EXTENSION OPERATORS FOR SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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By
Muhammed Altun
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assist. Prof. Dr. Alexander Goncharov (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Zafer Nurlu

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Mefharet Kocatepe

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. H. Turgay Kaptanoğlu

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assist. Prof. Dr. Aurelian Gheondea

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet B. Baray
Director of the Institute

ABSTRACT

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Muhammed Altun
Ph.D. in Mathematics
Supervisor: Assist. Prof. Dr. Alexander Goncharov
September, 2005

We start with a review of known linear continuous extension operators for the spaces of Whitney functions. The most general approach belongs to Pawłucki and Pleśniak. Their operator is continuous provided that the compact set, where the functions are defined, has Markov property. In this work, we examine some model compact sets having no Markov property, but where a linear continuous extension operator exists for the space of Whitney functions given on these sets. Using local interpolation of Whitney functions we can generalize the Pawłucki-Pleśniak extension operator. We also give an upper bound for the Green function of domains complementary to generalized Cantor-type sets, where the Green function does not have the Hölder continuity property. And, for spaces of Whitney functions given on multidimensional Cantor-type sets, we give the conditions for the existence and non-existence of a linear continuous extension operator.

Keywords: Extension operator, Green function, Markov inequality, infinitely differentiable functions, polynomial interpolation.

ÖZET

SONSUZ TÜREVLENEBİLİR FONKSİYON UZAYLARI İÇİN GENİŞLETME OPERATÖRLERİ

Muhammed Altun
Matematik Bölümü, Doktora
Tez Yöneticisi: Yard. Doç. Dr. Alexander Goncharov
Eylül, 2005

Whitney fonksiyon uzayları için üretilmiş, şimdiye kadar bilinen lineer, sürekli genişletme operatörleri ile ilgili bir inceleme vererek başladık. Bu operatörler arasında en genel olan operatör Pawłucki ve Pleśniak'a ait olanıdır. Pawłucki-Pleśniak operatörünün sürekli olması, kompakt kümenin Markov özelliğine sahip olmasına bağlıdır. Ondan dolayı bu calışmada, Markov özelliğinin olmadığı, fakat bu kümelerde tanımlanmış Whitney fonksiyon uzayları için lineer, sürekli bir genişletme operatörünün var olduğu, bazı model kompakt kümeleri inceledik. Whitney fonksiyonlarının polinomlarla lokal interpolasyonunu kullanarak, Pawłucki-Pleśniak genişletme operatörünü genelleştirdik. Ayrıca, Green fonksiyonunun Hölder süreklilik özelliğini sağlamadığı bazı genelleştirilmiş Cantor kümeleri için Green fonksiyonuna üstten sınırlandırma yaptık. Son olarak, çok boyutlu Cantor kümelerinde tanımlanmış Whitney fonksiyon uzaylarında, lineer, sürekli bir genişletme operatörünün var olma ve olmama durumları için gerekli şartları verdik.

Anahtar sözcükler: Genişletme operatörü, Green fonksiyonu, Markov eşitsizliği, sonsuz türevlenebilir fonksiyonlar, polinom interpolasyonu.

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Contents

Chapter 1

Introduction

Let U be an open set of $\mathbb{R}^n$. We denote by $\mathcal{E}^m(U)$ (respectively $\mathcal{E}(U)$) the algebra of m times continuously differentiable (respectively infinitely differentiable) functions in U, with the topology of uniform convergence of functions and all their partial derivatives on compact subsets of U. This is the topology defined by the seminorms

$$|f|_m^K = \sup\left\{ |\frac{\partial^{|k|} f}{\partial x^k}(x)| : x \in K, |k| \leq m \right\},$$

where K is a compact subset of U (and m runs through $\mathbb{N}$ in the C^∞ case). Here $x = (x_1, ..., x_n)$, k denotes a multiindex $k = (k_1, ..., k_n) \in \mathbb{N}^n, |k| = k_1 + ... + k_n$ and

$$\frac{\partial^{|k|}}{\partial x^k} = \frac{\partial^{|k|}}{\partial x_1^{k_1} ... \partial x_n^{k_n}}.$$

We will sometimes use m for either a nonnegative integer or $+\infty$ and write $\mathcal{E}^{+\infty}(U) = \mathcal{E}(U)$

When is a function f, defined in a closed subset X of $\mathbb{R}^n$, the restriction of a C^m function in $\mathbb{R}^n$ ([48],[49])? And when can we extend the function f in a continuous linear way? The existence of an extension operator in the C^∞ case was first proved by Mityagin [28] and Seeley [38].

Let $\mathcal{E}^m([0, \infty))$ denote the space of continuous functions g in $[0, \infty)$ such that g is C^m in $(0, \infty)$ and all derivatives of $g|(0, \infty)$ extend continuously to $[0, \infty)$.

Then $\mathcal{E}^m([0,\infty))$ has the structure of a Frechet space defined by the seminorms

$$|g|_m^K = \sup\{|g^{(k)}(y)| : y \in K, |k| \leq m\},$$

where K is a compact subset of $[0,\infty)$ (and m runs through $\mathbb{N}$ in the C^∞ case), and where $g^{(k)}$ denotes the continuation of $(d^k/dy^k)(g|(0,\infty))$ to $[0,\infty)$.

The following theorem gives the extension operator for the half space $[0,\infty)$, and from the proof we can see how the problem gets complicated when we pass from finite m to the case $m = \infty$.

Theorem 1.1 *There is a continuous linear extension operator*

$$E : \mathcal{E}^m([0,\infty)) \longrightarrow \mathcal{E}^m(\mathbb{R})$$

such that $E(g)|[0,\infty) = g$ *for all* $g \in \mathcal{E}^m([0,\infty))$.

Proof: Our problem is to define $E(g)(y)$ when $y < 0$. If $m = 0$ we can define $E(g)(y)$ by reflection in the origin : $E(g)(y) = g(-y), y < 0$. If $m = 1$ we can use a weighted sum of reflections. Consider

$$E(g)(y) = a_1 g(b_1 y) + a_2 g(b_2 y), \quad y < 0$$

Where $b_1, b_2 < 0$. Then $E(g)$ determines a C^1 extension of g provided that the limiting values of $E(g)(y)$ and $E(g)'(y)$ agree with those of $g(-y)$ and $g'(-y)$ as $y \longrightarrow 0-$; in other words if

$$a_1 + a_2 = 1$$
$$a_1 b_1 + a_2 b_2 = 1$$

For distinct $b_1, b_2 < 0$ these equations have a unique solution a_1, a_2.This extension is due to Lichtenstein [24].

Hestenes [21] remarked that the same technique works for any $m < \infty$: a weighted sum of m reflections leads to solving a system of linear equations determined by a Vandermonde matrix.

If $m = \infty$, we can use an infinite sum of reflections [38]:

$$E(g)(y) = \sum_{k=1}^{\infty} a_k \phi(b_k y) g(b_k y), \quad y < 0,$$

where $\{a_k\}, \{b_k\}$ are sequences satisfying

$$\begin{array}{ll} (1) & b_k < 0, b_k \longrightarrow -\infty \quad \text{as} \quad k \longrightarrow \infty; \\ (2) & \displaystyle\sum_{k=1}^{\infty} |a_k||b_k|^n < \infty \quad \text{for all} \quad n \geq 0; \\ (3) & \displaystyle\sum_{k=1}^{\infty} a_k b_k^n = 1 \quad \text{for all} \quad n \geq 0 \end{array}$$

and ϕ is a C^∞ function such that $\phi(y) = 1$ if $0 \leq y \leq 1$ and $\phi(y) = 0$ if $y \geq 2$. In fact condition (1) guarantees that the sum is finite for each $y < 0$. Condition (2) shows that all derivatives converge as $y \longrightarrow 0-$ uniformly in each bounded set, and (3) shows that the limits agree with those of the derivatives of $g(y)$ as $y \longrightarrow 0+$. The continuity of the extension operator also follows from (2).

It is easy to choose sequences $\{a_k\}, \{b_k\}$ satisfying the above conditions. We can take $b_k = -2^k$ and choose a_k using a theorem of Mittag Leffler : there exists an entire function $\sum_{k=1}^{\infty} a_k z^k$ taking arbitrary values (here $(-1)^n$) for a sequence of distinct points (here 2^n) provided that the sequence does not have a finite accumulation point. $\square$

It is clear that Seeley's extension operator actually provides a simultaneous extension of all classes of differentiability.

Mitiagin [28] presented an extension operator for a closed interval in $\mathbb{R}$. Mitiagin in his work proved the fact that the Chebyshev Polynomials $T_n(x) = \cos(n \cos^{-1} x)$ form a basis in the space $C^\infty[-1, 1]$ i.e., for $\Psi(t) \in C^\infty[-1, 1]$ and

$$\xi_n = \frac{1}{\pi} \int_{-1}^{1} \frac{\Psi(x) \cos(n \cos^{-1} x)}{\sqrt{1 - x^2}} dx$$

we have that

$$\Psi(x) = \sum_{n=0}^{\infty} \xi_n T_n(x) \text{ in } C^\infty[-1, 1].$$

A linear transformation of the argument sets up an isomorphism between the spaces $C^\infty[-1,1]$ and $C^\infty[a,b]$, $-\infty < a, b < \infty$; therefore the correspondingly transformed Chebishev polynomials form a basis in the space $C^\infty[a,b]$.

Mitiagin constructs in [28] special extensions $\tilde{T}_n$ for the polynomials $T_n(x)$ and defines the operator $M : C^\infty[-1,1] \longrightarrow C^\infty[-2,2]$ by

$$(M\Psi)(x) = \sum_{n=1}^{\infty} \xi_n(x)(\tilde{T}_n)(x)$$

and by using an infinitely differentiable function $l_0(t)$ on the whole straight line such that

$$l_0(t) \equiv 1 \quad |t| \leq 1 \text{ and } l_0(t) \equiv 0 \quad |t| \geq 1 + \frac{1}{4}$$

he defines the continuous linear extension operator $M' : C^\infty[-1,1] \longrightarrow C^\infty(-\infty,\infty)$ by

$$(M'\Phi)(x) = (M\Phi)(x)l_0(x).$$

1.1 Whitney jets and Whitney's Extension theorem

When we are speaking of extension operators it is important to examine the classical extension theorem of Whitney [48]. Let U be an open subset of $\mathbb{R}^n$, and X a closed subset of U. Whitney's theorem asserts that a function F^0 defined in X is the restriction of a C^m function in U ($m \in \mathbb{N}$ or $m = +\infty$) provided there exists a sequence $(F^k)_{|k|\leq m}$ of functions defined in X which satisfies certain conditions that arise naturally from Taylor's formula.

First we consider $m \in \mathbb{N}$. By a *jet of order* m on X we mean a set of continuous functions $F = (F^k)_{|k|\leq m}$ on X. Here k denotes a multiindex $k = (k_1, ..., k_n) \in \mathbb{N}^n$. Let $J^m(X)$ be the vector space of jets of order m on X. We write

$$|F|_m^K = \sup\{|F^k(x)| : x \in K, |k| \leq m\}$$

if K is a compact subset of X, and $F(x) = F^0(x)$.

There is a linear mapping $J^m : \mathcal{E}^m(U) \longrightarrow J^m(X)$ which associates to each $f \in \mathcal{E}^m(U)$ the jet

$$J^m(f) = \left(\frac{\partial^{|k|} f}{\partial x^k} \bigg| X \right)_{|k| \leq m}$$

For each k with $|k| \leq m$, there is a linear mapping $D^k : J^m(X) \longrightarrow J^{m-|k|}(X)$ defined by $D^k F = (F^{k+l})_{|l| \leq m-|k|}$. We also denote by D^k the mapping of $\mathcal{E}^m(U)$ into $\mathcal{E}^{m-|k|}(U)$ given by

$$D^k f = \frac{\partial^{|k|} f}{\partial x^k}$$

This will not cause any problem since

$$D^k \circ J^m = J^{m-|k|} \circ D^k$$

If $a \in X$ and $F \in J^m(X)$, then the *Taylor polynomial (of order m)* of F at a is the polynomial

$$T_a^m F(x) = \sum_{|k| \leq m} \frac{F^k(a)}{k!} (x-a)^k$$

of degree $\leq m$. Here $k! = k_1!...k_n!$. We define $R_a^m F = F - J^m(T_a^m F)$, so that

$$(R_a^m F)^k(x) = F^k(x) - \sum_{|l| \leq m-|k|} \frac{F^{k+l}(a)}{l!} .(x-a)^l$$

if $|k| \leq m$.

Definition 1.2 *A jet $F \in J^m(X)$ is a* Whitney jet of class C^m *on X if for each $|k| \leq m$*

$$(R_x^m F)^k(y) = o(|x-y|^{m-|k|}) \tag{1.1}$$

as $|x-y| \longrightarrow 0$, $x, y \in X$.

Let $\mathcal{E}^m(X) \subset J^m(X)$ be the subspace of Whitney jets of class C^m. $\mathcal{E}^m(X)$ is a Frechet space with the seminorms

$$\|F\|_m^K = |F|_m^K + \sup \left\{ \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} : x, y \in K, x \neq y, |k| \leq m \right\},$$

where $K \subset X$ is compact.

Two more equivalent systems of seminorms could be used to identify the topology in $\mathcal{E}^m(X)$, which are:

$$\|F\|_m^K = |F|_m^K + \sup\left\{\sum_{|k|\leq m} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} : x, y \in K, x \neq y\right\},$$

and the other is

$$\|F\|_m^K = \max\left\{|F|_m^K, \sup\left\{\frac{|R_x^{m-|k|} F^k(y)|}{|x-y|^{m-|k|}} : x, y \in K, x \neq y, |k| \leq m\right\}\right\}.$$

Remark 1.3 *If $F \in J^m(U)$ and for all $x \in U, |k| \leq m$ we have*

$$\lim_{y \longrightarrow x} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} = 0$$

then there exists $f \in \mathcal{E}^m(U)$ such that $F = J^m(f)$. This simple converse of Taylor's theorem shows that the two spaces we have denoted by $\mathcal{E}^m(U)$ are equivalent. On $\mathcal{E}^m(U)$, the topologies defined by the seminorms $|.|_m^K$, $\|.\|_m^K$ are equivalent (by the open mapping theorem).

Theorem 1.4 *(Whitney [48]) There is a continuous linear mapping*

$$W : \mathcal{E}^m(X) \longrightarrow \mathcal{E}^m(U)$$

such that $D^k W(F)(x) = F^k(x)$ if $F \in \mathcal{E}^m(X)$, $x \in X$, $|k| \leq m$, and $W(F)\big|(U - X)$ is C^∞.

Remark 1.5 *The condition (1.1) cannot be weakened to*

$$\lim_{y \longrightarrow x} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} = 0 \tag{1.2}$$

for all $x \in X$, $|k| \leq m$.

For example let A be the set of points (using one variable) $x = 0, 1/2^s$ and $1/2^s + 1/2^{2s}$ $(s = 1, 2, ...)$. Set $f(x) = 0$ at $x = 0$ and $1/2^s$ and $f(x) = 1/2^{2s}$ at

$x = 1/2^s + 1/2^{2s}$.Set $f^1(x) \equiv 0$ in A. The above condition is satisfied but there's no extension of $f(x)$ which has continuous first derivative.

The proof of Theorem 1.4 is based on the following fundamental lemma (Whitney partition of unity) [48].

Lemma 1.6 *Let K be a compact subset of $\mathbb{R}^n$. There exist a countable family of functions $\Phi_l \in \mathcal{E}(\mathbb{R}^n - K)$, $l \in I$, such that*
(1) $\{supp\Phi_l\}_{l\in I}$ is locally finite: in fact each x belongs to at most 3^n of the $supp\Phi_l$'s,
(2) $\Phi_l \geq 0$ for all $l \in I$, and $\sum_{l\in I} \Phi_l = 1$, $x \in \mathbb{R}^n - K$,
(3) $2d(supp\Phi_l, K) \geq diam(supp\Phi_l)$ for all $l \in I$,
(4) there exist constants C_k depending only on k and n, such that if $x \in \mathbb{R}^n - K$, then

$$|D^k\Phi_l(x)| \leq C_k \left(1 + \frac{1}{d(x,K)^{|k|}}\right).$$

The proof of Theorem 1.4 can be done by a simple partition of unity argument it is enough to assume $U = \mathbb{R}^n$ and $X = K$, a compact subset of $\mathbb{R}^n$. Let $\{\Phi_l\}_{l\in I}$ be a Whitney partition of unity on $\mathbb{R}^n - K$.

For each $l \in I$, choose $a_l \in K$ such that

$$d(\mathrm{supp}\Phi_l, K) = d(\mathrm{supp}\Phi_l, a_l).$$

Let $F \in \mathcal{E}^m(K)$. Define a function $f = W(F)$ on $\mathbb{R}^n$ by

$$f(x) = F^0(x) \quad x \in K \text{ and } f(x) = \sum_{l\in I} \Phi_l(x) T^m_{a_l} F(x) \quad x \notin K$$

Clearly $f = W(F)$ depends linearly on F, and is C^∞ on $\mathbb{R}^n - K$. We must show that f is C^m, $D^k f|_K = F^k$, $|k| \leq m$, and W is continuous. If $|k| \leq m$, we write

$$f^k(x) = D^k f(x), \quad x \notin K.$$

By a *modulus of continuity* we mean a continuous increasing function $\alpha : [0,\infty) \to [0,\infty)$ such that $\alpha(0) = 0$ and α is concave downwards. There exists a modulus of continuity α such that

$$|(R^m_a F)^k(x)| \leq \alpha(|x-a|) \cdot |x-a|^{m-|k|}$$

for all $a, x \in K$, $|k| \leq m$, and

$$\alpha(t) = \alpha(\text{diam}K), \qquad t \geq \text{diam}K,$$

$$||F||_m^k = |F|_m^k + \alpha(\text{diam}K).$$

In fact, define $\beta : [0, \infty) \to [0, \infty)$ by $\beta(0) = 0$ and

$$\beta(t) = \sup\left\{\frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} : x, y \in K, x \neq y, |x-y| \leq t, |k| \leq m\right\} \quad t \geq 0.$$

Then β is increasing and continuous at 0. We get α from the convex envelope of the positive t-axis and the graph of β.

