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MÖBIUS-INVARIANT HARMONIC FUNCTION SPACES ON THE UNIT DISC

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Abstract. We investigate and identify Möbius-invariant harmonic function spaces on the unit disc. We derive their fundamental properties and establish connections among various topologies on them. We show that the harmonic Bloch space b^∞ is the "maximal" and the Besov space b^1_{-2} is the minimal invariant complete seminormed space. There is only one invariant semi-Hilbert space and it is the harmonic Dirichlet space b^2_{-2}.

1. Introduction

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc and let $\operatorname{Aut}(\mathbb{D})$, the automorphism group, be the group of holomorphic, bijective maps of $\mathbb{D}$. Let (E, ρ_E) be a complete seminormed linear space of holomorphic functions on $\mathbb{D}$. Roughly speaking, (E, ρ_E) is called Möbius invariant if for all $f \in E$ and $\varphi \in \operatorname{Aut}(\mathbb{D})$, the composition $f \circ \varphi$ is in E and $\rho_E(f \circ \varphi) \sim \rho_E(f)$. Möbius-invariant holomorphic function spaces have been studied intensively in the past. It is shown in [11] that the Bloch space $\mathcal{B}^\infty$ is "maximal", and in [2] that the Besov space B^1 is the minimal invariant space. In [1] invariant semi-Hilbert spaces are considered and it is shown that the Dirichlet space is the unique Möbius-invariant semi-Hilbert space. In higher dimensions analogous results are obtained for the unit ball in $\mathbb{C}^n$ in [10,15], and for the polydisc in [8].

Consider now the *harmonic* analogue of the above problem. A Möbius transformation of $\hat{\mathbb{R}}^n := \mathbb{R}^n \cup \{\infty\}$ is a finite composition of reflections (inversions) in planes and spheres. Let $\mathbb{B}_n \subset \mathbb{R}^n$ be the real unit ball and

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let $\mathcal{M}(\mathbb{B}_n)$ be the group of Möbius transformations that map $\mathbb{B}_n$ to itself. If u is harmonic on $\mathbb{B}_n$ and $\varphi \in \mathcal{M}(\mathbb{B}_n)$, the composition $u \circ \varphi$ need not be harmonic. Nevertheless, if we multiply $u \circ \varphi$ with a correction factor, the function

$$\mathcal{K}_\varphi(u)(x) = \Big(\frac{1-|\varphi(x)|^2}{1-|x|^2}\Big)^{(n-2)/2} u(\varphi(x)) \quad (x \in \mathbb{B}_n)$$

is harmonic on $\mathbb{B}_n$. The first factor above stems from applying the Kelvin transform and because of this $\mathcal{K}_\varphi(u)$ is called the Kelvin–Möbius transform of u; for details see [9].

Note that when $n = 2$, the correction factor in the Kelvin–Möbius transform disappears and $\mathcal{K}_\varphi(u)$ is just $u \circ \varphi$. Because of this there are differences between the cases $n = 2$ and $n \geq 3$. For example, when $n \geq 3$, there are *normed* spaces containing unbounded functions that are Kelvin–Möbius invariant, but this is not true when $n = 2$ and to obtain interesting invariant spaces it is necessary to allow *seminorms*. Due to this fact even the definitions of invariant spaces are different in the cases $n = 2$ and $n \geq 3$. The case $n \geq 3$ is worked out in [9], where the minimal and the maximal invariant spaces are determined and the unique invariant Hilbert space is identified. The purpose of this work is to settle the remaining case $n = 2$.

Our results are similar to those in the holomorphic setting given, for example, in [1,2], which is to be expected since harmonic and holomorphic functions on the unit disc are intimately related. A close look at works on holomorphic spaces reveals that connections between convergence in a seminorm and pointwise convergence play essential roles in the proofs, although not always explicitly stated; for a careful treatment of these issues, see the recent monograph [6, Chapter 6]. We will start with slightly different but basic assumptions when defining Möbius-invariant harmonic spaces. More importantly, we will be meticulous in establishing the relations among various kinds of topologies, seminorm, norm, pointwise convergence, and make clear how and where the conditions on each are used in the proofs.

We endow $\mathbb{R}^2$ with the usual inner product $x \cdot y = x_1 y_1 + x_2 y_2$ for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ and with the induced norm $|x| = \sqrt{x \cdot x}$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ via the correspondence $z = x_1 + i x_2$ and freely pass between the real and complex notation. In the rest of this work we always have $n = 2$, and in real notation we denote the unit disc by $\mathbb{B} = \mathbb{B}_2$ and identify it with $\mathbb{D}$.

Let E be a linear space. A map $\rho_E \colon E \to [0, \infty)$ is called a *seminorm* on E if it satisfies the axioms of a norm except that $\rho_E(u)$ can be zero for non-zero $u \in E$. A map $\langle \cdot, \cdot \rangle_E \colon E \times E \to \mathbb{C}$ is called a *semi-inner product* on E if it satisfies the axioms of an inner product except that $\langle u, u \rangle_E$ can be zero for non-zero $u \in E$. The natural topology of a semi-inner product space is with respect to the induced seminorm $\rho_E(u) = \sqrt{\langle u, u\rangle_E}$. The seminormed space (E, ρ_E) is called *complete* if every Cauchy sequence in ρ_E is convergent in ρ_E. The pair $(E, \langle\cdot,\cdot\rangle_E)$ is called a *semi-Hilbert space* if it is complete with respect to its induced seminorm. Equivalently, if (E, ρ_E) is a seminormed space, we can let $Z = \{\, u \in E : \rho_E(u) = 0 \,\}$ and consider the quotient space $\breve{E} := E/Z$ and the corresponding $\breve{\rho}_{\breve{E}}$ on it. Then $(\breve{E}, \breve{\rho}_{\breve{E}})$ is a genuine normed space and is a Banach space if and only if (E, ρ_E) is complete. Similarly, $(E, \langle\cdot,\cdot\rangle_E)$ is a semi-Hilbert space if and only if $(\breve{E}, \langle\cdot,\cdot\rangle_{\breve{E}})$ is a Hilbert space. For more information, see [14, Section 1.2].

We now define a Möbius-invariant harmonic function space following [11] and [2]. Denote by $h(\mathbb{B})$ the space of all complex-valued harmonic functions on $\mathbb{B}$ endowed with the topology of uniform convergence on compact subsets. Let (E, ρ_E) be a seminormed space of harmonic functions on $\mathbb{B}$. A nonzero continuous linear functional L on $h(\mathbb{B})$ is called *decent on* E if $L|_E$ is also continuous in the seminorm topology. We denote by $\mathbf{1}$ the constant function whose value is 1. If $f\colon \mathbb{D} \to \mathbb{C}$ and $\theta \in [0, 2\pi]$, we define f_θ by $f_\theta(z) := f(e^{i\theta} z)$ for $z \in \mathbb{D}$.

Definition 1.1. Let E be a linear space of harmonic functions on $\mathbb{B}$ and ρ_E be a complete seminorm on E. We call (E, ρ_E) *Möbius invariant* if the following properties hold:

(i) $\mathbf{1} \in E$ and $\rho_E(\mathbf{1}) = 0$. We also assume E is nontrivial in the sense that it contains a nonconstant function.

(ii) There exists a decent linear functional on E.

(iii) For every $u \in E$ and $\varphi \in \mathcal{M}(\mathbb{B})$, $u \circ \varphi$ belongs to E and

$$\rho_E(u \circ \varphi) \le C \rho_E(u),$$

for some constant C independent of u and φ.

(iv) For each $u \in E$, the map $\theta \mapsto u_\theta$ is continuous from $[0, 2\pi]$ to (E, ρ_E).

If the constant in (iii) is $C = 1$, then we have $\rho_E(u \circ \varphi) = \rho_E(u)$ for every $u \in E$ and $\varphi \in \mathcal{M}(\mathbb{B})$ and in this case we call (E, ρ_E) *strictly* Möbius invariant.

If (E, ρ_E) is Möbius invariant, then $\rho_E(u \circ \varphi) \sim \rho_E(u)$ for all $u \in E$. Here, the notation $A \sim B$ means that $|A|/|B|$ is bounded above and below by positive constants that are independent of the parameters or functions involved. If $|A|/|B|$ is bounded above, we write $A \lesssim B$.

If (E, ρ_E) is Möbius invariant, then it is possible to define a new, equivalent seminorm on E with which E becomes strictly Möbius invariant; see [2, p. 111].

The condition (i) in Definition 1.1 does not appear in [1,2]. We use it in Lemma 5.3 and Lemma 6.2 below where we connect pointwise convergence and convergence in (E, ρ_E). Condition (iv) is needed in Proposition 5.5 which is the basis of the properties obtained at the end of Section 5.

The harmonic function spaces we are interested in are the Bergman–Besov spaces b^p_q $(q \in \mathbb{R})$ and the Bloch space b^∞. The *harmonic Bloch space* b^∞ consists of all $u \in h(\mathbb{B})$ such that the seminorm

$$\rho_{b^\infty}(u) := \sup_{x \in \mathbb{B}} (1 - |x|^2)|\nabla u(x)| < \infty, \tag{1}$$

where $\nabla u(x) = (u_{x_1}, u_{x_2})$ is the gradient of u and $u_{x_i} = \partial u / \partial x_i$, $i = 1, 2$. The space b^∞ satisfies the conditions (i)–(iii) of Definition 1.1, but not the condition (iv), which can be seen by taking $u(z) = \log(1 - z)$. Therefore b^∞ is not Möbius invariant in the sense of Definition 1.1. Nevertheless, it has the following maximality property which is the harmonic analogue of the main result of [11].

