Variations on a theme of Mirsky

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Let \( k \) and \( r \) be non-zero integers with \( r \geq 2 \). An integer is called \( r \)-free if it is not divisible by the \( r \)th power of a prime. A result of Mirsky states that there are infinitely many primes \( p \) such that \( p + k \) is \( r \)-free. In this paper, we study an additive Goldbach-type problem and prove two uniform distribution results using these primes. We also study certain properties of primes \( p \) such that \( p + a_1, \ldots, p + a_\ell \) are simultaneously \( r \)-free, where \( a_1, \ldots, a_\ell \) are non-zero integers and \( \ell \geq 1 \).

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1. Introduction and Statement of Results

Let \( k \) be a non-zero integer and \( r \geq 2 \) be an integer. Let \( \mathcal{P}_{r,k} \) denote the set of primes \( p \) such that \( p + k \) is positive and not divisible by an \( r \)th power of a prime. In [13] Mirsky showed that \( \mathcal{P}_{r,k} \) has positive density in the set of primes. More precisely, he showed that for every sufficiently large \( x \), the asymptotic formula

\[
\# (\mathcal{P}_{r,k} \cap [1, x]) = \prod_{p \mid k} \left(1 - \frac{1}{\phi(p^r)}\right) \frac{x}{\log^A x} + O \left( \frac{x}{\log^A x} \right)
\]

holds for any \( A > 0 \), where \( \phi \) is the Euler-totient function and \( \text{li}(x) = \int_2^x 1/(\log t)\,dt \). Mirsky’s result is based on inclusion and exclusion principle together with an

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application of the Siegel–Walfisz theorem; therefore, the error term here is very weak compared to the best error term in the prime number theorem due to Vinogradov and Korobov (cf. [10, 23]). Furthermore, assuming that there are no Siegel zeros, the result of Languasco [12] can be recast that the above error term can be taken as small as the standard error term in the prime number theorem (cf. [8, §18]). Following the work of Mirsky, it can be easily verified that under GRH for all Dirichlet $L$-functions the above error term can be taken $\ll x^{(r+1)/(2r)}(\log x)^{(r-2)/r}$.

In this paper, our first goal is to study primes that lie in the set $\cap_{i=1}^{r} P_{r,a_i}$, where $a_1, a_2, \ldots, a_r$ are non-zero integers, for a fixed $r \geq 2$. In Theorem 1.1 we give an asymptotic formula for the counting function of these primes. Similar results appeared in [11, 13], but our theorem allows the $\ell$ tuple and $\ell$ itself to vary. A similar result using the circle method was proved in [8] (not published) for primes in the set $\cap_{i=1}^{r} P_{r,a_i}$, where $2 \leq r_1 \leq \cdots \leq r_{\ell}$ are fixed. In [8], an asymptotic formula is given for the number of primes $p$ such that $p + 1$ and $p + 2$ are square-free under GRH.

Our second goal is to prove an asymptotic formula for the number of representations of a given integer as the sum of $s$ primes from $P_{r,k}$. Our third goal is to give two uniform distribution results using the primes in $P_{r,k}$ for a fixed non-zero integer $k$ and $r \geq 2$.

Let $a$ denote an $\ell$-tuple $(a_1, a_2, \ldots, a_{\ell})$ of non-zero integers, where $\ell \geq 1$. Define

$$\pi_r(x, a) := \# \{p \leq x : p + a_i \text{ is positive and } r\text{-free for each } i = 1, \ldots, \ell \}.$$ 

Put $|a| = \max_i |a_i|$, and

$$\mathcal{S}_r(a) = \prod_p \left(1 - \frac{\nu_p(a)}{\phi(p^s)}\right),$$

where $\nu_p(a)$ is the number of distinct residue classes $a_i$ modulo $p^s$ that are coprime to $p$. We call $a$ a permissible tuple of non-zero integers if there are infinitely many primes $p$ such that $p + a_1, \ldots, p + a_{\ell}$ are r-free. It is not hard to see that if $\nu_p(a) = \phi(p^s)$ for some $p$, then $a$ is not permissible. The next theorem establishes the converse.

**Theorem 1.1.** Let $r \geq 2$ be a fixed integer, and $a$ denote any $\ell$-tuple $(a_1, a_2, \ldots, a_{\ell})$ of non-zero integers, where $\ell \geq 1$. Given $0 < \varepsilon < (\log 2)^{-1}$ and $A > 0$, the estimate

$$\pi_r(x, a) - \mathcal{S}_r(a) \text{li}(x) \ll \ell(\log x)^{(r-1)|a|} + \text{li}(x)(\log x)^{-A}$$

holds uniformly for

$$1 \leq \ell < ((\log 2)^{-1} - \varepsilon) \log \log \log x$$

for every sufficiently large $x$. Here, the implied constant depends at most on $r, A$ and $\varepsilon$.

Under GRH for all Dirichlet $L$-functions the error estimate above is to be replaced by [24, 25], wherein the implied constant depends only on $r$. Furthermore, the largest range of $\ell$ and $|a|$ such that the error is still $x/(\log x)^2$ is given by [27].
Compared with the main results of [1, 3, 8], our result goes one step further as the parameters \( \ell \) and \( |a| \) in the above theorem are allowed to increase as a function of \( x \), which is not addressed in these papers.

**Remark 1.2.** By a simple argument, we can always find a permissible \( \ell \)-tuple for a given \( \ell \geq 2 \). This fact, as well as the lower bound

\[
\mathfrak{G}_r(a) \geq e^{-C(r)},
\]

for some \( C = C(r) > 0 \), whenever \( a \) is a permissible \( \ell \)-tuple, will be proved subsequent to the proof of Theorem 1.1.