Let Λ be a cube in $\mathbb{R}^n$ such that $K \subset \text{Int}\Lambda$. Let $\lambda = \sup_{x\in\Lambda} \text{d}(x, K)$. We have the following assertion from [48].

There exists a constant C depending only on m, n, λ such that if $|k| \leq m,\ a \in K,\ x \in \Lambda$, then

$$|f^k(x) - D^k T_a^m F(x)| \leq C\alpha(|x-a|) \cdot |x-a|^{m-|k|}. \tag{1.3}$$

Once (1.3) is established, the proof of the theorem can be completed as follows. Let (j) denote the multiindex whose j'th component is 1 as whose other components are 0. If $a \in K,\ x \notin K,\ |k| < m$, then

$$|f^k(x) - f^k(a) - \sum_{j=1}^{n}(x_j - a_j)f^{k+(j)}(a)| \leq$$

$$|f^k(x) - D^k T_a^m F(x)| + |D^k T_a^m F(x) - D^k T_a^m F(a) - \sum_{j=1}^{n}(x_j - a_j)D^{k+(j)}T_a^m F(a)|.$$

The first term on the right hand side is $o(|x-a|)$ by (1.3), while the second is $o(|x-a|)$ since $T_a^m F(x)$ is a polynomial. Hence f^k is continuously differentiable and $\frac{\partial f^k}{\partial x_j} = f^{k+(j)}$.

Applying (1.3) to a point $x \in \Lambda$ and a point $a \in K$ such that $d(x, K) = d(x, a)$, we have

$$\begin{aligned} |D^k f(x)| &\leq |D^k T_a^m F(x)| + C\alpha(\lambda)\lambda^{m-|k|} \\ &\leq \sum_{|i|\leq m-|k|} \frac{\lambda^{|i|}}{i!}|F|_m^K + C\lambda^{m-|k|}(||F||_m^K - |F|_m^K). \end{aligned}$$

Hence there is a constant C_λ (depending only on m, n, λ) such that

$$|W(F)|_m^\Lambda \leq C_\lambda ||F||_m^K.$$

In particular, W is a continuous linear operator.

Definition 1.7 *Let U be an open subset of $\mathbb{R}^n$ and X a closed subset of U. A* jet of infinite order *on X is a sequence of continuous functions $F = (F^k)_{k \in \mathbb{N}}$ on X. Let $J(X)$ be the space of such jets. For each $m \in \mathbb{N}$, there is a projection $\pi_m : J(X) \to J^m(X)$ associating to each jet $(F^k)_{k \in \mathbb{N}}$ the jet $(F^k)_{|k| \leq m}$. Let*

$$\mathcal{E}(X) = \bigcap_{m \in \mathbb{N}} \pi_m^{-1}(\mathcal{E}^m(X)).$$

An element of $\mathcal{E}(X)$ is a Whitney jet of class C^∞ *on X.*

$\mathcal{E}(X)$ is a Fréchet space, with the seminorms $|| \cdot ||_m^K$, where $m \in \mathbb{N}$ and $K \subset X$ is compact.

When we have perfect sets in $\mathbb{R}$, or C^∞-determining subsets of $\mathbb{R}^n$ for the closed subset given in the definition, the first element of the Whitney jet will describe the other elements. Which means, in such cases, functions will be in the front place. A compact set $K \subset \mathbb{R}^n$ is called C^∞-*determining* if for each $f \in C^\infty(\mathbb{R}^n)$, $f|_K = 0$ implies $f^{(k)}|_K = 0$ for all $k \in \mathbb{N}^n$.

Let us give an example of a function which is not Whitney (or not extendable). Let $K = \{0\} \cup \cup_{k=1}^\infty [a_k, b_k]$ such that $b_k > a_k$ and $[a_k, b_k] \cap [a_{k+1}, b_{k+1}] = \emptyset$ for $k = 1, 2, ...$ and $a_k \downarrow 0$. Now, define the function as $f(0) = 0$ and $f(x) = a_k$ for $x \in [a_k, b_k]$, $k = 1, 2, ...$ Since f is constant on any interval $[a_k, b_k]$, we have $f'(a_k) = 0$. If f is extendable to a function $\tilde{f} \in C^\infty(\mathbb{R})$, then by continuity $\tilde{f}'(0) = \lim_{k \to \infty} f'(a_k) = 0$. On the other hand, by the Mean-Value Theorem, for each $k = 1, 2, ...$ there exists a point $\xi_k \in (0, a_k)$ such that the extension $\tilde{f}'(\xi_k) = 1$ and hence we have $\tilde{f}'(0) = \lim_{k \to \infty} \tilde{f}'(\xi_k) = 1$. Therefore, $f \notin \mathcal{E}(K)$. In the same way for any $m \in \mathbb{N}$ one can construct $f \in \mathcal{E}^m(K) \backslash \mathcal{E}^{m+1}(K)$. Similar examples can be given also for Cantor type sets.

For K a closed subset of $\mathbb{R}^n$ and $m \in \mathbb{N}$, Whitney's extension theorem [48] gives an extension operator (a linear continuous extension operator) from the space $\mathcal{E}^m(K)$ of Whitney jets on K to the space $C^m(\mathbb{R}^n)$. In the case $m = \infty$ such an operator does not exist in general.

Definition 1.8 *For $K \subset \mathbb{R}^n$, K has the* Extension property *if there exists a linear continuous extension operator $L : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R}^n)$.*

The simplest example for a compact set which does not have the extension property is the set $K = \{0\} \subset \mathbb{R}$. Assume that there exists such a continuous extension operator L for $K = \{0\}$. Hence we have

$$\forall p \;\; \exists q, C \;\; : \;\; \|LF\|_p \leq C\|F\|_q \;\; \forall F \in \mathcal{E}(K).$$

Let $p = 0$, then we have q, C satisfying $\|LF\|_0 \leq C\|F\|_q \;\; \forall F \in \mathcal{E}(K)$.

Let $F = (F_i)_{i=0}^\infty$ with $F_{q+1} = 1$ and $F_i = 0$ for all $i \neq q + 1$.

It is easy to see that $\|F\|_q = 0$.

But of course $LF \neq 0$ since $LF^{(q+1)}(0) \neq 0$.

Then we get $0 < \|LF\|_0 \leq C\|F\|_q = 0$ which is a contradiction.

Generalizing this, it is easy to see that if $K \subset \mathbb{R}^n$ has isolated points then K has no extension property.

For $K = \{0\}$ any jet $f \in J(K)$ is a Whitney jet of class C^∞ (by Borel's theorem).

For any jet $f \in \mathcal{E}(X)$, an extension can be given by a telescoping series:

$$W(f) = W_0(f) + \sum_{m=1}^\infty [W_m(f) - W_{m-1}(f) - H_{m-1}]$$

where $\{H_m\}_{m=0}^\infty$ are C^∞ functions satisfying

$$|W_m(f) - W_{m-1}(f) - H_{m-1}|_{m-1} \leq \frac{1}{2^m},$$

and W_m is the Whitney extension operator for $\mathcal{E}^m(X)$, $m = 0, 1, ...$

1.2 Linear Topological Invariants

Let us denote by $\mathbb{K}$ either of the fields $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.9 *A $\mathbb{K}$-vector space F, endowed with a metric, is called* metric linear space*, if in F addition is uniformly continuous and scalar multiplication is continuous.*

A metric linear space F is said to be locally convex *if for each $a \in F$ and each neighborhood V of a there exists a convex neighborhood U of a with $U \subset V$.*

A complete, metric, locally convex space is called a Fréchet space*.*

Every normed space is a metric linear space and every Banach space is a Fréchet space.

$C^\infty(U)$ for U an open subset of $\mathbb{R}^n$, $C^\infty(\overline{U})$-the space of infinitely differentiable functions on an open bounded domain U which are uniformly continuous with all their derivatives, $\mathcal{E}(K)$ for K a compact subset of $\mathbb{R}^n$ and $A(U)$ for U an open domain in $\mathbb{C}^n$ are typical examples of non-normable Fréchet spaces.

Definition 1.10 *Let E be a locally convex space. A collection $\mathcal{U}$ of zero neighborhoods in E is called a* fundamental system of zero neighborhoods*, if for every zero neighborhood U there exists a $V \in \mathcal{U}$ and an $\epsilon > 0$ with $\epsilon V \subset U$.*

A family $(\|.\|_\alpha)_{\alpha \in A}$ of continuous seminorms on E is called a fundamental system of seminorms*, if the sets*

$$U_\alpha := \{x \in E : \|x\|_\alpha < 1\}, \quad \alpha \in A,$$

form a fundamental system of zero neighborhoods.

Notation 1.11 *Let E be a locally convex space which has a countable fundamental system of seminorms $(\|.\|_n)_{n \in \mathbb{N}}$. By passing over to $(max_{1 \le j \le n} \|.\|_j)_{n \in \mathbb{N}}$ one may assume that*

$$\|x\|_n \le \|x\|_{n+1} \qquad \forall x \in E, \quad n \in \mathbb{N}$$

holds. We call $(\|.\|_n)_{n\in\mathbb{N}}$ *an* increasing fundamental system.

Definition 1.12 *A sequence* $(e_j)_{j\in\mathbb{N}}$ *in a locally convex space* E *is called a* Schauder basis *of* E*, if for each* $x \in E$*, there is a uniquely determined sequence* $(\xi_j(x))_{j\in\mathbb{N}}$ *in* $\mathbb{K}$*, for which* $x = \sum_{j=1}^{\infty} \xi_j(x)e_j$ *is true. The maps* $\xi_j : E \longrightarrow \mathbb{K}, \ \ j \in \mathbb{N}$*, are called the* coefficient functionals *of the Schauder basis* $(e_j)_{j\in\mathbb{N}}$*. They are linear by the uniqueness stipulations.*

A Schauder basis $(e_j)_{j\in\mathbb{N}}$ *of* E *is called an* absolute basis*, if for each continuous seminorm* p *on* E *there is a continuous seminorm* q *on* E *and there is a* $C > 0$ *such that*

$$\sum_{j\in\mathbb{N}} |\xi_j(x)|p(e_j) \leq Cq(x) \quad \forall x \in E.$$

Let $A = (a_{ip})_{i\in I, p\in\mathbb{N}}$ be a matrix of real numbers such that $0 \leq a_{ip} \leq a_{ip+1}$. *Köthe space*, defined by the matrix A, is said to be the locally convex space $K(A)$ of all sequences $\xi = (\xi_i)$ such that

$$|\xi|_p := \sum_{i\in I} a_{ip}|\xi_i| < \infty \quad \forall p \in \mathbb{N}$$

with the topology, generated by the system of seminorms $\{|.|_p, p \in \mathbb{N}\}$. The set of indices I is supposed to be countable, but in general $I \neq \mathbb{N}$. This is convenient for applications, especially when multiple series are considered.

Definition 1.13 *Let* E *and* F *be locally convex spaces ; let us define*

$$L(E,F) := \{A : E \longrightarrow F : A \textit{ is linear and continuous } \}$$
$$L(E) := L(E,E) \textit{ and } E' := L(E,\mathbb{K})$$

E' *is called the* dual space*, of* E*.*

A linear map $A : E \longrightarrow F$ *is called an* isomorphism*, if* A *is a homomorphism.* E *and* F *are said to be* isomorphic*, if there exists an isomorphism* A *between* E *and* F*. Then we write* $E \simeq F$*.*

By the Dynin-Mityagin theorem (see for example [27]) every Fréchet space with absolute basis is isomorphic to some Köthe space. More precisely, If E is a Fréchet space, $\{e_i\}_{i\in I}$ is an absolute basis in E, and $\{\|.\|_p\}_{p\in\mathbb{N}}$ is an increasing sequence of seminorms, generating the topology of E, then E is isomorphic to the Köthe space, defined by the matrix $A = (a_{ip})$, where $a_{ip} = \|e_i\|_p$.

For example the space $C^\infty[-1, 1]$ is isomorphic to the Köthe space $s = K(n^p)$ (see [28]), the space $A(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, is isomorphic to $K(\exp(-n/p))$, the space $A(\mathbb{C})$ is isomorphic to $K(\exp(pn))$.

It is known ([9],[14],[41],[44],[54]) if the boundary of a domain D is smooth, Lipschitz or even Hölder, then the space $C^\infty(\overline{D})$ is isomorphic to the space s.

To examine whether two given linear topological spaces are isomorphic or not it is useful to deal with some properties of linear topological spaces which are invariant under isomorphisms. More precisely, if Σ is a class of linear topological spaces, Ω is a set with an equivalence relation $\sim$ and $\Phi : \Sigma \longrightarrow \Omega$ is a mapping, such that

$$X \simeq Y \implies \Phi(X) \sim \Phi(Y)$$

then Φ is called a *Linear Topological Invariant*. We say that the invariant Φ is complete on the class Σ if for any $X, Y \in \Sigma$

$$\Phi(X) \sim \Phi(Y) \implies X \simeq Y$$

First linear topological invariants connected with isomorphic classification of Fréchet spaces are due to A.N. Kolmogorov [23] and A. Pelczynski [30]. They introduced a linear topological invariant called *approximative dimension* and proved by its help that $A(D)$ is not isomorphic to $A(G)$ if $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$, $m \neq n$ and $A(\mathbb{D}^n)$ is not isomorphic to $A(\mathbb{C}^n)$, where $\mathbb{D}^n$ is the unit polydisc in $\mathbb{C}^n$. Later C. Bessaga, A. Pelczynsky, S. Rolewics [7] and B. Mitiagin [28] considered another linear topological invariant called *diametral dimension*, which turns out to be stronger and more convenient than the approximative dimension. V.Zahariuta [50, 51], introduced some general characteristics as generalizations of Mitiagin's invariants and some new invariants in terms of synthetic neighborhoods [52, 53].

Suppose X is a Fréchet space and $(\|.\|_p, p = 1, 2, ...)$ be a system of seminorms generating the topology of X. The following so called *Interpolation Invariants* are very important in structure theory of Fréchet spaces.

$$\begin{array}{ll} (DN) & \exists p \forall q \exists r, C : \|x\|_q^2 \leq C\|x\|_p \|x\|_r \quad x \in X; \\ (\Omega) & \forall p \exists q \forall r \exists \epsilon \exists C : \|x'\|_q^* \leq C(\|x'\|_p^*)^\epsilon (\|x'\|_r^*)^{1-\epsilon} \quad x' \in X'; \end{array}$$

The notations are due to D.Vogt [27]. (DN) means that the norm $||\cdot||_p$ dominates in the space X. V. Zahariuta uses the notations D_1, Ω_1 respectively.

We shall reformulate (DN) in an equivalent way in the following simple proposition. For the proof see for example [27].

Proposition 1.14 *A Fréchet space E with an increasing fundamental system $(\|.\|_k)_{k\in\mathbb{N}}$ of seminorms has the property (DN) if and only if the following holds:*

$$\exists p \ \forall q \ \forall \epsilon > 0 \ \exists r, C : \ \ \|x\|_q \leq C\|x\|_p^{1-\epsilon}\|x\|_r^\epsilon \tag{1.4}$$

for all $x \in E$.

(1.4) can be stated also as follows :

$$\exists p \ \forall q \ \forall \epsilon > 0 \ \exists r, C : \ \ \|x\|_q^{1+\epsilon} \leq C\|x\|_p\|x\|_r^\epsilon \tag{1.5}$$

for all $x \in E$.

(DN) is also equivalent to the following:

$$\exists p \ \forall q \ \exists r, C : \ \ \|x\|_q \leq t\|x\|_p + \frac{C}{t}\|x\|_r \quad t > 0 \tag{1.6}$$

Proposition 1.15 *The following statement is equivalent to DN:*

$$\forall R > 0 \, \forall q \, \exists r, C > 0 : |.|_q \ \ \leq \ \ t^R|.|_0 + \frac{C}{t}||.||_r, \ \ t > 0 \tag{1.7}$$

From [4] we have that the property DN is equivalent to the following:

$$\forall \epsilon \in (0,1) \, \forall q \, \exists r, C > 0 : |.|_q \leq C|.|_0^{1-\epsilon}.\|.\|_r^\epsilon$$

hence DN is equivalent to (1.7).

1.3 Tidten-Vogt Topological Characterization of the Extension Property

Let $(E_i, A_i)_{i\in\mathbb{Z}}$ be a sequence of linear spaces E_i and linear maps $A_i : E_i \longrightarrow E_{i+1}$. The sequence is said to be *exact at the position* E_i in case $R(A_{i-1}) = N(A_i)$. Here R denotes image and N denotes the kernel of the map. The sequence is said to be *exact*, if it is exact at each position. A *short sequence* is a sequence in which at most three successive spaces are different from $\{0\}$. We then write

$$0 \longrightarrow E \xrightarrow{A} F \xrightarrow{B} G \longrightarrow 0$$

Remark 1.16 *Let F be a Fréchet space and E be a closed subspace of F. Then E and F/E are likewise Fréchet spaces (see e.g. [27]). If $j : E \longrightarrow F$ is the inclusion and $q : F \longrightarrow F/E$ is the quotient map, then*

$$0 \longrightarrow E \xrightarrow{j} F \xrightarrow{q} F/E \longrightarrow 0$$

is a short exact sequence of Fréchet spaces.

