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Variations on a theme of Mirsky

Yıldırım Akbal*

Department of Mathematics
Atılım University, 06830 Gölbaşı, Ankara, Turkey
yildirim.akbal@gmail.com

Ahmet M. Güloğlu

Department of Mathematics
SA 131, Bilkent University, 06800 Bilkent, Ankara, Turkey
guloglua@fen.bilkent.edu.tr

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Let k and r be non-zero integers with $r \geq 2$. An integer is called r-free if it is not divisible by the rth power of a prime. A result of Mirsky states that there are infinitely many primes p such that $p+k$ is r-free. In this paper, we study an additive Goldbach-type problem and prove two uniform distribution results using these primes. We also study certain properties of primes p such that $p+a_1, \dots, p+a_\ell$ are simultaneously r-free, where $a_1, \dots, a_\ell$ are non-zero integers and $\ell \geq 1$.

Keywords: Hardy–Littlewood circle method; r-free shifted primes; Goldbach-type additive problems.

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1. Introduction and Statement of Results

Let k be a non-zero integer and $r \geq 2$ be an integer. Let $\mathcal{P}_{r,k}$ denote the set of primes p such that $p+k$ is positive and not divisible by an rth power of a prime. In [13] Mirsky showed that $\mathcal{P}_{r,k}$ has positive density in the set of primes. More precisely, he showed that for every sufficiently large x, the asymptotic formula

$$\#\left(\mathcal{P}_{r,k} \cap [1,x]\right) = \prod_{p \nmid k} \left(1 - \frac{1}{\phi(p^r)}\right) \mathrm{li}(x) + O\left(\frac{x}{\log^A x}\right)$$

holds for any $A > 0$, where ϕ is the Euler-totient function and $\mathrm{li}(x) = \int_2^x 1/(\log t)dt$. Mirsky's result is based on inclusion and exclusion principle together with an

*Corresponding author.

application of the Siegel–Walfisz theorem; therefore, the error term here is very weak compared to the best error term in the prime number theorem due to Vinogradov and Korobov (cf. [10, 23]). Furthermore, assuming that there are no Siegel zeros, the result of Languasco [12] can be recasted that the above error term can be taken as small as the standard error term in the prime number theorem (cf. [5, §18]). Following the work of Mirsky, it can be easily verified that under GRH for all Dirichlet L-functions the above error term can be taken $\ll x^{(r+1)/(2r)}(\log x)^{(r-2)/r}$.

In this paper, our first goal is to study primes that lie in the set $\cap_{i=1}^{\ell}\mathcal{P}_{r,a_i}$, where $a_1, a_2, \ldots, a_\ell$ are non-zero integers, for a fixed $r \geq 2$. In Theorem 1.1, we give an asymptotic formula for the counting function of these primes. Similar results appeared in [1, 3], but our theorem allows the ℓ tuple and ℓ itself to vary. A similar result using the circle method was proved in [8] (not published) for primes in the set $\cap_{i=1}^{\ell}\mathcal{P}_{r_i,a_i}$, where $2 \leq r_1 \leq \cdots \leq r_\ell$ are fixed. In [6], an asymptotic formula is given for the number of primes p such that $p+1$ and $p+2$ are square-free under GRH.

Our second goal is to prove an asymptotic formula for the number of representations of a given integer as the sum of $s \geq 3$ primes from $\mathcal{P}_{r,k}$.

Our third goal is to give two uniform distribution results using the primes in $\mathcal{P}_{r,k}$ for a fixed non-zero integer k and $r \geq 2$.

Let $\mathbf{a}$ denote an ℓ-tuple $(a_1, a_2, \ldots, a_\ell)$ of non-zero integers, where $\ell \geq 1$. Define

$$\pi_r(x, \mathbf{a}) := \#\{p \leq x : p + a_i \text{ is positive and } r\text{-free for each } i = 1, \ldots, \ell\}.$$

Put $|\mathbf{a}| = \max_i |a_i|$, and

$$\mathfrak{S}_r(\mathbf{a}) = \prod_p \left(1 - \frac{\nu_p(\mathbf{a})}{\phi(p^r)}\right),$$

where $\nu_p(\mathbf{a})$ is the number of distinct residue classes a_i modulo p^r that are coprime to p. We call $\mathbf{a}$ a *permissible tuple* of non-zero integers if there are infinitely many primes p such that $p + a_1, \ldots, p + a_\ell$ are r-free. It is not hard to see that if $\nu_p(\mathbf{a}) = \phi(p^r)$ for some p, then $\mathbf{a}$ is not permissible. The next theorem establishes the converse.

Theorem 1.1. *Let $r \geq 2$ be a fixed integer, and $\mathbf{a}$ denote any ℓ-tuple $(a_1, a_2, \ldots, a_\ell)$ of non-zero integers, where $\ell \geq 1$. Given $0 < \varepsilon < (\log 2)^{-1}$ and $A > 0$, the estimate*

$$\pi_r(x, \mathbf{a}) - \mathfrak{S}_r(\mathbf{a}) \operatorname{li}(x) \ll \ell(\log x)^{\frac{A+2}{r-1}(\ell-1)}|\mathbf{a}| + \operatorname{li}(x)(\log x)^{-A}$$

holds uniformly for

$$1 \leq \ell < ((\log 2)^{-1} - \varepsilon) \log\log\log x$$

for every sufficiently large x. Here, the implied constant depends at most on r, A and ε.

Under GRH for all Dirichlet L-functions the error estimate above is to be replaced by (2.6), *therein the implied constant depends only on r. Furthermore, the largest range of ℓ and $|\mathbf{a}|$ such that the error is still $< x/(\log x)^2$ is given by* (2.7).

Compared with the main results of [1, 3, 6, 8], our result goes one step further as the parameters ℓ and $|\mathbf{a}|$ in the above theorem are allowed to increase as a function of x, which is *not* addressed in these papers.

Remark 1.2. By a simple argument, we can always find a permissible ℓ-tuple for a given $\ell \geq 2$. This fact, as well as the lower bound

$$\mathfrak{S}_r(\mathbf{a}) \geq e^{-C\ell^{\frac{1}{r}}}$$

for some $C = C(r) > 0$, whenever $\mathbf{a}$ is a permissible ℓ-tuple, will be proved subsequent to the proof of Theorem 1.1.

Theorem 1.3. *Let $0 < \varepsilon < 1/(\log 2)$ and $r \geq 2$ be given. Then, for all sufficiently large H, the equation*

$$\sum_{|\mathbf{a}|\leq H} \mathfrak{S}_r(\mathbf{a}) - \left(\frac{2H}{\zeta(r)}\right)^{\ell} \ll (2H)^{\ell+\frac{1}{r}-1}\ell\left(\frac{15\log H}{\pi^2}\right)^{\frac{2^\ell-1}{r}} (\zeta(r)+1)^{\frac{(r-1)(\ell-1)}{r}} \tag{1.1}$$

holds uniformly for $1 \leq \ell \leq (1/\log 2 - \varepsilon)\log\log H$, where $\zeta(s)$ is the Riemann zeta function and the sum is taken over ℓ-tuples $\mathbf{a}$ with non-zero coordinates. The implied constant depends only on r and ε.

Theorem 1.4. *Let $0 < \varepsilon < (\log 2)^{-1}$, $r \geq 2$, and A with $A > 1/(r-1)$ be given. Then, for every sufficiently large x, the estimate*

$$\sum_{|\mathbf{a}|\leq H} \pi_r(x,\mathbf{a}) - \left(\frac{2H}{\zeta(r)}\right)^{\ell}\pi(x) \ll \frac{x\ell(2H)^{\ell-1+\frac{1}{r}}(\log H)^{\frac{2^\ell-1}{r}}(\zeta(r)+1)^{\frac{(\ell-1)(r-1)}{r}}}{\log^{1-1/r} x} + \frac{\ell\,(\log x)^{\frac{\ell-1}{r}}(2H)^{\ell+\frac{\ell-1}{r}}(\log H)^{\frac{(2^\ell-1)(\ell-1)}{r}}(H+x^{1/r})}{(1+\zeta(r))^{\frac{(\ell-1)^2}{r}}}$$

holds uniformly for

$$\begin{aligned}\log^A x \leq H &\leq \frac{x}{2.1\log x}\exp(-(\log x)^{1-\varepsilon\log 2}\log\log x),\\ \ell &\leq (1/\log 2 - \varepsilon)\log\log H,\end{aligned} \tag{1.2}$$

where the sum is taken over ℓ-tuples $\mathbf{a}$ with non-zero coordinates and the implied constant depends only on r and ε. Here, $\pi(x)$ denotes the number of primes not exceeding x.

Remark 1.5. Although a similar result can be obtained directly using Theorem 1.1, the latter gives a power saving in H.

Theorem 1.6. *For every sufficiently large integer N, the asymptotic formula*

$$\sum_{\substack{p_1,\dots,p_s\in\mathcal{P}_{r,k}\\ p_1+p_2+\cdots+p_s=N}} \prod_{1\le i\le s} \log p_i = \mathfrak{S}_{r,k}(N)\frac{N^{s-1}}{(s-1)!} + O\left(\frac{N^s}{\log^A N}\right) \tag{1.3}$$

holds for every $A>0$ and $s\ge 3$, where $\mathfrak{S}_{r,k}(N)$ is given by

$$\mathfrak{S}_{r,k}(N) = \sum_{q\ge 1}\sum_{\substack{a \bmod q\\ (a,q)=1}} \left(\sum_{\substack{1\le l\le q\\ (l,q)=1}} e\left(\frac{al}{q}\right) \sum_{\substack{d=1\\ (q,d^r)\mid k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])} \right)^s e\left(-\frac{aN}{q}\right). \tag{1.4}$$

The implied constant depends on s, r, k and A. Furthermore, $\mathfrak{S}_{r,k}(N) > C(s,r,k)$ for some positive absolute constant $C(s,r,k)$, provided that the parities of N and s are the same and in the case $r=2$ and k is odd, $4\nmid ks+N$ when N is odd, and $4\mid ks+N$ when N is even.

For the explicit formula of $\mathfrak{S}_{r,k}(N)$, see (2.29).

As we go through the proof of Theorem 1.6, the techniques that we develop allow us to give a uniform distribution result on the fractional parts of αp, where $p\in\mathcal{P}_{r,k}$. To do so, we first define irrational numbers that are of finite type. An irrational number α is called of *finite type* τ if

$$\tau = \sup\left\{\beta\in\mathbb{R} : \liminf_{\substack{q\to\infty\\ q\in\mathbb{N}}} q^\beta \|\alpha q\| = 0\right\} < \infty,$$

where the notation $\|x\|$ is used to denote the distance from the real number x to the nearest integer. We note here that by Dirichlet's approximation theorem one has $\tau\ge 1$. The celebrated theorems of Khinchin [9] and of Roth [19, 20] state that $\tau=1$ for almost all (in the sense of the Lebesque measure) real numbers and for all irrational algebraic numbers, respectively.

Theorem 1.7. *Given a real number x, let $\{x\}$ denote the fractional part of x. For any positive irrational number α and any real number β, the fractional parts of $\alpha p+\beta$ for $p\in\mathcal{P}_{r,k}$ are uniformly distributed in the unit interval; that is,*

$$\#\{p\le x : p\in\mathcal{P}_{r,k} \text{ and } a\le\{\alpha p+\beta\}\le b\} = (b-a)\#\mathcal{P}_{r,k}(x)(1+o(1)),$$

uniformly for $0\le a<b\le 1$.

Furthermore, if α is of finite type τ, then for every $\varepsilon>0$ and every $0\le a<b\le 1$,

$$\#\{p\le x : p\in\mathcal{P}_{r,k} \text{ and } a\le\{\alpha p+\beta\}\le b\} - (b-a)\#\mathcal{P}_{r,k}(x) \ll x^{1-\frac{r-1}{2r(\tau+1)+(r-1)\tau}+\varepsilon}.$$

The implied constant is independent of a and b.

We now give a corollary of Theorem 1.7. For any irrational number $\alpha>1$ and any real number β, the non-homogeneous Beatty sequence is defined by

$$\mathcal{B}_{\alpha,\beta} = \{n\in\mathbb{N} : n = \lfloor \alpha m+\beta\rfloor \text{ for some } m\in\mathbb{N}\backslash\{0\}\}.$$

Vinogradov showed that $\mathcal{B}_{\alpha,0}$ contains infinitely many primes (see [22, §XI]) when $\alpha > 1$ is irrational. The next corollary improves on Vinogradov's result.

Corollary 1.8. *Let $r \geq 2$. For any irrational number $\alpha > 1$, any real number β and any non-zero integer k, $\mathcal{P}_{r,k} \cap \mathcal{B}_{\alpha,\beta}$ is an infinite set.*

For $c > 1$, consider the set

$$\mathbb{N}_c = \{\lfloor m^c \rfloor : m \in \mathbb{N}\}.$$

Piatetski–Shapiro was the first to prove that there are infinitely many primes in $\mathbb{N}_c$ whenever $1 < c < 12/11$ (cf. [16]). The interested reader is invited to investigate the largest range of $c > 1$ such that $\mathcal{P}_{r,k} \cap \mathbb{N}_c$ is an infinite set.

1.1. *Preliminaries and notation*

1.1.1. *Notation*

Given a real number x, we write $e(x) = e^{2\pi i x}$, $\{x\}$ for the fractional part of x, $\lfloor x \rfloor$ for the greatest integer not exceeding x and $\lceil x \rceil$ for the smallest integer not less than x.

We recall that for functions F and real non-negative G, the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leq \alpha G$ holds for some constant $\alpha > 0$. Further, we use $F \asymp G$ to indicate that both $F \gg G$ and $F \ll G$ hold. The notations $F \ll_c G$ is use to denote the implied constant depends at most on c. In a slight departure from convention, we shall frequently use C to mean a positive number possibly different at each occurrence.

For positive integers a_1 and a_2, $[a_1, a_2]$ denotes their least common multiple and (a_1, a_2) denotes their greatest common divisor. $\tau(n)$ denotes the number of positive divisors of n, $\mu(n)$ denotes the usual Möbius function, for any positive integer k $\mu_k(n)$ is defined to be $\mu(n)$ if $(k, n) = 1$, and 0 otherwise, $\omega(n)$ denotes the number of positive divisors of n, and $\phi(n)$ denotes Euler's totient function.

