant ac, (c-b) > 1$$

Next case:



In that case,

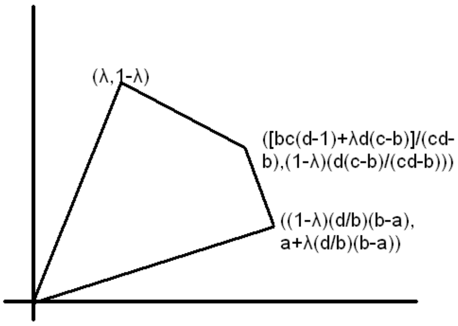
$$\begin{aligned} f^N(S,d) &= (a, \frac{d}{b}.(b-a)) && \text{if } \frac{b}{2} \geqslant a \\ &= (\frac{b}{2}, \frac{d}{2}) && \text{if } \frac{b}{2} < a, (2-d)c \geqslant b \\ &= (\frac{bc(d-1)}{cd-b}, \frac{d(c-b)}{cd-b}) && \text{if } \frac{b}{2} < a, (2-d)c < b, b \geqslant (2b-c)d \\ &= (\frac{c}{2}, \frac{1}{2}) && \text{if } \frac{b}{2} < a, b < (2b-c)d \end{aligned}$$

If Player 2 makes predonation,Then the upper line transforms to $u_2 = \frac{1-\lambda}{c-\lambda}(c-u_1)$. On this line:

$$\begin{aligned} \frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{c}{2} > u_1 \\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{c}{2} < u_1 \end{aligned}$$

The lower line transforms to $u_2 = \frac{d(1-\lambda)}{b-\lambda d}(b-u_1)$. On this line:

$$\begin{aligned}\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{b}{2} > u_1\\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{b}{2} < u_1\end{aligned}$$



Now,some facts:

$$\begin{aligned}\frac{c}{2} &\leq \frac{bc(d-1)+\lambda d(c-b)}{cd-b} \text{ iff } \lambda \geqslant \frac{c(b-d(2b-c))}{2d(c-b)}\\ \frac{b}{2} &\leq \frac{bc(d-1)+\lambda d(c-b)}{cd-b} \text{ iff } \lambda \geqslant \frac{b(c(2-d)-b)}{2d(c-b)}\\ \frac{b}{2} &\geqslant a+\frac{\lambda d}{b}(b-a) \qquad \text{iff } \frac{b(b-2a)}{2d(b-a)} \geqslant \lambda\end{aligned}$$

If $\frac{b}{2} \geqslant a$: If $\frac{b(b-2a)}{2d(b-a)} \geqslant 1$, then Player 2 gets $(1-\lambda)\frac{d}{b}(b-a)$. Thus, $\lambda^* = 0$. If $\frac{b(c(2-d)-b)}{2d(c-b)} \leq \frac{b(b-2a)}{2d(b-a)}$, by choosing $0 \leq \lambda \leq \frac{b(c(2-d)-b)}{2d(c-b)}$, Player 2 gets $\frac{d(1-\lambda)}{b-\lambda d}\frac{b}{2}$,increasing if $d > b$. In that case, we should check $\lambda = 0$ and $\lambda = \frac{b(c(2-d)-b)}{2d(c-b)}$. Calculations show that $\lambda^* = 0$. Easy to see in other cases $\lambda^* = 0$.

If $\frac{b}{2} < a, (2-d)c \geqslant b$, we have $\frac{b}{2} < a + \frac{\lambda d}{b}(b-a)$. By choosing $0 \leq \lambda \leq \frac{b(c(2-d)-b)}{2d(c-b)}(< \frac{b}{d})$, Player 2 gets $\frac{d(1-\lambda)}{b-\lambda d}\frac{b}{2}$,increasing if $d > b$.