Definition 1.17 *A seminorm p on a $\mathbb{K}$-vector space E is called a* Hilbert seminorm, *if there exists a semi-scalar product $\langle .,. \rangle$ on E with $p(x) = \sqrt{\langle x, x\rangle}$ for all $x \in E$.*

A Fréchet-Hilbert space *is a Fréchet space which has a fundamental system of Hilbert seminorms.*

The folowing theorem of D. Vogt from [27] is fundamental in the structure theory of Fréchet spaces.

Theorem 1.18 *(Splitting theorem) Let E, F and G be Fréchet-Hilbert spaces and let*

$$0 \longrightarrow F \xrightarrow{j} G \xrightarrow{q} E \longrightarrow 0$$

be a short exact sequence with continuous linear maps. If E has the property (DN) and F has the property (Ω), then the sequence splits, ie., q has a continuous linear right inverse and j has a continuous linear left inverse.

M. Tidten used the splitting theorem for the proof of the next theorem which tells that the extension property of K is equivalent to the property (DN) of $\mathcal{E}(K)$.

Theorem 1.19 *[41] A compact set K has the extension property iff the space $\mathcal{E}(K)$ has the property (DN).*

Let us make a sketch of the proof. For the proof of the sufficiency part assume that $\mathcal{E}(K)$ has the property (DN) and let L be a cube such that $K \subset L^o$. Now consider the short exact sequence

$$0 \longrightarrow \mathcal{F}(K, L) \xrightarrow{\ i\ } \mathcal{D}(L) \xrightarrow{\ q\ } \mathcal{E}(K) \longrightarrow 0$$

where $\mathcal{D}(L) = C_0^\infty(L)$ is the space of infinitely differentiable functions on L that vanish on the boundary of L together with all their derivatives, and $\mathcal{F}(K, L) = \{f \in \mathcal{D}(L) : f|_K \equiv 0\}$.

By [41] we have that $\mathcal{F}(K, L)$ has property (Ω) $\forall$ compact $K \subset L^o$. Hence we can apply the splitting theorem. This means that there exists an operator ψ, a continuous linear right inverse of q, $\psi : \mathcal{E}(K) \longrightarrow \mathcal{D}(L)$ where obviously $(\psi f)|_K = f$ for $f \in \mathcal{E}(K)$, that is the operator ψ is an extension operator.

On the other hand if there exists an extension operator ψ, then $q \circ \psi = Id_{\mathcal{E}(K)}$ and $\psi \circ q$ is a continuous projection of $\mathcal{D}(L)$ onto $\mathcal{E}(K)$. We know that $\mathcal{D}(L)$ is isomorphic to s, hence $\mathcal{E}(K)$ is a complemented subspace of s, therefore $\mathcal{E}(K)$ has (DN), since the property (DN) is inherited by subspaces.

1.4 Polynomial interpolation

If one decides to approximate a function $f \in C[a, b]$ by a polynomial

$$p(x) = \sum_{i=0}^{n} c_i x^i, \qquad a \leq x \leq b,$$

one has the problem of specifying the coefficients $\{c_i : i = 0, 1, ..., n\}$. The most straightforward method is to calculate the value of f at $(n + 1)$ distinct points

$\{x_i : i = 0, 1, ..., n\}$ of $[a, b]$, and to satisfy the equations

$$p(x_i) = f(x_i), \qquad i = 0, 1, ..., n. \tag{1.8}$$

In this case p is called the *interpolating polynomial* to f at the points $\{x_i : i = 0, 1, ..., n\}$. We note that there are as many conditions as coefficients, and the following well-known theorem shows that they determine $p \in \mathcal{P}_n$ uniquely, where $\mathcal{P}_n$ denotes the set of all polynomials of degree n.

Theorem 1.20 *Let $\{x_i : i = 0, 1, ..., n\}$ be any set $(n+1)$ distinct points in $[a, b]$, and let $f \in C[a, b]$. Then there is exactly one polynomial $p \in \mathcal{P}_n$ that satisfies the equation (1.8).*

For $k = 0, 1, ..., n$, let l_k be the polynomial

$$l_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}, \qquad a \leq x \leq b. \tag{1.9}$$

We note that $l_k \in \mathcal{P}_n$ and that the equations

$$l_k(x_i) = \delta_{ki}, \qquad i = 0, 1, ..., n,$$

hold, where δ_{ki} has the value

$$\delta_{ki} = \begin{cases} 1, & k = i, \\ 0, & k \neq i. \end{cases}$$

Clearly,

$$p = \sum_{k=0}^{n} f(x_k) l_k \tag{1.10}$$

is in $\mathcal{P}_n$ and satisfies the required interpolation conditions (1.8).

We remark first that if we put

$$w(x) = (x - x_0)(x - x_1) \cdots (x - x_n),$$

then the *fundamental polynomials* $l_k(x) = l_k^n(x)$ can be written as

$$l_k(x) = \frac{w(x)}{(x - x_k)w'(x_k)}, \qquad k = 0, 1, ..., n.$$

This method is called the *Lagrange interpolation formula.* We write as

$$L_n f(x) = \sum_{k=0}^{n} f(x_k) l_k(x).$$

The uniqueness property allows us to regard the interpolation process as an operator from $C[a, b]$ to $\mathcal{P}_n$, which depends on the choice of the fixed points $\{x_i : i = 0, 1, ..., n\}$. The operator is a projection, and since the functions l_k ($k = 0, 1, ..., n$) are independent of f, equation (1.10) shows that the operator is linear.

The Lagrange interpolation formula provides some algebraic relations that are useful in later work. They come from our remark that the interpolation process is a projection operator. In particular, for $0 \le i \le n$, we let f be the function

$$f(x) = x^i, \qquad a \le x \le b,$$

in order to obtain from expression (1.10) the equation

$$\sum_{k=0}^{n} x_k^i l_k(x) = x^i, \qquad a \le x \le b.$$

The value $i = 0$ gives the identity

$$\sum_{k=0}^{n} l_k(x) = 1, \qquad a \le x \le b.$$

The choice of the interpolation points is very important for having the error function

$$e(x) = f(x) - p(x), \qquad a \le x \le b,$$

of smallest modulus. One of the most important interpolation points for the interval are the Chebyshev interpolation points, and they are found by making use of *Chebyshev polynomials.*

For the range $0 \le \theta \le \pi$, the Chebyshev polynomial of degree n is the function T_n that satisfies the equation

$$T_n(\cos\theta) = \cos(n\theta),$$

which is equivalent to the equation

$$T_n(x) = \cos(n \cos^{-1} x), \qquad -1 \leq x \leq 1.$$

Chebyshev polynomials have many applications in approximation theory. The zeros of $T_n(x)$ are the points

$$\xi_j = \xi_j^{(n)} = \cos \frac{2j-1}{n}\frac{\pi}{2}.$$

We see that they are all distinct and lie in the interval $[-1, 1]$.

Now, if we take zeros of the Chebyshev polynomial of degree n as the interpolation points, then we have

$$|l_j^n(x)| \leq \frac{4}{\pi}, \quad x \in [-1, 1], \qquad j = 0, ..., n$$

(see e.g. [36]). This is an effective bound in the sense that

$$\lim_{n\to\infty} \max\{|l_j^n(x)| : x \in [-1, 1]\} = \frac{4}{\pi}.$$

In the case of equally spaced points the bound depends on the number of the interpolation points and

$$\lim_{n\to\infty} \max\{|l_j^n(x)| : x \in [-1, 1]\} = \infty.$$

1.5 Divided differences

Let $\{x_i : i = 0, ..., n\}$ be any $(n + 1)$ distinct points of $[a, b]$, and let f be a function in $C[a, b]$. The coefficient of x^n in the polynomial $p \in \mathcal{P}_n$ that satisfies the interpolation conditions

$$p(x_i) = f(x_i), \qquad i = 0, ..., n,$$

is defined to be a *divided difference* of order n for the function f, and we use the notation $[x_0, ..., x_n]f$ for its value. We note that the order of a divided difference is one less than the number of arguments in the expression $[., ..., .]f$. Hence $[x_0]f$

is a divided difference of order zero, which by definition has the value $f(x_0)$. Moreover, when $n \geq 1$, it follows from equation (1.10) that the equation

$$[x_0, ..., x_n]f = \sum_{k=0}^{n} \frac{f(x_k)}{\prod_{j=0,j\neq k}^{n}(x_k - x_j)} = \sum_{k=0}^{n} \frac{f(x_k)}{w'(x_k)}$$

is satisfied. We see that the divided difference is linear in the function values $\{f(x_i) : i = 0, ..., n\}$.

It is often convenient to represent the divided difference $[x_0, ..., x_n]f$ as a value of the n-th derivative of the function f divided by the factor $n!$.

Theorem 1.21 *(see e.g. [35]) Let $f \in C^n[a, b]$ and let $\{x_i : i = 0, ..., n\}$ be a set of distinct points in $[a, b]$. Then there exists a point ξ, in the smallest interval that contains the points $\{x_i : i = 0, ..., n\}$, at which the equation*

$$[x_0, ..., x_n]f = f^{(n)}(\xi)/n!$$

is satisfied.

Another important theorem that justifies the name *divided differences* is the following:

Theorem 1.22 *The divided difference $[x_j, x_{j+1}, ..., x_{j+k+1}]f$ of order $(k+1)$ is related to the divided differences $[x_j, x_{j+1}, ..., x_{j+k}]f$ and $[x_{j+1}, x_{j+2}, ..., x_{j+k+1}]f$ of order k by the equation*

$$[x_j, x_{j+1}, ..., x_{j+k+1}]f = \frac{[x_{j+1}, x_{j+2}, ..., x_{j+k+1}]f - [x_j, x_{j+1}, ..., x_{j+k}]f}{(x_{j+k+1} - x_j)}.$$

For the proof see e.g. [35].

Chapter 2

Asymptotics of Green's Function for $\mathbb{C}_\infty \setminus K^{(\alpha)}$

2.1 Cantor type sets

Let α be given such that $1 < \alpha < 2$. Let the sequence $(l_k)_{k=0}^{\infty}$ be such that $l_0 = 1$, $l_1^{\alpha-1} < \frac{1}{2}$ and

$$l_{k+1} = l_k^{\alpha}$$

for $k \geq 1$. Let $\{I_k\}_{k=0}^{\infty}$ be a family of subsets of $[0,1]$ such that $I_0 = [0,1]$ and I_{k+1} i obtained from I_k by deleting the open concentric subinterval of length $l_k - 2l_{k+1}$ from each interval of I_k.

$$K = K^{(\alpha)} = \bigcap_{k=0}^{\infty} I_k$$

Then every set I_k consists of 2^k subintervals $I_{k,1}, ..., I_{k,2^k}$ of length l_k each.

As another notation the subintervals of I_k can be named as $I_{1,k}, ..., I_{2^k,k}$. In Chapter 3 this notation is preferred.

2.2 Green's function

Let $\mathbb{C}_\infty$ denote the extended complex numbers.

Definition 2.1 *For an open subset G of $\mathbb{C}_\infty$ a* Green's function *is a function $g : G \times G \to (-\infty, \infty]$ having the following properties:*

(a) *for each a in G the function $g(z) = g(z, a, G)$ is positive and harmonic on $G \setminus \{a\}$;*

(b) *for each $a \neq \infty$ in G, $z \to g(z, a) + \log|z - a|$ is harmonic in a neighborhood of a; if $\infty \in G$, $z \to g(z, \infty) - \log|z|$ is harmonic in a neighborhood of ∞;*

(c) *g is the smallest function from $G \times G$ into $(\infty, \infty]$ that satisfies properties (a) and (b).*

Definition 2.2 *If G is an open subset of $\mathbb{C}$, a function $u : G \to [-\infty, \infty)$ is* subharmonic *if u is upper semicontinuous and, for every closed disk $\bar{B}(a; r)$ contained in G, we have the inequality*

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta.$$

A set Z is a polar set *if there is a non-constant subharmonic function u on $\mathbb{C}$ such that $Z \subset \{z : u(z) = -\infty\}$.*

Green's function with a pole at infinity can also be defined with polynomials. For $G \subset \mathbb{C}_\infty$ let $K = \mathbb{C}_\infty \setminus G$, then

$$g_K(z) := g(z, \infty, G) = \sup\left\{\frac{\ln|p(z)|}{\deg\, p} : p \in \Pi, |p|_K \leq 1\right\}, \tag{2.1}$$

where Π here denotes the set of all polynomials. In fact, from the Bernstein theorem (see e.g.[45]) we have that $g_K(z) \geq \sup\{\frac{\ln|p(z)|}{\deg\, p} : p \in \prod, |p|_K \leq 1\}$. On the other hand, let us choose for every $n \in \mathbb{N}$ a monic polynomial $p_n(z)$ of degree n such that the set $\{z \in \mathbb{C} : |p_n(z)| \leq 1\}$ contains K. Then Green's function for the set $C_\infty \setminus \{z \in \mathbb{C} : |p_n(z)| > 1\}$ is $g_n(z) = n^{-1} \ln|p(z)|$. We can choose the sequence of polynomials $(p_n)_{n=1}^\infty$ such that the intersection of the corresponding level domains gives the set K. Then, using Proposition 9.8 of [11] we can conclude that (2.1) holds.

For Cantor-type sets we have the following theorem from [11].

Theorem 2.3 *Let* $\{I_k\}$ *be the sequence of compact sets formed of* 2^k *subintervals of length* l_k *and* $K = \cap_k I_k$ *is the Cantor-type set defined as in section 2.1. Then the set* K *is polar if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \log l_k^{-1} = \infty$$

By use of this theorem, we see that the Cantor set $K^{(\alpha)}$ is non-polar if and only if $1 < \alpha < 2$. So, Green's function for $K^{(\alpha)}$ is undefined when $\alpha \geq 2$.

2.3 Polynomial inequalities

When a compact set is given, could the derivative of a polynomial on the compact set be estimated by the norm of it on the compact set? This question was first answered by A. A. Markov in 1889 for the set $I = [-1, 1]$ as follows

$$\sup_{x \in I} |p'(x)| \leq (\deg p)^2 \sup_{x \in I} |p(x)|.$$

As a generalization of this, any compact K set is defined to have *Markov property* (or is a Markov set) if there exist positive constants M, m such that

$$\sup_{x \in K} |\nabla p(x)| \leq M (\deg p)^m \sup_{x \in K} |p(x)|$$

for all $p \in \Pi$. The Markov property is crucial for the method of Pawłucki and Pleśniak to construct a linear continuous extension operator. This method will be considered in the next section.

Markov property is related with the Hölder continuity of the Green function for the set in $\mathbb{R}$. Green's function of $\mathbb{C}_\infty \setminus K$ is defined to be *Hölder continuous* when there exist constants $C,\ \mu$ satisfying

$$g_K(z,\infty) \leq C\delta^\mu \quad \text{for dist } (z,K) \leq \delta \leq 1.$$

By Cauchy's integral formula, it can be proved that Hölder Continuity (HCP) of Green's function g_K implies Markov property of the compact set K. The problem of the inverse implication is still open.

Next inequality about polynomials is from the so called Bernstein theorem [45]. Let $K \in \mathbb{C}$ be a non-polar compact set (i.e. cap $K > 0$). Then for any polynomial p of degree n, we have for $z \in \mathbb{C}$,

$$|p(z)| \leq \exp(n \cdot g_K(z,\infty))|p|_K.$$

From this inequality we see that an upper bound for Green's function will give us a direct relation between the value of the polynomial in a neighborhood of a compact set and the norm of it. Moreover, by using Cauchy's integral formula, we can reach a Markov type inequality.

Theorem 2.4 *Suppose there exists a constant $C > 0$, and a continuous invertible function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, such that for Green's function we have $g_K(z,\infty) \leq C \cdot \varphi(\delta)$ where $\delta = dist(z,K)$. Then for any polynomial we have*

$$|p'|_K \leq C_1 \cdot \phi(deg\ p)|p|_K$$

for a constant $C_1 > 0$, and the function $\phi(x) = 1/\varphi^{-1}(\frac{1}{x})$.

Proof: Let $z \in K$ and let p be a polynomial of degree n, then by the Cauchy's integral formula

$$p'(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{p(\zeta)}{(\zeta - z)^2}d\zeta$$

where $\Gamma = \{\zeta \in \mathbb{C} : |\zeta - z| = \delta\}$. Then, by using the Bernstein theorem

$$|p'(z)| \leq \frac{1}{2\pi}\oint_\Gamma \frac{|p(\zeta)|}{\delta^2}d\zeta \leq \frac{1}{2\pi\delta^2}\oint_\Gamma \exp[n \cdot g_K(\zeta)]|p|_K d\zeta \leq \frac{1}{\delta}\exp[n \cdot C\varphi(\delta)]|p|_K$$

Now, choose δ so that $\varphi(\delta) = 1/n$ and the result of the theorem follows. $\square$

Corollary 2.5 *(HCP) of Green's function $g_K(z)$ implies Markov property of the set K.*

The simplest example of sets without Markov property is the point. Any set with isolated points has no Markov property. The closure of a plain domain with a sharp cusp is the first non-trivial example of non-Markov set (Zerner, [54]). Other non-trivial examples of sets without Markov property could be given by Cantor type type sets or set of intervals tending to a point. The classical Cantor set is constructed from a segment by successive deleting subintervals with a constant quotient of their lengths. Consider Cantor type sets with arbitrary ratio of lengths. Let $(l_k)_{k=0,1,...}$ be a given sequence such that for every $k \geq 1$

$$l_k < \frac{1}{2} l_{k-1} \quad \text{and } l_0 = 1.$$

Let $\{E_k\}_{k=0,1,...}$ be a family of subsets of $[0,1]$ such that every set E_k consists of 2^k intervals $I_{k,1}, ..., I_{k,2^k}$ of length l_k each, $E_0 = [0,1]$ and E_{k+1} is obtained by deleting the open concentric subinterval of length $l_k - 2l_{k+1}$ from each interval $I_{k,n}$, $n = 1, ..., 2^k$. Then the set

$$E = \bigcap_{k=0}^{\infty} \bigcup_{n=1}^{2^k} I_{k,n}$$

is called a *generalized Cantor set.* Examples of Cantor type sets without Markov property were given by Pleśniak [34], Bialas [8] and Jonsson [22]. Examples for sets formed of intervals tending to point, without Markov property were given by Goncharov [15], [16]. For Cantor-type sets we have the following theorem from [8].