**Theorem 1.3.** Let \( 0 < \epsilon < 1/(\log 2) \) and \( r \geq 2 \) be given. Then, for all sufficiently large \( H \), the equation

\[
\sum_{|a| \leq H} \mathfrak{G}_r(a) = \left( \frac{2H}{\zeta(r)} \right)^{\ell} A \ll (2H)^{\ell-\epsilon-1}(\log H)^{\frac{x^\ell}{1+\epsilon}}(\zeta(r)+1)^{\frac{(r-1)(\ell-1)}{r}}
\]

(1.1)

holds uniformly for \( 1 \leq \ell \leq (1/\log 2 - \epsilon) \log \log H \), where \( \zeta(s) \) is the Riemann zeta function and the sum is taken over \( \ell \)-tuples \( a \) with non-zero coordinates. The implied constant depends only on \( r \) and \( \epsilon \).

**Theorem 1.4.** Let \( 0 < \epsilon < (\log 2)^{-1} \), \( r \geq 2 \), and \( A \) with \( A > 1/(r-1) \) be given. Then, for every sufficiently large \( x \), the estimate

\[
\sum_{|a| \leq H} \pi_r(x, a) = \left( \frac{2H}{\zeta(r)} \right)^{\ell} \pi(x) \ll \frac{x\ell(2H)^{\ell-\epsilon-1}(\log H)^{\frac{x^\ell}{1+\epsilon}}(\zeta(r)+1)^{\frac{(r-1)(\ell-1)}{r}}}{\log^{1-1/r} x} + \frac{\ell (\log x)^{\frac{x^\ell}{1+\epsilon}}(2H)^{\ell-1}(\log H)^{\frac{(x^\ell)(\ell-1)}{r}}}{(H + x^{1/r})^{(1 + \zeta(r))^{\frac{(r-1)}{r}}}}
\]

holds uniformly for

\[
\log^A x \leq H \leq \frac{x}{2.1 \log x} \exp(-\log x)^{1-\epsilon} \log^2 \log x), \quad \ell \leq (1/\log 2 - \epsilon) \log \log H,
\]

(1.2)

where the sum is taken over \( \ell \)-tuples \( a \) with non-zero coordinates and the implied constant depends only on \( r \) and \( \epsilon \). Here, \( \pi(x) \) denotes the number of primes not exceeding \( x \).

**Remark 1.5.** Although a similar result can be obtained directly using Theorem 1.1, the latter gives a power saving in \( H \).
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**Theorem 1.6.** For every sufficiently large integer $N$, the asymptotic formula

$$
\sum_{\begin{subarray}{c}
p_1, \ldots, p_s \in \mathcal{P}_{r,k} \\
p_1 + p_2 + \cdots + p_s = N
\end{subarray}} \prod_{1 \leq i \leq s} \log p_i = \mathcal{S}_{r,k}(N) \frac{N^{s-1}}{(s-1)!} + O \left( \frac{N^s}{\log^A N} \right) \quad (1.3)
$$

holds for every $A > 0$ and $s \geq 3$, where $\mathcal{S}_{r,k}(N)$ is given by

$$
\mathcal{S}_{r,k}(N) = \sum_{q \geq 1 \ (a, q) = 1} \phi(q) \sum_{l \leq q} \frac{\mu_k(d)}{\phi(q, a q')} \sum_{q \rightarrow \infty} e \left( \frac{a}{q} \right) \sum_{d \mid k, d' \mid s} \frac{\alpha}{\phi(q, d')} \left( 1 + o \left( \frac{1}{\log q} \right) \right) \quad (1.4)
$$

The implied constant depends on $s, r, k$ and $A$. Furthermore, $\mathcal{S}_{r,k}(N) > C(s, r, k)$ for some positive absolute constant $C(s, r, k)$, provided that the parities of $N$ and $s$ are the same and in the case $r = 2$ and $k$ is odd, $4 \nmid ks + N$ when $N$ is odd, and $4 \mid ks + N$ when $N$ is even.

For the explicit formula of $\mathcal{S}_{r,k}(N)$, see (2.29).

As we go through the proof of Theorem 1.6, the techniques that we develop allow us to give a uniform distribution result on the fractional parts of $\alpha p$ when $p \in \mathcal{P}_{r,k}$. To do so, we first define irrational numbers that are of finite type. An irrational number $\alpha$ is called of finite type if

$$
\tau = \sup \left\{ \beta \in \mathbb{R} : \liminf_{q \to \infty} q^\beta ||\alpha q|| = 0 \right\} < \infty,
$$

where the notation $||x||$ is used to denote the distance from the real number $x$ to the nearest integer. We note here that by Dirichlet’s approximation theorem one has $\tau \geq 1$. The celebrated theorems of Khinchin [9] and of Roth [19, 20] state that $\tau = 1$ for almost all (in the sense of the Lebesgue measure) real numbers and for all irrational algebraic numbers, respectively.

**Theorem 1.7.** Given a real number $x$, let $\{x\}$ denote the fractional part of $x$. For any positive irrational number $\alpha$ and any real number $\beta$, the fractional parts of $ap + \beta$ for $p \in \mathcal{P}_{r,k}$ are uniformly distributed in the unit interval; that is,

$$
\# \{ p \leq x : p \in \mathcal{P}_{r,k} \text{ and } a \leq \{ap + \beta\} \leq b \} = (b - a)\#\mathcal{P}_{r,k}(x)(1 + o(1)),
$$

uniformly for $0 \leq a < b \leq 1$.