If $\frac{b}{2} < a, (2-d)c < b, b \geqslant (2b-c)d$: In that case, $\frac{b}{2} \geqslant \frac{bc(d-1)+\lambda d(c-b)}{cd-b}$ .Now, if $\frac{b(c(2-d)-b)}{2d(c-b)} \geqslant 1$, NBS yields $(1-\lambda)\frac{d(c-b)}{cd-b}$. If $\frac{b(c(2-d)-b)}{2d(c-b)} < 1$, if $\frac{c}{2} < 1$, by choosing $\frac{b(c(2-d)-b)}{2d(c-b)} < \lambda \leq \frac{c}{2}$, Player 2 gets $\frac{1-\lambda}{c-\lambda}.\frac{c}{2}$ increasing in λ if $c < 1$.. Choosing $\frac{c}{2} \leq \lambda \leq 1$ player 2 gets $1-\lambda$. Thus, we should check $\lambda = 0$ and $\lambda = \frac{c}{2}$. Calculations yield $\lambda^* = 0$. If $\frac{c}{2} \geqslant 1$, choosing $\frac{b(c(2-d)-b)}{2d(c-b)} < \lambda \leq \frac{c}{2}$, Player 2 gets $\frac{1-\lambda}{c-\lambda}.\frac{c}{2}$ decreasing in λ. Thus, $\lambda^* = 0$.

If $\frac{b}{2} < a, b < (2b-c)d$, we have $\frac{c}{2} \leq \frac{bc(d-1)+\lambda d(c-b)}{cd-b}$. If $\frac{c}{2} < 1$, by choosing $0 \leq \lambda \leq \frac{c}{2}$, player 2 gets $\frac{1-\lambda}{c-\lambda}.\frac{c}{2}$ increasing in λ if $c < 1$. If $\frac{c}{2} \geqslant 1$, choosing $0 \leq \lambda \leq 1 < \frac{c}{2}$ Player 2 gets $\frac{1-\lambda}{c-\lambda}.\frac{c}{2}$ decreasing in λ.

Thus, Player 2 becomes better off if $[\frac{b}{2} < a, (2-d)c \geqslant b, d > b]$, or $[\frac{b}{2} < a, b < (2b-c)d, c < 1]$

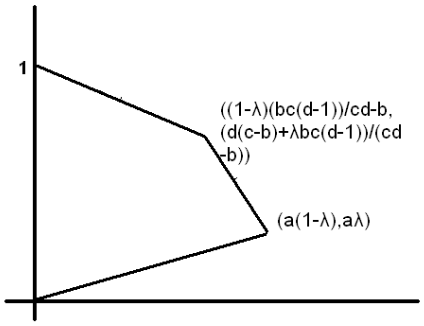
Now, let's check whether player 1 can become better off via predonation.

the upper line transforms to $u_2 = 1 + \frac{\lambda c-1}{c(1-\lambda)}u_1$. On this line:

$$\begin{aligned}
\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{c(1-\lambda)}{2(1-\lambda c)} > u_1 \\
\frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{c(1-\lambda)}{2(1-\lambda c)} < u_1
\end{aligned}$$

The lower line transforms to $u_2 = d + \frac{\lambda b-d}{b(1-\lambda)}u_1$. On this line:

$$\begin{aligned}
\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{db(1-\lambda)}{2(d-\lambda b)} > u_1 \\
\frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{db(1-\lambda)}{2(d-\lambda b)} < u_1
\end{aligned}$$



Now,some facts:

$$\begin{aligned}
\frac{c(1-\lambda)}{2(1-\lambda c)} &\geqslant (1-\lambda)\frac{bc(d-1)}{cd-b} \text{ iff } \lambda \geqslant \frac{-b+d(2b-c)}{2bc(d-1)} \\
\frac{db(1-\lambda)}{2(d-\lambda b)} &\geqslant (1-\lambda)\frac{bc(d-1)}{cd-b} \text{ iff } \lambda \geqslant \frac{d(c(d-2)+b)}{2bc(d-1)} \\
\frac{db(1-\lambda)}{2(d-\lambda b)} &\geqslant a(1-\lambda) \qquad \text{ iff } \frac{d(2a-b)}{2ab} \leq \lambda
\end{aligned}$$

If $\frac{b}{2} \geqslant a$: In that case $\frac{db(1-\lambda)}{2(d-\lambda b)} \geqslant a(1-\lambda)$. Thus, yielding $a(1-\lambda)$,hence $\lambda^* = 0$.