THEOREM 1.2. *Let (E, ρ_E) be a Möbius-invariant harmonic function space. Then $E \subset b^\infty$ and there is a $C > 0$ such that $\rho_{b^\infty}(u) \le C \rho_E(u)$ for every $u \in E$.*

REMARK 1.3. The conditions (i) and (iv) of Definition 1.1 are actually not needed in the above theorem. That is, if (E, ρ_E) satisfies conditions (ii) and (iii) of Definition 1.1, then E is continuously contained in b^∞.

We next define harmonic Bergman–Besov spaces for the whole range $q \in \mathbb{R}$. For this we need some preparation. Let dA be the normalized area measure on $\mathbb{B}$ and for $q \in \mathbb{R}$, let

$$dA_q(x) := c_q (1 - |x|^2)^q \, dA(x).$$

The measure dA_q is finite only when $q > -1$ and in this case we set c_q so that $A_q(\mathbb{B}) = 1$. When $q \le -1$, we set $c_q = 1$. The case $q = -2$ is especially important for us since $dA_{-2}(x) = (1 - |x|^2)^{-2} \, dA(x)$ is the Möbius-invariant measure on $\mathbb{B}$. For $1 \le p < \infty$, we denote the Lebesgue classes with respect to dA_q by $L^p(dA_q)$. To denote higher-order partial derivatives we use multi-indices, where a multi-index $\alpha = (\alpha_1, \alpha_2)$ is a pair of nonnegative integers and $|\alpha| := \alpha_1 + \alpha_2$. For a smooth function f we write

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.$$

So by definition, a function u is harmonic if $(\partial^{(2,0)} + \partial^{(0,2)})u = 0$.

DEFINITION 1.4. Let $1 \le p < \infty$ and $q \in \mathbb{R}$. Pick a nonnegative integer N so that

$$q + pN > -1. \tag{2}$$

The *harmonic Bergman–Besov space* b^p_q consists of all $u \in h(\mathbb{B})$ such that for every multi-index α with $|\alpha| = N$,

$$(1-|x|^2)^N \partial^\alpha u \in L^p(dA_q).$$

A norm on b^p_q is

$$\|u\|^p_{b^p_q} = \sum_{0\le|\alpha|<N} |\partial^\alpha u(0)|^p + \sum_{|\alpha|=N} \int_{\mathbb{B}} \big|(1-|x|^2)^N \partial^\alpha u(x)\big|^p \, dA_q(x). \tag{3}$$

The space b^p_q does not depend on the choice of N as long as (2) is satisfied. The norm $\|\cdot\|_{b^p_q}$ in (3) depends on N but we do not indicate this since different choices of N give rise to equivalent norms; see [7] for details. In the region $q > -1$, choosing $N = 0$ shows $b^p_q = h(\mathbb{B}) \cap L^p(dA_q)$, the *weighted harmonic Bergman space*. The region $q \le -1$ is the Besov region, where we need to consider derivatives of u. We also endow b^p_q with the seminorm

$$\rho_{b^p_q}(u) := \|u - u(0)\|_{b^p_q}$$

which vanishes on constants. The next theorem determines all Möbius-invariant harmonic Bergman–Besov spaces.

Theorem 1.5. *Let $1 \le p < \infty$ and $q \in \mathbb{R}$. Then $(b^p_q, \rho_{b^p_q})$ is Möbius invariant if and only if $q = -2$.*

In particular, there are no Möbius-invariant standard weighted Bergman spaces. In all papers dealing with holomorphic spaces, no such only-if-type result is proved.

Taking $p = 2$ in Theorem 1.5 shows that the space b^2_{-2} is a Möbius-invariant semi-Hilbert space. It is the harmonic Dirichlet space endowed with the seminorm (choosing $N = 1$)

$$\rho^2_{b^2_{-2}}(u) = \int_{\mathbb{B}} |\nabla u(x)|^2 \, dA(x) \tag{4}$$

which is induced by the semi-inner product

$$\langle u, v\rangle_{b^2_{-2}} = \int_{\mathbb{B}} \nabla u(x) \cdot \overline{\nabla v(x)} \, dA(x).$$

The Dirichlet space $(b^2_{-2}, \langle\cdot,\cdot\rangle_{b^2_{-2}})$ is actually strictly Möbius invariant and it is the only strictly Möbius-invariant semi-Hilbert space. This is our next theorem.

Theorem 1.6. (i) *The semi-Hilbert space $(b^2_{-2}, \langle\cdot,\cdot\rangle_{b^2_{-2}})$ is strictly Möbius invariant.*

(ii) *If $(H, \langle\cdot,\cdot\rangle_H)$ is a strictly Möbius-invariant semi-Hilbert space, then $H = b^2_{-2}$ and $\langle\cdot,\cdot\rangle_H = C\langle\cdot,\cdot\rangle_{b^2_{-2}}$ for some $C > 0$.*

The Möbius-invariant spaces b^p_{-2} $(1 \le p < \infty)$ grow larger as p increases; see [7, Proposition 13.3]. This suggests that b^1_{-2} might be the smallest Möbius-invariant space. The next theorem shows that this is true.

THEOREM 1.7. *The space b^1_{-2} is the smallest Möbius-invariant space. More precisely, if (E, ρ_E) is Möbius invariant, then $b^1_{-2} \subset E$ and there exists a $C > 0$ such that $\rho_E(u) \le C\rho_{b^1_{-2}}(u)$ for every $u \in b^1_{-2}$.*

Finally, we consider subspaces of the Fréchet space $h(\mathbb{B})$ and show that the only Möbius-invariant closed subspaces are the trivial ones.

THEOREM 1.8. *Let A be a closed subspace of $h(\mathbb{B})$. If $u \circ \varphi$ is in A for every $u \in A$ and $\varphi \in \mathcal{M}(\mathbb{B})$, then $A = \{0\}$, $A = \{constants\}$ or $A = h(\mathbb{B})$.*

The proofs of Theorems 1.2 and 1.8 are in Section 3. The proof of Theorem 1.5 is in Section 4, of Theorem 1.6 in Section 6, and of Theorem 1.7 in Section 7. In Section 5, we collect together general properties of Möbius-invariant harmonic function spaces not covered by the above theorems. In Section 2, we provide background on Möbius transformations, harmonic functions, and Bergman–Besov spaces.

2. Preliminaries

In this section we first recall some elementary facts about Möbius transformations and harmonic functions on the unit disc. We then review some properties of harmonic and holomorphic Bergman–Besov spaces that we use in the sequel.

2.1. Möbius transformations. For $a \in \mathbb{D}$, let

$$\varphi_a(z) := \frac{a - z}{1 - \bar{a}z} \quad (z \in \mathbb{D}). \tag{5}$$

The group $\mathrm{Aut}(\mathbb{D})$ is generated by φ_a's and rotations; if $\varphi \in \mathrm{Aut}(\mathbb{D})$, then $\varphi = e^{i\theta}\varphi_a$ for some $\theta \in [0, 2\pi]$ and $a \in \mathbb{D}$. Elements of $\mathrm{Aut}(\mathbb{D})$ are holomorphic, so they are orientation preserving. On the other hand, $\mathcal{M}(\mathbb{B})$ consists of compositions of reflections in lines and circles and its elements can be orientation preserving or reversing. Throughout this work we reserve the letter γ for the conjugation function

$$\gamma(z) = \bar{z}.$$

The group $\mathcal{M}(\mathbb{B})$ is generated by φ_a's, rotations and γ. If $\varphi \in \mathcal{M}(\mathbb{B})$ is a composition of an even number of reflections, then φ is holomorphic and $\varphi \in \mathrm{Aut}(\mathbb{D})$. If $\varphi \in \mathcal{M}(\mathbb{B})$ is a composition of an odd number of reflections, then $\bar{\varphi}$ is holomorphic and $\varphi = \gamma \circ \psi$ for some $\psi \in \mathrm{Aut}(\mathbb{D})$. If we write $\overline{\mathrm{Aut}}(\mathbb{D}) := \{\gamma \circ \varphi : \varphi \in \mathrm{Aut}(\mathbb{D})\}$, then

$$\mathcal{M}(\mathbb{B}) = \mathrm{Aut}(\mathbb{D}) \cup \overline{\mathrm{Aut}}(\mathbb{D}). \tag{6}$$

We record the following elementary lemma for future reference.

LEMMA 2.1. *Let $\varphi \in \mathrm{Aut}(\mathbb{D})$. Then there exists $\hat{\varphi} \in \mathrm{Aut}(\mathbb{D})$ such that*

$$\gamma \circ \varphi = \hat{\varphi} \circ \gamma.$$

PROOF. If $\varphi = e^{i\theta}\varphi_a$, take $\hat{\varphi} = e^{-i\theta} \circ \varphi_{\bar{a}}$. $\square$

2.2. Harmonic functions on the unit disc. A harmonic function u on $\mathbb{D}$ is the sum of a holomorphic and a conjugate holomorphic function in the form

$$u(z) = \sum_{m=0}^{\infty} a_m z^m + \sum_{m=1}^{\infty} b_m \bar{z}^m =: f(z) + g(\bar{z}) = f(z) + g \circ \gamma(z), \tag{7}$$

where the series converge absolutely and uniformly on compact subsets of $\mathbb{D}$, and f and g are holomorphic on $\mathbb{D}$. We use this form many times and state it as a lemma.