We put

$$\mathcal{P}_{r,a_1,\ldots,a_k} = \{p \text{ prime} : p + a_i \text{ is positive and } r\text{-free for each } i = 1, \ldots, k\}.$$

For any set S and any real number x, $S(x)$ denotes the set of elements of S not exceeding x.

1.1.2. *Preliminaries*

We first start by defining Ramanujan sums frequently used in the proof of Theorem 1.6. For the proof of the following result, see [17, Hilfssatz 2].

Lemma 1.9. *Let a and q be positive integers such that $(a, q) = 1$ and let d be a positive divisor of q. Let k be an integer with $(k, d) = 1$. The Ramanujan sums* *defined by*

$$c_q(a,d) = \sum_{\substack{r \bmod q\\ (r,q)=1\\ d\mid r-k}} e\left(\frac{ar}{q}\right)$$

satisfy

$$c_d(a,q) = \begin{cases} \mu\left(\frac{q}{d}\right) e\left(\frac{auk}{d}\right) & \text{if } \left(d, \frac{q}{d}\right) = 1,\\ 0 & \text{else,} \end{cases} \tag{1.5}$$

where u is the solution of the congruence $\frac{q}{d}u \equiv 1 \bmod d$.

When $d = 1$, we get the classical Ramanujan sums. For the proof of the following identity, see [5, §24].

Lemma 1.10. *Let a be a non-zero integer and let q be a positive integer. Then*

$$c_q(a) = \sum_{\substack{r \bmod q\\ (r,q)=1}} e\left(\frac{ar}{q}\right) = \mu\left(q/(q,a)\right)\frac{\phi(q)}{\phi(q/(q,a))}.$$

Lemma 1.11. *For any multiplicative functions $f_1, f_2, \ldots, f_k$, the arithmetical function L_k defined by*

$$L_k(n) = \sum_{\substack{d_1\mid n,\ldots,d_k\mid n\\ [d_1,d_2,\ldots,d_k]=n}} f_1(d_1)f_2(d_2)\cdots f_k(d_k)$$

is multiplicative.

Proof. Assume that $(m,n)=1$. For any divisor d_i of nm, we write $d_i = e_i r_i$, where $e_i \mid n$ and $r_i \mid m$. Note that such e_i and d_i are unique. Therefore,

$$L_k(nm) = \sum_{\substack{e_i r_i\mid mn\\ i=1,\ldots,k\\ [e_1r_1,e_2r_2,\ldots,e_kr_k]=mn}} f_1(e_1r_1)f_2(e_2r_2)\cdots f_k(e_kr_k).$$

At this point one can prove by induction on $k \geq 2$ that for such (e_i, r_i)

$$[e_1r_1,\ldots,e_kr_k] = [e_1,\ldots,e_k][r_1,\ldots,r_k].$$

Using multiplicativity of f_i one has

$$L_k(mn) = \sum_{\substack{e_i\mid n, r_i\mid m\\ i=1,\ldots,k\\ [e_1,\ldots,e_k]=m\\ [r_1,\ldots,r_k]=n}} f_1(e_1)f_1(r_1)f_2(e_2)f_2(r_2)\cdots f_k(e_k)f_k(r_k) = L_k(m)L_k(n),$$

hence the claim follows. □

Lemma 1.12. *For any integer $\ell > 1$ and any positive integer n, define*

$$g_\ell(n) = \sum_{\substack{d_1,\ldots,d_\ell \geq 1 \\ [d_1,d_2,\ldots,d_\ell]=n}} |\mu(d_1)\mu(d_2)\cdots\mu(d_\ell)|.$$

Then for any real number $z \geq 1$ and any integer $r \geq 2$, one has the following upper bounds:

$$\sum_{n>z} \frac{g_\ell(n)}{n^r} < \frac{e \cdot (1+\log 3z)^{2^\ell - 1}}{z^{r-1}},$$

$$\sum_{n>z} \frac{g_\ell(n)}{\phi(n^r)} < \frac{e\pi^2}{3} \frac{\left(15\pi^{-2}\left(1+\log 3z\right)\right)^{2^\ell-1}}{z^{r-1}}. \tag{1.6}$$

Proof. Let l be a real number in $(0, r-1)$ to be determined. By Rankin's trick

$$\sum_{n>z} \frac{g_\ell(n)}{n^r} \leq \frac{1}{z^l} \sum_{n=1}^{\infty} \frac{g_\ell(n)}{n^{r-l}}.$$

Since $g_\ell(n) \leq \tau^\ell(n) \ll n^\varepsilon$ for fixed ℓ, the latter sum converges. By Lemma 1.11, g_ℓ is multiplicative and is supported only on square-free numbers. Furthermore, $g_\ell(p) = 2^\ell - 1$. Using the inequality $1 + nx \leq (1+x)^n$, which holds for every positive integer n and every real number $x > -1$, one has

$$\prod_p \left(1 + \frac{2^\ell - 1}{p^{r-l}}\right) \leq \prod_p \left(1 + \frac{1}{p^{r-l}}\right)^{2^\ell - 1} = \left(\frac{\zeta(r-l)}{\zeta(2(r-l))}\right)^{2^\ell-1}.$$

Since $\zeta(\sigma) < \frac{\sigma}{\sigma-1}$ for $\sigma > 1$ (cf [15, Corollary 1.14]), it follows that

$$\sum_{n>z} \frac{g_\ell(n)}{n^r} < \frac{\left(1 + (r-l-1)^{-1}\right)^{2^\ell-1}}{z^l}.$$

Choosing $l = r - 1 - (\log 3z)^{-1} > 0$, the first upper bound follows.

As for the second upper bound, we record the inequality

$$\frac{1}{\phi(n^r)} = \frac{1}{n^r} \prod_{p|n} \left(1 + \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right)^{-1} < \frac{\pi^2}{6n^r} \sum_{d|n} \frac{|\mu(d)|}{d},$$

which yields

$$\sum_{n>z} \frac{g_\ell(n)}{\phi(n^r)} < \frac{\pi^2}{6} \sum_{nm>z} \frac{g_\ell(nm)|\mu(m)|}{n^r m^{r+1}} \leq \frac{\pi^2}{6} \sum_{m=1}^{\infty} \frac{g_\ell(m)}{m^{r+1}} \sum_{n>z/m} \frac{g_\ell(n)}{n^r}.$$

Here, we have used the inequality $g_\ell(nm) \leq g_\ell(n) g_\ell(m)$ and that $|\mu(m)|$ can be ignored since g_ℓ is supported on square-free integers. By the first inequality in

(1.6), it follows that

$$\begin{aligned}\sum_{m\le z}\frac{g_\ell(m)}{m^{r+1}}\sum_{n>z/m}\frac{g_\ell(n)}{n^r} &< \frac{e\,(1+\log 3z)^{2^\ell-1}}{z^{r-1}}\sum_{m\le z}\frac{g_\ell(m)}{m^2}\\ &\le \frac{e\,(1+\log 3z)^{2^\ell-1}}{z^{r-1}}\prod_p\left(1+\frac{2^\ell-1}{p^2}\right)\\ &\le \frac{e\left(\zeta(2)\zeta(4)^{-1}\,(1+\log 3z)\right)^{2^\ell-1}}{z^{r-1}}.\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{m>z}\frac{g_\ell(m)}{m^{r+1}}\sum_{n>z/m}\frac{g_\ell(n)}{n^r} &= \prod_p\left(1+\frac{2^\ell-1}{p^r}\right)\sum_{m>z}\frac{g_\ell(m)}{m^{r+1}}\\ &< \frac{e\left(\zeta(2)\zeta(4)^{-1}\,(1+\log 3z)\right)^{2^\ell-1}}{z^{r}}.\end{aligned}$$

Combining the last two inequalities and using $\zeta(4)=\pi^4/90$, we get the desired result. □

Lemma 1.13. *Let $N\ge 2$ be a real number. Assume that $|\alpha-\frac{a}{q}|\le\frac{2}{N}$ with $(a,q)=1$, then*

$$\sum_{\substack{p\le N\\ p\equiv b \bmod d}}(\log p)e(\alpha p)\ll\left(\frac{(q,d)N}{dq^{1/2}}+\frac{q^{1/2}N^{1/2}}{(q,d)^{1/2}}+\frac{N^{4/5}}{d^{2/5}}\right)\log^3 N.$$

Proof. This result follows from [2] noting that the contribution of prime powers $p^r\le N$ with $r\ge 2$ is absorbed by the third term above. □

Lemma 1.14 (Erdős-Turán inequality [11, Chap. 2. Eq. 2.42]). *Let $\{t_1,t_2,\ldots,t_K\}$ be a set of real numbers. Suppose that $\mathcal{I}\subset[0,1)$. Then,*

$$\#\{1\le i\le K:\{t_i\}\in\mathcal{I}\}-K|\mathcal{I}|\ll\frac{K}{H}+\sum_{1\le h\le H}\frac{1}{h}\left|\sum_{i\le K}e(ht_i)\right|$$

for any $H\ge 1$. The constant in the O-term is absolute.

Lemma 1.15. *Let $x>2$ be a real number, k a non-zero integer with $1\le|k|<x-1$, f an arithmetic function such that $|f(n)|\le 1$. Then, for any positive integer Q, any integer a coprime to Q, and any real number $1<z\le(x+k)^{1/r}$, one has*

$$\sum_{\substack{p\in\mathcal{P}_{r,k}(x)\\ p\equiv a \bmod Q}}f(p)-\sum_{\substack{1\le d\le z\\ (d,k)=1}}\mu(d)\sum_{\substack{p\le x\\ p\equiv a \bmod Q\\ p\equiv -k \bmod d^r}}f(p)\ll E_k(x,z,Q)+(x+k)^{1/r}+|k|,$$

where the implied constant is absolute, the term $|k|$ can be dropped if $k > 0$ and

$$E_k(x, z, Q) := x \sum_{z<d\le(x+k)^{1/r}} \frac{|\mu(d)|}{[Q, d^r]}.$$

Proof. We start by noting that

$$\sum_{d^r|n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } r\text{-free}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\sum_{\substack{p\in\mathcal{P}_{r,k}(x) \\ p\equiv a \bmod Q}} f(p) = \sum_{1\le d\le(x+k)^{1/r}} \mu(d) \sum_{\substack{-k<p\le x \\ p\equiv a \bmod Q \\ p\equiv -k \bmod d^r}} f(p).$$

For each fixed d, the contribution of primes $p \le -k$, in case $k < -1$ and only in that case, is trivially estimated by $1 - k/d^r$, which summing over d gives the bound $(x + k)^{1/r} - k$. Then, we remove the terms with $(d, k) > 1$ using the estimate

$$\sum_{\substack{1\le d\le(x+k)^{1/r} \\ (d,k)>1}} \sum_{\substack{p\le x \\ p\equiv a \bmod Q \\ p\equiv -k \bmod d^r}} 1 \le (x + k)^{1/r}.$$

We therefore have

$$\sum_{\substack{p\in\mathcal{P}_{r,k}(x) \\ p\equiv a \bmod Q}} f(p) = \sum_{\substack{1\le d\le(x+k)^{1/r} \\ (d,k)=1}} \mu(d) \sum_{\substack{p\le x \\ p\equiv a \bmod Q \\ p\equiv -k \bmod d^r}} f(p) + O((x + k)^{1/r} + |k|).$$

By the Chinese Remainder Theorem, the contribution of the terms with $d > z$ is at most

$$\begin{aligned} \left| \sum_{\substack{z<d\le(x+k)^{1/r} \\ (d,k)=1}} \mu(d) \sum_{\substack{p\le x \\ p\equiv a \bmod Q \\ p\equiv -k \bmod d^r}} f(p) \right| &\le \sum_{\substack{z<d\le(x+k)^{1/r} \\ (d,k)=1 \\ (Q,d^r)|a+k}} |\mu(d)| \sum_{\substack{n\le x \\ n\equiv c(a,k) \bmod [Q,d^r]}} 1 \\ &< \sum_{\substack{z<d\le(x+k)^{1/r} \\ (d,k)=1 \\ (Q,d^r)|a+k}} |\mu(d)| \left(1 + \frac{x}{[Q, d^r]}\right) \\ &< (x + k)^{1/r} + E_k(x, z, Q). \end{aligned}$$

This completes the proof. □

Next, we give an analogue of the Siegel–Walfisz theorem for $\mathcal{P}_{r,k}$ which will be needed in the proof of Theorems 1.6 and 1.7. Such a result was proved in [4] for a fixed modulus. Hence, as it stands the next result is of particular interest on its own.

Lemma 1.16. *For positive integers a and Q such that $(a, Q) = 1$, define*

$$\mathfrak{E}(x; a, Q) = \sum_{\substack{p\in\mathcal{P}_{r,k}(x)\\ p\equiv a \bmod Q}} \log p - x \sum_{\substack{d\geq 1\\ (Q,d^r)|a+k\\ (d,k)=1}} \frac{\mu(d)}{\phi([Q, d^r])}.$$

Then for any fixed positive numbers A, B, the upper bound

$$\mathfrak{E}(x; a, Q) \ll \frac{x}{\log^B X} \tag{1.7}$$

holds uniformly for every $Q \leq (\log x)^A$, where the implied constant depends only on A, B, k and r.

Furthermore, assuming GRH for all Dirichlet L-functions one has

$$\mathfrak{E}(x; a, Q) \ll (x^{1/2} + x^{(r+1)/(2r)} 2^{\omega(Q)/r} Q^{-1/r^2}) \log^2 x \tag{1.8}$$

for every $Q \geq 1$, where the implied constant depends only on k and r. If there is no prime $p \nmid k$ such that $p^r \mid Q$ and $p^r \mid a + k$, then

$$\sum_{\substack{d=1\\ (Q,d^r)|a+k\\ (d,k)=1}}^{\infty} \frac{\mu(d)}{\phi([Q, d^r])} \gg \frac{1}{Q}, \tag{1.9}$$

where the implied constant depends only on r.

Remark 1.17. Note that the estimate (1.8) is nontrivial in the range $Q \ll x^{1/2-1/(2(r+1))-\varepsilon}$.