If $\frac{b}{2} < a, (2-d)c \geqslant b$, we have $\frac{db(1-\lambda)}{2(d-\lambda b)} \geqslant (1-\lambda)\frac{bc(d-1)}{cd-b}$ and $\frac{c(1-\lambda)}{2(1-\lambda c)} \geqslant (1-\lambda)\frac{bc(d-1)}{cd-b}$. If $c < 1$, we have $\frac{1}{c} > 1$ thus $\frac{d}{b} > 1$. By choosing $0 \leq \lambda < \frac{d(2a-b)}{2ab}$, NBS yields $\frac{db(1-\lambda)}{2(d-\lambda b)}$ to player 1 which is increasing if $b > d$. If $\lambda \geqslant \frac{d(2a-b)}{2ab}$, Player 1 gets $a(1-\lambda)$. Thus, if $d \geqslant b$ Player 1 can not become better off. If,on the other hand, $d < b$, Player 1 becomes better off.

Thus, Player 1 becomes better off if $b > d$.

If $\frac{b}{2} < a, (2-d)c < b, b \geqslant (2b-c)d$: In that case, $\frac{c(1-\lambda)}{2(1-\lambda c)} \geqslant (1-\lambda)\frac{bc(d-1)}{cd-b}$. By choosing $0 \leq \lambda \leq \frac{d(c(d-2)+b)}{2bc(d-1)}$ player 1 gets $(1-\lambda)\frac{bc(d-1)}{2bc(d-1)}$. If $\frac{d(c(d-2)+b)}{2bc(d-1)} > 1$, then we are done. If $\frac{d(c(d-2)+b)}{2bc(d-1)} < 1$: Choosing $\frac{d(c(d-2)+b)}{2bc(d-1)} \leq \lambda \leq \frac{d(2a-b)}{2ab}$, player 1 gets $\frac{db(1-\lambda)}{2}$. Thus, $\lambda^* = 0$.

If $\frac{b}{2} < a, b < (2b-c)d$, choosing $0 \leq \lambda \leq \frac{-b+d(2b-c)}{2bc(d-1)}$ NBS yields $\frac{c(1-\lambda)}{2(1-\lambda c)}$ to player 1. Thus, if $c > 1$ Player 1 can become better off. Easy to verify when $c \leq 1$, Player 1 can not become better off.

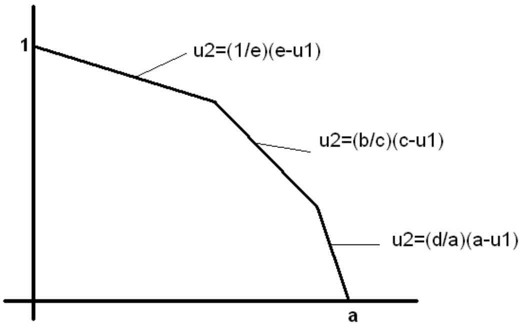
Thus, Player 1 becomes better off if

$$\frac{b}{2} < a, (2-d)c \geqslant b, b > d$$

or

$$\frac{b}{2} < a, b < (2b-c)d, c > 1$$

Final case:



In that case, if $2c \leq e, f^N(S,d)$

$$\begin{aligned}
&= (\frac{(b-1)ec}{eb-c}, \frac{b(e-c)}{eb-c}) \quad \text{if } e(2-b) < c, b(2c-a) < cd \\
&= (\frac{a}{2}, \frac{d}{2}) \qquad\qquad\quad \text{if } b(2c-a) \geqslant cd \\
&= (\frac{c}{2}, \frac{b}{2}) \qquad\qquad\qquad \text{if } b(2c-a) < cd, e(2-b) \geqslant c, ab \leq d(2a-c) \\
&= (\frac{(d-b)ac}{cd-ab}, \frac{bd(c-a)}{cd-ab}) \quad \text{if } \ b(2c-a) < cd, e(2-b) \geqslant c, ab > d(2a-c)
\end{aligned}$$