Furthermore, if $\alpha$ is of finite type $\tau$, then for every $\epsilon > 0$ and every $0 \leq a < b \leq 1$,

$$
\# \{p \leq x : p \in \mathcal{P}_{r,k} \text{ and } a \leq \{ap + \beta\} \leq b \} - (b - a)\#\mathcal{P}_{r,k}(x) \\
\ll x^{1 - \frac{1}{(\tau + 1)(r + 1) - 1}} + \epsilon.
$$

The implied constant is independent of $a$ and $b$.

We now give a corollary of Theorem 1.7. For any irrational number $\alpha > 1$ and any real number $\beta$, the non-homogeneous Beatty sequence is defined by

$$
\mathcal{B}_{\alpha, \beta} = \{ n \in \mathbb{N} : n = [\alpha m + \beta] \text{ for some } m \in \mathbb{N}\setminus\{0\} \}.\]
Vinogradov showed that $B_{\alpha,0}$ contains infinitely many primes (see [22, §XI]) when $\alpha > 1$ is irrational. The next corollary improves on Vinogradov’s result.

**Corollary 1.8.** Let $r \geq 2$. For any irrational number $\alpha > 1$, any real number $\beta$, and any non-zero integer $k$, $\mathcal{P}_{r,k} \cap B_{\alpha,\beta}$ is an infinite set.

For $c > 1$, consider the set
$$N_c = \{\lfloor mc \rfloor : m \in \mathbb{N}\}.$$ Piatetski–Shapiro was the first to prove that there are infinitely many primes in $N_c$ whenever $1 < c < 12/11$ (cf. [16]). The interested reader is invited to investigate the largest range of $c > 1$ such that $\mathcal{P}_{r,k} \cap N_c$ is an infinite set.

### 1.1. Preliminaries and notation

#### 1.1.1. Notation

Given a real number $x$, we write $e(x) = e^{2\pi ix}$, $\{x\}$ for the fractional part of $x$, $[x]$ for the greatest integer not exceeding $x$ and $\lfloor x \rfloor$ for the smallest integer not less than $x$.

We recall that for functions $F$ and real non-negative $G$, the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leq \alpha G$ holds for some constant $\alpha > 0$. Further, we use $F \asymp G$ to indicate that both $F \gg G$ and $F \ll G$ hold. The notations $F \ll_c G$ is use to denote the implied constant depends at most on $c$. In a slight departure from convention, we shall frequently use $C$ to mean a positive number possibly different at each occurrence.

For positive integers $a_1$ and $a_2$, $[a_1, a_2]$ denotes their least common multiple and $(a_1, a_2)$ denotes their greatest common divisor. $\tau(n)$ denotes the number of positive divisors of $n$, $\mu(n)$ denotes the usual Möbius function, for any positive integer $k$ $\mu_k(n)$ is defined to be $\mu(n)$ if $(k, n) = 1$, and 0 otherwise, $\omega(n)$ denotes the number of positive divisors of $n$, and $\phi(n)$ denotes Euler’s totient function.

We put
$$\mathcal{P}_{r,a_1,\ldots,a_k} = \{p \text{ prime} : p + a_i \text{ is positive and } r\text{-free for each } i = 1, \ldots, k\}.$$ For any set $S$ and any real number $x$, $S(x)$ denotes the set of elements of $S$ not exceeding $x$.

#### 1.1.2. Preliminaries

We first start by defining Ramanujan sums frequently used in the proof of Theorem [17]. For the proof of the following result, see [17, Hilfssatz 2].

**Lemma 1.9.** Let $a$ and $q$ be positive integers such that $(a, q) = 1$ and let $d$ be a positive divisor of $q$. Let $k$ be an integer with $(k, d) = 1$. The Ramanujan sums
defined by
\[
c_q(a, d) = \sum_{\substack{r \mod q \\ (r, q) = 1}} d \left\lfloor \frac{ar}{q} \right\rfloor
\]
satisfy
\[
c_d(a, q) = \begin{cases} 
\mu \left( \frac{q}{d} \right) e \left( \frac{au}{d} \right) \frac{\phi(q)}{\phi(q/(q,a))} & \text{if } (d, q) = 1, \\
0 & \text{else},
\end{cases}
\tag{1.5}
\]
where \( u \) is the solution of the congruence \( qu \equiv 1 \mod d \).

When \( d = 1 \), we get the classical Ramanujan sums. For the proof of the following identity, see [5, §24].

Lemma 1.10. Let \( a \) be a non-zero integer and let \( q \) be a positive integer. Then
\[
c_q(a) = \sum_{\substack{r \mod q \\ (r, q) = 1}} e \left( \frac{ar}{q} \right) = \mu \left( \frac{q}{d} \right) e \left( \frac{au}{d} \right) \frac{\phi(q)}{\phi(q/(q,a))}.
\]

Lemma 1.11. For any multiplicative functions \( f_1, f_2, \ldots, f_k \), the arithmetical function \( L_k \) defined by
\[
L_k(n) = \sum_{\substack{d_1|n, \ldots, d_k|n \\ [d_1, d_2, \ldots, d_k] = n}} f_1(d_1)f_2(d_2) \cdots f_k(d_k)
\]
is multiplicative.