Theorem 2.6 *If there exists a limit (finite or infinite) of the sequence $(l_k/l_{k+1})_{k=0,1,...}$ and E is a generalized Cantor set associated with $(l_k)_{k=0,1,...}$, then the following conditions are equivalent*

(i) *E satifies (HCP),*

(ii) *E satifies Markov property,*

(iii) *the limit of the sequence $(l_k/l_{k+1})_{k=0,1,...}$ is finite.*

Some more general form of Cantor set is when each interval of E_k includes n_k intervals of E_{k+1}. In [4] such Cantor sets were considered for the geometric characterization of extension property.

Examples for sets formed of intervals tending to point, without Markov property were given by Goncharov [15], [16]. Let $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$ such that $K \subset [0,1]$. $I_k = [a_k, b_k]$. Let $\delta_k = \frac{1}{2}(b_k - a_k), h_k = a_k - b_{k+1}$. $2\delta_k \leq h_k$ and $\delta_k \downarrow 0$, $a_k \downarrow 0$. Let $b_k \leq C\delta_k$ where C is a constant. Let $R > 1$ such that $\delta_{k+1} \geq \delta_k^R$. For these sets, it is given in [18] an explicit form of extension operator by use of the basis elements of $\mathcal{E}(K)$. In Chapter 3 we give an explicit form of an extension operator for generalized Cantor type sets without Markov property. And in Chapter 4 we give an explicit form of an extension operator for sets formed of intervals tending to a point, having no Markov property.

Another important inequality related to polynomials is given by the following theorem of Jackson (e.g. [43]).

Theorem 2.7 *Let f defined on the finite segment $I = [a,b]$ and has an q-th continuous derivative, then for $n > q$*

$$dist_I(f, \mathcal{P}_n) \leq M_q \left(\frac{b-a}{n}\right)^q w(f^{(q)}; \frac{b-a}{n})$$

where M_q is a constant depending only on q and w is the modulus of continuity.

2.4 Green's function of domains complementary to Cantor-type sets

We want to find an upper bound for Green's function of the set $\mathbb{C}_\infty \setminus K^{(\alpha)}$ with a pole at infinity, in the case Green's function exists ($1 < \alpha < 2$). The lower bound for Green's function can be obtained from the representation (2.1). To find the upper bound we will use the local interpolation of polynomials. The upper bound will lead us to a Markov-type inequality for the set $K^{(\alpha)}$.

Let $K^{(\alpha)}$ be defined as in the section 2.1. Every set I_k consists of 2^k subintervals $I_{k,1}, ..., I_{k,2^k}$ of length l_k each. Let $t_{k,j} = \min\{t : t \in I_{k,j}\}$. Let $L_j^{Nk,1}(z)$ be the Lagrange fundamental polynomials corresponding to $t_{(N+1)k,1}, ..., t_{(N+1)k,2^{Nk}} \in I_{k,1} \cap K$. For $j = 1, 2, ..., 2^{Nk}$ let

$$L_j^{Nk,1}(z) = \prod_{n=1,n\neq j}^{2^{Nk}} \left(\frac{z - t_{(N+1)k,n}}{t_{(N+1)k,j} - t_{(N+1)k,n}}\right)$$

It is easy to see that these points are the left endpoints of the intervals $I_{(N+1)k,1}, ..., I_{(N+1)k,2^{Nk}}$ which can be obtained from $I_{k,1}$ after Nk steps. Here N is supposed to be positive rational number with denominator k. In a similar way define $L_j^{Nk,2}(z)$ to be Lagrange fundamental polynomials corresponding to the next 2^{Nk} points, which are $\{t_{(N+1)k,2^{Nk}+1}, t_{(N+1)k,2^{Nk}+2}, ..., t_{(N+1)k,2\cdot 2^{Nk}}\} \subset I_{k,2} \cap K$. And so in general for $1 \leq i \leq 2^k$, $L_j^{Nk,i}(z)$ are Lagrange fundamental polynomials corresponding to the points from $I_{k,i} \cap K$.

Let

$$M = \lceil \frac{\ln 2}{\ln(2/\alpha)} \rceil,$$

where for any $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the least integer that is larger than x.

Lemma 2.8 *Given $k \in \mathbb{Z}^+$, and i such that $1 \leq i \leq 2^k$ let $l_{M(k+1)} < dist(z, K \cap I_{k,i}) \leq l_{Mk}$ for $z \in \mathbb{C}$. Then*

$$|L_j^{Nk,i}(z)| \leq \exp[2^{(N+1-M)k-1} + 2^{Nk+1-\alpha^{k-1}} + 2 \cdot 2^{Nk-(\alpha-1)\alpha^{Mk-2}}] \cdot l_1^A,$$

where $A = -[\frac{\alpha-1}{2-\alpha}]\alpha^{Mk-1}2^{(N+1-M)k} - \alpha^{Mk-1} + [\frac{1}{2-\alpha}]\alpha^{(N+1)k-1}$.

Proof: Without loss of of generality let $i = 1$. Suppose $N + 1 > M$

$$L_j^{Nk,1}(z) = \prod_{n=1,n\neq j}^{2^{Nk}} \left(\frac{z - t_{(N+1)k,n}}{t_{(N+1)k,j} - t_{(N+1)k,n}}\right)$$

Since $\text{dist}(z, K \cap I_{k,i}) \leq l_{Mk}$ we have

$$\left|\prod_{n=1,n\neq j}^{2^{Nk}} \left(z - t_{(N+1)k,n}\right)\right| \leq$$

$$(l_{Mk}+l_{(N+1)k-1})(l_{Mk}+l_{(N+1)k-2})^2...(l_{Mk}+l_{Mk})^{2^{(N+1-M)k-1}}...(l_{Mk}+l_k)^{2^{Nk-1}}$$

and

$$\left|\prod_{n=1,n\neq j}^{2^{Nk}}\left(t_{(N+1)k,j}-t_{(N+1)k,n}\right)\right| \geq$$

$$(l_{(N+1)k-1}-2l_{(N+1)k})(l_{(N+1)k-2}-l_{(N+1)k-1})^2...(l_k-2l_{k+1})^{2^{Nk-1}}.$$

Then

$$\begin{aligned}
& (l_{Mk}+l_{(N+1)k-1})(l_{Mk}+l_{(N+1)k-2})^2...(l_{Mk}+l_{Mk+1})^{2^{(N+1-M)k-2}} \\
= \;& l_{Mk}^{[1+2+...+2^{(N+1-M)k-2}]}(1+\frac{l_{(N+1)k-1}}{l_{Mk}})...(1+\frac{l_{Mk+1}}{l_{Mk}})^{2^{(N+1-M)k-2}} \\
\leq \;& l_1^{\alpha^{Mk-1}[2^{(N+1-M)k-1}-1]}(1+l_1^{\alpha^{(N+1)k-2}-\alpha^{Mk-1}})...(1+l_1^{\alpha^{Mk}-\alpha^{Mk-1}})^{2^{(N+1-M)k-2}} \\
\leq \;& l_1^{\alpha^{Mk-1}[2^{(N+1-M)k-1}-1]}(1+\frac{1}{2^{\alpha^{(N+1)k-2}-\alpha^{Mk-1}}})...(1+\frac{1}{2^{\alpha^{Mk}-\alpha^{Mk-1}}})^{2^{(N+1-M)k-2}}.
\end{aligned}$$

Since $l_1<1/2$ and $1+\epsilon<\exp\epsilon$, we get

$$\begin{aligned}
& (l_{Mk}+l_{(N+1)k-1})(l_{Mk}+l_{(N+1)k-2})^2...(l_{Mk}+l_{Mk+1})^{2^{(N+1-M)k-2}} \\
\leq \;& l_1^{\alpha^{Mk-1}[2^{(N+1-M)k-1}-1]}\exp[2^{\alpha^{Mk-1}-\alpha^{(N+1)k-2}}+2^{\alpha^{Mk-1}-\alpha^{(N+1)k-3}+1}+... \\
& ...+2^{\alpha^{Mk-1}-\alpha^{Mk}+(N+1-M)k-2}] \\
\leq \;& l_1^{\alpha^{Mk-1}[2^{(N+1-M)k-1}-1]}\exp[2^{\alpha^{Mk-1}-\alpha^{Mk}+(N+1-M)k-1}] \qquad (2.2)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
& (l_{Mk}+l_{Mk})^{2^{(N+1-M)k-1}}...(l_{Mk}+l_k)^{2^{Nk-1}} \\
= \;& 2^{2^{(N+1-M)k-1}}l_{Mk}^{2^{(N+1-M)k-1}}(1+\frac{l_{Mk}}{l_{Mk-1}})^{2^{(N+1-M)k}}l_{Mk-1}^{2^{(N+1-M)k}}...(1+\frac{l_{Mk}}{l_k})^{2^{Nk-1}}l_k^{2^{Nk-1}} \\
= \;& 2^{2^{(N+1-M)k-1}}l_1^{[\alpha^{Mk-1}2^{(N+1-M)k-1}+\alpha^{Mk-2}2^{(N+1-M)k}+...+\alpha^{k-1}2^{Nk-1}]}\,. \\
& \cdot(1+l_1^{\alpha^{Mk-1}-\alpha^{Mk-2}})^{2^{(N+1-M)k}}\cdots(1+l_1^{\alpha^{Mk-1}-\alpha^{k-1}})^{2^{Nk-1}} \\
\leq \;& 2^{2^{(N+1-M)k-1}}l_1^{\alpha^{k-1}2^{(N+1-M)k-1}[\frac{2^{(M-1)k+1}-\alpha^{(M-1)k+1}}{2-\alpha}]}\,. \\
& \cdot(1+\frac{1}{2^{\alpha^{Mk-1}-\alpha^{Mk-2}}})^{2^{(N+1-M)k}}\cdots(1+\frac{1}{2^{\alpha^{Mk-1}-\alpha^{k-1}}})^{2^{Nk-1}} \\
\leq \;& 2^{2^{(N+1-M)k-1}}l_1^{\alpha^{k-1}2^{(N+1-M)k-1}[\frac{2^{(M-1)k+1}-\alpha^{(M-1)k+1}}{2-\alpha}]}. \\
& .\exp[2^{\alpha^{Mk-2}-\alpha^{Mk-1}+(N+1-M)k}+...+2^{\alpha^{k-1}-\alpha^{Mk-1}+Nk-1}] \\
\leq \;& 2^{2^{(N+1-M)k-1}}l_1^{\alpha^{k-1}2^{(N+1-M)k-1}[\frac{2^{(M-1)k+1}-\alpha^{(M-1)k+1}}{2-\alpha}]}\,. \\
& \cdot\exp[2^{\alpha^{Mk-2}-\alpha^{Mk-1}+Nk}] \qquad (2.3)
\end{aligned}$$

and in a similar way

$$\begin{aligned}
& (l_{(N+1)k-1} - 2l_{(N+1)k})...(l_k - 2l_{k+1})^{2^{Nk-1}} \\
= \ & l_1^{\alpha^{k-1}[\frac{2^{Nk}-\alpha^{Nk}}{2-\alpha}]}(1 - 2l_1^{\varepsilon\alpha^{(N+1)k-2}})...(1 - 2l_1^{\varepsilon\alpha^{k-1}})^{2^{Nk-1}} \\
\geq \ & l_1^{\alpha^{k-1}[\frac{2^{Nk}-\alpha^{Nk}}{2-\alpha}]}(1 - \frac{2}{2^{\alpha^{(N+1)k-2}}})...(1 - \frac{2}{2^{\alpha^{k-1}}})^{2^{Nk-1}} \\
\geq \ & l_1^{\alpha^{k-1}[\frac{2^{Nk}-\alpha^{Nk}}{2-\alpha}]} \exp -[2^{1-\alpha^{(N+1)k-2}} + 2^{2-\alpha^{(N+1)k-3}} + ... + 2^{Nk-\alpha^{k-1}}] \\
\geq \ & l_1^{\alpha^{k-1}[\frac{2^{Nk}-\alpha^{Nk}}{2-\alpha}]} \exp[-2^{Nk+1-\alpha^{k-1}}] \qquad (2.4)
\end{aligned}$$

Combining (2.2),(2.3) and (2.4) we have

$$\begin{aligned}
|L_j^{Nk,1}(z)| \ \leq \ & l_1^{-[\frac{\alpha-1}{2-\alpha}]\alpha^{Mk-1}2^{(N+1-M)k} - \alpha^{Mk-1} + [\frac{1}{2-\alpha}]\alpha^{(N+1)k-1}} \cdot 2^{2^{(N+1-M)k-1}} \cdot \\
& \cdot \exp[2^{Nk+1-\alpha^{k-1}} + 2^{\alpha^{Mk-1}-\alpha^{Mk}+(N+1-M)k-1} + 2^{\alpha^{Mk-2}-\alpha^{Mk-1}+Nk}] \\
\leq \ & l_1^{-[\frac{\alpha-1}{2-\alpha}]\alpha^{Mk-1}2^{(N+1-M)k} - \alpha^{Mk-1} + [\frac{1}{2-\alpha}]\alpha^{(N+1)k-1}} \cdot \\
& \cdot \exp[2^{(N+1-M)k-1} + 2^{Nk+1-\alpha^{k-1}} + 2 \cdot 2^{Nk-(\alpha-1)\alpha^{Mk-2}}]
\end{aligned}$$

Let now $N + 1 \leq M$, then $\mathrm{dist}(z, K \cap I_{k,i}) \leq l_{Mk} \leq l_{(N+1)k}$ and we have

$$\left| \prod_{n=1, n\neq j}^{2^{Nk}} \left(z - t_{(N+1)k,n}\right) \right| \leq$$

$$\begin{aligned}
\leq \ & (l_{Mk} + l_{(N+1)k-1})(l_{Mk} + l_{(N+1)k-2})^2...(l_{Mk} + l_k)^{2^{Nk-1}} \\
= \ & l_1^{[\alpha^{(N+1)k-2} + 2\alpha^{(N+1)k-3} + ... + 2^{Nk-1}\alpha^{k-1}]} \cdot \\
& \cdot(1 + \frac{l_{Mk}}{l_{(N+1)k-1}})(1 + \frac{l_{Mk}}{l_{(N+1)k-2}})^2 \cdots (1 + \frac{l_{Mk}}{l_k})^{2^{Nk-1}} \\
\leq \ & l_1^{\alpha^{k-1}[\frac{2^{Nk}-\alpha^{Nk}}{2-\alpha}]}(1 + \frac{1}{2^{\alpha^{Mk-1}-\alpha^{(N+1)k-2}}}) \cdots (1 + \frac{1}{2^{\alpha^{Mk-1}-\alpha^{k-1}}})^{2^{Nk-1}} \\
\leq \ & l_1^{\alpha^{k-1}[\frac{2^{Nk}-\alpha^{Nk}}{2-\alpha}]} \exp[2^{\alpha^{(N+1)k-2}-\alpha^{Mk-1}} + \cdots + 2^{Nk-1+\alpha^{k-1}-\alpha^{Mk-1}}] \\
\leq \ & l_1^{\alpha^{k-1}[\frac{2^{Nk}-\alpha^{Nk}}{2-\alpha}]} \exp[2^{Nk+\alpha^{Mk-2}-\alpha^{Mk-1}}]
\end{aligned}$$

In this case, where $N+1 \leq M$ the term $(l_{(N+1)k-1} - 2l_{(N+1)k})...(l_k - 2l_{k+1})^{2^{Nk-1}}$ will not be effected. Hence, using (2.4) we reach to the same bound for $N + 1 \leq M$.
$\square$

Theorem 2.9 *We have the following upper bound for Green's function of the Cantor set* K *with a pole at infinity.*

$$g_K(z) \le C\left(\ln\frac{1}{\delta}\right)^{1-\frac{M-1}{M}\cdot\left[\frac{\ln 2}{\ln\alpha}\right]}$$

for some constant $C > 0$ *depending only on* K.