Proof. Write

$$\begin{aligned}\mathfrak{E}(x; a, Q) = &\sum_{\substack{p\in\mathcal{P}_{r,k}(x)\\ p\equiv a \bmod Q}} \log p - \sum_{\substack{1\leq d\leq z\\ (Q,d^r)|a+k\\ (d,k)=1}} \mu(d) \sum_{\substack{p\leq x\\ p\equiv a \bmod Q\\ p\equiv -k \bmod d^r}} \log p\\ &- x \sum_{\substack{d>z\\ (Q,d^r)|a+k\\ (d,k)=1}} \frac{\mu(d)}{\phi([Q, d^r])} - \sum_{\substack{1\leq d\leq z\\ (Q,d^r)|a+k\\ (d,k)=1}} \mu(d)\\ &\times \left(\frac{x}{\phi([Q, d^r])} - \sum_{\substack{p\leq x\\ p\equiv a \bmod Q\\ p\equiv -k \bmod d^r}} \log p \right).\end{aligned}$$

On choosing $f(p) = \frac{\log p}{\log x}$ in Lemma 1.15, one has

$$\sum_{\substack{p \in \mathcal{P}_{r,k}(x) \\ p \equiv a \bmod Q}} \log p - \sum_{\substack{1 \le d \le z \\ (d,k)=1}} \mu(d) \sum_{\substack{p \le x \\ p \equiv a \bmod Q \\ p \equiv -k \bmod d^r}} \log p$$

$$\ll x \log x \sum_{z < d \le (x+k)^{1/r}} \frac{|\mu(d)|}{[Q, d^r]} + x^{1/r} \log x \tag{1.10}$$

for every $1 < z \le (x+k)^{1/r}$, where the implied constant depends only on k and r. To handle the first error term on the right-hand side of (1.10), we use Rankin's trick to get

$$\sum_{z < d \le (x+k)^{1/r}} \frac{|\mu(d)|}{[Q, d^r]} \le \frac{1}{Q} \sum_{d > z} \frac{(Q, d^r)|\mu(d)|}{d^r} \le \frac{1}{Qz^\tau} \sum_{d=1}^{\infty} \frac{(Q, d^r)|\mu(d)|}{d^{r-\tau}} \tag{1.11}$$

for any τ satisfying $r - \tau > 1$. It is clear that the above sum can be expanded into the Euler product

$$\sum_{d=1}^{\infty} \frac{(Q, d^r)|\mu(d)|}{d^{r-\tau}} = \prod_p \left(1 + \frac{(Q, p^r)}{p^{r-\tau}}\right) = \prod_{p \mid Q} \left(1 + \frac{(Q, p^r)}{p^{r-\tau}}\right) \prod_{p \nmid Q} \left(1 + \frac{1}{p^{r-\tau}}\right).$$

As we did in the proof of Lemma 1.12 we have with $\tau = r - 1 - 1/\log(2z)$

$$\prod_{p \nmid Q} \left(1 + \frac{1}{p^{r-\tau}}\right) < \zeta(r - \tau) < 1 + \frac{1}{r - \tau - 1} < 1 + \log(2z),$$

where in the second inequality, we used $\zeta(\sigma) < \sigma/(\sigma - 1)$. As for the product over primes $p \mid Q$, we claim that

$$\prod_{p \mid Q} \left(1 + \frac{(Q, p^r)}{p^{r-\tau}}\right) < 2^{\omega(Q)} Q^{1 - \frac{1}{r}}.$$

This product is multiplicative in Q. Hence, it suffices to confirm this inequality for prime powers. Take $Q = p^l$ with $l < r$. Then

$$1 + \frac{(Q, p^r)}{p^{r-\tau}} < 1 + \frac{(p^l, p^r)}{p} \le 2p^{l-1} < 2p^{l - l/r}.$$

The case $r \le l$ is proved similarly. Thus, combining the inequalities above we arrive at

$$\sum_{z < d \le (x+k)^{1/r}} \frac{|\mu(d)|}{[Q, d^r]} < \frac{2^{\omega(Q)} Q^{-\frac{1}{r}} (1 + \log(2z))}{z^{\tau - 1}}.$$

The upper bound

$$\sum_{d > z} \frac{|\mu(d)|}{\phi([Q, d^r])} \ll \frac{Q^{1-1/r} 2^{\omega(Q)} (1 + \log 2z)}{\phi(Q) z^{r-1}}$$

can be proved similarly as in (1.11), thus we arrive at

$$\mathfrak{E}(x;a,Q) \ll \sum_{\substack{d\le z\\ (d,k)=1\\ (Q,d^r)\mid a+k}} \left| \sum_{\substack{p\le x\\ p\equiv a \bmod Q\\ p\equiv -k \bmod d^r}} \log p - \frac{x}{\phi([Q,d^r])} \right| + x^{1/r}\log x$$
$$+ \frac{2^{\omega(Q)} x\log^2 x}{Q^{1/r} z^{r-1}} + \frac{Q^{1-1/r}2^{\omega(Q)}\log x}{\phi(Q) z^{r-1}}$$

for every $1 < z \ll x^{1/r}$.

For the claim (1.7), we assume $Q \le \log^A x$ for some $A > 0$. Put $z = (\log x)^{\frac{B+2}{r-1}}$ for any positive number B. Then $Qd^r \le (\log x)^{A+r(B+2)/(r-1)}$, thus by the Siegel–Walfisz theorem (cf. [15, §11]) one has

$$\sum_{\substack{p\le x\\ p\equiv a \bmod Q\\ p\equiv -k \bmod d^r}} \log p - \frac{x}{\phi([Q,d^r])} \ll x\exp(-c\sqrt{\log x})$$

for some constant $c > 0$, uniformly for all $d \le z$ and $Q \le \log^A x$ provided that $(Q,d^r)\mid a+k$. Thus, we deduce the estimate

$$\mathfrak{E}(x;a,Q) \ll x\exp(-c'\sqrt{\log x})) + x^{1/r}\log x$$
$$+\frac{2^{\omega(Q)}x}{Q^{1/r}(\log x)^B} + \frac{Q^{1-1/r}2^{\omega(Q)}x}{\phi(Q)(\log x)^{B+1}}$$

for some $c' > 0$, proving (1.7) on noting that $2^{\omega(Q)} \le \tau(Q) \ll Q^{\varepsilon}$ and $\phi(Q) \gg Q/(\log\log 5Q)$ for $Q \ge 1$.

As for (1.8), we assume GRH for all Dirichlet L-functions so that the estimate

$$\sum_{\substack{p\le x\\ p\equiv a \bmod Q\\ p\equiv -k \bmod d^r}} \log p - \frac{x}{\phi([Q,d^r])} \ll x^{1/2}\log^2 x$$

holds uniformly for all positive integers Q and d such that $(Q,d^r)\mid a+k$ (see [15, Corollary 13.8]). This yields

$$\mathfrak{E}(x;a,Q) \ll zx^{1/2}\log^2 x + \frac{2^{\omega(Q)}x\log^2 x}{Q^{1/r}z^{r-1}} + \frac{Q^{1-1/r}2^{\omega(Q)}x\log x}{\phi(Q)z^{r-1}}$$

for every $1 < z \ll x^{1/r}$. For $Q \le x^{1/2}$, we use $\phi(Q) \gg Q/(\log\log 5Q)$ to eliminate the third term and then use [7, Lemma 2.4.] with $1 < z \ll_k x^{1/r}$ and this yields (1.8). For $Q > x^{1/2}$, using (1.9) and trivially estimating $\mathfrak{E}(x;a,Q)$ yield (1.8).

Finally, using the identity $\phi(a)\phi(b) = \phi((a,b))\phi([a,b])$, we see that

$$\sum_{\substack{d=1\\(Q,d^r)\,|\,a+k\\(d,k)=1}}^{\infty} \frac{\mu(d)}{\phi([Q,d^r])} = \frac{1}{\phi(Q)} \prod_{\substack{p\nmid k\\(p^r,Q)|(a+k)}} \left(1 - \frac{\phi((p^r,Q))}{p^{r-1}(p-1)}\right)$$

$$\geq \frac{1}{\phi(Q)} \prod_{p|Q}\left(1-\frac{1}{p}\right) \prod_{p\nmid Qk}\left(1-\frac{1}{p^{r-1}(p-1)}\right) \gg \frac{1}{Q}$$

provided that the sum, and hence the corresponding Euler product, is non-zero.

□

2. Proof of Main Theorems

2.1. *Proof of Theorem* 1.1

Assume that we are given $(a_1,\ldots,a_\ell) \in (\mathbb{Z}\backslash\{0\})^\ell$ with $|\mathbf{a}| < x-1$ where $|\mathbf{a}| = \max_i |a_i|$. For any set S, we put $\chi(S,x) = 1$ if $x \in S$ and 0 otherwise.

For any $1 \leq z \leq (x-|\mathbf{a}|)^{1/r}$, we shall show that

$$\pi_r(x,\mathbf{a}) - \sum_{\substack{d_1,\ldots,d_\ell\leq z\\(d_i,a_i)=1\\(d_i,d_j)^r|a_i-a_j\\1\leq i\neq j\leq \ell}} \mu(d_1)\mu(d_2)\cdots\mu(d_\ell) \sum_{\substack{p\leq x\\p\equiv -a_i \bmod {d_i}^r\\i=1,\ldots,\ell}} 1$$

$$\ll \ell z^{\ell-1}\left(|\mathbf{a}| + x^{1/r}\right) + \frac{x\ell\,(1+\log 3z)^{2^\ell-1}}{z^{r-1}}, \tag{2.1}$$

where the implied constant depends only on r. To do this, we use Lemma 1.15 iteratively and prove that for each $1 \leq l \leq \ell$

$$\pi_r(x,\mathbf{a}) - \sum_{\substack{d_1,\ldots,d_l\leq z\\(d_i,a_i)=1\\(d_i,d_j)^r|a_i-a_j\\1\leq i\neq j\leq \ell}} \mu(d_1)\mu(d_2)\cdots\mu(d_l) \sum_{\substack{p\leq x\\p\equiv -a_i \bmod {d_i}^r\\i=1,\ldots,l}} f_{l+1}(p)$$

$$\ll l z^{l-1}\left(|\mathbf{a}| + x^{1/r}\right) + \frac{xl\,(1+\log 3z)^{2^l-1}}{z^{r-1}}, \tag{2.2}$$

where the implied constant depends only on r and $f_l(p) = \chi(\mathcal{P}_{r,a_l,\ldots,a_\ell}, p)$ if $l \leq \ell$ and $f_l(p) = 1$ if $l > \ell$. For $l = 1$,

$$\pi_r(x,\mathbf{a}) = \sum_{p\leq x} \chi(\mathcal{P}_{r,a_l,\ldots,a_\ell}, p) = \sum_{p\in\mathcal{P}_{r,a_1}(x)} f_2(p).$$

Applying Lemma 1.15 and assuming $z \leq (x-|\mathbf{a}|)^{1/r}$ yield

$$\pi_r(x,\mathbf{a}) = \sum_{\substack{1\leq d_1\leq z\\(d_1,a_1)=1}} \mu(d_1) \sum_{\substack{p\leq x\\p\equiv -a_1 \bmod d_1^r}} f_2(p) + O\left(\frac{x}{z^{r-1}} + (x+a_1)^{1/r} + |a_1|\right),$$

where $f_2(p) = \chi(\mathcal{P}_{r,a_2,\dots,a_\ell}, p)$ if $\ell \geq 2$ and $f_2(p) = 1$ if $\ell = 1$. This establishes (2.2) with $l = 1$.

Assume (2.2) holds for some l. If $l = \ell$, we are done. Otherwise, assume that $l < \ell$. Then, $f_{l+1}(p) = \chi(\mathcal{P}_{r,a_{l+1},\dots,a_\ell}, p)$, and we have by the Chinese Remainder Theorem

$$\sum_{\substack{p\leq x\\ p\equiv -a_i \bmod {d_i}^r\\ i=1,\dots,l}} f_{l+1}(p) = \sum_{\substack{p\in\mathcal{P}_{r,k}(x)\\ p\equiv a \bmod Q}} f_{l+2}(p),$$

where a depends on $a_1, \dots, a_l$, $k = -a_{l+1}$ and $Q = [d_1, \dots, d_l]^r$. By Lemma 1.15

$$\pi_r(x,\mathbf{a}) - \sum_{\substack{d_1,\dots,d_{l+1}\leq z\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\leq i\neq j\leq l+1}} \mu(d_1)\mu(d_2)\cdots\mu(d_{l+1}) \sum_{\substack{p\leq x\\ p\equiv -a_i \bmod {d_i}^r\\ i=1,\dots,l+1}} f_{l+2}(p)$$

$$\ll (l+1)z^l\left(|\mathbf{a}| + x^{1/r}\right) + \frac{xl\,(1+\log 3z)^{2^l-1}}{z^{r-1}} + x\sum_{\substack{d_1,\dots,d_l\leq z\\ d_{l+1}>z}} \frac{|\mu(d_1)\cdots\mu(d_{l+1})|}{[d_1,\dots,d_{l+1}]^r}.$$

For the last term, we have

$$\sum_{\substack{d_1,\dots,d_l\leq z\\ d_{l+1}>z}} \frac{|\mu(d_1)\cdots\mu(d_{l+1})|}{[d_1,\dots,d_{l+1}]^r} \leq \sum_{d>z}\frac{g_{l+1}(d)}{d^r} < \frac{e\cdot(1+\log 3z)^{2^{l+1}-1}}{z^{r-1}}.$$

Hence, (2.2) follows for each $1 \leq l \leq \ell$, and taking $l = \ell$ yields (2.1).