Proof. Assume that \( (m, n) = 1 \). For any divisor \( d_i \) of \( nm \), we write \( d_i = e_i r_i \), where \( e_i \mid n \) and \( r_i \mid m \). Note that such \( e_i \) and \( d_i \) are unique. Therefore,
\[
L_k(nm) = \sum_{\substack{e_1 r_1|mn \\ \ldots, e_k r_k|mn}} f_1(e_1 r_1)f_2(e_2 r_2) \cdots f_k(e_k r_k).
\]
At this point one can prove by induction on \( k \geq 2 \) that for such \( (e, r_i) \)
\[
[e_1 r_1, \ldots, e_k r_k] = [e_1, \ldots, e_k] [r_1, \ldots, r_k].
\]
Using multiplicativity of \( f_i \) one has
\[
L_k(mn) = \sum_{\substack{e_i|n, r_i|m \\ \ldots, e_k=r_k|n}} f_1(e_1)f_1(r_1)f_2(e_2)f_2(r_2) \cdots f_k(e_k)f_k(r_k) = L_k(m)L_k(n),
\]
and hence the claim follows.
Lemma 1.12. For any integer $\ell > 1$ and any positive integer $n$, define

$$g_{\ell}(n) = \sum_{d_1, \ldots, d_{\ell} \geq 1 \atop \prod d_i = n} |\mu(d_1)\mu(d_2)\cdots\mu(d_{\ell})|.$$  

Then for any real number $z \geq 1$ and any integer $r \geq 2$, one has the following upper bounds:

$$\sum_{n > z} \frac{g_{\ell}(n)}{n^r} < \frac{e \cdot (1 + \log 3z)^{2\ell - 1}}{z^{r-1}},$$  

$$\sum_{n > z} \frac{g_{\ell}(n)}{\phi(n^r)} < \frac{e n^2 (15\pi^{-2}(1 + \log 3z))^{2\ell - 1}}{3z^{r-1}}.$$  

(1.6)

Proof. Let $l$ be a real number in $(0, r - 1)$ to be determined. By Rankin’s trick

$$\sum_{n > z} \frac{g_{\ell}(n)}{n^r} \leq \frac{1}{z^l} \sum_{n=1}^{\infty} \frac{g_{\ell}(n)}{n^{r-l}}.$$  

Since $g_{\ell}(n) \leq \tau^\ell(n) \ll n^\ell$ for fixed $\ell$, the latter sum converges. By Lemma 1.11, $g_{\ell}$ is multiplicative and is supported only on square-free numbers. Furthermore, $g_{\ell}(p) = 2^\ell - 1$. Using the inequality $1 + nx \leq (1 + x)^n$, which holds for every positive integer $n$ and every real number $x > -1$, one has

$$\prod_{p} \left(1 + \frac{2^\ell - 1}{p^{r-l}}\right) \leq \prod_{p} \left(1 + \frac{1}{p^{r-l}}\right)^{2^\ell - 1} = \left(\zeta(r-l)\right)^{2^\ell - 1}.$$  

Since $\zeta(\sigma) < \frac{\sigma}{\sigma-1}$ for $\sigma > 1$ (cf. [15 Corollary 1.14]), it follows that

$$\sum_{n > z} \frac{g_{\ell}(n)}{n^r} < \left(1 + (r - l - 1)^{-1}\right)^{2^\ell - 1}.$$  

Choosing $l = r - 1 - (\log 3z)^{-1} > 0$, the first upper bound follows.

As for the second upper bound, we record the inequality

$$\frac{1}{\phi(n^r)} = \frac{1}{n^r} \prod_{p \mid n} \left(1 + \frac{1}{p} \right)^{-1} < \frac{\pi^2}{6n^r} \sum_{d \mid n} \frac{|\mu(d)|}{d},$$  

which yields

$$\sum_{n > z} \frac{g_{\ell}(n)}{\phi(n^r)} < \frac{\pi^2}{6} \sum_{nm > z} \frac{g_{\ell}(nm)|\mu(n)|}{n^r m^{r+1}} \leq \frac{\pi^2}{6} \sum_{m=1}^{\infty} \frac{g_{\ell}(m)}{m^{r+1}} \sum_{n > z/m} \frac{g_{\ell}(n)}{n^r}.$$  

Here, we have used the inequality $g_{\ell}(nm) \leq g_{\ell}(n)g_{\ell}(m)$ and that $|\mu(m)|$ can be ignored since $g_{\ell}$ is supported on square-free integers. By the first inequality in
Lemma 1.14 (Erdős-Turán inequality [11, Chap. 2. Eq. 2.42]).

By (1.6), it follows that

$$\sum_{m \leq z} \frac{g_e(m)}{m^{r+1}} \sum_{n > z/m} \frac{g_e(n)}{n^r} < \frac{e (1 + \log 3z)^{2\ell - 1}}{z^{r-1}} \sum_{m \leq z} \frac{g_e(m)}{m^r}$$

$$\leq \frac{e (1 + \log 3z)^{2\ell - 1}}{z^{r-1}} \prod_p \left(1 + \frac{2^\ell - 1}{p^r}\right)$$

$$\leq \frac{\left(\zeta(2)\zeta(4)^{-1}(1 + \log 3z)\right)^{2\ell - 1}}{z^{r-1}}.$$

Finally,

$$\sum_{m > z} \frac{g_e(m)}{m^{r+1}} \sum_{n > z/m} \frac{g_e(n)}{n^r} = \prod_p \left(1 + \frac{2^\ell - 1}{p^r}\right) \sum_{m > z} \frac{g_e(m)}{m^{r+1}}$$

$$\leq \frac{\left(\zeta(2)\zeta(4)^{-1}(1 + \log 3z)\right)^{2\ell - 1}}{z^r}.$$