Proof: Take $p \in \mathcal{P}_n$ such that $|p|_K \le 1$. Given $z \in \mathbb{C}$ such that $\delta := \mathrm{dist}(z, K) \le l_M$. Choose $k \in \mathbb{N}$ so that $l_{M(k+1)} < \delta \le l_{Mk}$. We choose i from $\{1, ..., 2^k\}$ such that $\delta = \mathrm{dist}(z, K \cap I_{k,i})$. And let N be a rational number such that Nk is integer satisfying $2^{Nk-1} \le n < 2^{Nk}$. Then

$$p(z) = L_{Nk,i}p(z) = \sum_{j=(i-1)2^{Nk}+1}^{i2^{Nk}} p(t_{(N+1)k,j})L_j^{Nk,i}(z).$$

Since $|p|_K \le 1$ we have $|p(t_{(N+1)k,j})| \le 1$ for all j that appears in the sum. Hence by use of the lemma

$$\begin{aligned}|p(z)| \;&\le\; 2^{Nk} l_1^{-\left[\frac{\alpha-1}{2-\alpha}\right]\alpha^{Mk-1}2^{(N+1-M)k}-\alpha^{Mk-1}+\left[\frac{1}{2-\alpha}\right]\alpha^{(N+1)k-1}}\cdot\\ &\quad\cdot\exp\left[2^{(N+1-M)k-1} + 2^{Nk+1-\alpha^{k-1}} + 2\cdot 2^{Nk-(\alpha-1)\alpha^{Mk-2}}\right].\end{aligned}$$

Then we have

$$\begin{aligned}\frac{\ln|p(z)|}{\deg\, p} \;&\le\; \frac{Nk + 2^{(N+1-M)k-1} + 2^{Nk+1-\alpha^{k-1}} + 2^{Nk+1-(\alpha-1)\alpha^{Mk-2}}}{2^{Nk-1}} +\\ &\quad+\frac{\left[\left(\frac{\alpha-1}{2-\alpha}\right)\alpha^{Mk-1}2^{(N+1-M)k} + \alpha^{Mk-1} - \left(\frac{1}{2-\alpha}\right)\alpha^{(N+1)k-1}\right]\ln\frac{1}{l_1}}{2^{Nk-1}}\end{aligned}$$

After some cancellations the inequality above can be written in the following form.

$$\begin{aligned}\frac{\ln|p(z)|}{\deg\, p} \;&\le\; \frac{Nk + \left[\alpha^{Mk-1} - \left(\frac{1}{2-\alpha}\right)\alpha^{(N+1)k-1}\right]\ln\frac{1}{l_1}}{2^{Nk-1}} +\\ &\quad+2^{(1-M)k} + 2^{2-\alpha^{k-1}} + 2^{2-(\alpha-1)\alpha^{Mk-2}} + \left(\frac{\alpha-1}{2-\alpha}\right)\alpha^{Mk-1}2^{(1-M)k+1}\ln\frac{1}{l_1}\end{aligned}$$

The first summand on the right is negative for large enough N. Let $N_0 \in \mathbb{N}$ be the number such that for $N \geq N_0$ this negativity occurs. Then for $N \geq N_0$ we have

$$\frac{\ln|p(z)|}{\deg\, p} \leq 2^{(1-M)k} + 2^{2-\alpha^{k-1}} + 2^{2-(\alpha-1)\alpha^{Mk-2}} + \left(\frac{\alpha-1}{2-\alpha}\right)\alpha^{Mk-1}2^{(1-M)k+1}\ln\frac{1}{l_1}$$

Here the last term is the effective when k is large (or δ is small). Hence there exists a constant C_0 such that

$$\frac{\ln|p(z)|}{\deg\, p} \leq C_0\alpha^{Mk}2^{(1-M)k}\ln\frac{1}{l_1}$$

We have $l_{M(k+1)} < \delta \leq l_{Mk}$, from this relation it will not be so difficult to reach the following inequality for k.

$$k \leq \frac{1}{M}\left[\frac{\ln\left(\frac{\ln\delta}{\ln l_1}\right)}{\ln\alpha} + 1\right] \leq k+1$$

Then using the right part of this inequality we have

$$\begin{aligned}\frac{\ln|p(z)|}{\deg\, p} &\leq C_0\left(\frac{\alpha^M}{2^{M-1}}\right)^{\frac{1}{M}\left[\frac{\ln\left(\frac{\ln\delta}{\ln l_1}\right)}{\ln\alpha}+1\right]-1}\ln\frac{1}{l_1} \leq C_0\frac{\alpha^{\left[\frac{\ln\left(\frac{\ln\delta}{\ln l_1}\right)}{\ln\alpha}\right]}\alpha^{1-M}}{2^{\frac{M-1}{M}\cdot\left[\frac{\ln\left(\frac{\ln\delta}{\ln l_1}\right)}{\ln\alpha}\right]}2^{\frac{M-1}{M}+1-M}}\ln\frac{1}{l_1}\\ &\leq C_0\frac{\left(\frac{\ln\delta}{\ln l_1}\right)\alpha^{1-M}}{2^{\frac{M-1}{M}\cdot\left[\frac{\ln\left(\frac{\ln\delta}{\ln l_1}\right)}{\ln\alpha}\right]}2^{\frac{M-1}{M}+1-M}}\ln\frac{1}{l_1} = C_0\frac{\alpha^{1-M}}{2^{\frac{M-1}{M}\cdot\left[\frac{\ln\left(\frac{\ln\delta}{\ln l_1}\right)}{\ln\alpha}\right]}2^{\frac{M-1}{M}+1-M}}\ln\frac{1}{\delta}.\end{aligned}$$

Hence there exists a constant C_1 such that

$$\frac{\ln|p(z)|}{\deg\, p} \leq C_1\ln\frac{1}{\delta}2^{-\frac{M-1}{M}\cdot\left[\frac{\ln\left(\frac{\ln\delta}{\ln l_1}\right)}{\ln\alpha}\right]} = C_1\ln\frac{1}{\delta}\left(\frac{\ln\delta}{\ln l_1}\right)^{-\frac{M-1}{M}\cdot\left[\frac{\ln 2}{\ln\alpha}\right]}.$$

We can write this last expression as a function of only δ then we will have

$$\frac{\ln|p(z)|}{\deg\, p} \leq C_2\left(\ln\frac{1}{\delta}\right)^{1-\frac{M-1}{M}\cdot\left[\frac{\ln 2}{\ln\alpha}\right]}$$

where C_2 is a constant depending only on l_1 and α. We see that this last inequality does not depend on the interval which z is closest to. Also it does not depend to the degree of the polynomial except it is great enough. Now using the form of the Green function (2.1) which is defined by polynomials, we get

$$g_K(z) \le C_2 \left(\ln \frac{1}{\delta}\right)^{1-\frac{M-1}{M}\cdot\left[\frac{\ln 2}{\ln \alpha}\right]}.$$

$\square$

Corollary 2.10 *Let* p *be any polynomial of degree* n*. Then, there exist constants* $C, \mu > 0$ *such that*

$$|p'|_K \le C \cdot \exp[n^\mu] \cdot |p|_K.$$

Proof: By using Theorem 2.3 we have

$$|p'|_K \le C \cdot \exp[n^{1/(\frac{M-1}{M}\cdot\left[\frac{\ln 2}{\ln \alpha}\right]-1)}] \cdot |p|_K.$$

$\square$

Chapter 3

Extension by means of local interpolation

In [29] (see also [32], [33]) Pawłucki and Pleśniak suggested an explicit construction of the extension operator for a rather wide class of compact sets preserving Markov's inequality. In [15] and later in [18] compact sets K were presented without Markov's Property, but such that the space $\mathcal{E}(K)$ admits the extension operator. Here we deal with the generalized Cantor-type sets $K^{(\alpha)}$, that have the extension property for $1 < \alpha < 2$ by [18], but are not Markov sets for any $\alpha > 1$ due to Pleśniak [33] and Białas [8]. The extension operator in [29] was given in the form of a telescoping series containing Lagrange interpolation polynomials with the Fekete-Leja system of knots. This operator is continuous in the Jackson topology τ_J, which is equivalent to the natural topology τ of the space $\mathcal{E}(K)$, provided that the compact set K admits Markov's inequality. Here, following [20], we interpolate the functions from $\mathcal{E}(K^{(\alpha)})$ locally and show that the modified operator is continuous in τ.

3.1 Jackson topology

For a perfect compact set K on the line, $\mathcal{E}(K)$ denotes the space of all functions f on K extendable to some $\tilde{f} \in C^\infty(\mathbb{R})$. The space $\mathcal{E}(K)$ can be identified with the quotient space $C^\infty(I)/Z$, where I is an interval containing K (let $I = [0,1]$) and $Z = \{F \in C^\infty(I) : F|_K \equiv 0\}$. By the Whitney theorem ([48]) the quotient topology τ can be given by the norms

$$\| f \|_q = |f|_q + \sup \left\{ |(R_y^q f)^{(k)}(x)| \cdot |x-y|^{k-q};\ x, y \in K, x \neq y, k = 0, 1, ...q \right\},$$

$q = 0, 1, ...,$ where $|f|_q = \sup\{|f^{(k)}(x)| : x \in K, k \leq q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder.

Following Zerner [54], Pleśniak [32] introduced in $\mathcal{E}(K)$ the following seminorms

$$d_{-1}(f) = |f|_0,\ d_0(f) = E_0(f),\ d_k(f) = \sup_{n \geq 1} n^k E_n(f)$$

for $k = 1, 2, \cdots$. Here $E_k(f)$ denotes the best approximation to f on K by polynomials of degree at most k. For a perfect set $K \subset \mathbb{R}$ the *Jackson topology* τ_J, given by (d_k), is Hausdorff. By the Jackson theorem the topology τ_J is well-defined and is not stronger than τ.

The characterization of analytic functions on a compact set K in terms of (d_k) was considered in [5]; for the spaces of ultradifferentiable functions see [12].

We remark that for any perfect set K the space $(\mathcal{E}(K), \tau_J)$ has the dominating norm property:

$$\exists p\ \forall q\ \exists r,\ C > 0 :\ d_q^2(f) \leq C\ d_p(f)\, d_r(f) \quad \text{for all} \quad f \in \mathcal{E}(K).$$

In fact, let n_k be such that $d_k(f) = n_k^k\, E_{n_k}(f)$. Then, $d_p(f) \geq n_q^p\, E_{n_q}(f)$ and $d_r(f) \geq n_q^r\, E_{n_q}(f)$, so we have the desired condition with $r = 2\,q$.

Tidten proved in [41] that the space $\mathcal{E}(K)$ admits an extension operator if and only if it has the property (DN). Clearly, the completion of the space with the property (DN) also has the dominating norm. Therefore, the Jackson topology is not generally complete. Moreover, it is not complete in the cases of compact sets from [15],[18] in spite of the fact that the corresponding spaces have the (DN) property. Indeed, by Th.3.3 in [32] the topologies τ and τ_J coincide for $\mathcal{E}(K)$ if and only if the compact set K satisfies the Markov Property (see [29]-[33] for the definition) and this is possible if and only if the extension operator, presented in [29], [32] and [33] is continuous in τ_J. We do not know the distribution of the Fekete points for Cantor-type sets, therefore we can not check the continuity of the Pawłucki and Pleśniak operator in the natural topology. Instead, following [20], we will interpolate the functions from $\mathcal{E}(K)$ locally.

3.2 The Pawłucki and Pleśniak extension operator

Following [29] let us explain the Pawłucki and Pleśniak extension operator for the (UPC) compact subsets of $\mathbb{R}^n$.

Definition 3.1 *A subset X on $\mathbb{R}$ is said to be* uniformly polynomially cuspidal *(UPC) if there exists positive constants M and m and a positive integer d such that for each point $x \in \bar{X}$, one may choose a polynomial map $h_x : \mathbb{R} \to \mathbb{R}^n$ of degree at most d satisfying the following conditions.*
(i) $h_x((0,1]) \subset X$ *and* $h_x(0) = x$,
(ii) $dist(h_x(t), \mathbb{R}^n - X) \geq Mt^m \ \forall x \in X$ *and* $t \in (0,1]$.

When X is a (UPC) compact subset of $\mathbb{R}^n$, then Siciak's extremal function of X has (HCP). Siciak's extremal function [39] is the generalized Green's function for the multidimensional case. So we also have Markov property for (UPC) compact sets.

Let the set of monomials $e_1, ..., e_{m_k}$ be a basis of the space $\mathcal{P}_k$ where $m_k = \binom{n+k}{k}$. Let $t^{(k)} = \{t_1, ..., t_k\}$ be a system of k points of $\mathbb{R}^n$. Consider the Vandermonde determinant

$$V(t^{(k)}) = \det[e_j(t_i)]$$

where $i, j \in \{1, ..., k\}$. If $V(t^{(k)}) \neq 0$ we have

$$l_j(x, t^{(k)}) = V(t_1, ..., t_{j-1}, x, t_{j+1}, ..., t_k)/V(t^{(k)})$$

as the lagrange fundamental polynomials and we get the following Lagrange interpolation formula [39]. If $p \in \mathcal{P}_k$ and $t^{(m_k)}$ i a system of m_k points of $\mathbb{R}^n$ such that $V(t^{(m_k)}) \neq 0$, then

$$p(x) = \sum_{j=1}^{m_k} p(t_j) l_j(x, t^{(m_k)})$$

for $x \in \mathbb{R}^n$. Let X be a compact subset of $\mathbb{R}^n$. A system $t^{(k)}$ of k points $t_1, ..., t_k$ of X is called *Fekete-Leja system of extremal points* of X of order k if $V(t^{(k)}) \geq V(s^{(k)})$ for all systems $s^{(k)} = \{s_1, ..., s_k\} \subset X$. Observe that if $t^{(k)}$ is a system of extremal points of X such that $V(t^{(k)}) \neq 0$, then

$$l_j(x, t^{(k)}) \leq 1$$

on X for $j = 1, ..., k$. Let

$$L_k f(x) = \sum_{j=1}^{m_k} f(t_j) l_j(x, t^{(m_k)}),$$

which is the Lagrange interpolation polynomial of f of degree k.Suppose f is continuous on X. Let p_k be any polynomial of degree k such that $|f - p_k|_X = \mathrm{dist}_X(f, \mathcal{P}_k)$. Then we have

$$\begin{aligned} |f - L_k f|_X &\leq |f - p_k| + |L_k f - L_k p_k|_X \\ &\leq (m_k + 1)|f - p_k|_X \leq 4k^n \mathrm{dist}_X(f, \mathcal{P}_k) \end{aligned}$$

Now, let X be a (UPC) compact subset of $\mathbb{R}^n$. Let $\epsilon_0 = 1$ and for each $k \geq 1$, set $\epsilon_k = (1/(C_1 k))^{1/\mu}$, where the constants C_1 and μ are chosen so that Siciak extremal function satisfies (HCP) and $C_1 \geq 1$. For $k = 0, 1, ...,$ define C^∞ functions u_k on $\mathbb{R}^n$ such that $u_k = 1$ in a neighborhood of X, $u_k = 0$ if $\mathrm{dist}(x, X) \geq \epsilon_k$, and for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}_+^n$, $|D^\alpha u_k(x)| \leq C_\alpha \epsilon_k^{-|\alpha|}$, with some constants C_α depending only on α. Given $f \in C^\infty(X)$, the extension operator is defined by Pawłucki and Pleśniak as follows:

$$Lf = u_1 L_1 f + \sum_{k=1}^{\infty} u_k (L_{k+1} f - L_k f)$$

This is a C^∞ function on $\mathbb{R}^n$ where the restriction to X is equal to f. If $\alpha \in \mathbb{Z}_+^n$, then by using (HCP) and Markov property we get

$$\begin{aligned}
|D^\alpha Lf|_{\mathbb{R}^n} &\leq |D^\alpha(u_1L_1f)|_{X\epsilon_1} + \sum_{k=1}^{\infty}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}|D^\beta u_k D^{\alpha-\beta}(L_{k+1}f - L_kf)|_{X\epsilon_k} \\
&\leq C_2|f|_X + \sum_{k=1}^{\infty}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}C_\beta\epsilon_k^{-|\beta|}(1 + C_1\epsilon_k^\mu)^k|D^{\alpha-\beta}(L_{k+1}f - L_kf)|_X \\
&\leq C_2|f|_X + \sum_{k=1}^{\infty}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}C_\beta(C_1k)^{|\beta|/\mu}(1 + 1/k)^kCk^{r|\alpha-\beta|}|(L_{k+1}f - L_kf)|_X \\
&\leq C_2|f|_X + C_3\sum_{k=1}^{\infty}k^{s|\alpha|+n}\mathrm{dist}(f, \mathcal{P}_k) \\
&\leq C_2d_{-1}(f) + C_4d_{s|\alpha|+n+2}(f),
\end{aligned}$$

where $s = \max(1/\mu, r)$ and the constants C_2, C_3, C_4 depend only on α, X and n. Now, the continuity of the operator follows from the equivalence of the topologies τ and τ_J for Markov sets.

3.3 Extension operator for $\mathcal{E}(K^{(\alpha)})$

Let $(l_s)_{s=0}^\infty$ be a sequence such that $l_0 = 1$, $0 < 2l_{s+1} < l_s$, $s \in \mathbb{N}$. Let K be the Cantor set associated with the sequence (l_s) that is $K = \bigcap_{s=0}^\infty E_s$, where $E_0 = I_{1,0} = [0,1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,s}$ of length l_s and E_{s+1} is obtained by deleting the open concentric subinterval of length $l_s - 2l_{s+1}$ from each $I_{j,s}$, $j = 1, 2, ...2^s$.

Fix $1 < \alpha < 2$ and l_1 with $2l_1^{\alpha-1} < 1$. We will denote by $K^{(\alpha)}$ the Cantor set associated with the sequence (l_n), where $l_0 = 1$ and $l_{n+1} = l_n^\alpha = ... = l_1^{\alpha^n}$ for $n \geq 1$. In this case $K^{(\alpha)}$ has no Markov property by Theorem 2.6, so we can not see if the Pawłucki-Pleśniak extension operator [29] is continuous or not.

In the notations of [4] we consider the set $K_2^{(\alpha)}$. The construction of operator for the case $K_n^{(\alpha)}$ with $\alpha < n$ is quite similar, so we can restrict ourselves to $n = 2$.

Let us fix $s, m \in \mathbb{N}$ and take $N = 2^m - 1$. The interval $I_{1,s}$ covers 2^{m-1}

basic intervals of the length l_{s+m-1}. Then $N+1$ endpoints (x_k) of these intervals give us the interpolating set of Lagrange interpolation polynomial $L_N(f,x,I_{1,s}) = \sum_{k=1}^{N+1} f(x_k)\,\omega_k(x)$, corresponding to the interval $I_{1,s}$. Here $\omega_k(x) = \frac{\Omega_{N+1}(x)}{(x-x_k)\Omega'_{N+1}(x_k)}$ with $\Omega_{N+1}(x) = \Pi_{k=1}^{N+1}(x-x_k)$. In the case $2^m < N+1 < 2^{m+1}$ we use the same procedure as in [20] to include new $N+1-2^m$ endpoints of the basic intervals of the length l_{s+m} into the interpolation set. The polynomials $L_N(f,x,I_{j,s})$, corresponding to other basic intervals, are taken in the same manner.