We now proceed to prove the first upper bound in Theorem 1.1. Let $z = (\log x)^A$ for some $A > 0$ to be determined. Put $q = [d_1, d_2, \dots, d_\ell]$. Then, Brun–Titchmarsh inequality (cf. [14, Theorem 3.9])

$$\pi(x;q,a) \leq \frac{2x}{\phi(q)\log(x/q)} \quad (1 \leq q < x)$$

gives

$$\left|\sum_{\substack{d_1,\dots,d_\ell\leq z\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\leq i\neq j\leq \ell\\ q>z}} \mu(d_1)\cdots\mu(d_\ell) \sum_{\substack{p\leq x\\ p\equiv -a_i \bmod {d_i}^r\\ i=1,\dots,\ell}} 1\right|$$

$$\leq 2x \sum_{\substack{d_1,\dots,d_\ell\leq z\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\leq i\neq j\leq \ell\\ q>z}} \frac{|\mu(d_1)\cdots\mu(d_\ell)|}{\phi(q^r)\log(x/q^r)} \leq 2x\sum_{z<n\leq z^\ell}\frac{g_\ell(n)}{\phi(n^r)\log(x/n^r)},$$

provided that $q^r < x$, which holds since $q^r \le z^{r\ell} < x$ for sufficiently large x. Then, by (1.6) we see that the last term is

$$\ll \frac{x\left(15\pi^{-2}\left(1+\log 3z\right)\right)^{2^\ell-1}}{z^{r-1}\log x}. \tag{2.3}$$

For $q \le z$, we can apply the Siegel–Walfisz theorem (cf. [15, Corollary 11.19]) to get

$$\begin{aligned}
&\sum_{\substack{d_1,\ldots,d_\ell\le z\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\ \ 1\le i\ne j\le \ell\\ q\le z}} \mu(d_1)\cdots\mu(d_\ell) \sum_{\substack{p\le x\\ p\equiv -a_i \bmod {d_i}^r\\ i=1,\ldots,\ell}} 1\\
&= \sum_{\substack{d_1,\ldots,d_\ell\le z\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\le i\ne j\le \ell\\ q\le z}} \mu(d_1)\cdots\mu(d_\ell)\left(\frac{\mathrm{li}(x)}{\phi(q^r)} + O\left(x\exp(-C\sqrt{\log x})\right)\right)\\
&= \mathrm{li}(x)\sum_{\substack{d_1,\ldots,d_\ell\le z\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\le i\ne j\le \ell\\ q\le z}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{\phi(q^r)} + O\left(z^\ell x\exp(-C\sqrt{\log x})\right)
\end{aligned} \tag{2.4}$$

for some constant $C = C(A, r)$ that depends only on A and r. The first term can be replaced by

$$\sum_{\substack{d_1,\ldots,d_\ell\le z\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\le i\ne j\le \ell}} \mu(d_1)\mu(d_2)\cdots\mu(d_\ell)\left(\frac{\mathrm{li}(x)}{\phi(q^r)}\right)$$

with the same error in (2.3). Note that for every $\ell \ge 1$, the estimate

$$\begin{aligned}
&\left|\sum_{\substack{d_i\le z\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\le i\ne j\le \ell}} \frac{\mu(d_1)\mu(d_2)\cdots\mu(d_\ell)}{\phi([d_1,d_2,\ldots,d_\ell]^r)} - \sum_{\substack{d_i\ge 1\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\le i\ne j\le \ell}} \frac{\mu(d_1)\mu(d_2)\cdots\mu(d_\ell)}{\phi([d_1,d_2,\ldots,d_\ell]^r)}\right|\\
&\le \ell\sum_{n>z}\frac{g_\ell(n)}{\phi(n^r)} \ll \ell\frac{\left(15\pi^{-2}\left(1+\log 3z\right)\right)^{2^\ell-1}}{z^{r-1}}
\end{aligned} \tag{2.5}$$

holds, where the last inequality follows from (1.6). Hence, combining the estimates in (2.1), (2.3)–(2.5), we conclude that

$$\pi_r(x,\mathbf{a}) - \mathrm{li}(x) \sum_{\substack{d_i\geq 1\\ (d_i,a_i)=1\\ (d_i,d_j)^r \mid a_i-a_j\\ 1\leq i\neq j\leq \ell}} \frac{\mu(d_1)\mu(d_2)\cdots\mu(d_\ell)}{\phi([d_1,d_2,\ldots,d_\ell]^r)}$$

$$\ll \ell z^{\ell-1}\big(|\mathbf{a}| + x^{1/r}\big) + \frac{x\ell\,(1+\log 3z)^{2^\ell-1}}{z^{r-1}}$$

$$+ \mathrm{li}(x)\ell\frac{\left(15\pi^{-2}\,(1+\log 3z)\right)^{2^\ell-1}}{z^{r-1}} + z^\ell x\exp(-C\sqrt{\log x}).$$

Given $B>0$, we now choose $A=(B+2)/(r-1)$. Then, for sufficiently large x,

$$\pi_r(x,\mathbf{a}) - \mathrm{li}(x) \sum_{\substack{d_i\geq 1\\ (d_i,a_i)=1\\ (d_i,d_j)^r \mid a_i-a_j\\ 1\leq i\neq j\leq \ell}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{\phi([d_1,d_2,\ldots,d_\ell]^r)} \ll \ell(\log x)^{\frac{B+2}{r-1}(\ell-1)}|\mathbf{a}| + \frac{\mathrm{li}(x)}{(\log x)^B},$$

uniformly for $\ell \leq ((\log 2)^{-1}-\varepsilon)\log\log\log x$ and every sufficiently large x depending only on ε, A and r.

Note that we assumed above that $|\mathbf{a}|\leq x-z^r$. However, this condition can be removed since for larger $|\mathbf{a}|$, the claim of the theorem becomes trivial.

Now, we assume GRH. For any given modulus $f\geq 1$, we have (cf. [15, Corollary 13.8])

$$\pi(x;f,a) = \frac{\mathrm{li}(x)}{\phi(f)} + O(x^{1/2}\log x).$$

Hence, by (2.1) we obtain

$$\pi_r(x,\mathbf{a}) - \sum_{\substack{d_1,\ldots,d_\ell\leq z\\ (d_i,a_i)=1\\ (d_i,d_j)^r \mid a_i-a_j\\ 1\leq i\neq j\leq \ell}} \mu(d_1)\cdots\mu(d_\ell)\left(\frac{\mathrm{li}(x)}{\phi(q^r)} + O\left(x^{1/2}\log x\right)\right)$$

$$\ll \ell z^{\ell-1}(|\mathbf{a}| + x^{1/r}) + \frac{x\ell\,(1+\log 3z)^{2^\ell-1}}{z^{r-1}}.$$

Using (2.5) we end up with

$$\pi_r(x,\mathbf{a}) - \mathrm{li}(x) \sum_{\substack{d_i\geq 1\\ (d_i,a_i)=1\\ (d_i,d_j)^r \mid a_i-a_j\\ 1\leq i\neq j\leq \ell}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{\phi([d_1,d_2,\ldots,d_\ell]^r)}$$

$$\ll \ell z^{\ell-1}(|\mathbf{a}| + x^{1/r}) + \frac{x\ell\,(\log x)^{2^\ell-1}}{z^{r-1}}\left(1+\frac{(15/\pi^2)^{2^\ell-1}}{\log x}\right) + z^\ell x^{1/2}\log x.$$

We choose

$$z = \left(x^{1/2}\ell\,(\log x)^{2^\ell-2}\left(1+\frac{(15/\pi^2)^{2^\ell-1}}{\log x}\right)\right)^{\frac{1}{\ell+r-1}}$$

so as to balance the last two error terms above. Then, we have

$$\pi_r(x,\mathbf{a}) - \mathrm{li}(x) \sum_{\substack{d_i\ge 1\\(d_i,a_i)=1\\(d_i,d_j)^r\mid a_i-a_j\\1\le i\ne j\le \ell}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{\phi([d_1,d_2,\ldots,d_\ell]^r)}$$

$$\ll \ell\left(x^{1/2}\ell\,(\log x)^{2^\ell-2}\left(1+\frac{(15/\pi^2)^{2^\ell-1}}{\log x}\right)\right)^{\frac{\ell-1}{\ell+r-1}}(|\mathbf{a}|+x^{1/r})$$

$$+\left(\ell\,(\log x)^{2^\ell-2}\left(1+\frac{(15/\pi^2)^{2^\ell-1}}{\log x}\right)\right)^{\frac{\ell}{\ell+r-1}} x^{\frac{2\ell+r-1}{2(\ell+r-1)}}\log x. \qquad (2.6)$$

The above error terms are easily seen to be $< x/\log^2 x$ for sufficiently large x provided that

$$\ell < \frac{1}{\log 2}\log\left(\frac{1}{(\log(15\pi^{-2})+\log\log x)}\left(\frac{\log 2}{2\log\log x}\log x - \log\log x\right)\right),$$

$$|\mathbf{a}| < \frac{x}{\ell z^{\ell-1}\log^2 x} = \frac{x}{\ell\log^2 x}\left(x^{1/2}\ell\,(\log x)^{2^\ell-2}\left(1+\frac{(15/\pi^2)^{2^\ell-1}}{\log x}\right)\right)^{-\frac{\ell-1}{\ell+r-1}}. \qquad (2.7)$$

Note that with this bound on $|\mathbf{a}|$, we also have $|\mathbf{a}| \le x - z^r$, as needed.

We now show that

$$\mathfrak{S}_r(\mathbf{a}) = \prod_p \left(1-\frac{\nu_p(\mathbf{a})}{\phi(p^r)}\right) = \sum_{\substack{d_i\ge 1\\(d_i,a_i)=1\\(d_i,d_j)^r\mid a_i-a_j\\1\le i\ne j\le \ell}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{\phi([d_1,d_2,\ldots,d_\ell]^r)}. \qquad (2.8)$$

Since

$$\sum_{\substack{d_1\mid n,\ldots,d_\ell\mid n\\ [d_1,\ldots,d_\ell]=n}} 1 \le \tau(n)^\ell, \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{\tau(n)^\ell}{\phi(n^r)} < \infty,$$

we can write

$$\sum_{\substack{d_i\ge 1\\(d_i,a_i)=1\\(d_i,d_j)^r\mid a_i-a_j\\1\le i\ne j\le \ell}} \frac{\mu(d_1)\cdots\mu(d_\ell)}{\phi([d_1,d_2,\ldots,d_\ell]^r)} = \sum_{n\ge 1}\frac{H_\ell(n)}{\phi(n^r)},$$

where $H_1(n) = \mu_{a_1}(n)$ and for $\ell > 1$

$$H_\ell(n) = \sum_{\substack{d_1,\ldots,d_\ell \geq 1 \\ [d_1,\ldots,d_\ell]=n}} \mu_{a_1}(d_1)\cdots\mu_{a_\ell}(d_\ell)\rho(d_1, d_2, \ldots, d_\ell).$$

Here,

$$\rho(d_1, \ldots, d_\ell) = \prod_{1\leq i<j\leq \ell} \psi(d_i, d_j), \quad \psi(d_i, d_j) = \begin{cases} 1 & \text{if } (d_i, d_j)^r \mid a_i - a_j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mu_a(d)$ is multiplicative. Since ψ is multiplicative on each of its components, using a method similar to the one used in the proof of Lemma 1.11, one can show that $H_\ell(n)$ is multiplicative and is supported only on square-free numbers. This gives

$$\sum_{n\geq 1} \frac{H_\ell(n)}{\phi(n^r)} = \prod_p \left(1 + \frac{H_\ell(p)}{\phi(p^r)}\right).$$

It remains to determine $H_\ell(p)$ for each prime p. We prove below that $-H_\ell(p)$ gives the number of distinct a_i's modulo p^r each of which are coprime to p. We can assume that $\ell > 1$.

Since $[d_1, \ldots, d_\ell] = p$, each $d_i \in \{1, p\}$. If $d_i = p$ for some $1 \leq i \leq \ell$, then

$$\mu_{a_1}(d_1)\mu_{a_2}(d_2)\cdots\mu_{a_\ell}(d_\ell)\rho(d_1, d_2, \ldots, d_\ell) \neq 0$$

iff $(p \nmid a_i) \wedge \forall\, j, d_j = p \Rightarrow a_i \equiv a_j \bmod p^r$. Hence, if $p \mid a_i$ for some i, then we must have $d_i = 1$ and these terms have no effect on $H_\ell(p)$. This means we can ignore these indices by renaming a_i's if necessary and replacing ℓ with a smaller number ℓ' and then consider $H_{\ell'}$. Thus, we shall assume below that $p \nmid a_i$ for any i.

The sets

$$V_p(k) = \{i : 1 \leq i \leq \ell, a_i \equiv k \bmod p^r\} \quad (0 \leq k < p^r)$$

provide a partition of the set of indices $\{1, \ldots, \ell\}$, and we can consider the terms of $H_\ell(p)$ coming from each non-empty set $V_p(k)$. Now, assume $V_p(k)$ is non-empty for some k (necessarily coprime to p), and put $v = \#V_p(k)$. For each $1 \leq s \leq v$, choosing s d_i's equal to p and the rest equal to 1 gives a contribution of $(-1)^s$. Since there are $\binom{v}{s}$ such choices for each s, each non-empty $V_p(k)$ contributes

$$\sum_{s=1}^{v} \binom{v}{s} (-1)^s = (1-1)^v - 1 = -1.$$

Hence, $-H_\ell(p)$ gives the number of non-empty sets $V_p(k)$ for $0 \leq k < p^r$, as claimed above and in Theorem 1.1.

We next construct a permissible ℓ-tuple and obtain the aforementioned lower bound in Remark 1.2.

Let $\ell \geq 2$ be given. If $\ell < \phi(2^r)$, then any ℓ-tuple $\mathbf{a}$ works since $\nu_p(\mathbf{a}) \leq \ell < \phi(2^r) \leq \phi(p^r)$ for every prime $p \geq 2$. Otherwise, choose n as the largest integer satisfying

$$\phi(p_n^r) \leq \ell < \phi(p_{n+1}^r),$$

where p_n stands for the nth prime. Pick $\mathbf{a} = (k_1, k_2, \ldots, k_\ell)$, where $k_i = 1 + i(p_1 p_2 \cdots p_n)^r$ for $i = 1, \ldots, \ell$. It now follows that for such a tuple

$$1 = \nu_{p_j}(\mathbf{a}) < 2 \leq \phi(p_j^r) \quad j = 1, \ldots, n$$

and $\nu_p(\mathbf{a}) \leq \ell < \phi(p^r)$ for $p \geq p_{n+1}$.