Combining the last two inequalities and using \(\zeta(4) = \pi^4/90\), we get the desired result. 

\[\square\]

Lemma 1.13. Let \(N \geq 2\) be a real number. Assume that \(|\alpha - \frac{d}{q}| \leq \frac{1}{4}\) with \((a, q) = 1\), then

$$\sum_{\substack{p \leq N \quad \text{mod} \; d \quad \text{or} \; b \mod \; d \quad \text{or} \; d \mod \; N \leq 3}} (\log p)e(\alpha p) \ll \left(\frac{(q, d)N}{dq^{1/2}} + \frac{q^{1/2}N^{1/2}}{(q, d)^{1/2}} + \frac{N^{1/5}}{d^{2/5}}\right) \log^3 N.$$

**Proof.** This result follows from [2] noting that the contribution of prime powers \(p^r \leq N\) with \(r \geq 2\) is absorbed by the third term above. 

\[\square\]

Lemma 1.14 (Erdős-Turán inequality [11, Chap. 2. Eq. 2.42]). Let \(\{t_1, t_2, \ldots, t_K\}\) be a set of real numbers. Suppose that \(\mathcal{I} \subset [0, 1)\). Then,

$$\#\{1 \leq i \leq K : \{t_i\} \in \mathcal{I}\} \ll \frac{K}{H} + \sum_{1 \leq b \leq H} \frac{1}{b} \left|\sum_{i \leq K} e(bt_i)\right|$$

for any \(H \geq 1\). The constant in the \(O\)-term is absolute.

**Lemma 1.15.** Let \(x > 2\) be a real number, \(k\) a non-zero integer with \(1 \leq |k| < x - 1\), \(f\) an arithmetic function such that \(|f(n)| \leq 1\). Then, for any positive integer \(Q\), any integer a coprime to \(Q\), and any real number \(1 < z \leq (x + k)^{1/r}\), one has

$$\sum_{\substack{p \leq x \quad \text{mod} \; D \quad \text{or} \; p \equiv 0 \mod \; Q \quad \text{or} \; p \equiv -k \mod \; d^r}} f(p) - \sum_{\substack{1 \leq d \leq z \quad (d, k) = 1}} \mu(d) \sum_{\substack{p \leq x \quad \text{mod} \; D \quad \text{or} \; p \equiv 0 \mod \; Q \quad \text{or} \; p \equiv -k \mod \; d^r}} f(p) \ll E_k(x, z, Q) + (x + k)^{1/r} + |k|,$$
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where the implied constant is absolute, the term $|k|$ can be dropped if $k > 0$ and

$$E_k(x, z, Q) := x \sum_{z < d \leq (x+k)^{1/r}} \frac{|\mu(d)|}{[Q,d^r]}.$$ 

**Proof.** We start by noting that

$$\sum_{d' | n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } r\text{-free,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\sum_{p \in P_{r, k}(x)} f(p) = \sum_{1 \leq d \leq (x+k)^{1/r}} \mu(d) \sum_{p \equiv a \mod Q, p \equiv -k \mod d'} f(p).$$

For each fixed $d$, the contribution of primes $p \leq -k$, in case $k < -1$ and only in that case, is trivially estimated by $1 - k/d'$, which summing over $d$ gives the bound $(x+k)^{1/r} - k$. Then, we remove the terms with $(d,k) > 1$ using the estimate

$$\sum_{1 \leq d \leq (x+k)^{1/r}} \sum_{p \leq x, p \equiv a \mod Q, p \equiv -k \mod d'} 1 \leq (x+k)^{1/r}.$$ 

We therefore have

$$\sum_{p \in P_{r, k}(x)} f(p) = \sum_{1 \leq d \leq (x+k)^{1/r}} \mu(d) \sum_{p \equiv a \mod Q, p \equiv -k \mod d'} f(p) + O((x+k)^{1/r} + |k|).$$

By the Chinese Remainder Theorem, the contribution of the terms with $d > z$ is at most

$$\sum_{z < d \leq (x+k)^{1/r}} \mu(d) \sum_{p \leq x, p \equiv -k \mod d'} f(p) \leq \sum_{z < d \leq (x+k)^{1/r}} \mu(d) \sum_{n \equiv \alpha(a,k) \mod [Q,d^r]} 1$$

$$< \sum_{z < d \leq (x+k)^{1/r}} \mu(d) \left( 1 + \frac{x}{[Q,d^r]} \right)$$

$$< (x+k)^{1/r} + E_k(x, z, Q).$$

This completes the proof. 

Next, we give an analogue of the Siegel–Walfisz theorem for $P_{r,k}$ which will be needed in the proof of Theorems 1.6 and 1.7. Such a result was proved in [4] for
a fixed modulus. Hence, as it stands the next result is of particular interest on its own.

Lemma 1.16. For positive integers $a$ and $Q$ such that $(a, Q) = 1$, define

$$E(x; a, Q) = \sum_{\substack{p \leq x \mid a \mod Q \equiv a \mod Q \equiv -k \mod d}} \log p - x \sum_{\substack{d \geq 1 \mid (Q, d) = 1}} \frac{\mu(d)}{\phi([Q, d])}.$$

Then for any fixed positive numbers $A, B$, the upper bound

$$E(x; a, Q) \ll \frac{x}{\log B} X$$

(1.7)

holds uniformly for every $Q \leq (\log x)^A$, where the implied constant depends only on $A, B$. Furthermore, assuming GRH for all Dirichlet $L$-functions one has

$$E(x; a, Q) \ll (x^{1/2} + x^{(r+1)/(2r)}2^{\omega(Q)/r}Q^{-1/r^2}) \log^2 x$$

(1.8)

for every $Q \geq 1$, where the implied constant depends only on $A, B, k$ and $r$. If there is no prime $p \mid k$ such that $p^r \mid Q$ and $p^r \mid a + k$, then

$$\sum_{\substack{d \geq 1 \mid (Q, d) = 1 \mid a + k \mid (d,k) = 1}} \frac{\mu(d)}{\phi([Q, d])} \gg \frac{1}{Q},$$

(1.9)

where the implied constant depends only on $r$.