LEMMA 2.2. *If u is harmonic on $\mathbb{D}$, then there exist functions f and g holomorphic on $\mathbb{D}$ such that $u = f + g \circ \gamma$. The functions f and g are unique up to a constant summand.*

As stated in Definition 1.1, for Möbius-invariant spaces we use seminorms that vanish on constants. So when we refer to Lemma 2.2, the constant summand will have no effect and it will not be important which f or g we use.

Let

$$\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big) \tag{8}$$

be the Wirtinger operators. Then

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \quad \text{and} \quad \frac{\partial}{\partial x_2} = i\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\Big). \tag{9}$$

If u is harmonic on $\mathbb{D}$ and f and g are as in Lemma 2.2, then

$$\frac{\partial u}{\partial z} = f'(z) \quad \text{and} \quad \frac{\partial u}{\partial \bar{z}} = g'(\bar{z}). \tag{10}$$

A harmonic function composed on the right with a holomorphic or conjugate holomorphic function remains harmonic.

A continuous linear functional L on the Fréchet space $h(\mathbb{B})$ is continuous if and only if there exists a compact $F \subset \mathbb{B}$ and a constant $C > 0$ such that

$$|L(u)| \leq C \sup\big\{|u(x)| : x \in F\big\} \quad (u \in h(\mathbb{B})). \tag{11}$$

If (E, ρ_E) is a seminormed space of harmonic functions and L is a linear functional on E, then L is continuous if and only if there exists a constant $C > 0$ such that

$$|L(u)| \leq C \rho_E(u) \quad (u \in E). \tag{12}$$

2.3. Holomorphic and harmonic Bergman–Besov spaces. Let $H(\mathbb{D})$ be the set of all holomorphic functions on $\mathbb{D}$. Definition of holomorphic Bergman–Besov spaces B^p_q $(q \in \mathbb{R})$ is similar to that of harmonic ones.

Definition 2.3. Let $1 \leq p < \infty$ and $q \in \mathbb{R}$. Pick a nonnegative integer N so that $q + pN > -1$. The *holomorphic Bergman–Besov space* B^p_q consists of all $f \in H(\mathbb{D})$ such that $(1-|z|^2)^N f^{(N)} \in L^p(dA_q)$. A norm on B^p_q is

$$\|f\|^p_{B^p_q} = \sum_{k=0}^{N-1} |f^{(k)}(0)|^p + \int_{\mathbb{D}} \big|(1-|z|^2)^N f^{(N)}(z)\big|^p \, dA_q(z).$$

Again, the space B^p_q does not depend on the choice of N as long as $q + pN > -1$ and different choices of N give rise to equivalent norms. For $q > -1$, choosing $N = 0$ shows $B^p_q = H(\mathbb{D}) \cap L^p(dA_q)$ is the holomorphic weighted Bergman space. We endow B^p_q also with the seminorm $\rho_{B^p_q}(f) := \|f - f(0)\|_{B^p_q}$.

The following lemma shows how taking derivatives moves a function from one Bergman–Besov space to another in both holomorphic and harmonic cases.

Lemma 2.4. (i) *Suppose $f \in H(\mathbb{D})$ and $m \geq 1$. Then $f \in B^p_q$ if and only if $f^{(m)} \in B^p_{q+pm}$ and*

$$\|f\|_{B^p_q} \sim \sum_{k=0}^{m-1} |f^{(k)}(0)| + \|f^{(m)}\|_{B^p_{q+pm}}.$$

(ii) *Suppose $u \in h(\mathbb{B})$ and $m \geq 1$. Then $u \in b^p_q$ if and only if $\partial^\alpha u \in b^p_{q+pm}$ for every multi-index α with $|\alpha| = m$ and*

$$\|u\|_{b^p_q} \sim \sum_{|\alpha|<m} |\partial^\alpha u(0)| + \sum_{|\alpha|=m} \|\partial^\alpha u\|_{b^p_{q+pm}}.$$

In Definition 1.4 (and similarly in Definition 2.3), partial derivatives can be replaced with the more convenient radial derivatives. If $f \in H(\mathbb{D})$, then the radial derivative $\mathcal{R}f$ of f is defined as $\mathcal{R}f(z) := zf'(z)$. So, if $f(z) = \sum_{m=0}^{\infty} c_m z^m$, then

$$\mathcal{R}f(z) = \sum_{m=1}^{\infty} m c_m z^m.$$

For $k \geq 2$, we define $\mathcal{R}^k f := \mathcal{R}^{k-1}\mathcal{R}f = \sum_{m=1}^{\infty} m^k c_m z^m$. If $u \in h(\mathbb{B})$, then the radial derivative $\mathcal{R}_h u$ of u is defined as $\mathcal{R}_h u(x) := x \cdot \nabla u(x)$. If u is written as in (7), then

$$\mathcal{R}_h u(z) = \sum_{m=1}^{\infty} m\left(a_m z^m + b_m \bar{z}^m\right).$$

For $k \geq 2$, we define $\mathcal{R}_h^k u := \mathcal{R}_h^{k-1}\mathcal{R}_h u = \sum_{m=1}^{\infty} m^k (a_m z^m + b_m \bar{z}^m)$.

Lemma 2.5. *Let $p \geq 1$ and $q \in \mathbb{R}$. Pick $N \geq 1$ so that $q + pN > -1$.*

(i) *$f \in B_q^p$ if and only if $(1-|z|^2)^N \mathcal{R}^N f \in L^p(dA_q)$ and*

$$\|f\|_{B_q^p}^p \sim |f(0)|^p + \int_{\mathbb{D}} \left|(1-|z|^2)^N \mathcal{R}^N f(z)\right|^p dA_q(z).$$

(ii) *$u \in b_q^p$ if and only if $(1-|x|^2)^N \mathcal{R}_h^N u \in L^p(dA_q)$ and*

$$\|u\|_{b_q^p}^p \sim |u(0)|^p + \int_{\mathbb{B}} \left|(1-|x|^2)^N \mathcal{R}_h^N u(x)\right|^p dA_q(x).$$

Parts (i) of Lemmas 2.4 and 2.5 are well known. For their parts (ii), see [7]. When $q > -1$, it is clear that $B_q^p \subset b_q^p$. This continues to hold when $q \leq -1$.

Lemma 2.6. *Let $1 \leq p < \infty$, $q \in \mathbb{R}$ and f be holomorphic on $\mathbb{D}$. Then*

$$\|f\|_{B_q^p} \sim \|f\|_{b_q^p} \sim \|f \circ \gamma\|_{b_q^p} \quad \text{and} \quad \rho_{B_q^p}(f) \sim \rho_{b_q^p}(f) \sim \rho_{b_q^p}(f \circ \gamma).$$

Proof. If $q > -1$, since the measure dA_q is invariant under conjugation,

$$\|f \circ \gamma\|_{b_q^p} = \int_{\mathbb{D}} |f(\bar{z})|^p \, dA_q(z) = \int_{\mathbb{D}} |f(z)|^p \, dA_q(z) = \|f\|_{b_q^p} = \|f\|_{B_q^p}.$$

If $q \leq -1$, pick $N \geq 1$ so that $q + pN > -1$. Note that since f is holomorphic, $\mathcal{R}f = \mathcal{R}_h f$. Thus, using both parts of Lemma 2.5,

$$\|f\|_{b_q^p} \sim |f(0)|^p + \int_{\mathbb{B}} \left|(1-|x|^2)^N \mathcal{R}_h^N f(x)\right|^p dA_q(x)$$
$$= |f(0)|^p + \int_{\mathbb{D}} \left|(1-|z|^2)^N \mathcal{R}^N f(z)\right|^p dA_q(z) \sim \|f\|_{B_q^p}.$$

The equivalence $\|f\circ\gamma\|_{b^p_q}\sim\|f\|_{b^p_q}$ follows from $\mathcal{R}^N_h(f\circ\gamma)=\mathcal{R}^N_h f\circ\gamma$ and the conjugation invariance of dA_q. For the assertion about seminorms just replace f with $f-f(0)$. □

For a proof of the following estimate see [12, Proposition 1.4.10].

Lemma 2.7. *For $z\in\mathbb{D}$, $t>-1$ and $c\in\mathbb{R}$, we have*

$$\int_{\mathbb{D}}\frac{(1-|w|^2)^t}{|1-z\bar{w}|^{2+t+c}}\,dA(w)\sim\begin{cases}\frac{1}{(1-|z|^2)^c} & \text{if } c>0,\\ 1+\log\frac{1}{1-|z|^2} & \text{if } c=0,\\ 1 & \text{if } c<0.\end{cases}$$

3. The maximal space

In this section we prove Theorems 1.2 and 1.8 which are the harmonic counterparts of the Theorem and Corollary 2 of [11]. In the holomorphic case the proof of the Rubel–Timoney theorem is based on the lemma below. We denote the *holomorphic Bloch space* by $\mathcal{B}^\infty$ which consists of all $f\in H(\mathbb{D})$ such that the seminorm

$$\rho_{\mathcal{B}^\infty}(f):=\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty.$$

Lemma 3.1 [11, p. 46]. *Let $f\in H(\mathbb{D})$ and $m\geq 1$. If*

$$\sup_{\varphi\in\mathrm{Aut}(D)}\left|(f\circ\varphi)^{(m)}(0)\right|<\infty,$$

then $f\in\mathcal{B}^\infty$ and

$$\rho_{\mathcal{B}^\infty}(f)\lesssim\sup_{\varphi\in\mathrm{Aut}(D)}\left|(f\circ\varphi)^{(m)}(0)\right|.$$

Our first aim is to obtain an analogue of the above lemma for harmonic functions. As a preliminary result, we first show the following simple connection between the harmonic Bloch space b^∞ and the holomorphic Bloch space $\mathcal{B}^\infty$.