As for the lower bound, we consider a permissible ℓ-tuple $\mathbf{a}$, where ℓ is a large number. Let p_n be the largest prime such that

$$\phi(p_n^r) \leq \ell < \phi(p_{n+1}^r).$$

We now determine the size of n in terms of ℓ. As a consequence of Chebyshev's inequalities (see [21]), $p_n \asymp n \log 2n$ for every $n \geq 1$. Actually cooperating with [18, Theorem 3] together with some numerical computation with PARI/GP, one has

$$0.3n \log(2n) \leq p_n \leq 3n \log(2n) \tag{2.9}$$

for every $n \geq 1$. Therefore,

$$\phi(p_n^r) = p_n^r - p_n^{r-1} \asymp (n \log 2n)^r$$

for every $n \geq 1$, where the implied constants depend only on r. Hence,

$$(n \log 2n)^r \asymp \phi(p_n^r) \leq \ell < \phi(p_{n+1}^r) \asymp ((n+1) \log 2(n+1))^r.$$

This yields

$$\ell \asymp p_n^r \asymp (n \log 2n)^r, \tag{2.10}$$

where the implied constants depend only on r. Let $m > n + 1$ be an integer to be determined in terms of n. Then as $m \to \infty$

$$\prod_{p \leq p_m} \left(1 - \frac{\nu_p(\mathbf{a})}{\phi(p^r)}\right) \geq \frac{1}{\phi(2^r \cdots p_m^r)}$$

$$> \frac{1}{2^r \cdots p_m^r} = e^{-r \sum_{p \leq p_m} \log p} = e^{-r p_m (1 + o(1))}, \tag{2.11}$$

where in the last step we used the prime number theorem in the form

$$\sum_{p \leq p_m} \log p = p_m (1 + o(1)), \quad (m \to \infty).$$

We next write

$$\prod_{p > p_m} \left(1 - \frac{\nu_{\mathbf{p}}(\mathbf{a})}{\phi(p^r)}\right) \geq \prod_{p > p_m} \left(1 - \frac{\ell}{\phi(p^r)}\right).$$

Let $p > p_m$. Then, since $\ell < \phi(p_{n+1}^r)$, it follows by (2.9) that

$$\ell/\phi(p^r) < (p_{n+1}/p_m)^r \le 10^r \left((n+1)\log(2n+2)/(m\log 2m)\right)^r$$

for any $m > n+1$. Choose $m = \left\lceil (3/2)^{1/r} 20 \right\rceil n$ so that

$$\left(\frac{10(n+1)\log(2n+2)}{m\log(2m)}\right)^r < \frac{2}{3}$$

holds. This choice of m yields $p_m \asymp p_n$ for every $n \ge 1$ as well.

This gives rise to the fact that for all $p > p_m$, the inequality $\ell/\phi(p^r) < 2/3$ holds. We next use the lower bound $\log(1-x) > -\frac{3\log 3}{2}x$ being valid for $0 < x < 2/3$. Therefore, we have

$$\begin{aligned}\sum_{p>p_m} \log\left(1 - \frac{\ell}{\phi(p^r)}\right) &\ge -\frac{3\log 3\ell}{2} \sum_{p>p_m} \frac{\ell}{\phi(p^r)} \\ &\ge -\frac{3\log 3}{2} \sum_{k>p_m} \frac{\ell}{k^{r-1}(k-1)} > -\frac{\ell C}{(p_m)^{r-1}},\end{aligned}$$

for some positive constant C, where, in the last inequality, we used the fact that

$$\sum_{k>K} \frac{1}{k^r} < \frac{1}{K^{r-1}},$$

which is valid for each integer $K \ge 1$. Furthermore, we can write

$$\frac{\ell}{(p_m)^{r-1}} = \frac{\ell {p_n}^{r-1}}{{p_n}^{r-1} p_m^{r-1}} \asymp \frac{\ell}{{p_n}^{r-1}} \asymp \ell^{1/r},$$

where in the last two steps we used (2.10) together with the observation that $p_m \asymp p_n$. Furthermore,

$$\prod_{p>p_m} \left(1 - \frac{\ell}{\phi(p^r)}\right) = e^{\sum_{p>p_m} \log\left(1-\frac{\ell}{\phi(p^r)}\right)} \ge e^{-C\ell^{1/r}}$$

for some $C > 0$. Also with this choice of m and (2.10), the left-hand side of (2.11) is easily seen to be $\ge e^{-C\ell^{1/r}}$ for some positive constant C not necessarily the same as above. Therefore the lower bound in (1.2) now follows.

2.2. *Proof of Theorem* 1.3

Using (2.5) and (2.8), we have for every $z, H \ge 1$

$$\sum_{|\mathbf{a}|\le H} \mathfrak{S}_r(\mathbf{a}) - \sum_{d_1,\ldots,d_\ell \le z} \frac{\mu(d_1)\mu(d_2)\cdots\mu(d_\ell)}{\phi([d_1,d_2,\ldots,d_\ell]^r)} \sum_{\substack{0<|a_i|\le H \\ (d_i,a_i)=1 \\ (d_i,d_j)^r | a_i - a_j \\ 1\le i\ne j\le \ell}} 1$$

$$\ll \frac{\ell(2H)^\ell \left(15\pi^{-2}\left(1+\log 3z\right)\right)^{2^\ell - 1}}{z^{r-1}}. \tag{2.12}$$

Recall that the first sum above runs over ℓ-tuples $\mathbf{a}$ with non-zero entries. Writing

$$\sum_{\substack{0<|a_i|\le H\\(d_i,a_i)=1\\(d_i,d_j)^r|a_i-a_j\\1\le i\ne j\le \ell}} 1 = \sum_{\substack{m_i \bmod {d_i}^r\\(m_i,d_i)=1\\(d_i,d_j)^r|m_i-m_j\\1\le i\ne j\le \ell}} \sum_{\substack{0<|a_i|\le H\\a_i\equiv m_i \bmod {d_i}^r\\i=1,\dots,\ell}} 1, \tag{2.13}$$

and using the estimate

$$\left|\sum_{\substack{n\le H\\n\equiv a \bmod d}} 1 - \frac{H}{d}\right| \le 1,$$

being valid for all $H \ge 1$ and $1 \le a \le d$, we get

$$\sum_{\substack{0<|a_i|\le H\\a_i\equiv m_i \bmod {d_i}^r\\i=1,\dots,\ell}} 1 = \prod_{i=1}^{\ell}\left(\frac{2H}{{d_i}^r} + 2\theta_i\right)$$

$$= \frac{(2H)^\ell}{(d_1 d_2\cdots d_\ell)^r} + O(2^\ell E(d_1,d_2,\dots,d_\ell)), \tag{2.14}$$

where $|\theta_i| \le 1$ and

$$E(d_1,d_2,\dots,d_\ell) = 1 + \sum_{i=1}^{\ell}\frac{H}{{d_i}^r} + \sum_{\substack{1\le i,j\le \ell\\i\ne j}}\frac{H^2}{(d_i d_j)^r}$$

$$+ \cdots + \sum_{i=1}^{\ell}\frac{H^{\ell-1}}{(d_1 d_2\cdots d_{i-1}d_{i+1}\cdots d_\ell)^r}.$$

Note here that the implied constant in the O-term in (2.14) is independent of m_i's. Inserting (2.14) into (2.13) and using the fact that

$$\sum_{\substack{m_i \bmod {d_i}^r\\(m_i,d_i)=1\\(d_i,d_j)^r|m_i-m_j\\1\le i\ne j\le \ell}} 1 = \phi([d_1,\dots,d_\ell]^r),$$

which follows by the Chinese Remainder Theorem, we obtain

$$\sum_{\substack{0<|a_i|\le H\\(d_i,a_i)=1\\(d_i,d_j)^r|a_i-a_j\\1\le i\ne j\le \ell}} 1 = \sum_{\substack{m_i \bmod {d_i}^r\\(m_i,d_i)=1\\(d_i,d_j)^r|m_i-m_j\\1\le i\ne j\le \ell}} \left(\frac{(2H)^\ell}{(d_1 d_2\cdots d_\ell)^r} + O(2^\ell E(d_1,d_2,\dots,d_\ell))\right)$$

$$= \phi([d_1,\dots,d_\ell]^r)\left(\frac{(2H)^\ell}{(d_1 d_2\cdots d_\ell)^r} + O(2^\ell E(d_1,d_2,\dots,d_\ell))\right). \tag{2.15}$$

We have

$$\sum_{d_1,\ldots,d_\ell\le z} E(d_1,\ldots,d_\ell) \le \sum_{i=0}^{\ell} z^{\ell-i}\binom{\ell}{i}H^i\zeta(r)^i - H^\ell\zeta(r)^\ell$$
$$= (H\zeta(r)+z)^\ell - (H\zeta(r))^\ell \le z\ell\,(H\zeta(r)+z)^{\ell-1}, \tag{2.16}$$

where the last inequality follows by the mean value theorem. Inserting (2.15) and (2.16) into (2.12), we conclude that

$$\sum_{|\mathbf{a}|\le H}\mathfrak{S}_r(\mathbf{a}) - (2H)^\ell\left(\sum_{d\le z}\frac{\mu(d)}{d^r}\right)^\ell \ll \ell z 2^\ell\,(H\zeta(r)+z)^{\ell-1}$$
$$+\frac{\ell(2H)^\ell\left(15\pi^{-2}\,(1+\log 3z)\right)^{2^\ell-1}}{z^{r-1}}.$$

By the mean value theorem and the inequality

$$\left|\zeta(r)^{-1} - \sum_{d\le z}\frac{\mu(d)}{d^r}\right| \le \frac{1}{(r-1)\lfloor z\rfloor^{r-1}},$$

we have

$$\left(\sum_{d\le z}\frac{\mu(d)}{d^r}\right)^\ell - \zeta(r)^{-\ell} \ll \frac{\ell\zeta(r)^{\ell-1}}{(r-1)z^{r-1}}.$$

Hence, assuming $1\le z< H/(3e)$ it follows that

$$\sum_{|\mathbf{a}|\le H}\mathfrak{S}_r(\mathbf{a}) - \left(\frac{2H}{\zeta(r)}\right)^\ell \ll z\ell 2^\ell H^{\ell-1}\,(\zeta(r)+1)^{\ell-1} + \frac{\ell(2H)^\ell\zeta(r)^{\ell-1}}{(r-1)z^{r-1}}$$
$$+\frac{\ell(2H)^\ell\left(15\pi^{-2}\,(\log H)\right)^{2^\ell-1}}{z^{r-1}}.$$

We can omit the second term on the right when H is large as it is dominated by the third term. Thus, choosing

$$z = \left(\frac{H\left(15\pi^{-2}\,(\log H)\right)^{2^\ell-1}}{(1+\zeta(r))^{\ell-1}}\right)^{1/r}$$

to balance the two remaining terms, we arrive at the claimed result in (1.1) since for $1\le\ell\le(1/\log 2-\varepsilon)\log\log H$ our assumption above that $1\le z<H/(3e)$ is satisfied for H sufficiently large.

2.3. *Proof of Theorem* 1.4

Let A be a fixed real number with $A > 1/(r-1)$. Assume that $\log^A x \le H < x$. By (2.1) we can write

$$\sum_{|\mathbf{a}|\le H} \pi_r(x,\mathbf{a}) - \sum_{p\le x}\sum_{d_1,\ldots,d_\ell\le z} \mu(d_1)\mu(d_2)\cdots\mu(d_\ell) \sum_{\substack{0<|a_i|\le H\\ a_i\equiv -p \bmod {d_i}^r\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\le i\ne j\le \ell}} 1$$

$$\ll \ell z^{\ell-1}(2H)^\ell((2H)+x^{1/r}) + \frac{x\ell(2H)^\ell\,(1+\log 3z)^{2^\ell-1}}{z^{r-1}} \qquad (2.17)$$

for every $1 \le z \le (x-H)^{1/r}$. For $p \le x$, we obtain by (2.14) and (2.16) that

$$\sum_{d_1,\ldots,d_\ell\le z} \mu(d_1)\mu(d_2)\cdots\mu(d_\ell) \sum_{\substack{0<|a_i|\le H\\ a_i\equiv -p \bmod {d_i}^r\\ (d_i,a_i)=1\\ (d_i,d_j)^r|a_i-a_j\\ 1\le i\ne j\le \ell}} 1$$

$$= \sum_{\substack{d_1,\ldots,d_\ell\le z\\ p\nmid d_i}} \mu(d_1)\mu(d_2)\cdots\mu(d_\ell) \sum_{\substack{0<|a_i|\le H\\ a_i\equiv -p \bmod {d_i}^r}} 1$$

$$= \left(2H\sum_{\substack{d\le z\\ p\nmid d}} \frac{\mu(d)}{d^r}\right)^\ell + O(z\ell 2^\ell\,(H\zeta(r)+z)^{\ell-1}). \qquad (2.18)$$

By the mean value theorem and the inequality

$$\left|\sum_{\substack{d\le z\\ p\nmid d}} \frac{\mu(d)}{d^r} - \zeta(r)^{-1}\left(1-\frac{1}{p^r}\right)^{-1}\right| \le \frac{1}{(r-1)\lfloor z\rfloor^{r-1}},$$

we obtain

$$\left(\sum_{\substack{d\le z\\ p\nmid d}} \frac{\mu(d)}{d^r}\right)^\ell = \left(1-\frac{1}{p^r}\right)^{-\ell}\zeta(r)^{-\ell} + O\left(\frac{\ell(2\zeta(r))^{\ell-1}}{z^{r-1}}\right). \qquad (2.19)$$

Therefore, combining (2.18) and (2.19) and inserting the result into (2.17) gives

$$\sum_{0<|\mathbf{a}|\le H}\pi_r(x,\mathbf{a})-\left(\frac{2H}{\zeta(r)}\right)^{\ell}\sum_{p\le x}\left(1-\frac{1}{p^r}\right)^{-\ell}$$
$$\ll \ell z^{\ell-1}(2H)^{\ell}((2H)+x^{1/r})+\frac{x\ell(2H)^{\ell}\left(1+\log 3z\right)^{2^{\ell}-1}}{z^{r-1}}$$
$$+\frac{xz\ell(2H)^{\ell-1}\left(\zeta(r)+1\right)^{\ell-1}}{\log x}+\frac{x\ell(2H)^{\ell}(2\zeta(r))^{\ell-1}}{z^{r-1}\log x}$$

for every $2\le z\le \min\{(x-H)^{1/r},H/(3e)\}$. Finally noting that

$$\sum_{p\le x}\left(1-\frac{1}{p^r}\right)^{-\ell}=\pi(x)+O(\ell 2^{\ell}),$$

where the implied constant depends only on $r\ge 2$, we conclude that

$$\sum_{0<|\mathbf{a}|\le H}\pi_r(x,\mathbf{a})-\left(\frac{2H}{\zeta(r)}\right)^{\ell}\pi(x)\ll \ell z^{\ell-1}(2H)^{\ell}\left((2H)+x^{1/r}\right)$$
$$+\frac{x\ell(2H)^{\ell}\left(\log H\right)^{2^{\ell}-1}}{z^{r-1}}+\frac{xz\ell(2H)^{\ell-1}\left(\zeta(r)+1\right)^{\ell-1}}{\log x}$$

for every $2\le z\le \min\{(x-H)^{1/r},H/(3e)\}$. We choose

$$z=\left(\frac{2H\left(\log H\right)^{2^{\ell}-1}\log x}{(1+\zeta(r))^{\ell-1}}\right)^{1/r}$$

so that the last two terms are

$$\ll\frac{x\ell(2H)^{\ell-1+\frac{1}{r}}(\log H)^{\frac{2^{\ell}-1}{r}}\left(\zeta(r)+1\right)^{\frac{(\ell-1)(r-1)}{r}}}{\log^{1-1/r}x}.$$

The proof follows after a quick check that $z\le\min\{(x-H)^{1/r},H/(3e)\}$ holds whenever (1.2) is satisfied.