Given $\delta > 0$, and a compact set E we take the $C^\infty-$ function $u(\cdot,\delta,E)$ with the properties: $u(\cdot,\delta,E) \equiv 1$ on E, $u(x,\delta,E) = 0$ for $\mathrm{dist}(x,E) > \delta$ and $|u|_p \le c_p\,\delta^{-p}$, where the constant c_p depends only on p. Let $(c_p)\uparrow$.

Fix $n_s = [s\ \log_2\alpha]$ for $s \ge \log 4/\log\alpha$, $n_s = 2$ for the previous values of s and $\delta_{N,\,s} = l_{s+[\log_2 N]}$ for $N \ge 2$. Here $[a]$ denotes the greatest integer in a.

Let $N_s = 2^{n_s} - 1$ and $M_s = 2^{n_{s-1}-1} - 1$ for $s \ge 1, M_0 = 1$. Consider the operator from [20]

$$L(f,x) = L_{M_0}(f,x,I_{1,0})\,u(x,\delta_{M_0+1,\,0},I_{1,0}\cap K)$$
$$+\sum_{s=0}^{\infty}\Big\langle\sum_{j=1}^{2^s}\ \sum_{N=M_s+1}^{N_s}[L_N(f,x,I_{j,s}) - L_{N-1}(f,x,I_{j,s})]\,u(x,\delta_{N,\,s},I_{j,s}\cap K)$$
$$+\sum_{j=1}^{2^{s+1}}[L_{M_{s+1}}(f,x,I_{j,s+1}) - L_{N_s}(f,x,I_{[\frac{j+1}{2}],\,s})]\,u(x,\delta_{N_s,\,s},I_{j,\,s+1}\cap K)\Big\rangle.$$

We call the sums $\sum_{N=M_s+1}^{N_s}\cdots$ the *accumulation sums*. For fixed j (without loss of generality let $j=1$) we represent the term in the last sum in the telescoping form

$$-\sum_{N=2^{n_s-1}}^{2^{n_s}-1}[L_N(f,x,I_{1,s}) - L_{N-1}(f,x,I_{1,s})]\,u(x,l_{s+n_s-1},I_{1,s+1}\cap K) \qquad (3.1)$$

and will call this the *transition sum*. Here the interpolation set for the polynomial $L_N(f,x,I_{1,s})$ consists of all endpoints of the basic subintervals of the length l_{s+n_s-1} on $I_{1,s+1}$ and some (from 0 for $N = 2^{n_s-1}-1$ to all for $N = 2^{n_s}-1$) endpoints of the basic subintervals of the same length on $I_{2,s+1}$.

Clearly, the operator L is linear. Let us show that it extends the functions from $\mathcal{E}(K^{(\alpha)})$.

Lemma 3.2 *For any* $f \in \mathcal{E}(K^{(\alpha)}),\ \ x \in K^{(\alpha)}$ *we have* $L(f,x) = f(x)$.

Proof: By telescoping effect

$$L(f,x) = \lim_{s\to\infty} L_{M_s}(f,x,I_{j,s}), \tag{3.2}$$

where $j = j(s)$ is chosen in a such way, that $x \in I_{j,s}$.

We will denote temporarily $n_{s-1}-1$ by n. Then $M_s = 2^n - 1$. Arguing as in [20], for any q, $1 \le q \le M_s$, we have the bound

$$|\, L_{M_s}(f,x,I_{j,s}) - f(x)| \le ||\, f\,||_q \sum_{k=1}^{2^n} |\, x - x_k|^{\,q}\, |\, \omega_k(x)\,|. \tag{3.3}$$

For the denominator of $|\,\omega_k(x)\,|$ we get

$$|\, x_k - x_1| \cdots |\, x_k - x_{k-1}| \cdot |\, x_k - x_{k+1}| \cdots |\, x_k - x_{M_s+1}| \ge l_{n+s-1}\,(l_{n+s-2} - 2\,l_{n+s-1})^2 \cdot$$

$$(l_{n+s-3} - 2\,l_{n+s-2})^4 \cdots (l_s - 2\,l_{s+1})^{2^{\,n-1}} = l_{n+s-1} \cdot l_{n+s-2}^2 \cdots l_s^{2^{\,n-1}} \cdot A,$$

where $A = \Pi_{k=1}^{n-1}\,(1 - 2\frac{l_{s+k}}{l_{s+k-1}})^{2^{n-k}}$.

Clearly, $\ln A > -\sum_{k=1}^{n-1} 2^{n-k+2}\frac{l_{s+k}}{l_{s+k-1}}$ for large enough s. Since $\frac{l_{s+k}}{l_{s+k-1}} < \frac{l_{s+k-1}}{l_{s+k-2}}$ and $2^n \le \frac{1}{2}\,\alpha^{s-1}$, we have $\ln A > -2^{n+2}\, l_s^{\alpha-1} > -1$.

On the other hand, the numerator of $|\,\omega_k(x)\,|$ multiplied by $|\, x - x_k|^{\,q}$ gives the bound

$$|\, x - x_k|^{\,q-1}\, \Pi_1^{2^n}\, |\, x - x_k| \le l_s^{q-1} \cdot l_{n+s} \cdot l_{n+s-1} \cdot l_{n+s-2}^2 \cdots l_s^{2^n-1}.$$

Hence, the sum in (3.3) may be estimated from above by $e\ 2^n\ l_{n+s}\, l_s^{q-1}$, which approaches 0 as s becomes large. Therefore, the limit in (3.2) exists and equals to $f(x)$. $\Box$

3.4 Continuity of the operator

Theorem 3.3 *Let $1 < \alpha < 2$. The operator $L : \mathcal{E}(K^{(\alpha)}) \longrightarrow C^\infty(\mathbb{R})$, given in Section 3.3, is a continuous linear extension operator.*

Proof: Let us prove that the series, representing the operator L, uniformly converges together with any of its derivative.

For any $p \in \mathbb{N}$ let $q = 2^v - 1$ be such that $(2/\alpha)^v > p + 4$. Given q let s_0 satisfy the following conditions: $s_0 \geq 2v + 3$ and $\alpha^m \geq m$ for $m \geq n_{s_0 - 1}.$

Suppose the points $(x_k)_1^{N+1}$ are arranged in ascending order. Let us write the divided difference $[x_1, \cdots, x_{N+1}]f$ in the form

$$[x_1, \cdots, x_{N+1}]f = \sum_{k=1}^{N-q+1} A_k^{(q+1)} [x_k, \cdots, x_{k+q}]f.$$

By using the recurrence relation given in Theorem 1.22, we can easily see that there are $\binom{N-q}{k-1}$ different ways to obtain $[x_k, \cdots, x_{k+q}]f$ from $[x_1, \cdots, x_{N+1}]f$. And so we have the inequality

$$|A_k^{(q+1)}| \leq \binom{N-q}{k-1} \max \prod_{m=1}^{N-q} |x_{a(m)} - x_{b(m)}|^{-1}.$$

Therefore using Theorem 1.21 we have the following bound :

$$|\ [x_1, \cdots, x_{N+1}]f\ | \leq 2^{N-q}\, ||f||_q\ (\min \Pi_{m=1}^{N-q} |\, x_{a(m)} - x_{b(m)} |\,)^{-1}, \tag{3.4}$$

where min is taken over all $1 \leq j \leq N + 1 - q$ and all possible chains of strict embeddings $[x_{a(0)}, \cdots, x_{b(0)}] \subset [x_{a(1)}, \cdots, x_{b(1)}] \subset \cdots \subset [x_{a(N-q)}, \cdots, x_{b(N-q)}]$ with $a(0) = j$, $b(0) = j + q, \cdots, a(N - q) = 1$, $b(N - q) = N + 1$; here given $a(k)$, $b(k)$ we take $a(k + 1) = a(k)$, $b(k + 1) = b(k) + 1$ or $a(k + 1) = a(k) - 1$, $b(k + 1) = b(k)$. The length of the first interval in the chain is not included in the product in (3.4), which we denote in the sequel by Π.

For $s \geq s_0$ and for any $j \leq 2^s$ we consider the corresponding term of the accumulation sum. By the Newton form of interpolation operator we get

$$L_N(f, x, I_{j,s}) - L_{N-1}(f, x, I_{j,s}) = [x_1, \cdots, x_{N+1}]f \cdot \Omega_N(x),$$

where $\Omega_N(x) = \Pi_1^N(x - y_k)$ with the set $(y_k)_1^N$ consisting of all points $(x_k)_1^{N+1}$ except one.

Thus we need to estimate $|\,[x_1, \cdots, x_{N+1}]f\,| \cdot |(\,\Omega_N \cdot u(x, \delta_{N,\,s}, I_{j,s} \cap K))^{(p)}|$ from above. Here $M_s+1 \leq N \leq N_s$, that is $2^{m-1} \leq N < 2^m$ for some $m = n_{s-1}, \cdots, n_s$ and $\delta_{N,\,s} = l_{s+m-1}$. The interpolation set $(x_k)_1^{N+1}$ consists of all endpoints of the basic intervals of length l_{s+m-2} (inside the interval $I_{j,s}$) and some (possibly all for $N = 2^m - 1$) endpoints of the basic intervals of length l_{s+m-1}. For simplicity we take $j = 1$. In this case $x_1 = 0$, $x_2 = l_{s+m-1}$, $x_3 = l_{s+m-2} - l_{s+m-1}$ or $x_3 = l_{s+m-2}$, etc.

Since $\mathrm{dist}(x, I_{1,s} \cap K) \leq l_{s+m-1}$, we get $|\,\Omega_N^{(i)}(x)\,| \leq \frac{N!}{(N-i)!}\,\Pi_{k=i+1}^N(l_{s+m-1} + y_k)$. Therefore, $|\,(\Omega_N \cdot u)^{(p)}| \leq \sum_{i=0}^p \binom{p}{i}\, c_{p-i}\, l_{s+m-1}^{i-p}\, N^i\, \Pi_{k=i+1}^N(l_{s+m-1} + y_k) \leq$

$2^p\, c_p\, l_{s+m-1}^{-p}\, \Pi_{k=1}^N(l_{s+m-1} + y_k) \cdot max_{i \leq p}\, B_i$, with $B_0 = 1$, $B_1 = N$, $B_2 = N^2/2, \cdots, B_i = N^2/2 \cdot (N\, l_{s+m-1})^{i-2}\, (l_{s+m-1} + y_3)^{-1} \cdots (l_{s+m-1} + y_i)^{-1}$ for $i \geq 3$. To estimate B_3, we note that $l_{s+m-1} + y_3 \geq l_{s+m-2}$, $N\, l_{s+m-1} < 2^m\, l_{s+m-2}^{\alpha} \leq l_{s+m-2}$, as $2^m\, l_{s+m-2}^{\alpha-1} = 2^m\, l_{s-2}^{(\alpha-1)\alpha^m} < 2^m\, l_1^{(\alpha-1)\alpha^m} < 2^m(\frac{1}{2})^{\alpha^m} \leq 1$, due to the choice of s_0. Therefore, B_3, and all the more B_i for $i > 3$, is less than B_2. On the other hand, $l_{s+m-1} + y_k < y_{k+1}$, $k \leq N-1$, as l_{s+m-1} is a mesh of the net $(y_k)_1^N$, and $l_{s+m-1} + y_N < 2 l_s$. This implies that,

$$|\,(\Omega_N \cdot u)^{(p)}| \leq 2^p\, c_p\, N^2\, l_{s+m-1}^{-p}\, l_s\, \Pi_{k=2}^N y_k \leq 2^p\, c_p\, N^2\, l_{s+m-1}^{-p-1}\, l_s\, \Pi_{k=2}^{N+1} x_k. \tag{3.5}$$

To apply (3.4), for $1 \leq j \leq N+1-q$ we consider $q+1$ consecutive points $(x_{j+k})_{k=0}^q$ from $(x_k)_1^{N+1}$. Every interval of the length l_{s+k} contains from $2^{m-k-1}+1$ to 2^{m-k} points x_k. Therefore the interval of the length $l_{s+m-v-1}$ contains more than $q+1$ points. In order to minimize the product Π, we have to include intervals containing large gaps of the set $K^{(\alpha)}$in the chain $[x_j, \cdots, x_{j+q}] \subset \cdots \subset [x_1, \cdots, x_{N+1}]$ as late, as possible, that is all $q+1$ points must belong to $I_{j,s+m-v-1}$ for some j. By the symmetry of the set $K^{(\alpha)}$ we again can take $j = 1$. The interval of the length l_{s+m-v} contains at most 2^v points, whence for any choice of $q+1$ points in succession, all values that make up the product Π, are not smaller than the length of the gap $h_{s+m-v-1} := l_{s+m-v-1} - 2\, l_{s+m-v}$. Therefore, $\Pi \geq h_{s+m-v-1}^{J-q-1}\, \Pi_{J+1}^{N+1} x_k$, where J, $J \leq 2^{v+1}$, is the number of points x_k on $I_{1,s+m-v-1}$.

Further, $J - q - 1 \leq 2^v$ and

$$\frac{x_{q+2} \cdots x_J}{h_{s+m-v-1}^{J-q-1}} \leq \Big(\frac{l_{s+m-v-1}}{l_{s+m-v-1} - 2\, l_{s+m-v}}\Big)^{2^v} < exp\,(2^v\, 4 l_{s+m-v-1}^{\alpha-1}). \tag{3.6}$$

Since $l_{s+m-v-1}^{\alpha-1} = l_1^{(\alpha-1)(s+m-v-2)} < 2^{-s+v}$, we see that the fraction above is smaller than 2, due to the choice of s_0. It follows that $\Pi \geq \frac{1}{2}\,\Pi_{q+2}^{N+1} x_k$ and $|\,[x_1, \cdots, x_{N+1}]f\,| \leq 2^{N-\,q-1}\,||f||_q\;(x_{q+2} \cdots x_{N+1})^{-1}$.

Combining this with (3.5) we have

$$|\,[x_1, \cdots, x_{N+1}]f\,| \cdot |(\,\Omega_N \cdot u)^{(p)}| \leq c_p\, N^2\, 2^N\, l_s\; l_{s+m-1}^{-p-1}\, \Pi_{k=2}^{q+1} x_k.$$

Our next goal is to evaluate $\Pi_{k=2}^{q+1} x_k$ in terms of l_{s+m-1}. Estimating roughly all x_k, $k > 2$, that are not endpoints of the basic intervals of length l_{s+m-2}, from above by $l_{s+m-v-1}$, we get

$$\Pi_{k=2}^{q+1} x_k \leq l_{s+m-1}\, l_{s+m-2}\, l_{s+m-3}^2 \cdots l_{s+m-v}^{2^{v-2}}\, l_{s+m-v-1}^{2^{v-1}-1} = l_{s+m-1}^{\kappa}$$

with $\kappa = 1 + \frac{1}{\alpha} + \frac{2}{\alpha^2} + \cdots + \frac{2^{v-1}}{\alpha^v} - \frac{1}{\alpha^v} > (2/\alpha)^v - 1$.

Therefore,

$$|\,[x_1, \cdots, x_{N+1}]f\,| \cdot |(\,\Omega_N \cdot u)^{(p)}| \leq c_p\, N^2\, 2^N\; l_{s+m-1}^2,$$

since $\kappa + \alpha^{-m+1} - p - 1 > 2$, due to the choice of q.

On the one hand, $2^N\; l_{s+m-1} < 2^{2^m}\, l_1^{\alpha^{s+m-2}} < 2^{2^{n_s} - \alpha^s} \leq 1$, as $m \geq 2$ and $l_1 < 1/2$. On the other hand, the accumulation sum contains $N_s - M_s < N_s$ terms. Therefore,

$$|\,(\sum_{N=M_s+1}^{N_s} \cdots)^{(p)}| \leq c_p\, N_s^3\; l_s,$$

which is a term of convergent with respect to s series, as is easy to see. We neglect the sum with respect to j, because for fixed x at most one term of this sum does not vanish.

The same proof works for terms of the transition sums. The sum (3.1) does not vanish only for x at a short distance to $I_{1,s+1} \cap K$. For this reason the arguments of the estimation of $|\Omega_N^{(i)}(x)|$ remain valid. On the other hand, if we want to minimize the product of lengths of intervals, constituent the chain $[x_j, \cdots, x_{j+q}] \subset \cdots \subset [x_1, \cdots, x_{N+1}]$, then we have to take $x_j, \cdots, x_{j+q}$ in the interval $I_{1,s+1}$. Therefore the bound (3.6) and the followings still go. Thus the operator L is well-defined and continuous. $\Box$

Remark. It is a simple matter to find a sequence of functions divergent in τ that converges in the Jackson topology. It is interesting that the same sequence can destroy the Markov inequality. Given $s \in \mathbb{N}$ let $N = 2^s$ and $P_N(x) = (l_{s-1} \cdot l_{s-2}^2 \cdots l_0^{2^{s-1}})^{-1} \Pi_{j=1}^N (x - c_{j,s})$, where $c_{j,s}$ is a midpoint of the interval $I_{j,s}$. Then $\frac{1}{s} \ln(|P_N'(0)|/|P_N|_0) \to \infty$ as $s \to \infty$, contrary to the Markov property. On the other hand, $E_n(P_N) \le |P_N|_0$ for $n < N$. Then for any k we get $d_k(P_N) \le N^k \, |P_N|_0 \le 2^{s\,k} \, l_s \to 0$, as $s \to \infty$. But $P_N'(0) \nrightarrow 0$, so the sequence (P_N) diverges.

Chapter 4

Extension for another model case

Here we consider a compact set of the form of a sequence of closed intervals convergent to a point. The spaces of Whitney functions on compact sets of this type were considered in [15], [16], [19]. For ultradifferentiable classes of functions on such sets, an extension operator was given by Beaugendre [6]. Under suitable choice of parameters, these sets are the first examples of compact sets without Markov property, but with the extension property. So, in such cases the Pawłucki-Pleśniak extension method can not be applied.