If $2c > e$:

$$\begin{aligned}
&= (\frac{1}{2}, \frac{e}{2}) \qquad\qquad \text{if } (2c-e)b \geqslant c \\
&= (\frac{(b-1)ec}{eb-c}, \frac{b(e-c)}{eb-c})\text{if}(2c-e)b < c, e(2-b) < c, b(2c-a) < cd \\
&= (\frac{a}{2}, \frac{d}{2}) \qquad \text{if } (2c-e)b < c, b(2c-a) \geqslant cd \\
&= (\frac{c}{2}, \frac{b}{2})\text{if } (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab \leq d(2a-c)
\end{aligned}$$

and

$$= (\frac{(d-b)ac}{cd-ab}, \frac{bd(c-a)}{cd-ab})\text{if } (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab > d(2a-c)$$

If Player 2 makes predonation,Then the upper line transforms to $u_2 = \frac{1-\lambda}{e-\lambda}(e - u_1)$. On this line:

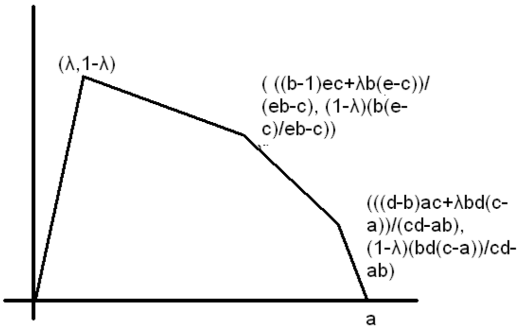
$$\begin{aligned}
\frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{e}{2} > u_1 \\
\frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{e}{2} < u_1
\end{aligned}$$

The middle line transforms to $u_2 = \frac{b(1-\lambda)}{c-\lambda b}(c - u_1)$. On this line:

$$\begin{aligned} \frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{c}{2} > u_1 \\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{c}{2} < u_1 \end{aligned}$$

The lower line transforms to $u_2 = \frac{d(1-\lambda)}{a-\lambda d}(a - u_1)$. On this line:

$$\begin{aligned} \frac{\partial(u_1u_2)}{\partial u_1} &> 0 \text{ if } \frac{a}{2} > u_1 \\ \frac{\partial(u_1u_2)}{\partial u_1} &< 0 \text{ if } \frac{a}{2} < u_1 \end{aligned}$$



Now,some facts:

$$\begin{aligned} \frac{e}{2} &\geqslant \frac{(b-1)ec+\lambda b(e-c)}{be-c} && \text{iff } \lambda \leq \frac{e(b(e-2c)+c)}{2b(e-c)} \\ \frac{c}{2} &\geqslant \frac{(b-1)ec+\lambda b(e-c)}{be-c} && \text{iff } \lambda \leq \frac{c(e(2-b)-c)}{2b(e-c)} \\ \frac{c}{2} &> \frac{(d-b)ac+\lambda bd(c-a)}{cd-ab} && \text{iff } \frac{c(d(1-2a)+ab)}{2bd(c-a)} > \lambda \\ \frac{a}{2} &< \frac{(d-b)ac+\lambda bd(c-a)}{cd-ab} && \text{iff } \frac{a(b(2c-a)-cd)}{2bd(c-a)} < \lambda \end{aligned}$$

If $2c \leq e, e(2-b) < c, b(2c-a) < cd$, we have $\frac{a}{2} < \frac{(d-b)ac+\lambda bd(c-a)}{cd-ab}$, $\frac{c}{2} \leq \frac{(b-1)ec+\lambda b(e-c)}{be-c}$. In that case if $e \in [\lambda, \frac{(b-1)ec+\lambda b(e-c)}{be-c}]$, $f^N(S,d) = (\frac{e}{2}, \frac{e}{2}\frac{1-\lambda}{e-\lambda})$. If $\frac{e}{2} < \lambda$, $f^N(S,d) = (\lambda, 1-\lambda)$. If $\frac{e}{2} > \frac{(b-1)ec+\lambda b(e-c)}{be-c}$, NBS yields $(1-\lambda)\frac{b(e-c)}{be-c}$. Now, $e \in [\lambda, \frac{(b-1)ec+\lambda b(e-c)}{be-c}]$ iff $\frac{e}{2} \geqslant \lambda \geqslant \frac{e(b(e-2c)+c)}{2b(e-c)} > 0$. If $\frac{e(b(e-2c)+c)}{2b(e-c)} \geqslant 1$, we have $\lambda^* = 0$. If $\frac{e}{2} \geqslant 1 \geqslant \frac{e(b(e-2c)+c)}{2b(e-c)}$, by choosing $\frac{e(b(e-2c)+c)}{2b(e-c)} < \lambda < 1$, Player