2.4. *Proof of Theorem* 1.6

We shall make use of the circle method. Throughout the proof, we fix $k\in\mathbb{Z}\backslash\{0\}$ and $r\ge 2$. For $\alpha\in\mathbb{R}$, we define the following auxiliary function:

$$f_{r,k}(\alpha,N)=\sum_{p\in\mathcal{P}_{r,k}(N)}(\log p)e(\alpha p).$$

Set $R_s(N)=\#\{(p_1,p_2,\ldots,p_s)\in(\mathcal{P}_{r,k})^s : N=p_1+p_2+\cdots+p_s\}$. Then

$$R_s(N)=\int_0^1\left(f_{r,k}(\alpha,N)\right)^s e(-\alpha N)d\alpha.$$

Let $\mathcal{L} = \log^A N$, where $A > 0$ is to be determined. Then we set

$$\mathfrak{M} = \bigcup_{1 \le q \le \mathcal{L}} \bigcup_{\substack{0 < a \le q \\ (a,q)=1}} \mathfrak{M}(a, q), \tag{2.20}$$

where $\mathfrak{M}(a, q) = \left[\frac{a}{q} - \frac{\mathcal{L}}{qN}, \frac{a}{q} + \frac{\mathcal{L}}{qN}\right]$ and we define the minor arc as $\mathfrak{m} = [0, 1)\backslash\mathfrak{M}$. As the integrand is of period 1, we can think of $\mathfrak{M}(1, 1)$ as the interval $\left[-\frac{\mathcal{L}}{N}, \frac{\mathcal{L}}{N}\right]$. We first do the minor arc analysis.

Lemma 2.1. *Suppose that* $|\alpha - a/q| < 2/N$ *with* $q \ge 1$ *and* $(a, q) = 1$. *Then,*

$$f_{r,k}(\alpha, N) \ll 2^{\omega(q)} q^{-1/2} N \log^3 N + N^{(r+1)/2r} q^{(r-1)/2r} \log^{(3r-2)/r} N + \begin{cases} N^{5/6} \log^{8/3} N & \text{if } r = 2, \\ N^{4/5} \log^3 N & \text{if } r > 2. \end{cases}$$

The implied constant depends at most on k *and* r.

Proof. We prove only the case $r > 2$ since the case $r = 2$ follows similarly. We may assume that $q \le N/\log^4 N$, otherwise the above inequality is worse than the trivial estimate of $f_{r,k}(\alpha, N)$.

By Lemma 1.15 with the choices $Q = 1$ and the arithmetic function $\frac{\log p}{\log N} e(\alpha p)$, one has

$$f_{r,k}(\alpha, N) = \sum_{\substack{d \le z \\ (d,k)=1}} \mu(d) \sum_{\substack{p \le N \\ p \equiv -k \,(\mathrm{mod}\ d^r)}} (\log p) e(\alpha p) + O\left(\frac{N \log N}{z^{r-1}}\right)$$

for any $1 < z \ll_k N^{1/r}$, where the implied constant depends on k and r. By Lemma 1.13, one has

$$f_{r,k}(\alpha, N) \ll \sum_{d \le z} \mu^2(d) \left(\frac{(q, d^r)N}{d^r q^{1/2}} + \frac{q^{1/2} N^{1/2}}{(q, d^r)^{1/2}} + \frac{N^{4/5}}{d^{2r/5}}\right) \log^3 N + \frac{N \log N}{z^{r-1}}.$$

Here, the trivial upper bound

$$\sum_{d \le z} \frac{1}{(d^r, q)^{1/2}} \le \sum_{d \le z} 1 \le z$$

and the less trivial upper bound

$$\sum_{d \le z} \mu^2(d) \frac{(d^r, q)}{d^r} \le \prod_{p \le z} \left(1 + \frac{(p^r, q)}{p^r}\right) \ll \prod_{p \mid q} \left(1 + \frac{(p^r, q)}{p^r}\right) \ll 2^{\omega(q)}$$

yield

$$f_{r,k}(\alpha, N) \ll \frac{2^{\omega(q)}N}{q^{1/2}} \log^3 N + \left(q^{1/2}N^{1/2}\log^3 N\right) z + N^{4/5}\log^3 N + \frac{N\log N}{z^{r-1}}.$$

Appropriate choice of $1 \leq z \ll_k N^{1/r}$ via [7, Lemma 2.4] together with the initial assumption $q \leq N/\log^4 N$ gives the desired result. □

Take $\alpha \in \mathfrak{m}$. By Dirichlet's approximation theorem, there are co-prime integers a and q with $1 \leq a \leq q \leq N/\mathcal{L}$ such that $|\alpha - \frac{a}{q}| < \frac{\mathcal{L}}{qN}$. Since $\alpha \in \mathfrak{m}$, we must have $\mathcal{L} < q$, yielding by Lemma 2.1 and the divisor bounds $2^{\omega(q)} \ll \tau(q) \ll q^{\varepsilon}$,

$$\sup_{\alpha\in\mathfrak{m}} |f_{r,k}(\alpha, N)| \ll \frac{N}{\log^{A_r} N},$$

where for $r \geq 2$,

$$A_r = \frac{(r-1)}{2r}A - \frac{(3r-2)}{r} \tag{2.21}$$

if $A > 4$ and N is large. It follows now by a standard argument

$$\int_{\mathfrak{m}} (f_{r,k}(\alpha, N))^s e(-\alpha N) \ll \frac{N^{s-1}}{(\log N)^{(s-2)A_r}}. \tag{2.22}$$

This estimate is nontrivial if $(s-2)A_r > 0$ and this forces

$$A > 6 + 2/(r-1) > 8.$$

Lemma 2.2. *Let $\alpha \in \mathfrak{M}(a, q)$ and let $\beta = \alpha - \frac{a}{q}$. Then,*

$$f_{r,k}(\alpha, N) - \sum_{\substack{l \bmod q\\(l,q)=1}} e\left(\frac{al}{q}\right) \sum_{\substack{d=1\\(q,d^r)|k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])} \sum_{n\leq N} e(\beta n) \ll \frac{N}{\log^B N} \tag{2.23}$$

for any fixed $B > 0$, where the implied constant depends only on A, B, k and r. Furthermore,

$$\sum_{\substack{l \,(\bmod q)\\(l,q)=1}} e\left(\frac{al}{q}\right) \sum_{\substack{d=1\\(q,d^r)|k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])} \ll \frac{2^{\omega(q)}}{\phi(q)}, \tag{2.24}$$

where the constant in (2.24) *is independent of k.*

Proof of Lemma 2.2. To prove (2.24), we swap the summations and use (1.5) to get

$$\sum_{\substack{1\leq l\leq q\\(l,q)=1\\(q,d^r)|k+l}} e\left(\frac{al}{q}\right) \ll 1,$$

which then yields

$$\sum_{\substack{l\ (\mathrm{mod}\ q)\\(l,q)=1}} e\left(\frac{al}{q}\right) \sum_{\substack{d=1\\(q,d^r)|k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])}$$

$$\ll \frac{1}{\phi(q)} \prod_{p|q}\left(1+\frac{\phi((q,p^r))}{\phi(p^r)}\right)\prod_{p\nmid q}\left(1+\frac{1}{\phi(p^r)}\right) \ll \frac{2^{\omega(q)}}{\phi(q)}.$$

Next, we write

$$f_{r,k}(\alpha,N) = \sum_{\substack{1\le l\le q\\(l,q)=1}} \sum_{\substack{p\in\mathcal{P}_{r,k}(N)\\p\equiv l \bmod q}} e(\alpha p)\log p + O\left(\sum_{p|q}\log p\right)$$

$$= \sum_{\substack{1\le l\le q\\(l,q)=1}} e(al/q) \sum_{\substack{p\in\mathcal{P}_{r,k}(N)\\p\equiv l \bmod q}} e(\beta p)\log p + O(\omega(q)\log q).$$

Given any $B' > A$, by Lemma 1.16 and partial integration we obtain

$$\begin{aligned}\sum_{\substack{p\in\mathcal{P}_{r,k}(N)\\p\equiv l \bmod q}} e(\beta p)\log p = &\sum_{\substack{d\ge 1\\(q,d^r)|l+k}} \frac{\mu_k(d)}{\phi([q,d^r])}\int_{\sqrt{N}}^{N} e(\beta x)dx\\ &+ \int_{\sqrt{N}}^{N} e(\beta x)d\mathfrak{E}(x;l,q) + O(\sqrt{N})\\ = &\sum_{\substack{d\ge 1\\(q,d^r)|l+k}} \frac{\mu_k(d)}{\phi([q,d^r])}\left(\sum_{n\le N} e(\beta n) + O(\mathcal{L}/q+\sqrt{N})\right)\\ &+ O\left(\frac{N\mathcal{L}}{q\log^{B'} N}\right)\end{aligned}$$

for sufficiently large N, since

$$\sum_{1\le n\le N} e(\beta n) = \int_{\sqrt{N}}^{N} e(\beta x)dx + 2\pi i\beta\int_0^N e(\beta x)\{x\}dx + O(\sqrt{N}).$$

Hence, choosing $B' = B + A$ gives (2.23). □

We next establish the asymptotic formula in (1.3) briefly. We write

$$R_s(N) = \int_0^1 (f_{r,k}(\alpha,N))^s e(-\alpha N)d\alpha = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right)(f_{r,k}(\alpha,N))^s e(-\alpha N)d\alpha,$$

where $\mathfrak{M}$ and $\mathfrak{m}$ are defined in (2.20). Since we have (2.22), we shall look at only the major arc analysis; that is, we shall determine the asymptotic behavior of

$$\int_{\mathfrak{M}} (f_{r,k}(\alpha,N))^s d\alpha = \sum_{q\le\mathcal{L}} \sum_{\substack{a \bmod q\\(a,q)=1}} \int_{\mathfrak{M}(a,q)} (f_{r,k}(\alpha,N))^s e(-\alpha N)d\alpha.$$

By Lemma 2.2 together with the upper bound in (2.24), it follows that for $\alpha \in \mathfrak{M}(a,q)$

$$\begin{aligned}(f_{r,k}(\alpha,N))^s &= \left(\sum_{\substack{l(\bmod q)\\(l,q)=1}} e\left(\frac{al}{q}\right) \sum_{\substack{d=1\\(q,d^r)|k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])} \sum_{n\le N} e(\beta n) + O\left(\frac{N}{\log^B N}\right)\right)^s \\ &= \left(\sum_{\substack{l(\bmod q)\\(l,q)=1}} e\left(\frac{al}{q}\right) \sum_{\substack{d=1\\(q,d^r)|k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])} \sum_{n\le N} e(\beta n)\right)^s + O\left(\frac{N^s}{\log^B N}\right).\end{aligned}$$

This, after an obvious change of variable, yields

$$\begin{aligned}&\sum_{q\le\mathcal{L}} \sum_{\substack{a \bmod q\\(a,q)=1}} \int_{\mathfrak{M}(a,q)} (f_{r,k}(\alpha,N))^s e(-\alpha N) d\alpha \\ &\quad= \sum_{q\le\mathcal{L}} \sum_{\substack{a \bmod q\\(a,q)=1}} \left(\sum_{\substack{l \bmod q\\(l,q)=1}} e\left(\frac{al}{q}\right) \sum_{\substack{d=1\\(q,d^r)|k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])}\right)^s e\left(-\frac{aN}{q}\right) \mathcal{J}(N,\mathcal{L}/qN) \\ &\quad+ O\left(\frac{N^{s-1}}{\log^{B-2A} N}\right), \qquad (2.25)\end{aligned}$$

where

$$\mathcal{J}(N,t) = \int_{-t}^{t} \left(\sum_{n\le N} e(\beta n)\right)^s e(-\beta N) d\beta$$

for any $0 < t \le 1/2$. Recalling the simple upper bound

$$\sum_{n\le x} e(\beta n) \ll \min\left\{x, \frac{1}{||\beta||}\right\},$$

it follows that

$$\int_{-1/2}^{-t} \left(\sum_{n\le N} e(\beta n)\right)^s e(-\beta N) d\beta + \int_{t}^{1/2} \left(\sum_{n\le N} e(\beta n)\right)^s e(-\beta N) d\beta \ll \frac{1}{t^{s-1}}.$$

Therefore,

$$\mathcal{J}(N,\mathcal{L}/qN) - \mathcal{J}(N,1/2) \ll \frac{q^{s-1}N^{s-1}}{\mathcal{L}^{s-1}}.$$

Replacing $\mathcal{J}(N,\mathcal{L}/qN)$ by $\mathcal{J}(N,1/2)$ in (2.25) using the upper bound (2.24) introduces an error

$$\begin{aligned} &\ll \frac{N^{s-1}}{\mathcal{L}^{s-1}}\sum_{q\le\mathcal{L}}\frac{2^{s\omega(q)}q^{s-1}}{\phi^{s-1}(q)} \ll \frac{N^{s-1}}{\mathcal{L}^{s-1}}\sum_{q\le\mathcal{L}}2^{s\omega(q)}\left(\prod_{p\mid q}\frac{p}{p-1}\right)^{s-1}\\ &\ll \frac{N^{s-1}}{\mathcal{L}^{s-1}}\sum_{q\le\mathcal{L}}\tau(q)^{2s-1} \ll \frac{N^{s-1}}{\mathcal{L}^{s-1}}\mathcal{L}(\log\mathcal{L})^{2^{2s-1}-1}, \end{aligned}$$

where we used

$$\sum_{n\le x}\tau(n)^k \ll x(\log x)^{2^k-1}.$$

Therefore, combining (2.25) and this estimate we derive

$$\begin{aligned} &\int_{\mathfrak{M}}(f_{r,k}(\alpha,N))^s\,d\alpha - \mathfrak{S}_{r,k}(N,\mathcal{L})\mathcal{J}(N,1/2)\\ &\qquad \ll \frac{N^{s-1}}{\log^{B-2A}N} + \frac{N^{s-1}}{\mathcal{L}^{s-2}}(\log\mathcal{L})^{2^{2s-1}-1}, \end{aligned}$$

where

$$\mathfrak{S}_{r,k}(N,\mathcal{L}) = \sum_{q\le\mathcal{L}}\sum_{\substack{a \bmod q\\ (a,q)=1}}\left(\sum_{\substack{l \bmod q\\ (l,q)=1}} e\left(\frac{al}{q}\right)\sum_{\substack{d=1\\ (q,d^r)\mid k+l}}^{\infty}\frac{\mu_k(d)}{\phi([q,d^r])}\right)^s e\left(-\frac{aN}{q}\right).$$

Now, using the upper bound in (2.24) and Rankin's trick, one has

$$\mathfrak{S}_{r,k}(N,\mathcal{L}) - \mathfrak{S}_{r,k}(N) \ll \sum_{q>\mathcal{L}}\frac{2^{s\omega(q)}}{\phi^{s-1}(q)} < \frac{1}{\mathcal{L}^{s'}}\sum_{q=1}^{\infty}\frac{2^{s\omega(q)}q^{s'}}{\phi^{s-1}(q)}$$

for any $s' < s-2$, where the singular series $\mathfrak{S}_{r,k}(N)$ is given by (1.4).