Lemma 3.2. *Let u be harmonic on $\mathbb{D}$ and f and g be as in Lemma 2.2. Then $u\in b^\infty$ if and only if $f,g\in\mathcal{B}^\infty$ and*

$$\rho_{b^\infty}(u)\sim\rho_{\mathcal{B}^\infty}(f)+\rho_{\mathcal{B}^\infty}(g).$$

Proof. It is clear that $\rho_{b^\infty}(u)$ defined in (1) is equivalent to

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\big(|u_{x_1}(z)|+|u_{x_2}(z)|\big)$$

which, by (8) and (9), is equivalent to

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\Big(\Big|\frac{\partial u}{\partial z}(z)\Big|+\Big|\frac{\partial u}{\partial \bar z}(z)\Big|\Big).$$

Finally, by (10), the above quantity is equivalent to $\rho_{\mathcal{B}^\infty}(f)+\rho_{\mathcal{B}^\infty}(g)$. □

LEMMA 3.3. *Let $u\in h(\mathbb{B})$ and $m\ge 1$. If*

$$\sup_{\varphi\in\mathcal{M}(\mathbb{B})}\Big|\frac{\partial^m(u\circ\varphi)}{\partial z^m}(0)\Big|<\infty,$$

then $u\in b^\infty$ and

$$\rho_{b^\infty}(u)\lesssim\sup_{\varphi\in\mathcal{M}(\mathbb{B})}\Big|\frac{\partial^m(u\circ\varphi)}{\partial z^m}(0)\Big|.$$

PROOF. Write $u=f+g\circ\gamma$ as in Lemma 2.2. We first deal with f. Let $\varphi\in\operatorname{Aut}(\mathbb{D})$. By Lemma 2.1, there exists $\hat\varphi\in\operatorname{Aut}(\mathbb{D})$ such that

$$u\circ\varphi=f\circ\varphi+g\circ\gamma\circ\varphi=f\circ\varphi+g\circ\hat\varphi\circ\gamma.$$

Since the second function $g\circ\hat\varphi\circ\gamma$ is conjugate holomorphic we have

$$\frac{\partial^m(u\circ\varphi)}{\partial z^m}=\frac{d^m(f\circ\varphi)}{dz^m}.$$

Then by hypothesis and the fact that $\operatorname{Aut}(\mathbb{D})\subset\mathcal{M}(\mathbb{B})$,

$$\sup_{\varphi\in\operatorname{Aut}(\mathbb{D})}\big|(f\circ\varphi)^{(m)}(0)\big|\le\sup_{\varphi\in\mathcal{M}(\mathbb{B})}\Big|\frac{\partial^m(u\circ\varphi)}{\partial z^m}(0)\Big|<\infty.$$

Lemma 3.1 shows that $f\in\mathcal{B}^\infty$ and

$$\rho_{\mathcal{B}^\infty}(f)\lesssim\sup_{\varphi\in\mathcal{M}(\mathbb{B})}\Big|\frac{\partial^m(u\circ\varphi)}{\partial z^m}(0)\Big|.$$

To deal with g, let $\varphi\in\operatorname{Aut}(\mathbb{D})$. Then $\psi=\gamma\circ\varphi\in\mathcal{M}(\mathbb{B})$. By Lemma 2.1, $\gamma\circ\varphi=\hat\varphi\circ\gamma$ for some $\hat\varphi\in\operatorname{Aut}(\mathbb{D})$ and $u\circ\psi=f\circ\hat\varphi\circ\gamma+g\circ\varphi$. Since the first function $f\circ\hat\varphi\circ\gamma$ is conjugate holomorphic,

$$\frac{\partial^m(u\circ\psi)}{\partial z^m}=\frac{d^m(g\circ\varphi)}{dz^m}$$

and

$$\sup_{\varphi\in\operatorname{Aut}(\mathbb{D})}\big|(g\circ\varphi)^{(m)}(0)\big|\le\sup_{\psi\in\mathcal{M}(\mathbb{B})}\Big|\frac{\partial^m(u\circ\psi)}{\partial z^m}(0)\Big|<\infty.$$

Lemma 3.1 shows that $g \in \mathcal{B}^\infty$ and

$$\rho_{\mathcal{B}^\infty}(g) \lesssim \sup_{\psi \in \mathcal{M}(\mathbb{B})} \left| \frac{\partial^m (u \circ \psi)}{\partial z^m}(0) \right|.$$

The desired result now follows from Lemma 3.2. □

We follow the approach of [13] and obtain Theorems 1.2 and 1.8 as consequences of the following theorem.

THEOREM 3.4. *Let L be a non-zero continuous linear functional on $h(\mathbb{B})$. If*

$$K = \sup\{|L(u \circ \varphi)| : \varphi \in \mathcal{M}(\mathbb{B})\} < \infty$$

for some $u \in h(\mathbb{B})$, then $u \in b^\infty$ and $\rho_{b^\infty}(u) \le C_L K$, where the constant C_L depends on L but is independent of u.

PROOF. Since $L \neq 0$ on $h(\mathbb{D})$, it does not equal 0 on the whole set $\{\mathbf{1}, z^m, \bar{z}^m : m \ge 1\}$. We can assume that there exists $m_0 \ge 0$ such that $L(z^{m_0}) \neq 0$ since, if necessary, we can replace L with $\tilde{L}$, where $\tilde{L}(v) = L(v \circ \gamma)$.

Define a new linear functional $\mathcal{L}$ on $h(\mathbb{D})$ by

$$\mathcal{L}(v) := \frac{1}{2\pi} \int_0^{2\pi} L(v_\theta) e^{-im_0\theta}\, d\theta.$$

Using condition (11) it is straightforward to show that $\mathcal{L}$ is continuous on $h(\mathbb{D})$. It is also elementary to verify that the following properties hold:

$$\begin{cases} \mathcal{L}(z^{m_0}) = L(z^{m_0}) \neq 0; & \\ \mathcal{L}(z^m) = 0, & m \ge 0,\ m \neq m_0; \\ \mathcal{L}(\bar{z}^m) = 0, & m \ge 1. \end{cases} \tag{13}$$

Suppose the hypothesis of the theorem holds for u. Then

$$|\mathcal{L}(u \circ \varphi)| \le K \quad (\varphi \in \mathcal{M}(\mathbb{B})). \tag{14}$$

Write u as a series as in (7). By the uniform convergence of this series on compact subsets of $\mathbb{D}$, the continuity of $\mathcal{L}$ on $h(\mathbb{D})$, and (13),

$$\mathcal{L}(u) = \sum_{m=0}^\infty a_m \mathcal{L}(z^m) + \sum_{m=1}^\infty b_m \mathcal{L}(\bar{z}^m) = a_{m_0} \mathcal{L}(z^{m_0}) = a_{m_0} L(z^{m_0})$$

and since

$$a_{m_0} = \frac{1}{m_0!} \frac{\partial^{m_0} u}{\partial z^{m_0}}(0),$$

we deduce

$$\Big|\frac{\partial^{m_0} u}{\partial z^{m_0}}(0)\Big| \leq \frac{m_0!}{|L(z^{m_0})|}\,|\mathcal{L}(u)|.$$

Replacing u by $u \circ \varphi$ and using (14) we obtain

$$\Big|\frac{\partial^{m_0} (u\circ\varphi)}{\partial z^{m_0}}(0)\Big| \leq \frac{m_0!}{|L(z^{m_0})|}K$$

for all $\varphi \in \mathcal{M}(\mathbb{B})$. If $m_0 \geq 1$, then $u \in b^\infty$ and $\rho_{b^\infty}(u) \leq C_L K$ by Lemma 3.3. If $m_0 = 0$, then u is a bounded harmonic function and an application of Cauchy's estimate for harmonic functions ([3, p. 33]) shows $u \in b^\infty$ and $\rho_{b^\infty}(u) \leq C_L K$. □

Proof of Theorem 1.2. Let L be a decent linear functional on E. So $L \neq 0$ is continuous on $h(\mathbb{B})$ and (12) holds. Pick $u \in E$. For every $\varphi \in \mathcal{M}(\mathbb{B})$, by (12) and the Möbius invariance of E,

$$|L(u\circ\varphi)| \lesssim \rho_E(u\circ\varphi) \lesssim \rho_E(u).$$

Theorem 3.4 implies $u \in b^\infty$ and $\rho_{b^\infty}(u) \lesssim \rho_E(u)$. □

Proof of Theorem 1.8. Suppose $A \neq h(\mathbb{B})$. By the Hahn-Banach theorem, there exists a non-zero continuous linear functional L on $h(\mathbb{B})$ which vanishes on A. If $u \in A$, then

$$\sup\big\{|L(u\circ\varphi)| : \varphi \in \mathcal{M}(\mathbb{B})\big\} = 0,$$

since $u \circ \varphi \in A$ for every $\varphi \in \mathcal{M}(\mathbb{B})$. Theorem 3.4 shows that $\rho_{b^\infty}(u) = 0$ and so u must be constant. □