Remark 1.17. Note that the estimate (1.8) is nontrivial in the range $Q \ll x^{1/2 - 1/(2(r+1)) - \epsilon}$.

Proof. Write

$$E(x; a, Q) = \sum_{\substack{p \leq x \mid a \mod Q \equiv a \mod Q \equiv -k \mod d'}} \log p - \sum_{\substack{d \geq 1 \mid (Q, d) = 1 \mid a + k \mid (d,k) = 1}} \frac{\mu(d)}{\phi([Q, d])}$$

$$- x \sum_{\substack{d > z \mid (Q, d') = 1 \mid a + k \mid (d,k) = 1}} \frac{\mu(d)}{\phi([Q, d'])} - \sum_{\substack{d \geq 1 \mid (Q, d) = 1 \mid a + k \mid (d,k) = 1}} \mu(d)$$

$$\times \left( \frac{x}{\phi([Q, d'])} - \sum_{\substack{p \leq x \mid a \mod Q \equiv a \mod Q \equiv -k \mod d'}} \log p \right).$$
On choosing \( f(p) = \frac{\log p}{\log x} \) in Lemma 1.15, one has
\[
\sum_{p \in \mathcal{P}_{x,k}(x)} \log p - \sum_{\substack{1 \leq d \leq z \\ (d,k) = 1}} \mu(d) \sum_{\substack{p \leq z \\ p \equiv \alpha \mod Q}} \log p
\]
\[
\ll x \log x \sum_{z < d \leq (x + k)^{1/r}} \frac{|\mu(d)|}{[Q,d^s]} + x^{1/r} \log x \quad (1.10)
\]
for every \( 1 < z \leq (x + k)^{1/r} \), where the implied constant depends only on \( k \) and \( r \). To handle the first error term on the right-hand side of (1.10), we use Rankin’s trick to get
\[
\sum_{z < d \leq (x + k)^{1/r}} \frac{|\mu(d)|}{[Q,d^s]} \leq \frac{1}{Q} \sum_{d > z} (Q,d^r)|\mu(d)| \leq \frac{1}{Q} \sum_{d = 1}^{\infty} (Q,d^r)|\mu(d)| \quad (1.11)
\]
for any \( \tau \) satisfying \( r - \tau > 1 \). It is clear that the above sum can be expanded into the Euler product
\[
\prod_{d = 1}^{\infty} \frac{(Q,d^r)|\mu(d)|}{d^{r-\tau}} = \prod_{p | Q} \left( 1 + \frac{(Q,p^r)}{p^{r-\tau}} \right) = \prod_{p | Q} \left( 1 + \frac{(Q,p^r)}{p^{r-\tau}} \right) \prod_{p | Q} \left( 1 + \frac{1}{p^{r-\tau}} \right).
\]
As we did in the proof of Lemma 1.12, we have with \( \tau = r - 1 - 1/\log(2z) \)
\[
\prod_{p | Q} \left( 1 + \frac{1}{p^{r-\tau}} \right) < \zeta(r - \tau) < 1 + \frac{1}{r - \tau - 1} < 1 + \log(2z),
\]
where in the second inequality, we used \( \zeta(\sigma) < \sigma/(\sigma - 1) \). As for the product over primes \( p | Q \), we claim that
\[
\prod_{p | Q} \left( 1 + \frac{(Q,p^r)}{p^{r-\tau}} \right) < 2^{\omega(Q)} Q^{1 - \epsilon}.
\]
This product is multiplicative in \( Q \). Hence, it suffices to confirm this inequality for prime powers. Take \( Q = p^l \) with \( l < r \). Then
\[
1 + \frac{(Q,p^r)}{p^{r-\tau}} < 1 + \frac{(p^l,p^r)}{p} \leq 2p^{l-1} < 2p^{l-1/r}.
\]
The case \( r \leq l \) is proved similarly. Thus, combining the inequalities above we arrive at
\[
\sum_{z < d \leq (x + k)^{1/r}} \frac{|\mu(d)|}{[Q,d^s]} < \frac{2^{\omega(Q)} Q^{-\epsilon}(1 + \log(2z))}{z^{r-1}}.
\]
The upper bound
\[
\sum_{d > z} \frac{|\mu(d)|}{\sigma([Q,d^s])} \ll \frac{Q^{1-1/r} 2^{\omega(Q)} (1 + \log 2z)}{\phi(Q) z^{r-1}}.
\]
can be proved similarly as in (1.11), thus we arrive at

\[
\mathcal{E}(x; a, Q) \ll \sum_{d \leq z} \sum_{\substack{p \leq x \atop p \equiv a \mod Q \atop p \equiv -k \mod d'}} \log p - \frac{x}{\phi([Q, d'])} + x^{1/r} \log x \\
+ 2^{-(Q)x} \frac{2^z x}{Q^{1/r}z^{r-1}} + \frac{Q^{1-1/r}2^{-(Q)x} \log x}{\phi(Q)z^{r-1}}
\]

for every \(1 < z \ll x^{1/r}\).