Using $\phi(q) = q\prod_{p\mid q}(1-1/p)$, we see that

$$\begin{aligned} \sum_{q=1}^{\infty}\frac{2^{s\omega(q)}q^{s'}}{\phi^{s-1}(q)} &\le \sum_{q=1}^{\infty}\frac{2^{(2s-1)\omega(q)}}{q^{s-s'-1}} < \prod_p\left(1+\frac{4^s}{p^{s-s'-1}}\right)\\ &< \zeta(s-s'-1)^{4^s} < \left(1+\frac{1}{s-s'-2}\right)^{4^s}. \end{aligned}$$

Choosing $s' = s-2-\frac{1}{\log\mathcal{L}}$, it follows that

$$\mathfrak{S}_{r,k}(N,\mathcal{L}) - \mathfrak{S}_{r,k}(N) \ll \frac{(\log\mathcal{L})^{4^s}}{\mathcal{L}^{s-2}}.$$

Since $\mathcal{J}(N,1/2)$ counts the number of ways to express N as the sum of s positive integers, we have

$$\mathcal{J}(N,1/2) = \frac{(N-1)!}{(N-s)!(s-1)!} = \frac{N^{s-1}}{(s-1)!} + O(N^{s-2}),$$

where the constant in the error term certainly depends on s. Finally, noting

$$\mathfrak{S}_{r,k}(N) \ll \sum_{q=1}^{\infty} \frac{2^{s\omega(q)}}{\phi^{s-1}(q)} \ll 1,$$

and using (2.22) we arrive at

$$\begin{aligned} R_s(N) - \frac{\mathfrak{S}_{r,k}(N)N^{s-1}}{(s-1)!} &\ll N^{s-2} + \frac{N^{s-1}(\log\mathcal{L})^{4^s}}{\mathcal{L}^{s-2}} \\ &+ \frac{N^{s-1}}{\mathcal{L}^{s-2}}(\log\mathcal{L})^{2^{2s-1}-1} + \frac{N^{s-1}}{\log^{B-2A} N} + \frac{N^{s-1}}{(\log N)^{(s-2)A_r}}, \end{aligned}$$

where A_r is defined in (2.21). This proves (1.3) in Theorem 1.6. We next prove the rest of Theorem 1.6.

Proposition 2.3. *As a function of q, the following function is multiplicative in q:*

$$F(q,l) = \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-1} \sum_{\substack{d=1 \\ (q,d^r)\mid k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])}; \tag{2.26}$$

that is, for every integer l and non-zero integer k, $F(q_1q_2,l) = F(q_1,l)F(q_2,l)$ whenever $(q_1,q_2)=1$.

Proof of Proposition 2.3. We have

$$\begin{aligned} F(q,l) &= \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-1} \sum_{\substack{d=1 \\ (q,d^r)\mid k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])} \\ &= \frac{1}{\phi(q)} \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-1} \prod_p \left(1 + \frac{\mu_k(p)\theta(q,p^r)\phi(q,p^r)}{\phi(p^r)}\right), \end{aligned}$$

where $\theta(q,d)=1$ if $(q,d) \mid k+l$ and 0 otherwise. Hence,

$$F(q,l) = \frac{1}{\phi(q)} \prod_{p\mid q} \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-1} \prod_{p\mid q} \left(1 + \frac{\mu_k(p)\theta(q,p^r)\phi(q,p^r)}{\phi(p^r)}\right).$$

Since θ is multiplicative in q and d, the result follows. ∎

Lemma 2.4 (Non-vanishing of the Singular Series). *For any non-zero integer k and any integer $s \geq 3$, the singular series $\mathfrak{S}_{r,k}(N)$ given by (1.4) satisfies $\mathfrak{S}_{r,k}(N) > C$ for some $C > 0$ independent of N, provided that the parities of N*

and s are the same and in the case k is odd and $r = 2$, one also needs $4 \nmid ks + N$ when N is odd, and $4 \mid ks + N$ when N is even.

Proof. Define

$$b_{r,N}(q) = \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-s} \sum_{\substack{0<a\le q \\ (a,q)=1}} \left(\sum_{\substack{1\le l\le q \\ (l,q)=1}} e\left(\frac{al}{q}\right) \sum_{\substack{d=1 \\ (q,d^r)\mid k+l}}^{\infty} \frac{\mu_k(d)}{\phi([q,d^r])} \right)^s e \times \left(\frac{-aN}{q}\right).$$

We shall show that, for fixed r and N, b is a multiplicative function of q, and then investigate properties of b at prime powers. First, we write

$$b_{r,N}(q) = \sum_{\substack{0<a\le q \\ (a,q)=1}} \left(\sum_{\substack{0<l\le q \\ (l,q)=1}} e\left(\frac{al}{q}\right) F(q,l) \right)^s e\left(\frac{-aN}{q}\right), \tag{2.27}$$

where $F(q, l)$ is given by (2.26). Assume that $q = q_1 q_2$, where $(q_1, q_2) = 1$. Then,

$$b_{r,N}(q_1q_2) = \sum_{\substack{0<a\le q_1q_2 \\ (a,q_1q_2)=1}} \left(\sum_{\substack{0<l\le q_1q_2 \\ (l,q_1q_2)=1}} e\left(\frac{al}{q_1q_2}\right) F(q_1q_2,l) \right)^s e\left(\frac{-aN}{q_1q_2}\right).$$

Given $l \bmod q_1q_2$ and $(l, q_1q_2) = 1$, there are unique $l_1 \bmod q_1$ and $l_2 \bmod q_2$ with $(l_1, q_1) = 1$ and $(l_2, q_2) = 1$ such that

$$l \equiv l_1q_2 + l_2q_1 \mod q_1q_2.$$

Therefore, by Proposition 2.3

$$\begin{aligned}
\sum_{\substack{0<l\le q_1q_2 \\ (l,q_1q_2)=1}} e\left(\frac{al}{q_1q_2}\right) F(q_1q_2,l) &= \sum_{\substack{0<l_1\le q_1 \\ (l_1,q_1)=1}} \sum_{\substack{0<l_2\le q_2 \\ (l_2,q_2)=1}} e\left(\frac{a(l_1q_2+l_2q_1)}{q_1q_2}\right) \\
&\quad \times F(q_1,l_1q_2)F(q_2,l_2q_1) \\
&= \sum_{\substack{0<l_1\le q_1 \\ (l_1,q_1)=1}} e\left(\frac{al_1q_2}{q_1}\right) F(q_1,l_1q_2) \sum_{\substack{0<l_2\le q_2 \\ (l_2,q_2)=1}} e \\
&\quad \times \left(\frac{al_2q_1}{q_2}\right) F(q_2,l_2q_1).
\end{aligned}$$

Similarly, picking $a_i \bmod q_i$ with $(a_i, q_i) = 1$ such that $a \equiv a_1 q_2 + a_2 q_1 \bmod q_1 q_2$, and replacing $l_1 q_2$ by l_1 and $l_2 q_1$ by l_2, it follows that

$$\begin{aligned} b_{r,N}(q_1 q_2) &= \sum_{\substack{0<a_1\le q_1\\(a_1,q_1)=1}} \left(\sum_{\substack{0<l_1\le q_1\\(l_1,q_1)=1}} e\left(\frac{a_1 l_1}{q_1}\right) F(q_1, l_1) \right)^s e\left(\frac{-a_1 N}{q_1}\right) \\ &\quad \cdot \sum_{\substack{0<a_2\le q_2\\(a_2,q_2)=1}} \left(\sum_{\substack{0<l_2\le q_2\\(l_2,q_2)=1}} e\left(\frac{a_2 l_2}{q_2}\right) F(q_2, l_2) \right)^s e\left(\frac{-a_2 N}{q_2}\right) \\ &= b_{r,N}(q_1) b_{r,N}(q_2). \end{aligned}$$

This proves the desired multiplicativity of $b_{r,N}$. We shall now investigate $b_{r,N}$ at prime powers. We write

$$\begin{aligned} F(p^m, l) &= \prod_{q \text{ prime}} \left(1 + \frac{\mu_k(q)}{\phi(q^r)}\right)^{-1} \sum_{\substack{d=1\\(p^m, d^r)\mid k+l}}^{\infty} \frac{\mu_k(d)}{\phi([p^m, d^r])} \\ &= \frac{1}{\phi(p^m)} \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-1} \left(1 + \frac{\mu_k(p)\phi(p^m, p^r)\rho((p^m, p^r), l)}{\phi(p^r)}\right), \end{aligned} \tag{2.28}$$

where $\rho(v, l) = 1$ if $v \mid (l + k)$ and equals 0 otherwise. Note that for $p \mid k$, we have $F(p^m, l) = \frac{1}{\phi(p^m)}$, in which case Lemma 1.10 gives

$$b_{r,N}(p^m) = \frac{1}{\phi^s(p^m)} \sum_{\substack{0<a\le p^m\\(a,p)=1}} \left(\sum_{\substack{0<l\le p^m\\(l,p)=1}} e\left(\frac{al}{p^m}\right) \right)^s e\left(\frac{-aN}{p^m}\right) = \frac{\mu^s(p^m) c_{p^m}(N)}{\phi^s(p^m)}.$$

By absolute convergence we can therefore write

$$\begin{aligned} \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-s} \mathfrak{S}_{r,k}(N) &= \sum_{q\ge 1} b_{r,N}(q) \\ &= \prod_{p\mid k} \left(1 + \frac{(-1)^s c_p(N)}{\phi^s(p)}\right) \prod_{p\nmid k} (1 + b_{r,N}(p) + b_{r,N}(p^2) \\ &\quad + \cdots). \end{aligned}$$

For any $m \geq 1$ and $p \nmid k$, we obtain using (2.27) and (2.28) that

$$\begin{aligned} b_{r,N}(p^m) &= \frac{1}{\phi^s(p^m)}\left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \\ &\quad \times \sum_{\substack{0<a\leq p^m \\ (a,p)=1}} \left(\sum_{\substack{0<l\leq p^m \\ (l,p)=1}} e\left(\frac{al}{p^m}\right)\left(1 - \frac{p^{\min(m,r)}\rho(p^{\min(m,r)}, l)}{p^r}\right)\right)^s e \\ &\quad \times \left(\frac{-aN}{p^m}\right). \end{aligned}$$

For $m \geq 2$,

$$\sum_{\substack{0<l\leq p^m \\ (l,p)=1}} e\left(\frac{al}{p^m}\right) = \mu(p^m) = 0.$$

Therefore, for $2 \leq m \leq r$ and $p \nmid k$,

$$\begin{aligned} b_{r,N}(p^m) = \frac{(-1)^s}{p^{s(r-m)}\phi^s(p^m)}\left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \\ \sum_{\substack{0<a\leq p^m \\ (a,p)=1}} \left(\sum_{\substack{0<l\leq p^m \\ (l,p)=1 \\ l\equiv -k \bmod p^m}} e\left(\frac{al}{p^m}\right)\right)^s e\left(\frac{-aN}{p^m}\right). \end{aligned}$$

Note that since the sum over l reduces to one term only and $p^{s(r-m)}\phi^s(p^m) = \phi^s(p^r)$, we get

$$\begin{aligned} b_{r,N}(p^m) &= \frac{(-1)^s}{\phi^s(p^r)}\left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \sum_{\substack{0<a\leq p^m \\ (a,p)=1}} e\left(\frac{-aks - aN}{p^m}\right) \\ &= \frac{(-1)^s}{\phi^s(p^r)}\left(1 - \frac{1}{\phi(p^r)}\right)^{-s} c_{p^m}(ks + N). \end{aligned}$$

If $m > r$, then by (1.5)

$$\sum_{\substack{0<l\leq p^m \\ (l,p)=1 \\ l\equiv -k \bmod p^r}} e\left(\frac{al}{p^m}\right) = 0$$

since $(p^{m-r}, p^r) \neq 1$. Hence, we get $b_{r,N}(p^m) = 0$ when $m > r$ and $p \nmid k$. It remains to treat the case $m = 1$ and $p \nmid k$. This time we have

$$b_{r,N}(p) = \frac{1}{\phi^s(p)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \sum_{\substack{0<a\leq p \\ (a,p)=1}} \left(\sum_{\substack{0<l\leq p \\ (l,p)=1}} e\left(\frac{al}{p}\right) \left(1 - \frac{\rho(p,l)}{p^{r-1}}\right)\right)^s e\left(\frac{-aN}{p}\right).$$

Here,

$$\sum_{\substack{0<l\leq p \\ (l,p)=1}} e\left(\frac{al}{p}\right) \left(1 - \frac{\rho(p,l)}{p^{r-1}}\right) = -1 - \frac{1}{p^{r-1}} \sum_{\substack{0<l\leq p \\ (l,p)=1 \\ l\equiv -k \bmod p}} e\left(\frac{al}{p}\right)$$

$$= -1 - \frac{1}{p^{r-1}} e\left(\frac{-ak}{p}\right).$$

It follows by the binomial theorem that

$$b_{r,N}(p) = \frac{(-1)^s}{\phi^s(p)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \sum_{\substack{0<a\leq p \\ (a,p)=1}} \sum_{i=0}^{s} \binom{s}{i} \frac{1}{p^{i(r-1)}} e\left(\frac{-aki}{p}\right) e$$
$$\times \left(\frac{-aN}{p}\right).$$