4. Möbius-invariant harmonic Bergman–Besov spaces

In this section we prove Theorem 1.5 and thereby determine all Möbius-invariant harmonic Bergman–Besov spaces. In the holomorphic case, when $1 \leq p < \infty$ and $q = -2$, the spaces B^p_{-2} are Möbius invariant. This is clear when $1 < p < \infty$, since if we choose $N = 1$ in Definition 2.3,

$$\rho_{B^p_{-2}}(f) = \int_{\mathbb{D}} |f'(z)|^p\,(1-|z|^2)^{p-2}\,dA(z)$$

and for $\varphi \in \mathrm{Aut}(\mathbb{D})$,

$$\rho_{B^p_{-2}}(f\circ\varphi) = \rho_{B^p_{-2}}(f) \tag{15}$$

by the identity

$$|\varphi'(z)| = \frac{1-|\varphi(z)|^2}{1-|z|^2}.$$

If $p=1$, choosing $N=2$ in Definition 2.3 shows

$$\rho_{B^1_{-2}}(f) = |f'(0)| + \int_{\mathbb{D}} |f''(z)|\,dA(z),$$

and it is shown in [2, Theorem 8] that $\rho_{B^1_{-2}}(f\circ\varphi) \sim \rho_{B^1_{-2}}(f)$ for $\varphi \in \mathrm{Aut}(\mathbb{D})$.

In the proof of Theorem 1.5 we will use the Möbius invariance of B^p_{-2} and to do this we first establish a connection between holomorphic Bergman–Besov spaces B^p_q and harmonic Bergman–Besov spaces b^p_q.

Lemma 4.1. *Let $1 \le p < \infty$ and $q \in \mathbb{R}$. Let $u \in h(\mathbb{B})$ be written in the form $u = f + g\circ\gamma$ as in Lemma 2.2. Then $u \in b^p_q$ if and only if $f, g \in B^p_q$ and*

$$\rho_{b^p_q}(u) \sim \rho_{B^p_q}(f) + \rho_{B^p_q}(g).$$

Proof. Suppose $f, g \in B^p_q$. Then

$$\rho_{b^p_q}(u) = \rho_{b^p_q}(f + g\circ\gamma) \le \rho_{b^p_q}(f) + \rho_{b^p_q}(g\circ\gamma) \sim \rho_{B^p_q}(f) + \rho_{B^p_q}(g),$$

where the last equivalence follows from Lemma 2.6. Thus $u \in b^p_q$ and the estimate $\rho_{b^p_q}(u) \lesssim \rho_{B^p_q}(f) + \rho_{B^p_q}(g)$ holds.

To see the other direction, let $u \in b^p_q$. By Lemma 2.4(ii), $u_{x_i} \in b^p_{q+p}$ and

$$\|u_{x_i}\|_{b^p_{q+p}} \lesssim \|u - u(0)\|_{b^p_q} \quad (i = 1, 2).$$

Then by (8) and (10), $\partial u/\partial z = f'$ and $\partial u/\partial \bar{z} = g'\circ\gamma$ are in b^p_{q+p} and

$$\|f'\|_{b^p_{q+p}} \lesssim \|u - u(0)\|_{b^p_q} \quad \text{and} \quad \|g'\circ\gamma\|_{b^p_{q+p}} \lesssim \|u - u(0)\|_{b^p_q}.$$

Applying Lemma 2.6 shows that f' and g' are in the holomorphic space B^p_{q+p} and

$$|f'\|_{B^p_{q+p}} \lesssim \|u - u(0)\|_{b^p_q} \quad \text{and} \quad \|g'\|_{B^p_{q+p}} \lesssim \|u - u(0)\|_{b^p_q}.$$

Finally, by Lemma 2.4(i), f and g are in B^p_q and

$$\|f - f(0)\|_{B^p_q} \sim \|f'\|_{B^p_{q+p}} \lesssim \|u - u(0)\|_{b^p_q} \quad \text{and} \quad \|g - g(0)\|_{B^p_q} \lesssim \|u - u(0)\|_{b^p_q}.$$

Thus $\rho_{B^p_q}(f) + \rho_{B^p_q}(g) \lesssim \rho_{b^p_q}(u)$. $\square$

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. To see the if part we check the conditions stated in Definition 1.1. The conditions (i), (ii) and (iv) actually hold for every $(b^p_q, \rho_{b^p_q})$, $q \in \mathbb{R}$. It is clear that $\mathbf{1} \in b^p_q$ and for the decent linear functional one can take, for example, $u \mapsto u_{x_1}(0)$. Verifying condition (iv) for $q > -1$ is straightforward. When $q \le -1$, take N so that $q + pN > -1$. Then $\mathcal{R}^N_h u \in b^p_{q+pN}$ and condition (iv) follows from the fact that $\mathcal{R}^N_h u_\theta = (\mathcal{R}^N_h u)_\theta$. It remains to verify that condition (iii) holds when $q = -2$.

Let $u \in b^p_{-2}$ and write it in the form $u = f + g \circ \gamma$ as in Lemma 2.2. By Lemma 4.1, f and g are in B^p_{-2}. Let $\varphi \in \mathcal{M}(\mathbb{B})$. Then by (6), either $\varphi \in \operatorname{Aut}(\mathbb{D})$ or $\varphi \in \overline{\operatorname{Aut}}(\mathbb{D})$. If $\varphi \in \operatorname{Aut}(\mathbb{D})$, by Lemma 2.1, there exists $\hat{\varphi} \in \operatorname{Aut}(\mathbb{D})$ such that

$$u \circ \varphi = f \circ \varphi + g \circ \gamma \circ \varphi = f \circ \varphi + g \circ \hat{\varphi} \circ \gamma.$$

The right-hand side is a representation of $u \circ \varphi$ in the form stated in Lemma 2.2 and we have

$$\rho_{b^p_{-2}}(u \circ \varphi) \sim \rho_{B^p_{-2}}(f \circ \varphi) + \rho_{B^p_{-2}}(g \circ \hat{\varphi}) \sim \rho_{B^p_{-2}}(f) + \rho_{B^p_{-2}}(g) \sim \rho_{b^p_{-2}}(u),$$

where the first and the last equivalences follow from Lemma 4.1 and the second one follows from the Möbius invariance of B^p_{-2}. If $\varphi \in \overline{\operatorname{Aut}}(\mathbb{D})$, then $\varphi = \gamma \circ \psi$ for some $\psi \in \operatorname{Aut}(\mathbb{D})$ and exchanging the roles of f and g, we again obtain $\rho_{b^p_{-2}}(u \circ \varphi) \sim \rho_{b^p_{-2}}(u)$. Thus condition (iii) of Definition 1.1 is satisfied and b^p_{-2} is Möbius invariant.

We next show the only if part, that is, if $q \neq -2$, then condition (iii) in Definition 1.1 does not hold. For $1/2 < r < 1$, let $f_r(z) := 1 - rz$. Then

$$\rho_{b^p_q}(f_r) = \rho_{b^p_q}(1 - rz) = r\rho_{b^p_q}(z) \sim 1. \tag{16}$$

With φ_r as in (5), we have $f_r \circ \varphi_r(z) = (1 - r^2)/(1 - rz)$ and by Lemma 2.6,

$$\rho_{b^p_q}(f_r \circ \varphi_r) \sim \rho_{B^p_q}(f_r \circ \varphi_r) = (1 - r^2)\rho_{B^p_q}\Big(\frac{1}{1 - rz}\Big).$$

To compute the right-hand side, pick $N \ge 1$ so that $q + pN > -1$. Since the k^{th} derivative of $1/(1 - rz)$ at $z = 0$ is $k!r^k$, by Definition 2.3 we have

$$\rho^p_{b^p_q}(f_r \circ \varphi_r) \sim (1 - r^2)^p \bigg(\sum_{k=1}^{N-1} (k!r^k)^p + \int_{\mathbb{D}} \frac{(1 - |z|^2)^{q+pN}}{|1 - rz|^{p(N+1)}}\, dA(z)\bigg).$$

Set $M_r = \sum_{k=1}^{N-1} (k!r^k)^p$ and note that it stays bounded as $r \to 1^-$. We estimate the above integral with $c = p(N + 1) - (q + pN) - 2 = p - q - 2$ in Lemma 2.7.

If $c > 0$, then

$$\rho^p_{b^p_q}(f_r \circ \varphi_r) \sim (1-r^2)^p M_r + (1-r^2)^{q+2}$$

and the right-hand side approaches 0 or ∞ as $r \to 1^-$ when $q \neq -2$. If $c = 0$, then $\rho^p_{b^p_q}(f_r \circ \varphi_r)$ approaches 0 as $r \to 1^-$ since $(1-r^2)^p$ dominates $\log 1/(1-r^2)$. If $c < 0$, then $\rho^p_{b^p_q}(f_r \circ \varphi_r)$ approaches 0 as $r \to 1^-$.