2 will get $\frac{e}{2}\frac{1-\lambda}{e-\lambda}$. If $1 \geqslant \frac{e}{2}$, if $e < 1$, then $\lambda^* = \frac{e}{2}$ or $\lambda^* = 0$. Calculations show that $\lambda^* = 0$.

If $2c \leq e, b(2c-a) \geqslant cd$: if $e(2-b) \leq c, \frac{c}{2} \leq \frac{(b-1)ec+\lambda b(e-c)}{be-c}$. In that case this reduces to case 1. Thus, $\lambda^* = 0$. If $e(2-b) > c$, by choosing $0 \leq \lambda \leq \frac{a(b(2c-a)-cd)}{2bd(c-a)}$, NBS yields $(\frac{a}{2}.\frac{d(1-\lambda)}{a-\lambda d})$, increasing in λ if $a < d$. Thus, player 2 makes better off if $2c \leq e, b(2c-a) \geqslant cd, e(2-b) > c, a < d$.

If $2c \leq e, b(2c-a) < cd, e(2-b) \geqslant c, ab \leq d(2a-c)$, we have $\frac{a}{2} < \frac{(d-b)ac+\lambda bd(c-a)}{cd-ab}$. By choosing $0 \leq \lambda \leq \frac{c(e(2-b)-c)}{2b(e-c)}$, player 2 gets $\frac{b(1-\lambda)}{c-\lambda b}.\frac{c}{2}$ increasing in λ if $b > c$. Thus,player 2 becomes better off.

If $2c \leq e, b(2c-a) < cd, e(2-b) \geqslant c, ab > d(2a-c)$, we have to check $\lambda = 0$ and $\lambda = \frac{c(e(2-b)-c)}{2b(e-c)}$. Calculations yield $\lambda^* = 0$.

If $2c > e, (2c-e)b \geqslant c$, we have $\frac{e}{2} \leq \frac{(b-1)ec+\lambda b(e-c)}{be-c}$. If $\frac{e}{2} < \lambda$, NBS yields $(1-\lambda)$, if $\frac{e}{2} \geqslant \lambda$, $f^N(S,d) = (\frac{e}{2}, \frac{e}{2}.\frac{1-\lambda}{e-\lambda})$. If $e \geqslant 1$, we have $\lambda^* = 0$. If $e < 1$, we should check $\lambda = 0$ and $\lambda = \frac{e}{2}$. Then, we have $\lambda^* = \frac{e}{2}$. Thus, if $2c > e, (2c-e)b \geqslant c, e < 1$, Player 2 becomes better off.

The remaining cases reduce to the first four case.Thus, Player 2 becomes better off via predonation if

$$\begin{aligned}
[2c &\leq e, b(2c-a) \geqslant cd, e(2-b) > c, a < d] \text{ or} \\
[2c &\leq e, (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab \leq d(2a-c), b > c] \text{ or} \\
[2c &> e, (2c-e)b \geqslant c, e < 1] \text{ or} \\
[2c &> e, (2c-e)b < c, b(2c-a) \geqslant cd, e(2-b) > c, a < d] \text{ or} \\
[2c &> e, (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab \leq d(2a-c), b > c]
\end{aligned}$$

By similar calculations player 1 becomes better off via predonation if

$$
\begin{array}{rcl}
[2c & \leq & e, b(2c-a) \geqslant cd, e(2-b) > c, a > d] \text{ or} \\
[2c & \leq & e, (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab \leq d(2a-c), b < c] \text{ or} \\
[2c & > & e, (2c-e)b < c, b(2c-a) \geqslant cd, e(2-b) > c, a > d] \text{ or} \\
[2c & > & e, (2c-e)b < c, b(2c-a) < cd, e(2-b) \geqslant c, ab \leq d(2a-c), b < c]
\end{array}
$$