This yields

$$b_{r,N}(p) = \frac{(-1)^s}{\phi^s(p)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \sum_{i=0}^{s} \binom{s}{i} \frac{1}{p^{i(r-1)}} \sum_{\substack{0<a\leq p \\ (a,p)=1}} e\left(\frac{-a(ki+N)}{p}\right)$$

$$= \frac{(-1)^s}{\phi^s(p)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \sum_{i=0}^{s} \binom{s}{i} \frac{1}{p^{i(r-1)}} c_p(ki+N).$$

To wrap up, one has

$$\mathfrak{S}_{r,k}(N) = \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^s \prod_{p\mid k} \left(1 + \frac{(-1)^s c_p(N)}{\phi^s(p)}\right)$$
$$\times \prod_{p\nmid k} \left(1 + \frac{(-1)^s}{\phi^s(p)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \left(\sum_{i=0}^{s} \binom{s}{i} \frac{1}{p^{i(r-1)}} c_p(ki+N) \right.\right.$$
$$\left.\left. + \frac{1}{p^{s(r-1)}} (c_{p^2}(ks+N) + \cdots + c_{p^r}(ks+N)) \right)\right). \quad (2.29)$$

One important observation here is

$$\lim_{r\to\infty} \mathfrak{S}_{r,k}(N) = \prod_p \left(1 + \frac{(-1)^s c_p(N)}{\phi^s(p)}\right).$$

From [5, §26], we have the identity

$$\sum_{d|q} c_d(N) = \begin{cases} q & \text{if } q \mid N, \\ 0 & \text{if } q \nmid N. \end{cases}$$

Therefore, we have

$$c_{p^2}(ks+N) + \cdots + c_{p^r}(ks+N)) = p^r \rho(p^r, k(s-1)+N) - c_p(ks+N) - 1.$$

For $p > 2$, we note that

$$\left| 1 + \frac{(-1)^s c_p(N)}{\phi^s(p)} \right| \geq 1 - (p-1)^{1-s} \geq 3/4.$$

Therefore the factors in (2.29) corresponding to odd $p \mid k$ do not vanish. On the other hand, for $p = 2$, this factor vanishes if and only if s and N have different parities.

Assume now that $p \nmid k$. Since $c_p(ik+N) \leq p-1$ for all i, the expression

$$\frac{(-1)^s}{\phi^s(p^r)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \left(p^{s(r-1)} \sum_{i=0}^{s} \binom{s}{i} \frac{1}{p^{i(r-1)}} c_p(ki+N) + (c_{p^2}(ks+N) + \cdots + c_{p^r}(ks+N)) \right) \tag{2.30}$$

has absolute value

$$\leq h(p,r,s) := \frac{(p-1)(1+p^{r-1})^s + p^r - p}{(p^r - p^{r-1} - 1)^s}.$$

Since $s \geq 3$ and $r \geq 2$ are fixed, $h(p,r,s) \ll p^{-s+1}$ for every large p. Hence, the factors in $\mathfrak{S}_{r,k}(N)$ corresponding to large $p \nmid k$ do not vanish. Therefore, it is enough to show that $h(p,r,s) < 1$ for every $s \geq 3, r \geq 2$ and small primes p.

When $p \geq 3$ and $r \geq 2$ are fixed, then h is decreasing as a function of s, since for $p \geq 3$ and $r \geq 2$

$$\frac{p^{r-1}+1}{p^r - p^{r-1} - 1} < 1.$$

Hence, $h(p,r,s) \leq h(p,r,3)$. Define

$$f(r) = \frac{1 + p^{r-1}}{p^r - p^{r-1} - 1}, \quad \text{and} \quad g(r) = \frac{p^r - p}{(p^r - p^{r-1} - 1)^3}.$$

Since

$$f'(r) = -\frac{p^r \log p}{(p^r - p^{r-1} - 1)^2} < 0,$$

$$g'(r) = -\frac{p^r \log p (2p^r - 2p^{r-1} - 3p + 4)}{(p^r - p^{r-1} - 1)^4} < 0$$

for $r \geq 2$ and each fixed $p \geq 2$, we conclude that $h(p,r,3)$ is decreasing as a function of r. For $r = 2$, we see that

$$h(p,2,3) = \frac{p^4 + 2p^3 + p^2 - 3p - 1}{p^6 - 3p^5 + 5p^3 - 3p - 1} < 1$$

for all $p > 3$. This shows $h(p,r,s) < h(p,2,3) < 1$ for all $r \geq 2, s \geq 3$, and $p > 3$. Furthermore, we note that $h(3,3,3) < 1$. Thus, $h(3,r,s) < 1$ for all $r,s \geq 3$.

It remains to investigate (2.30) in the following cases:

(1) $p, s = 3, r = 2$ and $3 \nmid k$,
(2) $p = 2, s \geq 3, r \geq 2$ and $2 \nmid k$.

In the first case, (2.30) becomes

$$-\frac{1}{5^3}\left(\sum_{i=0}^{2}\binom{3}{i}3^{3-i}c_3(ki+N) + c_9(3k+N) + c_3(3k+N)\right).$$

An easy computation reveals that the maximum value of this expression has absolute value $37/125$ for $1 \leq k \leq 9$ with $(k,9) = 1$ and $1 \leq N \leq 9$.

We can now work on the case $p = 2$. Assume now $2 \nmid k$ and N is odd. In this case, $c_2(ki+N) = (-1)^{i+1}$. Therefore.

$$\sum_{i=0}^{s}\binom{s}{i}\frac{1}{2^{i(r-1)}}c_2(ki+N) = -\left(1 - \frac{1}{2^{r-1}}\right)^s.$$

Inserting this in (2.30), we get

$$\begin{aligned}
&= \frac{(-1)^s}{(2^{r-1}-1)^s}\left(-2^{s(r-1)}\left(1 - \frac{1}{2^{r-1}}\right)^s + 2^r\rho(2^r, k(s-1)+N) + (-1)^s - 1\right)\\
&= (-1)^s\left(-1 + \frac{2^r\rho(2^r, k(s-1)+N) + (-1)^s - 1}{(2^{r-1}-1)^s}\right).
\end{aligned}$$

This expression equals -1, and hence the corresponding local factor in (2.29) vanishes, only if s is even, or if s is odd, and $r = 2$ and $2^r \mid (ks+N)$.

If N is even, $c_2(ki+N) = (-1)^i$. This time (2.30) becomes

$$(-1)^s\left(1 + \frac{2^r\rho(2^r, k(s-1)+N) + (-1)^{s+1} - 1}{(2^{r-1}-1)^s}\right).$$

Therefore, the corresponding factor in (2.29) vanishes only if s is odd or if s is even, and $r = 2$ and $2^r \nmid (ks+N)$. This completes the proof. □

2.5. *Proof of Theorem* 1.7

We borrow the notation $f_{r,k}(\alpha, N)$ from the proof of Theorem 1.6. Let $\alpha > 0$ be an irrational number. For the first claim, it will be sufficient to show that

$$f_{r,k}(\alpha, N) = o(N), \tag{2.31}$$

as this will imply by partial summation that

$$\sum_{p \in \mathcal{P}_{r,k}(N)} e(h(\alpha p + \beta)) = o(|\mathcal{P}_{r,k}(N)|) \tag{2.32}$$

for any fixed non-zero integer h and any real number β. This shows by Weyl's criterion (see [11, Chap. 1, Theorem 2.1]) that the fractional parts $\{\alpha p + \beta\}$ with $p \in \mathcal{P}_{r,k}$ are uniformly distributed modulo 1.

To prove (2.31), we assume, shifting by an appropriate integer if needed, that $\alpha \in (0,1)$. Let A be a sufficiently large number. By Dirichlet's approximation theorem, we find a reduced rational number a/q satisfying

$$\left|\alpha - \frac{a}{q}\right| < \frac{\log^A N}{qN}$$

with $1 \le q \le N/\log^A N$. Note that $q \to \infty$ as $N \to \infty$ since α is irrational. If $q \ge \log^A N$, then the result follows by Lemma 2.1. Otherwise, by Lemma 2.2 it follows that

$$f_{r,k}(\alpha, N) \ll \frac{N 2^{\omega(q)}}{\phi(q)} + \frac{N}{\log N} \ll \frac{N 2^{\omega(q)} \log\log q}{q} + \frac{N}{\log N},$$

yielding $\limsup_{N\to\infty} |f_{r,k}(\alpha, N)|/N = 0$. This proves the first assertion in Theorem 1.7.

To prove the second assertion, we assume that $\alpha > 0$ is of finite type $\tau \ge 1$. Hence, for a given $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, \alpha) > 0$ such that $\|\alpha n\| \ge C n^{-(\tau+\varepsilon)}$ for every positive integer n.

Let $h \in [1, 2(CN)^{\frac{1}{\tau+\varepsilon}}]$ be an integer. By Dirichlet's approximation theorem, there exist co-prime integers a, q with $1 \le q \le N/Q$ such that $|\alpha h - a/q| < Q/(qN)$, where Q is to be chosen in $[1, (CN)^{\frac{1}{\tau+1+\varepsilon}}(2/h)^{\frac{\tau+\varepsilon}{\tau+1+\varepsilon}}]$. Since

$$C(hq)^{-(\tau+\varepsilon)} \le \|\alpha h q\| \le |\alpha h q - a| < Q/N,$$

we find

$$q > h^{-1}(CN/Q)^{\frac{1}{\tau+\varepsilon}} \ge Q/2,$$

last inequality by our choice of the upper limit of Q. Hence, it follows that $|\alpha h - a/q| < 2/N$ and we can use Lemma 2.1 to get

$$f_{r,k}(\alpha, N) N^{-\varepsilon} \ll h^{\frac{1}{2}} Q^{\frac{1}{2\tau}} N^{1-\frac{1}{2\tau}} + N Q^{-\frac{r-1}{2r}} + \begin{cases} N^{\frac{5}{6}} & \text{if } r = 2, \\ N^{\frac{4}{5}} & \text{if } r > 2, \end{cases}$$

where we used the elementary inequality $2^{\omega(q)} \le \tau(q) \ll (N/Q)^{\varepsilon/2}$.

Using [7, Lemma 2.4] to choose Q optimally, we obtain

$$\begin{aligned} f_{r,k}(\alpha h, N) N^{-\varepsilon} &\ll h^{\frac{1}{2}} N^{1-\frac{1}{2\tau}} + h^{\frac{(r-1)(\tau+\varepsilon)}{2r(\tau+1+\varepsilon)}} N^{1-\frac{r-1}{2r(\tau+1+\varepsilon)}} \\ &\quad + h^{\frac{(r-1)\tau}{2(r+(r-1)\tau)}} N^{1-\frac{r-1}{2(r+(r-1)\tau)}} + \begin{cases} N^{\frac{5}{6}} & \text{if } r = 2, \\ N^{\frac{4}{5}} & \text{if } r > 2, \end{cases} \\ &\ll N((h^{\tau+\varepsilon}/N)^{\frac{1}{2\tau}} + (h^{\tau+\varepsilon}/N)^{\frac{r-1}{2r(\tau+1+\varepsilon)}} \\ &\quad + (h^{\tau+\varepsilon}/N)^{\frac{r-1}{2(r+(r-1)\tau)}}) + \begin{cases} N^{\frac{5}{6}} & \text{if } r = 2, \\ N^{\frac{4}{5}} & \text{if } r > 2. \end{cases} \end{aligned}$$

We can assume that $h \le N^{1/(\tau+\varepsilon)}/2$ by taking a smaller C if necessary. Hence, the second term dominates the first and the third terms, and we can write

$$f_{r,k}(\alpha h, N)N^{-\varepsilon} \ll h^{\frac{(r-1)(\tau+\varepsilon)}{2r(\tau+1+\varepsilon)}} N^{1-\frac{r-1}{2r(\tau+1+\varepsilon)}} + \begin{cases} N^{\frac{5}{6}} & \text{if } r=2, \\ N^{\frac{4}{5}} & \text{if } r>2. \end{cases}$$

Note that if $h \gg N^{1/(\tau+\varepsilon)}$, then the above inequality is trivially satisfied. Therefore, the inequality holds for all non-zero integers h.

Finally, by using Lemma 1.14 subsequent to an application of partial summation, it follows that

$$\begin{aligned} &\#\{p \le x : p \in \mathcal{P}_{r,k} \text{ and } a \le \{\alpha p + \beta\} \le b\} - (b-a)\#\mathcal{P}_{r,k}(x) \\ &\ll \frac{x}{H \log x} + \frac{x^{\varepsilon}}{\log x}\left(H^{\frac{(r-1)(\tau+\varepsilon)}{2r(\tau+1+\varepsilon)}} x^{1-\frac{r-1}{2r(\tau+1+\varepsilon)}} + \log H \begin{cases} x^{\frac{5}{6}} & \text{if } r=2 \\ x^{\frac{4}{5}} & \text{if } r>2 \end{cases}\right) \\ &\ll x^{2\varepsilon}\left(xH^{-1} + H^{\frac{(r-1)\tau}{2r(\tau+1)}} x^{1-\frac{r-1}{2r(\tau+1)}} + \begin{cases} x^{\frac{5}{6}} & \text{if } r=2 \\ x^{\frac{4}{5}} & \text{if } r>2 \end{cases}\right). \end{aligned}$$

Using [7, Lemma 2.4] with $H \in [1, x]$ gives the error stated in Theorem 1.7.

Proof of Corollary 1.8. We shall show that

$$\sum_{p \in (\mathcal{P}_{r,k} \cap \mathcal{B}_{\alpha,\beta})(x)} 1 \sim \frac{1}{\alpha}\mathcal{P}_{r,k}(x).$$

It is not hard to show that if $\alpha > 1$ irrational, then

$$p \in \mathcal{P}_{r,k} \cap \mathcal{B}_{\alpha,\beta} \iff p \in \mathcal{P}_{r,k} \quad \text{and} \quad 0 < \left\{\frac{p+1-\beta}{\alpha}\right\} \le \frac{1}{\alpha},$$

where equality can hold for at most one prime p. Hence, the proof of the corollary is apparent from Theorem 1.7. □

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