Thus, by (16), the estimate $\rho_{b^p_q}(f_r \circ \varphi_r) \sim \rho_{b^p_q}(f_r)$ can not hold when $q \neq -2$. $\square$

5. Properties of Möbius-invariant spaces

We begin this section by explaining why we allow seminorms in Definition 1.1. Suppose $(E, \|\cdot\|)$ is a *normed* space of harmonic functions and point evaluations are bounded on E. If E contains an unbounded function, then E cannot be Möbius invariant. To see this, let $u \in E$ and $a_m \in \mathbb{D}$ be such that $|u(a_m)| \to \infty$. Because point evaluation at 0 is bounded we have

$$|u(a_m)| = |u \circ \varphi_{a_m}(0)| \leq C\|u \circ \varphi_{a_m}\|.$$

This shows that $\|u \circ \varphi_{a_m}\| \to \infty$ and we cannot have $\|u \circ \varphi_{a_m}\| \sim \|u\|$. Note that the Besov spaces b^p_{-2}, except when $p = 1$, and the Bloch space b^∞ contain unbounded functions. Therefore to call these spaces Möbius invariant we need to give up requiring norms and allow seminorms vanishing on constants. The downside of this approach is point evaluations are not bounded on these seminormed spaces and additional care is needed in dealing with convergence issues.

To establish a connection between pointwise convergence and convergence in a Möbius-invariant space we first introduce a norm on the seminormed space (E, ρ_E) that will make point evaluations bounded.

LEMMA 5.1. *Let (E, ρ_E) be a Möbius-invariant space of harmonic functions. Then*

(i) $\rho_E(u) = 0$ *if and only if u is constant,*

(ii) *for any $u \in E$ and any constant $C \in \mathbb{C}$, we have $\rho_E(u + C) = \rho_E(u)$.*

PROOF. Seminorm of a constant function is 0 by assumption (i) in Definition 1.1. If u is nonconstant, then $\rho_{b^\infty}(u) \neq 0$ and so $\rho_E(u) \neq 0$ by Theorem 1.2. $\square$

For a Möbius-invariant space (E, ρ_E), define

$$\|u\|_E := |u(0)| + \rho_E(u). \tag{17}$$

LEMMA 5.2. $(E, \|\cdot\|_E)$ *is a complete normed space.*

PROOF. That $\|\cdot\|_E$ is a norm follows from Lemma 5.1(i). To see the completeness, let (u_m) be a Cauchy sequence in $(E, \|\cdot\|_E)$. Then $(u_m(0))$ is Cauchy and (u_m) is Cauchy in (E, ρ_E). So, there exist $a \in \mathbb{C}$ such that $u_m(0) \to a$ and $u \in E$ such that $\rho_E(u_m - u) \to 0$. Let $\tilde{u} := u - u(0) + a$. Then $\tilde{u} \in E$ since E contains constants and $u_m \to \tilde{u}$ in $(E, \|\cdot\|_E)$ because $(u_m - \tilde{u})(0) = u_m(0) - a \to 0$ and

$$\rho_E(u_m - \tilde{u}) = \rho_E(u_m - u + u(0) - a) = \rho_E(u_m - u) \to 0$$

by Lemma 5.1(ii). □

For $x \in \mathbb{B}$, let $\Lambda_x \colon E \to \mathbb{C}$, $\Lambda_x(u) := u(x)$ be the point-evaluation functional.

LEMMA 5.3. *Let (E, ρ_E) be a Möbius-invariant space of harmonic functions and let $\|\cdot\|_E$ be as in (17). Then Λ_x is bounded on $(E, \|\cdot\|_E)$.*

PROOF. Fix $x \in \mathbb{B}$ and let $u \in E$. By Theorem 1.2, u belongs to b^∞ and $\rho_{b^\infty}(u) \lesssim \rho_E(u)$. Then

$$|u(x) - u(0)| = \left| \int_0^1 \nabla u(tx) \cdot x \, dt \right| \le \rho_{b^\infty}(u) \int_0^1 \frac{dt}{1 - t^2|x|^2} \lesssim \rho_{b^\infty}(u) \lesssim \rho_E(u).$$

Thus $|u(x)| \lesssim |u(0)| + \rho_E(u) = \|u\|_E$. □

Note that although Λ_x is continuous on $(E, \|\cdot\|_E)$, it is not continuous on (E, ρ_E) since that would violate (12) for a constant u. On the other hand, whereas $\rho_E(u \circ \varphi) \sim \rho_E(u)$, it is not true that $\|u \circ \varphi\|_E \sim \|u\|_E$ for all $\varphi \in \mathcal{M}(\mathbb{B})$. However, if we consider only the rotations and the conjugation, this equivalence is true.

LEMMA 5.4. *Let (E, ρ_E) be a Möbius-invariant space of harmonic functions and let $\|\cdot\|_E$ be as in (17). Then*

(i) $\|u_\theta\|_E \sim \|u\|_E$ *and* $\|u \circ \gamma\|_E \sim \|u\|_E$,

(ii) *for fixed $u \in E$, the map $\theta \mapsto u_\theta$ is continuous from $[0, 2\pi]$ to $(E, \|\cdot\|_E)$.*

PROOF. Part (i) is true because $u_\theta(0) = u(0)$ and $u \circ \gamma(0) = u(0)$. Part (ii) follows from Definition 1.1(iv) and the fact that $\|u_{\theta_1} - u_{\theta_2}\|_E = \rho_E(u_{\theta_1} - u_{\theta_2})$. □

We next show the harmonic analogues of Propositions 1 and 2 of [2]. The ideas are taken from [2], but more details are provided and we will be cautious about convergence issues.

PROPOSITION 5.5. *Let (E, ρ_E) be a Möbius-invariant space of harmonic functions, $u \in E$ and $m \in \mathbb{Z}$. Define the function U pointwise by*

$$U(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta} z) e^{-im\theta} \, d\theta \quad (z \in \mathbb{D}). \tag{18}$$

Then $U \in E$, $\|U\|_E \lesssim \|u\|_E$ *and* $\rho_E(U) \lesssim \rho_E(u)$.

Proof. We work in the Banach space $(E, \|\cdot\|_E)$ and consider the Bochner integral

$$V = \frac{1}{2\pi} \int_0^{2\pi} u_\theta \, e^{-im\theta} \, d\theta.$$

For Bochner integrals see, for example, [4, Appendix E]. The above integral exists by the completeness of $(E, \|\cdot\|_E)$ (Lemma 5.2) and the continuity in Lemma 5.4(ii). Hence $V \in E$ and by [4, Proposition E.5] and Lemma 5.4(i),

$$\|V\|_E \le \frac{1}{2\pi} \int_0^{2\pi} \|u_\theta \, e^{-im\theta}\|_E \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \|u_\theta\|_E \, d\theta \lesssim \frac{1}{2\pi} \int_0^{2\pi} \|u\|_E \, d\theta = \|u\|_E.$$

By Lemma 5.3, for every $z \in \mathbb{D}$, the point evaluation functional Λ_z is bounded on $(E, \|\cdot\|_E)$. Applying [4, Proposition E.11] we see that

$$V(z) = \Lambda_z(V) = \Lambda_z\left(\frac{1}{2\pi} \int_0^{2\pi} u_\theta \, e^{-im\theta} \, d\theta\right) = \frac{1}{2\pi} \int_0^{2\pi} u_\theta(z) \, e^{-im\theta} \, d\theta = U(z),$$

for all $z \in \mathbb{D}$. Thus $U = V$ and we conclude that $U \in E$ and $\|U\|_E \lesssim \|u\|_E$. To see $\rho_E(U) \lesssim \rho_E(u)$, we replace u with $u - u(0)$ and apply the same argument. □

Lemma 5.6. *Let* (E, ρ_E) *be a Möbius-invariant space of harmonic functions. Let* $u \in E$ *and write* u *as a series as in* (7). *If* $a_{m_0} \neq 0$ *for some* $m_0 \ge 1$, *then* $z^{m_0} \in E$ *and if* $b_{m_0} \neq 0$ *for some* $m_0 \ge 1$, *then* $\bar{z}^{m_0} \in E$.

Proof. If $a_{m_0} \neq 0$ for some $m_0 \ge 1$, define U as in (18). A calculation shows $U(z) = a_{m_0} z^{m_0}$ and $U \in E$ by Proposition 5.5. The argument for $\bar{z}^{m_0}$ is similar. □

Now we prove the harmonic counterpart of [2, Proposition 2].

Theorem 5.7. *A Möbius-invariant space of harmonic functions* (E, ρ_E) *contains all* z^m *and* $\bar{z}^m$, $m \ge 1$. *Moreover,* $\rho_E(z^m) \lesssim m\rho_E(z)$ *and* $\rho_E(\bar{z}^m) \lesssim m\rho_E(z)$.

Proof. By assumption E contains a nonconstant function. Let $u \in E$ be such that in the series representation (7) at least one of $a_{m_0} \neq 0$ or $b_{m_0} \neq 0$ with $m_0 \ge 1$. If $a_{m_0} \neq 0$, then by Lemma 5.6, $z^{m_0} \in E$. If $b_{m_0} \neq 0$, then by the same lemma, $\bar{z}^{m_0} \in E$. But then $z^{m_0} = \bar{z}^{m_0} \circ \gamma$ is in E. We conclude that $z^{m_0} \in E$ for some $m_0 \ge 1$.