For the claim (1.7), we assume \(Q \leq \log^A x\) for some \(A > 0\). Put \(z = (\log x)^{r+2}\) for any positive number \(B\). Then \(Qd' \leq (\log x)^{A+r(B+2)/(r-1)}\), thus by the Siegel–Walfisz theorem (cf. [15, §11]) one has

\[
\sum_{\substack{p \leq x \atop p \equiv a \mod Q \atop p \equiv -k \mod d'}} \log p - \frac{x}{\phi([Q, d'])} \ll x \exp(-c\sqrt[2]{\log x})
\]

for some constant \(c > 0\), uniformly for all \(d \leq z\) and \(Q \leq \log^A x\) provided that \((Q, d')|a + k\). Thus, we deduce the estimate

\[
\mathcal{E}(x; a, Q) \ll x \exp(-c'\sqrt[2]{\log x}) + x^{1/r} \log x \\
+ 2^{-(Q)x} \frac{2^z x}{Q^{1/r}z^{r-1}} + \frac{Q^{1-1/r}2^{-(Q)x} \log x}{\phi(Q)z^{r-1}}
\]

for some \(c' > 0\), proving (1.7) on noting that \(2^{-(Q)} \leq \tau(Q) \ll Q^c\) and \(\phi(Q) \gg Q/(\log \log 5Q)\) for \(Q \geq 1\).

As for (1.8), we assume GRH for all Dirichlet \(L\)-functions so that the estimate

\[
\sum_{\substack{p \leq x \atop p \equiv a \mod Q \atop p \equiv -k \mod d'}} \log p - \frac{x}{\phi([Q, d'])} \ll x^{1/2} \log^2 x
\]

holds uniformly for all positive integers \(Q\) and \(d\) such that \((Q, d')|a + k\) (see [15, Corollary 13.8]). This yields

\[
\mathcal{E}(x; a, Q) \ll x^{1/2} \log^2 x + \frac{2^{-(Q)x} \log^2 x}{Q^{1/r}z^{r-1}} + \frac{Q^{1-1/r}2^{-(Q)x} \log x}{\phi(Q)z^{r-1}}
\]

for every \(1 < z \ll x^{1/r}\). For \(Q \leq x^{1/2}\), we use \(\phi(Q) \gg Q/(\log \log 5Q)\) to eliminate the third term and then use [7, Lemma 2.4.] with \(1 < z \ll x^{1/r}\) and this yields (1.8). For \(Q > x^{1/2}\), using (1.9) and trivially estimating \(\mathcal{E}(x; a, Q)\) yield (1.8).
Finally, using the identity \( \phi(a)\phi(b) = \phi((a,b))\phi([a,b]) \), we see that
\[
\sum_{d=1}^{\infty} \frac{\mu(d)}{\phi([Q, d^\ell])} = \frac{1}{\phi(Q)} \prod_{p|Q} \left( 1 - \frac{\phi((p^r, Q))}{p^{r-1}(p-1)} \right)
\]
\[
\geq \frac{1}{\phi(Q)} \prod_{p|Q} \left( 1 - \frac{1}{p} \right) \prod_{p|Q} \left( 1 - \frac{1}{p^{r-1}(p-1)} \right) \gg \frac{1}{Q}
\]
provided that the sum, and hence the corresponding Euler product, is non-zero.

\[ \square \]

2. Proof of Main Theorems

2.1. Proof of Theorem 1.1

Assume that we are given \((a_1, \ldots, a_\ell) \in (\mathbb{Z}\setminus\{0\})^{\ell} \) with \(|a| < x - 1\) where \(|a| = \max_i |a_i|\). For any set \(S\), we put \(\chi(S, x) = 1\) if \(x \in S\) and 0 otherwise.

For any \(1 \leq z \leq (x - |a|)^{1/r}\), we shall show that
\[
\pi_r(x, a) - \sum_{d_1, \ldots, d_\ell \leq z} \mu(d_1) \mu(d_2) \cdots \mu(d_\ell) \sum_{p \equiv a_i \mod d_i, r} 1 
\ll \ell z^{\ell-1} \left( |a| + x^{1/r} \right) + \frac{x \ell (1 + \log 3z)^{2\ell-1}}{z^{r-1}}, \tag{2.1}
\]
where the implied constant depends only on \(r\). To do this, we use Lemma 11.15 iteratively and prove that for each \(1 \leq l \leq \ell\)
\[
\pi_r(x, a) - \sum_{d_1, \ldots, d_l \leq z} \mu(d_1) \mu(d_2) \cdots \mu(d_l) \sum_{p \equiv a_i \mod d_i, r} f_{l+1}(p) 
\ll l z^{l-1} \left( |a| + x^{1/r} \right) + \frac{x l (1 + \log 3z)^{2l-1}}{z^{r-1}}, \tag{2.2}
\]
where the implied constant depends only on \(r\) and \(f_1(p) = \chi(P, a_1, \ldots, a_\ell, p)\) if \(l = \ell\) and \(f_1(p) = 1\) if \(l > \ell\). For \(l = 1\),
\[
\pi_r(x, a) = \sum_{p \leq x} \chi(P, a_1, \ldots, a_\ell, p) = \sum_{p \in P_{r, a_1}(x)} f_2(p).
\]
Applying Lemma 11.14 and assuming \(z \leq (x - |a|)^{1/r}\) yield
\[
\pi_r(x, a) = \sum_{1 \leq d_1 \leq z} \mu(d_1) \sum_{p \leq x, p \equiv -a_1 \mod d_1, r} f_2(p) + O \left( \frac{x}{z^{r-1}} + (x + a_1)^{1/r} + |a_1| \right).
\]
where \( f_2(p) = \chi(\mathcal{P}_{r,a_2,\ldots,a_\ell},p) \) if \( \ell \geq 2 \) and \( f_2(p) = 1 \) if \( \ell = 1 \). This establishes (2.2) with \( l = 1 \).