Let $K \subset \mathbb{R}$ be a perfect compact set. $\mathcal{E}(K)$ is the space of infinitely differentiable Whitney functions on K. Let the norms $(|| \cdot ||_q)_{q=1}^{\infty}$ be given as in section 3.1.

For any set $S \subset \mathbb{R}$, let

$$|f|^S = \sup\{|f(x)| : x \in S\},$$

and for $\epsilon \geq 0$ let $S\epsilon$ denote the set $\{x \in \mathbb{R} : \mathrm{dist}(x, S) \leq \epsilon\}$.

Let $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$ such that $K \subset [0,1]$. $I_k = [a_k, b_k]$. Let $\delta_k = \frac{1}{2}(b_k - a_k), h_k = a_k - b_{k+1}$. $2\delta_k \leq h_k$ and $\delta_k \downarrow 0$, $a_k \downarrow 0$. Let $b_k = Bk\delta_k$ where B is a constant. Let $R > 1$ such that $\delta_{k+1} = \delta_k^R$. Then by [19] K has the extension property, we will see below that the set has no Markov property. In [16] for compact sets formed of intervals converging to a point, under some conditions for the parameters, a basis was given for the Whitney functions on such sets, and an extension operator was constructed by use of extensions of the basis elements. We can see that, the set K does not satisfy the conditions of having a basis in [16]. So, for $\mathcal{E}(K)$, we do not have any information about the basis, and the extension could not be given by using basis elements. Here, we give explicit form of an extension operator for $\mathcal{E}(K)$ by using local interpolation of functions by polynomials.

Let

$$l_j^N(x, I_k) = \prod_{i=1, i\neq j}^{N+1} \left(\frac{x - t_{N,i}^k}{t_{N,j}^k - t_{N,i}^k} \right)$$

$j = 1, ..., N+1$ and

$$l_j^N(x, I_k, 0) = \prod_{i=0, i\neq j}^{N} \left(\frac{x - t_{N-1,i}^k}{t_{N-1,j}^k - t_{N-1,i}^k} \right)$$

$j = 0, ..., N$ where $\{t_{N,1}^k, ..., t_{N,N+1}^k\}$ are the Chebyshev zeros of order $N+1$ of the interval I_k and $t_{N,0}^k = 0$ for all $k, N \in \mathbb{N}$. Define

$$L_N^{I_k} f(x) = \sum_{j=1}^{N+1} f(t_{N,j}^k) l_j^N(x, I_k).$$

which is the Lagrange interpolation polynomial of f of degree N of the points $\{t_{N,1}^k, ..., t_{N,N+1}^k\}$.

Let $J_k = [0, b_k]$ and $K_k = K \cap J_k$ and define

$$L_N^{J_k} f(x) = \sum_{j=o}^{N} f(t_{N-1,j}^{k-2}) l_j^N(x, I_{k-2}, 0).$$

which is the Lagrange interpolation polynomial of f of degree N of the points $\{t_{N-1,0}^{k-2}, ..., t_{N-1,N}^{k-2}\}$.

We have the following [36] upper bound for $|l_j^N(x, I_k)|$.

$$|l_j^N(x, I_k)| \leq \frac{4}{\pi}$$

for $x \in I_k$, $k \geq 1$ and $j \in \{1, 2, ..., N+1\}$. We will use the integer bound 2 instead of $\frac{4}{\pi}$. Next lemma gives an upper bound for $|l_j^k(x, I_{k-2}, 0)|$.

Lemma 4.1 *There exists a constant $C_1 > 0$ such that $|l_j^k(x, I_{k-2}, 0)| \leq C_1$ for $x \in J_k$ for all $k \geq 3$ and $j \in \{0, 1, ..., k\}$.*

Proof: Suppose $j = 0$, then

$$|l_0^k(x, I_{k-2}, 0)| = \left| \prod_{i=1}^{k} \left(\frac{x - t_{k-1,i}^{k-2}}{0 - t_{k-1,i}^{k-2}} \right) \right| = \prod_{i=1}^{k} \left(1 - \frac{x}{t_{k-1,i}^{k-2}} \right)$$

and hence $|l_0^k(x, I_{k-2}, 0)| \leq 1$ for $x \in J_k$.

Now, suppose $1 \leq j \leq k$. Then

$$\begin{aligned} |l_j^k(x, I_{k-2}, 0)| &= \prod_{i=0, i \neq j}^{k} \left| \frac{x - t_{k-1,i}^{k-2}}{t_{k-1,j}^{k-2} - t_{k-1,i}^{k-2}} \right| = \prod_{i=1, i \neq j}^{k} \left| \frac{x - t_{k-1,i}^{k-2}}{t_{k-1,j}^{k-2} - t_{k-1,i}^{k-2}} \right| \cdot \left| \frac{x}{t_{k-1,j}^{k-2}} \right| \\ &\leq \frac{b_{k-2}^{k-1}}{2(\frac{\delta_{k-2}}{2})^k \cdot |T'_{k,k-2}(t_{k-1,j}^{k-2})|} \cdot \frac{b_k}{a_{k-2}} \\ &\leq \frac{b_{k-2}^{k-1}}{(\frac{\delta_{k-2}}{2})^{k-1} \cdot k} \cdot \frac{b_k}{a_{k-2}} \leq \frac{(2Bk)^{k-1}}{k} \cdot \frac{Bk\delta_k}{\delta_{k-1}} \\ &\leq B^k (2k)^{k-1} \delta_1^{(R-1)R^{k-2}} \end{aligned}$$

using the fact that $|T'_{k,k-2}(t_{k-1,j}^{k-2})| > \frac{k}{\delta_{k-2}}$. The last expression above goes to zero as k goes to infinity. □

We have the following inequality by the Hölder continuity property of Green's function for domains complementary to closed intervals. Let $I = [x_0 - \delta, x_0 + \delta]$, then for some constant $C > 0$:

$$|p_n|^{I\epsilon} \leq \left[1 + C \left(\frac{\epsilon}{\delta} \right)^{\frac{1}{2}} \right]^n |p_n|^I$$

for any polynomial $p_n \in \mathcal{P}_n$.

Define $L_N f(x) = L_N^{I_k} f(x)$ for $x \in I_k \delta_k$ for $1 \leq k < N$, $L_N f(x) = L_N^{J_N} f(x)$ for $x \in J_N \delta_N$ and $L_N f(x) = 0$ elsewhere.

Let $\epsilon_{Nk} = \delta_k(\frac{1}{N^2})$. Let $u_N = u_{N1} + \cdots + u_{NN}$ where u_{Nk} is a C^∞ function such that for $k < N$ $u_{Nk} = 1$ for $x \in I_k$, $u_{Nk} \equiv 0$ for $\text{dist}(x, I_k) > \epsilon_{Nk}$ and

$|u_{Nk}|_p \leq D_p \epsilon_{Nk}^{-p}$. And $u_{NN} = 1$ for $x \in J_N$, $u_{NN} \equiv 0$ for $dist(x, J_N) > \epsilon_{NN}$ and $|u_{NN}|_p \leq D_p \epsilon_{NN}^{-p}$. Let here the constants D_p be of increasing order.

On the interval $I = [x_0 - \delta, x_0 + \delta]$ any polynomial $p_n \in \mathcal{P}_n$ satisfies the following Markov inequality as a generalization of the Markov inequality for the interval [-1,1].

$$|p_n'|^I \leq \frac{1}{\delta} n^2 |p_n|^I.$$

Let $f \in C^\infty(I)$ where $I = [x_0 - \delta, x_0 + \delta]$, then by Jackson's theorem [43] for any $n > q$ the following holds:

$$\text{dist}_I(f, \mathcal{P}_n) \leq M_q \left(\frac{2\delta}{n}\right)^q w(f^{(q)}; \frac{2\delta}{n})$$

where M_q is a constant depending only on q. Since $f^{(q)}$ is differentiable, for the modulus of continuity w we have

$$w(f^{(q)}; \frac{2\delta}{n}) \leq \frac{2\delta}{n} |f^{(q+1)}|^I.$$

Hence we have

$$\text{dist}_I(f, \mathcal{P}_n) \leq M_q \left(\frac{2\delta}{n}\right)^{q+1} |f^{(q+1)}|^I.$$

Lemma 4.2 *Let $f \in C^\infty(I)$ where $I = [x_0 - \delta, x_0 + \delta]$, and S be any closed subset of I, then for any $q \in \mathbb{N}$ such that for $n \geq q$ the following holds:*

$$dist_S(f, \mathcal{P}_n) \leq (2\delta)^q ||f||_q^S$$

Proof: For any $y \in S$ we have

$$\text{dist}_S(f, \mathcal{P}_n) \leq \text{dist}_S(f, \mathcal{P}_q) \leq |R_y^q f(x)|^S \leq (2\delta)^q ||f||_q^S$$

□

4.1 Extension operator for $\mathcal{E}(K)$

Define the operator as

$$Lf = u_1 L_1 f + \sum_{N=2}^{\infty} u_N (L_N f - L_{N-1} f). \tag{4.1}$$

Lemma 4.3 *$Lf(x) = f(x)$ for any $f \in \mathcal{E}(K)$ and $x \in K$.*

Proof:

$$Lf(x) = \lim_{N\to\infty} L_N f(x)$$

Suppose $x \in I_k$ for some $k \in \mathbb{N}$. Let $N > k$, choose $p_N \in \mathcal{P}_N$ such that $|f - p_N|^{I_k} = \mathrm{dist}_{I_k}(f, \mathcal{P}_N)$. Then we have

$$\begin{aligned} |L_N f(x) - f(x)| &= |L_N^{I_k} f(x) - f(x)| \\ &\leq |L_N^{I_k} f(x) - p_N(x)| + |p_N(x) - f(x)| \\ &\leq |L_N^{I_k}(f - p_N)(x)| + \mathrm{dist}_{I_k}(f, \mathcal{P}_N) \\ &\leq 2\sum_{j=1}^{N+1} |f(t_{N,j}^k) - p_N(t_{N,j}^k)| + \mathrm{dist}_{I_k}(f, \mathcal{P}_N) \\ &\leq (2N+3)\mathrm{dist}_{I_k}(f, \mathcal{P}_N). \end{aligned}$$

Here using the Jackson theorem we get to

$$|L_N f(x) - f(x)| \leq M_q(2N+3)\left(\frac{2\delta_k}{N}\right)^{q+1} |f|_{q+1}$$

for any q and for any $N \geq q$. Hence $\lim_{N\to\infty} |L_N f(x) - f(x)| = 0$ for $x > 0$. For $x = 0$ we have by definition $L_N f(0) = f(0)$ for all N. $\Box$

4.2 Continuity of the operator

Theorem 4.4 *The operator $L : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R})$, defined in (4.1) is continuous.*

Proof: For given $p \in \mathbb{N}$ let $q = 2\lceil R^3 \rceil p + 3$, then

$$(Lf)^{(p)} = (u_1 L_1 f)^{(p)} + \sum_{N=2}^{\infty}\sum_{i=0}^{p} \binom{p}{i} u_N^{(p-i)} \cdot (L_N f - L_{N-1} f)^{(i)}.$$

Hence

$$|(Lf)^{(p)}| \leq |(u_1 L_1 f)^{(p)}|^{J_1 \epsilon_{11}} +$$

$$\sum_{N=2}^{\infty}\sum_{i=0}^{p} \binom{p}{i} \max\left\{ |u_N^{(p-i)} \cdot (L_N f - L_{N-1} f)^{(i)}|^A : A \in \mathcal{A}_N \right\} \tag{4.2}$$

for $\mathcal{A}_N = \{I_1\epsilon_{N1}, ..., I_{N-1}\epsilon_{NN-1}, J_N\epsilon_{NN}\}$.

For $k \leq N-2$

$$\begin{aligned}
|(L_Nf - L_{N-1}f)^{(i)}|^{I_k\epsilon_{Nk}} &= |(L_N^{I_k}f - L_{N-1}^{I_k}f)^{(i)}|^{I_k\epsilon_{Nk}} \\
&\leq \left[1 + C\left(\frac{\epsilon_{Nk}}{\delta_k}\right)^{\frac{1}{2}}\right]^N |(L_N^{I_k}f - L_{N-1}^{I_k}f)^{(i)}|^{I_k} \\
&\leq \left[1 + C\left(\frac{\epsilon_{Nk}}{\delta_k}\right)^{\frac{1}{2}}\right]^N \delta_k^{-i}N^{2i}|L_N^{I_k}f - L_{N-1}^{I_k}f|^{I_k} \\
&\leq e^C\delta_k^{-i}N^{2i}|L_N^{I_k}f - L_{N-1}^{I_k}f|^{I_k}.
\end{aligned}$$

Choose $p_{N-1} \in \mathcal{P}_{N-1}$ such that $|f - p_{N-1}|^{I_k} = \mathrm{dist}_{I_k}(f, \mathcal{P}_{N-1})$, then

$$\begin{aligned}
|L_N^{I_k}f - L_{N-1}^{I_k}f|^{I_k} &\leq |L_N^{I_k}f - p_{N-1}|^{I_k} + |p_{N-1} - L_{N-1}^{I_k}f|^{I_k} \\
&\leq |L_N^{I_k}(f - p_{N-1})|^{I_k} + |L_{N-1}^{I_k}(f - p_{N-1})|^{I_k} \\
&\leq 2(2N+1)\mathrm{dist}_{I_k}(f, \mathcal{P}_{N-1}).
\end{aligned}$$

Hence for $k \leq N-2$

$$\begin{aligned}
|u_N^{(p-i)} \cdot (L_Nf - L_{N-1}f)^{(i)}|^{I_k\epsilon_{Nk}} &\leq D_{p-i}\delta_k^{-(p-i)}N^{2(p-i)}6e^C\delta_k^{-i}N^{2i+1}\mathrm{dist}_{I_k}(f, \mathcal{P}_{N-1}) \\
&\leq 6D_pe^C\delta_k^{-p}N^{2p+1}\mathrm{dist}_{I_k}(f, \mathcal{P}_{N-1}).
\end{aligned}$$

Let $N > q+1$, then

$$\begin{aligned}
|u_N^{(p-i)} \cdot (L_Nf - L_{N-1}f)^{(i)}|^{I_k\epsilon_{Nk}} &\leq 6D_pe^C\delta_k^{-p}N^{2p+1}M_q\left(\frac{2\delta_k}{N-1}\right)^{q+1}|f|_{q+1} \\
&\leq 24D_pe^CM_q2^{2q}N^{-2}|f|_{q+1}.
\end{aligned}$$

Similarly

$$\begin{aligned}
|(L_Nf - L_{N-1}f)^{(i)}|^{J_N\epsilon_{NN}} &\leq \left[1 + C\left(\frac{2\epsilon_{NN}}{b_N}\right)^{\frac{1}{2}}\right]^N (\frac{b_N}{2})^{-i}N^{2i}|L_N^{J_N}f - L_{N-1}^{J_{N-1}}f|^{J_N} \\
&\leq \left[1 + C\left(\frac{\epsilon_{NN}}{\delta_N}\right)^{\frac{1}{2}}\right]^N \delta_N^{-i}N^{2i}|L_N^{J_N}f - L_{N-1}^{J_{N-1}}f|^{J_N} \\
&\leq e^C\delta_N^{-i}N^{2i}(2N+1)C_1\mathrm{dist}_{K_{N-3}}(f, \mathcal{P}_{N-1}) \\
&\leq 3C_1e^C\delta_N^{-i}N^{2i+1}\mathrm{dist}_{K_{N-3}}(f, \mathcal{P}_{N-1}).
\end{aligned}$$

Then for $N \geq q+1$ we have

$$\begin{aligned}
|u_N^{(p-i)} \cdot (L_N f - L_{N-1} f)^{(i)}|^{J_N \epsilon_{NN}} &\leq D_p \delta_N^{-(p-i)} N^{2(p-i)} 3C_1 e^C \delta_N^{-i} N^{2i+1} \mathrm{dist}_{K_{N-3}}(f, \mathcal{P}_q) \\
&\leq 3C_1 e^C D_p \delta_N^{-p} N^{2p+1} \mathrm{dist}_{K_{N-3}}(f, \mathcal{P}_q) \\
&\leq 3C_1 e^C D_p \delta_N^{-p} N^{2p+1} b_{N-3}^q ||f||_q \\
&\leq 3C_1 e^C D_p \delta_{N-3}^{-R^3 p} N^{2p+1} (B(N-3)\delta_{N-3})^q ||f||_q \\
&\leq 3C_1 e^C D_p B^q \delta_1^{(q-R^3 p)R^{N-4}} N^{q+2p+1} ||f||_q
\end{aligned}$$

Hence there exists an integer N_0 such that for $N \geq N_0$:

$$|u_N^{(p-i)} \cdot (L_N f - L_{N-1} f)^{(i)}|^{J_N \epsilon_{NN}} \leq \frac{1}{N^2} ||f||_q.$$

And for $k = N-1$ we have

$$\begin{aligned}
|(L_N f - L_{N-1} f)^{(i)}|^{I_{N-1}\epsilon_{N(N-1)}} &\leq \left[1 + C\left(\frac{\epsilon_{N(N-1)}}{\delta_{N-1}}\right)^{\frac{1}{2}}\right]^N \delta_{N-1}^{-i} N^{2i} |L_N^{I_{N-1}} f - L_{N-1}^{J_{N-1}} f|^{I_{N-1}} \\
&\leq e^C \delta_{N-1}^{-i} N^{2i} [2(N+1) + C_1 N] \mathrm{dist}_{K_{N-3}}(f, \mathcal{P}_{N-1}) \\
&\leq (4 + C_1) e^C \delta_{N-1}^{-i} N^{2i+1} \mathrm{dist}_{K_{N-3}}(f, \mathcal{P}_{N-1}).
\end{aligned}$$

Therefore, similarly we can conclude that there exist an integer N_1 such that for $N \geq \max\{N_1, q+1\}$:

$$|u_N^{(p-i)} \cdot (L_N f - L_{N-1} f)^{(i)}|^{I_{N-1}\epsilon_{N(N-1)}} \leq \frac{1}{N^2} ||f||_q.$$

Now, let $N_2 = \max\{q+1, N_0, N_1\}$. Dividing the sum in (4.2) into two parts we have

$$|(Lf)^{(p)}| \leq C_2 |f|_0 + \sum_{N=2}^{N_2-1} + \sum_{N=N_2}^{\infty}$$

for some constant $C_2 > 0$. For the first sum there exists a constant $C_3 > 0$ such that

$$\sum_{N=2}^{N_2-1} \leq C_3 ||f||_q.$$

For the second sum

$$
\begin{aligned}
\sum_{N=N_2}^{\infty} &\leq ||f||_{q+1} \sum_{N=N_2}^{\infty} \sum_{i=0}^{p} \binom{p}{i} \max\left\{\frac{1}{N^2}, 24D_p e^C M_q 2^{2q} \frac{1}{N^2}\right\} \\
&\leq ||f||_{q+1} 24 D_p e^C M_q 2^{2q+p} \sum_{N=N_2}^{\infty} \frac{1}{N^2} \\
&\leq C_4 ||f||_q
\end{aligned}
$$

for some constant $C_4 > 0$. Hence, the operator (4.1) is continuous. $\square$

Following the idea from [16](Proposition 1) we can prove the following theorem.