Next, by Möbius invariance, $z^{m_0} \circ \varphi_r = \varphi_r^{m_0} \in E$ for every $-1 < r < 1$. In the series expansion of the holomorphic function $\varphi_r^{m_0}$, the coefficient of z is non-zero for $r \neq 0$ because $(\varphi_r^{m_0})'(0) = -m_0 r^{m_0-1}(1-r^2) \neq 0$. Thus $z \in E$ by Lemma 5.6, and $\varphi_r \in E$ $(-1 < r < 1)$ by Möbius invariance. Since

$$\varphi_r(z) = \frac{r-z}{1-rz} = (r-z)\sum_{m=0}^{\infty} r^m z^m = r - (1-r^2)\sum_{m=1}^{\infty} r^{m-1} z^m, \tag{19}$$

$z^m \in E$ for every $m \geq 1$ by Lemma 5.6.

We next estimate $\rho_E(z^m)$. By (19),

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_r(e^{i\theta}z)e^{-im\theta}\,d\theta = -(1-r^2)r^{m-1}z^m \quad (m \geq 1),$$

and so by Proposition 5.5,

$$(1-r^2)r^{m-1}\rho_E(z^m) \lesssim \rho_E(\varphi_r) \sim \rho_E(z).$$

Set $r^2 = (m-1)/m$. Then $1 - r^2 = 1/m$ and

$$\frac{1}{r^{m-1}} = \Big(\frac{m}{m-1}\Big)^{(m-1)/2} = \bigg(\Big(1+\frac{1}{m-1}\Big)^{m-1}\bigg)^{1/2} \leq e^{1/2}.$$

We conclude that $\rho_E(z^m) \lesssim e^{1/2} m \rho_E(z)$. The estimate of $\rho_E(\bar{z}^m)$ follows from Möbius invariance. □

COROLLARY 5.8. *A Möbius-invariant space of harmonic functions contains all harmonic polynomials.*

Finally we show that harmonic polynomials are dense.

LEMMA 5.9. *Let (E, ρ_E) be a Möbius-invariant space of harmonic functions. Then harmonic polynomials are dense in $(E, \|\cdot\|_E)$ and in (E, ρ_E).*

PROOF. We first work in the normed space $(E, \|\cdot\|_E)$. Pick $u \in E$ and represent u as a series as in (7). Let

$$K_m(\theta) := \sum_{k=-m}^{m}\Big(1 - \frac{|k|}{m+1}\Big)e^{-ik\theta} \quad (0 \leq \theta \leq 2\pi),$$

be the Fejér kernel. The Bochner integral

$$U_m = \frac{1}{2\pi}\int_0^{2\pi} u_\theta K_m(\theta)\,d\theta$$

belongs to E, and since point evaluations are bounded on $(E, \|\cdot\|_E)$ by Lemma 5.3, applying [4, Proposition E.11] shows that for all $z \in \mathbb{D}$,

$$U_m(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}z)\bigg(\sum_{k=-m}^{m}\Big(1-\frac{|k|}{m+1}\Big)e^{-ik\theta}\bigg)\,d\theta$$

$$= \sum_{k=0}^{m}\Big(1-\frac{k}{m+1}\Big)a_k z^k + \sum_{k=1}^{m}\Big(1-\frac{k}{m+1}\Big)b_k \bar{z}^k.$$

Thus U_m is a harmonic polynomial. Using the fact that $\{K_m\}$ is a positive approximate identity, it is straightforward to show $U_m \to u$ in $(E, \|\cdot\|_E)$ as $m \to \infty$. To see that harmonic polynomials are dense in (E, ρ_E), note that $\rho_E(U_m - u) \le \|U_m - u\|_E \to 0$. $\square$

6. Uniqueness of the semi-Hilbert space

In this section we show that the harmonic Dirichlet space b^2_{-2} is the unique *strictly* Möbius-invariant semi-Hilbert space. By Theorem 1.5, $(b^2_{-2}, \rho_{b^2_{-2}})$ is Möbius invariant. Let us first show that it is strictly Möbius invariant.

Lemma 6.1. *For all $u \in b^2_{-2}$ and $\varphi \in \mathcal{M}(\mathbb{B})$, the composition $u \circ \varphi$ is in b^2_{-2} and $\rho_{b^2_{-2}}(u \circ \varphi) = \rho_{b^2_{-2}}(u)$.*

Proof. Note that by (4), (8) and (9), we have

$$\rho^2_{b^2_{-2}}(u) = 2\int_{\mathbb{D}} \bigg(\Big|\frac{\partial u}{\partial z}(z)\Big|^2 + \Big|\frac{\partial u}{\partial \bar{z}}(z)\Big|^2 \bigg)\, dA(z).$$

Pick $u \in b^2_{-2}$ and let $u = f + g \circ \gamma$ as in Lemma 2.2. By (10) and conjugation invariance of dA,

$$\rho^2_{b^2_{-2}}(u) = 2\int_{\mathbb{D}} \big(|f'(z)|^2 + |g'(\bar{z})|^2\big)\, dA(z) = 2\int_{\mathbb{D}} \big(|f'(z)|^2 + |g'(z)|^2\big)\, dA(z).$$

Let $\varphi \in \mathcal{M}(\mathbb{B})$. Then either $\varphi \in \mathrm{Aut}(\mathbb{D})$ or $\varphi \in \overline{\mathrm{Aut}}(\mathbb{D})$. If $\varphi \in \mathrm{Aut}(\mathbb{D})$, as in the proof of Lemma 3.3, there exists $\hat{\varphi} \in \mathrm{Aut}(\mathbb{D})$ such that

$$u \circ \varphi = f \circ \varphi + g \circ \gamma \circ \varphi = f \circ \varphi + g \circ \hat{\varphi} \circ \gamma$$

and

$$\frac{\partial (u \circ \varphi)}{\partial z} = (f \circ \varphi)'(z) \quad \text{and} \quad \frac{\partial (u \circ \varphi)}{\partial \bar{z}} = (g \circ \hat{\varphi})'(\bar{z}).$$

Then

$$\begin{aligned} \rho^2_{b^2_{-2}}(u \circ \varphi) &= 2\int_{\mathbb{D}} \big(|(f \circ \varphi)'(z)|^2 + |(g \circ \hat{\varphi})'(\bar{z})|^2\big)\, dA(z) \\ &= 2\int_{\mathbb{D}} \big(|(f \circ \varphi)'(z)|^2 + |(g \circ \hat{\varphi})'(z)|^2\big)\, dA(z) \end{aligned}$$

$$= 2\int_{\mathbb{D}} \big(|f'(z)|^2 + |g'(z)|^2\big)\, dA(z) = \rho^2_{b^2_{-2}}(u),$$

where in the second equality we use conjugation invariance of dA and in the third equality we use (15) with $p = 2$. The case $\varphi \in \overline{\mathrm{Aut}}(\mathbb{D})$ is dealt with similarly as done a few times above. □

An elementary calculation shows that if u is written as a series as in (7), then

$$\rho^2_{b^2_{-2}}(u) = 2\sum_{m=1}^{\infty} m\big(|a_m|^2 + |b_m|^2\big). \tag{20}$$

We next show that b^2_{-2} is the only strictly Möbius-invariant semi-Hilbert space. In the proof, as in [1], we use the functions

$$1 - r\varphi_r(z) = \frac{1-r^2}{1-rz} = (1-r^2)\sum_{m=0}^{\infty} r^m z^m, \quad (-1 < r < 1,\ z \in \mathbb{D}). \tag{21}$$

The Taylor series above converges pointwise. Let us verify that it also converges in a Möbius-invariant space. As a preliminary we first show the following lemma.

Lemma 6.2. *Let (E, ρ_E) be a Möbius-invariant space of harmonic functions. Suppose (u_m) is convergent in the space (E, ρ_E) and $u_m \to u$ pointwise. Then the limit of (u_m) in (E, ρ_E) is u.*

Proof. With $\|\cdot\|_E$ as in (17), we pass to the normed space $(E, \|\cdot\|_E)$ where point evaluations are bounded by Lemma 5.3. The sequence (u_m) is Cauchy in $(E, \|\cdot\|_E)$ because (u_m) is Cauchy in (E, ρ_E) and $(u_m(0))$ is Cauchy. Then, by Lemma 5.2, there exists $v \in E$ such that $\|u_m - v\|_E \to 0$. By Lemma 5.3, $u_m \to v$ pointwise and thus $v = u$. This shows $u \in E$ and $\rho_E(u_m - u) \le \|u_m - u\|_E \to 0$. □

Proposition 6.3. *Let (E, ρ_E) be a Möbius-invariant space of harmonic functions. The series $(1-r^2)\sum_{m=0}^{\infty} r^m z^m$ in (21) converges to $1 - r\varphi_r$ in the space (E, ρ_E).*

Proof. By the estimate $\rho_E(z^m) \lesssim m$ in Theorem 5.7, the series

$$\sum_{m=0}^{\infty} r^m z^m$$

is Cauchy and therefore converges in the complete space (E, ρ_E). The proposition now follows from Lemma 6.2. □

Proof of Theorem 1.6(ii). As $\rho_H(u\circ\varphi)=\rho_H(u)$, by polarization, we have

$$\langle u\circ\varphi, v\circ\varphi\rangle_H=\langle u,v\rangle_H \quad (u,v\in H,\ \varphi\in\mathcal{M}(\mathbb{B})). \tag{22}$$

Taking $\varphi(z)=e^{i\theta}z$, a rotation, we see that

$$\langle z^m, z^k\rangle_H=\langle e^{im\theta}z^m, e^{ik\theta}z^k\rangle_H=e^{i(m-k)\theta}\langle z^m,z^k\rangle_H.$$

This shows $\langle z^m, z^k\rangle_H=0$ when $m\neq k$. Replacing one or both z with $\bar z$ we obtain a similar equality. We list all these below which include $\langle \mathbf{1},\mathbf{1}\rangle_H=0$.

$$\begin{cases}\langle z^m, z^k\rangle_H=0, & m\neq k\ (m,k\geq 0),\\ \langle \bar z^m, \bar z^k\rangle_H=0, & m\neq k\ (m,k\geq 0),\\ \langle z^m, \bar z^k\rangle_H=0, & (m,k\geq 0).\end{cases} \tag{23}$$

Also taking $\varphi(z)=\gamma(z)=\bar z$ shows

$$\langle \bar z^m, \bar z^m\rangle_H=\langle z^m, z^m\rangle_H \quad (m\geq 1). \tag{24}$$

As in [1], for $-1<r<1$, we compute $\langle 1-r\varphi_r, 1-r\varphi_r\rangle_H$ in two ways. One using Möbius invariance, the other by using the series expansion in Proposition 6.3. First,

$$\langle 1-r\varphi_r, 1-r\varphi_r\rangle_H=\langle 1-rz, 1-rz\rangle_H=r^2\langle z,z\rangle_H$$

by (22) and (23). Second, by Proposition 6.3 and (23),

$$\begin{aligned}\langle 1-r\varphi_r, 1-\varphi_r\rangle_H&=(1-r^2)^2\Big\langle\sum_{m=0}^{\infty}r^m z^m, \sum_{m=0}^{\infty}r^m z^m\Big\rangle_H\\ &=(1-r^2)^2\sum_{m=0}^{\infty}r^{2m}\langle z^m, z^m\rangle_H.\end{aligned}$$

Equating the above and using $\langle \mathbf{1},\mathbf{1}\rangle_H=0$, we see that for all $-1<r<1$,

$$r^2\langle z,z\rangle_H=(1-r^2)^2\sum_{m=0}^{\infty}r^{2m}\langle z^m,z^m\rangle_H$$

$$=r^2\langle z,z\rangle_H+\sum_{m=2}^{\infty}r^{2m}\big(\langle z^m,z^m\rangle_H-2\langle z^{m-1},z^{m-1}\rangle_H+\langle z^{m-2},z^{m-2}\rangle_H\big).$$

Thus for $m\geq 2$, we must have

$$\langle z^m,z^m\rangle_H-2\langle z^{m-1},z^{m-1}\rangle_H+\langle z^{m-2},z^{m-2}\rangle_H=0.$$

Solving this equation recursively we obtain

$$\langle z^m, z^m\rangle_H = m\langle z, z\rangle_H \quad (m \ge 2).$$

Let $\lambda := \sqrt{\langle z, z\rangle_H/2}$. Using also (24) we conclude that

$$\langle \bar{z}^m, \bar{z}^m\rangle_H = \langle z^m, z^m\rangle_H = \lambda^2 2m \quad (m \ge 0). \tag{25}$$

We now show $H = b^2_{-2}$ and $\rho_H(u) = \lambda\rho_{b^2_{-2}}(u)$ for every $u \in H$. By polarization this implies $\langle u, v\rangle_H = \lambda^2\langle u, v\rangle_{b^2_{-2}}$ for $u, v \in H$.

To establish a connection between convergence in the space H and pointwise convergence, we pass from semi-inner product to inner product. Define

$$\langle\langle u, v\rangle\rangle := \lambda^2 u(0)\overline{v(0)} + \langle u, v\rangle_H \quad (u, v \in H). \tag{26}$$

By Lemma 5.1 (i), $\langle\langle\cdot,\cdot\rangle\rangle$ is a true inner product with the corresponding norm

$$|||u|||^2 = \lambda^2|u(0)|^2 + \rho_H^2(u).$$

The above norm is slightly different from the one defined in (17), but is equivalent to it and therefore the properties obtained in Section 5 for $(E, \|\cdot\|_E)$ hold also for $(H, |||\cdot|||)$. Same is true for b^2_{-2} when endowed with the norm

$$\|u\|^2_{b^2_{-2}} = |u(0)|^2 + \rho^2_{b^2_{-2}}(u).$$

If $p(z) = \sum_{m=0}^M a_m z^m + \sum_{m=1}^M b_m \bar{z}^m$ is a harmonic polynomial, then by (20), (23) and (25),

$$|||p||| = \lambda\|p\|_{b^2_{-2}}. \tag{27}$$

To see that $H \subset b^2_{-2}$, pick $u \in H$. By Lemma 5.9, there is a sequence (p_m) of harmonic polynomials such that $|||p_m - u||| \to 0$ and $p_m(z) \to u(z)$ pointwise for every $z \in \mathbb{D}$ by Lemma 5.3. By (27), (p_m) is Cauchy in the complete space $(b^2_{-2}, \|\cdot\|_{b^2_{-2}})$ and so there exists $v \in b^2_{-2}$ such that $\|p_m - v\|_{b^2_{-2}} \to 0$. But then by Lemma 5.3, (p_m) converges pointwise to v and we conclude that $u = v \in b^2_{-2}$. In addition, by (27),

$$|||u||| = \lim_{m\to\infty} |||p_m||| = \lambda \lim_{m\to\infty} \|p_m\|_{b^2_{-2}} = \lambda\|u\|_{b^2_{-2}},$$

which implies $\rho_H(u) = \lambda\rho_{b^2_{-2}}(u)$. The inclusion $b^2_{-2} \subset H$ can be verified in the same way. $\square$

7. The minimal space

To show the minimality of b^1_{-2} we use atomic decomposition. Atomic decomposition of harmonic Bergman spaces b^p_q $(q > -1)$ is obtained in [5, Theorem 3] and this is extended to all $q \in \mathbb{R}$ in [7]. The following lemma follows from [7, Theorem 10.1] by taking $p = 1$, $q = -2$, $s = -1$, and using the formula in the last paragraph of [7, Section 14].

Lemma 7.1. *There exists a sequence $(a_m) \in \mathbb{D}$ such that the following holds. For every $u \in b^1_{-2}$, there is a sequence $(c_m) \in \ell^1$ with $\|(c_m)\|_{\ell^1} \sim \|u\|_{b^1_{-2}}$ such that*

$$u(z) = \sum_{m=1}^{\infty} c_m (1 - |a_m|^2)\Big(\frac{1}{1 - \bar{a}_m z} + \frac{1}{1 - a_m \bar{z}} - 1\Big). \tag{28}$$

The above series converges absolutely and uniformly on compact subsets of $\mathbb{D}$ and also in the norm $\|\cdot\|_{b^1_{-2}}$.

Proof of Theorem 1.7. Let $(a_m) \in \mathbb{D}$ be as asserted in Lemma 7.1. Pick arbitrary $u \in b^1_{-2}$ and let (c_m) be such that

$$\|(c_m)\|_{\ell^1} \sim \|u\|_{b^1_{-2}} \tag{29}$$

and (28) holds. By the identities

$$\frac{1 - |a_m|^2}{1 - \bar{a}_m z} = 1 - \bar{a}_m \varphi_{a_m}(z) \quad \text{and} \quad \frac{1 - |a_m|^2}{1 - a_m \bar{z}} = 1 - a_m \varphi_{\bar{a}_m}(\bar{z}),$$

we have

$$u(z) = \sum_{m=1}^{\infty} c_m \left(1 + |a_m|^2 - \bar{a}_m \varphi_{a_m}(z) - a_m \varphi_{\bar{a}_m}(\bar{z})\right). \tag{30}$$

The series on the right converges pointwise to u. Let us show that it also converges to u in the space (E, ρ_E). Writing $\varphi_{\bar{a}_m}(\bar{z}) = (\varphi_{\bar{a}_m} \circ \gamma)(z)$ and using first Lemma 5.1 (ii), then $|a_m| < 1$, and finally Möbius invariance, we obtain

$$\begin{aligned} &\rho_E\big(1 + |a_m|^2 - \bar{a}_m \varphi_{a_m} - a_m \varphi_{\bar{a}_m} \circ \gamma\big) \\ &= \rho_E\big(- \bar{a}_m \varphi_{a_m} - a_m \varphi_{\bar{a}_m} \circ \gamma\big) \le \rho_E(\varphi_{a_m}) + \rho_E(\varphi_{\bar{a}_m} \circ \gamma) \lesssim \rho_E(z). \end{aligned} \tag{31}$$

Because $(c_m) \in \ell^1$, the series in (30) is Cauchy (convergent) in E and applying Lemma 6.2 shows that u is in E. In addition, by (31) and (29),

$$\rho_E(u) = \rho_E\bigg(\sum_{m=1}^{\infty} c_m \big(1 + |a_m|^2 - \bar{a}_m \varphi_{a_m} - a_m \varphi_{\bar{a}_m} \circ \gamma\big)\bigg) \lesssim \sum_{m=1}^{\infty} |c_m| \lesssim \|u\|_{b^1_{-2}}.$$

To finish the proof all we need is to replace $\|u\|_{b^1_{-2}}$ above with $\rho_{b^1_{-2}}(u)$. This is easily done by replacing u with $u - u(0)$ and using Lemma 5.1(ii). $\square$

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