Theorem 4.5 *Let the constants $R \geq 2$ and $B \geq 6$, then the compact set K does not have the Markov property.*

Proof: Without loss of generality, let $\delta_k = \exp(-R^k)$, $k \in \mathbb{N}$. Fix $m \in \mathbb{N}$. Consider the polynomial

$$P(x) = x \cdot \prod_{k=1}^{m} \gamma_k \cdot \tilde{T}_{n_k,k}(x)$$

where $\gamma_k = \tilde{T}_{n_k,k}(0)$. Take $n_m = 1$, $n_k = R^{m+(m-1)+\cdots+(k+1)}$ for $k \leq m-1$.

Then $P'(0) = 1$ and deg $P = 1 + \sum_{k=1}^{m} n_k < R^{m^2}$. We will show that

$$|P(x)| \leq b_m, \qquad x \in K.$$

This implies the absence of Markov property for K since

$$1 \leq C R^{\mu m^2} m \exp(-R^m), \quad m \to \infty$$

is a contradiction for fixed C, μ.

Fix $x \in K$. If $x \leq b_m$, then $|\gamma_k \cdot \tilde{T}_{n_k,k}(x)| \leq 1$, $k = 1, 2, ..., m$, and the desired bound for $|P(x)|$ is obvious. Consider now $x \in I_j$, $1 \leq j \leq m-1$. Then

$$|P(x)| \leq b_j |\gamma_j| \prod_{k=j+1}^{m} |\gamma_k \cdot \tilde{T}_{n_k,k}(x)|,$$

since all other terms of the product are less than 1.

To estimate the remaining terms, we use the following bound from [28].

$$2^{n-1}(\Delta_k/\delta_k)^n < |\tilde{T}_{n,k}(x)| < 2^{n-1}(\Delta_k/\delta_k + 2)^n, \quad n > 0,$$

$\Delta_k = \operatorname{dist}(x, I_k)$. Therefore,

$$|\gamma_k \cdot \tilde{T}_{n_k,k}(x)| \leq (\frac{b_j}{a_k})^{n_k} = \left(\frac{Bj\delta_j}{(Bk-2)\delta_k}\right)^{n_k}$$

and

$$|\gamma_j| < 2(2Bj-4)^{-n_j}.$$

Hence,

$$|P(x)| < 2Bj\exp(-R^j)(2Bj-4)^{-n_j}\exp\sum_{k=j+1}^{m} n_k[R^k - R^j + \ln(\frac{Bj}{Bk-2})].$$

We have

$$R^j \geq R > \ln\frac{4}{3} > \ln(\frac{Bj}{Bj+B-2}) > \ln(\frac{Bj}{Bk-2}).$$

Using the relations

$$2\exp(-R^j) < 1, \quad n_k R^k = n_{K-1},$$

we have

$$\ln(|P(x)|/b_m) < \ln(Bm)\cdot[R^m - n_j\ln(2Bj-4) + \sum_{k=j}^{m-1} n_k]. \tag{4.3}$$

Using the estimates

$$R^m + \sum_{k=j}^{m-1} n_k \leq 2n_j, \quad \ln(2Bj-4) > 2$$

it follows the expression on the right of (4.3) is negative and $|P|^K \leq b_m$. Hence K does not have the Markov property. $\square$

Chapter 5

Extension property of Cantor sets in $\mathbb{R}^n$

There are several results about the existence of the extension property of compact sets in $\mathbb{R}$. In the multidimensional case only a few results are known. In [29] (see also [31], [33]) Pawłucki and Pleśniak suggested an explicit construction of the extension operator for a rather wide class of compact sets. For example if K is the closure of a domain with Hölder type boundary then it has the extension property (see e.g.[41]). On the other hand if K has a thin cusp then K does not have the extension property (see e.g. [17]).

In this chapter we will consider Cantor sets in $\mathbb{R}^n$. In the one-dimensional case perfect sets of class α were considered by Tidten [42] and he proved as a corollary that the classical Cantor set K has the extension property. Later Goncharov [18] gave necessary and sufficient conditions for the extension property of generalized Cantor sets of class α. In $\mathbb{R}^n$ we will show that some similar conditions can be given for the Cantor set which is formed by taking cross product of the one dimensional generalized Cantor sets (This chapter mostly contains some results of the M.S. thesis of the applicant).

In what follows we consider only C^∞-determining compact sets. Let K be a C^∞-determining compact set in $\mathbb{R}^n$. Then $\mathcal{E}(K)$ is the space of Whitney functions with the topology defined by the norms

$$||f||_q = |f|_q + \sup\left\{\frac{|(R_y^q f)^{(k)}(x)|}{|x-y|^{q-|k|}} : x, y \in K, x \neq y, |k| \leq q\right\},$$

$|k| = k_1 + ... + k_n,$

$q = 0, 1, ...,$ where $|f|_q = \sup\{|f^{(k)}(x)| : x \in K, |k| \leq q\}$ and

$$R_y^q f(x) = f(x) - T_y^q f(x) = f(x) - \sum_{|k|\leq q} \frac{f^{(k)}(y)}{k_1!...k_n!}(x_1 - y_1)^{k_1}...(x_n - y_n)^{k_n}$$

is the Taylor remainder. Let $K^{(\alpha)}$ be as in section 3.3.

Theorem 5.1 *[18] For $1 < \alpha$, $\alpha \neq 2$, the Cantor set $K^{(\alpha)}$ has the extension property if and only if $\alpha < 2$.*

5.1 Cantor type sets in $\mathbb{R}^n$ and the extension property

We see that the critical value of the parameter α for the one dimensional Cantor sets is $\alpha = 2$. We want to find the critical values for the set $K^{(\alpha_1)} \times K^{(\alpha_2)} \times ... \times K^{(\alpha_n)}$. Let for $i \leq n$ $K^{[\alpha_1,\dots,\alpha_i]}$ denote the set $K^{(\alpha_1)} \times K^{(\alpha_2)} \times ... \times K^{(\alpha_i)}$. For simplicity we will use the following notations:

$\|f\|_q^{(i)}$ denotes the q^{th} norm of $f \in \mathcal{E}(K^{[\alpha_1,\dots,\alpha_i]})$, $i \in \{1, 2, ..., n\}$.

For $x = (x_1, ..., x_n) \in K^{[\alpha_1,...,\alpha_n]}$ and $k = (k_1, ..., k_n) \in \mathbb{N}^n$ let

$$\begin{aligned}
\vec{x} &= (x_1, ..., x_n) \\
\vec{x}_i &= (x_i, ..., x_n) \\
\bar{x}_i &= (x_1, ..., x_i) \\
k! &= k_1!...k_n! \\
x^k &= x_1^{k_1}...x_n^{k_n} \\
\vec{x} \geq \vec{y} &\Leftrightarrow x_i \geq y_i \ \ \forall i \leq n \\
\vec{x} = \vec{y} &\Leftrightarrow x_i = y_i \ \ \forall i \leq n \\
\vec{x} > \vec{y} &\Leftrightarrow \vec{x} \geq \vec{y} \ \text{ and } \ \vec{x} \neq \vec{y}
\end{aligned}$$

Lemma 5.2 *Let* $f \in \mathcal{E}(K^{[\alpha_1,...,\alpha_n]})$*. For* $n \geq 2$ *fix* $c \in K^{[\alpha_2,...,\alpha_n]}$ *and let* $f_c(x) = f(x, c), x \in K^{(\alpha_1)}$*. Then*

$$\|f\|_q^{(n)} \geq \|f_c\|_q^{(1)}.$$

Lemma 5.3 *Let* $f \in \mathcal{E}(K^{[\alpha_1,...,\alpha_n]})$*. For* $n \geq 2$ *fix* $c \in K^{(\alpha_n)}$ *and let* $f_c^{(i)}(y) = \frac{\partial^i}{\partial x_n^i} f(y, c)$*,* $i \in \{1, 2, ..., n-1\}$*,* $y \in K^{[\alpha_1,...,\alpha_{n-1}]}$*. Then*

$$\|f\|_q^{(n)} \geq \|f_c^{(i)}\|_{q-i}^{(n-1)}.$$

The proofs of these lemmas are straightforward.

Theorem 5.4 $K^{[\alpha_1,...,\alpha_n]}$ *has the extension property for* $1 < \alpha_i < 2$*,* $i = 1, ..., n$*.*

Proof: We will prove by induction on n. We know the statement is true for $k = 1$. Now suppose the statement is true for $k \leq n - 1$. Then take

$$z_0 = (x_0, y_0) \in K^{[\alpha_1,...,\alpha_n]}$$

where $x_0 \in K^{[\alpha_1,...,\alpha_{n-1}]}$ and $y_0 \in K^{(\alpha_n)}$.

Fix q, fix $f \in \mathcal{E}(K^{[\alpha_1,...,\alpha_n]})$. Fix $k_2 \leq q$. Let $g_1(x) := f^{(\vec{0},k_2)}(x, y_0)$. Then $g_1(x) \in \mathcal{E}(K^{[\alpha_1,...,\alpha_{n-1}]})$.

Therefore by proposition 1.15 and by our induction assumption

$$\forall R > 0\ \exists r, C > 0 : |g_1|_q^{(n-1)} \leq t^R |g_1|_0^{(n-1)} + \frac{C}{t} \|g_1\|_r^{(n-1)}, \ \ t > 0\ .$$

So $\forall \vec{k_1} \in \mathbb{N}^{n-1}$ s.t. $|\vec{k_1}| \leq q - k_2$ we have

$$|f^{(\vec{k_1}, k_2)}(z_0)| \leq t^R \sup_{x \in K^{[\alpha_1, \ldots, \alpha_{n-1}]}} |f^{(\vec{0}, k_2)}(x, y_0)| + \frac{C}{t} \|g_1\|_r^{(n-1)}, \ \ t > 0\ . \tag{5.1}$$

Now let $g_2(y) := f(x, y)$ then $g_2(y) \in \mathcal{E}(K^{(\alpha_n)})$. Using our assumption again, if we fix x we will have

$$|f^{(\vec{0}, k_2)}(x, y_0)| \leq d^R \sup_{y \in K^{(\alpha_n)}} |f(x, y)| + \frac{C}{d} \|g_2\|_r^{(1)}, \ \ d > 0\ ,$$

then

$$\begin{aligned}(\sup_{x \in K^{[\alpha_1, \ldots, \alpha_{n-1}]}} |f^{(\vec{0}, k_2)}(x, y_0)|) &\leq (\sup_{x \in K^{[\alpha_1, \ldots, \alpha_{n-1}]}} (d^R \sup_{y \in K^{(\alpha_n)}} |f(x,y)| + \tfrac{C}{d} ||g_2||_r^{(1)})) \\ &\leq (d^R \sup_{(x,y)} |f(x,y)| + \tfrac{C}{d} \sup_x ||g_2||_r^{(1)})\end{aligned}$$

for all $d > 0$. By Lemma 5.2

$$\|g_2\|_r^{(1)} \leq \|f\|_r^{(n)}\ ,$$

and by Lemma 5.3

$$||g_1||_r^{(n-1)} \leq ||f||_{r+k_2}^{(n)} \leq ||f||_{2r}^{(n)}\ .$$

Then by (5.1)

$$|f^{(\vec{k_1}, k_2)}(z_0)| \leq t^R d^R |f|_0 + t^R \frac{C}{d} \|f\|_{2r} + \frac{C}{t} \|f\|_{2r}\ .$$

Now let $d = t^{R+1}$ then

$$|f^{(\vec{k_1}, k_2)}(z_0)| \leq t^{R^2 + 2R} |f|_0 + \frac{2C}{t} \|f\|_{2r} \ \ \forall t > 0\ .$$

□

The proof of the following lemma is not so difficult, but we decided to present at least one of the proofs of the lemmas given in this chapter.

Lemma 5.5 *Let $f \in \mathcal{E}(K^{[\alpha_1, \ldots, \alpha_n]})$ s.t. $f(x) = f(x_1, ..., x_n) = F(x_1)$, $F(x_1) \in \mathcal{E}(K^{(\alpha_1)})$ that is, f depends only on the first variable. Then $\|f\|_q^{(n)} = \|F\|_q^{(1)}$.*

Proof: Since $F^{(k_1,\vec{k}_2)}(x_1)=0$ for $\vec{k}_2>0$ we trivially have

$$|f|_q^{(n)}=|F|_q^{(1)} .$$

On the other hand we have

$$F^{(i_1,\vec{i}_2)}(x_1)-\sum_{j\geq i,|j|\leq q}\frac{F^{(j_1,\vec{j}_2)}(y_1)}{(j_1-i_1)!...(j_n-i_n)!}(x_1-y_1)^{j_1-i_1}...(x_n-y_n)^{j_n-i_n}=0$$

for $\vec{i}_2>0$ and $F^{(j_1,\vec{j}_2)}(x_1)=0$ for $\vec{j}_2>0$. Therefore

$$\begin{aligned}
S_q^n(f) &= \sup_{x,y,i}\left\{|\frac{(R_y^q f)^{(i)}(x)}{|x-y|^{q-|i|}}| : x,y\in K^{[\alpha_1,...,\alpha_n]}, x\neq y, |i|\leq q\right\}\\
&= \sup\left\{\frac{|F^{(i_1,\vec{i}_2)}(x_1)-\sum_{j\geq i,|j|\leq q}\frac{F^{(j_1,\vec{j}_2)}(y_1)}{(j-i)!}(x-y)^{j-i}|}{|x-y|^{q-|i|}}\right\}\\
&= \sup{}_{x,y,i_1}\left\{\frac{|F^{(i_1)}(x_1)-\sum_{j_1\geq i_1,\vec{j}_2\geq\vec{0},|j|\leq q}\frac{F^{(j_1,\vec{j}_2)}(y_1)}{(j_1-i_1)!\vec{j}_2!}(x_1-y_1)^{j_1-i_1}(\vec{x}_2-\vec{y}_2)^{\vec{j}_2}|}{|x-y|^{q-i_1}}\right\}\\
&\qquad\qquad for\ \ i_1\leq q\\
&= \sup\left\{\frac{|F^{(i_1)}(x_1)-\sum\frac{F^{(j_1)}(y_1)}{(j_1-i_1)!}(x_1-y_1)^{j_1-i_1}|}{(\sqrt{(x_1-y_1)^2+...+(x_n-y_n)^2})^{q-i_1}} : x\neq y, i_1\leq q\right\}\\
&= \sup\left\{\frac{|F^{(i_1)}(x_1)-\sum\frac{F^{(j_1)}(y_1)}{(j_1-i_1)!}(x_1-y_1)^{j_1-i_1}|}{|x_1-y_1|^{q-i_1}} : x_1,y_1\in\mathbb{R}, x_1\neq y_1, i_1\leq q\right\}\\
&= S_q^1(F) .
\end{aligned}$$

Hence we get $\|f\|_q^{(n)}=\|F\|_q^{(1)}$. $\square$

Theorem 5.6 $K^{[\alpha_1,...,\alpha_n]}$ *does not have the extension property if at least one of the* α_i*'s is greater than 2.*

Proof: Suppose without loss of generality $\alpha_1>2$. By the proof of Theorem 2 in [18] we have

$$\forall p\ \ \exists\epsilon\ \ \exists q\ \ \forall r>q\ \ \exists(f_m)\subset\mathcal{E}(K^{(\alpha_1)}) : \frac{\|f_m\|_p^{(1)}\|f_m\|_r^{(1)\epsilon}}{\|f_m\|_q^{(1)1+\epsilon}}\longrightarrow 0$$

as $n \longrightarrow \infty$. Now define $g_m(x_1, ..., x_n) = f_m(x_1)$. By Lemma 5.5 $\|g_m\|_q^{(n)} = \|f_m\|_q^{(1)}$. Hence we have

$$\forall p \ \ \exists \epsilon \ \ \exists q \ \ \forall r > q \ \ \exists (g_m) \subset \mathcal{E}(K^{[\alpha_1,...,\alpha_n]}) : \frac{\|g_m\|_p^{(n)} \|g_m\|_r^{(n)\epsilon}}{\|g_m\|_q^{(n)1+\epsilon}} \longrightarrow 0$$

as $n \longrightarrow \infty$, which shows the negation of (1.5). $\square$

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