Assume (2.2) holds for some \( l \). If \( l = \ell \), we are done. Otherwise, assume that \( l < \ell \). Then, \( f_{l+1}(p) = \chi(\mathcal{P}_{r,a_{l+1},\ldots,a_\ell},p) \), and we have by the Chinese Remainder Theorem

\[
\sum_{\substack{p \leq x \\ p \equiv -a \mod d_i}} f_{l+1}(p) = \sum_{\substack{p \leq x \\ p \equiv a \mod Q}} f_{l+2}(p),
\]

where \( a \) depends on \( a_1,\ldots,a_l, k = -a_{l+1} \) and \( Q = [d_1,\ldots,d_l]^r \). By Lemma 1.15

\[
\pi_r(x,a) = \sum_{d_1,\ldots,d_l \leq z} \mu(d_1)\mu(d_2)\cdots\mu(d_{l+1}) \sum_{\substack{p \leq x \\ p \equiv -a \mod d_i}} f_{l+2}(p)
\]

\[
\ll (l+1)z^l \left( |a| + x^{1/r} \right) + x \left( 1 + \log 3z \right)^{2r-1} + x \sum_{d_1,\ldots,d_{l+1} \leq z} \frac{|\mu(d_1)\cdots\mu(d_{l+1})|}{[d_1,\ldots,d_{l+1}]^r}.
\]

For the last term, we have

\[
\sum_{d_1,\ldots,d_{l+1} \leq z} \frac{|\mu(d_1)\cdots\mu(d_{l+1})|}{[d_1,\ldots,d_{l+1}]^r} \leq \sum_{d \leq z} \frac{g_{l+1}(d)}{d^r} < \frac{e \cdot (1 + \log 3z)^{2r-1}}{z^{r-1}}.
\]

Hence, (2.2) follows for each \( 1 \leq l \leq \ell \), and taking \( l = \ell \) yields (2.1).

We now proceed to prove the first upper bound in Theorem 1.1. Let \( z = (\log x)^A \) for some \( A > 0 \) to be determined. Put \( q = [d_1,\ldots,d_\ell] \). Then, Brun–Titchmarsh inequality (cf. [14, Theorem 3.9])

\[
\pi(x;q,a) \leq \frac{2x}{\phi(q) \log(x/q)} \quad (1 \leq q < x)
\]
gives

\[
\sum_{d_1,\ldots,d_\ell \leq z} \mu(d_1)\cdots\mu(d_\ell) \sum_{p \equiv -a \mod d_i} \frac{1}{d_i} \leq 2x \sum_{d_1,\ldots,d_\ell \leq z} \frac{|\mu(d_1)\cdots\mu(d_\ell)|}{\phi(q^\ell) \log(x/q^\ell)} \leq 2x \sum_{z/n \leq z} \frac{g_{\ell}(n)}{\phi(n^\ell) \log(x/n^\ell)}.
\]
provided that \( q^r < x \), which holds since \( q^r \leq z^{rt} < x \) for sufficiently large \( x \). Then, by (1.6) we see that the last term is
\[
< \frac{x \left( 15\pi^{-2} (1 + \log 3z) \right)^{2^r-1}}{z^{r-1} \log x}.
\] (2.3)

For \( q \leq z \), we can apply the Siegel–Walfisz theorem (cf. [15, Corollary 11.19]) to get
\[
\sum_{d_1, \ldots, d_t \leq z} \mu(d_1) \cdots \mu(d_t) \sum_{\substack{p \leq x \\
 p \equiv -a \mod d_i^r}} 1 \\
\sum_{d_i, d_j \neq 1} 1
\]
\[
= \sum_{d_1, \ldots, d_t \leq z} \mu(d_1) \cdots \mu(d_t) \left( \frac{\lim(x)}{\phi(q^r)} + O \left( x \exp(-C \sqrt{\log x}) \right) \right)
\]
\[
= \lim(x) \sum_{d_1, \ldots, d_t \leq z} \mu(d_1) \cdots \mu(d_t) \left( \frac{\lim(x)}{\phi(q^r)} \right) + O \left( z^t x \exp(-C \sqrt{\log x}) \right) \tag{2.4}
\]
for some constant \( C = C(A, r) \) that depends only on \( A \) and \( r \). The first term can be replaced by
\[
\sum_{d_1, \ldots, d_t \leq z} \mu(d_1) \mu(d_2) \cdots \mu(d_t) \left( \frac{\lim(x)}{\phi(q^r)} \right)
\]
with the same error in (2.4). Note that for every \( \ell \geq 1 \), the estimate
\[
\sum_{d_1 \leq z} \frac{\mu(d_1) \mu(d_2) \cdots \mu(d_t)}{\phi([d_1, d_2, \ldots, d_t]^t)} - \sum_{d_i \geq 1} \frac{\mu(d_1) \mu(d_2) \cdots \mu(d_t)}{\phi([d_1, d_2, \ldots, d_t]^t)}
\]
\[
\leq \ell \sum_{n > z} \frac{g_\ell(n)}{\phi(n^r)} \leq \ell \frac{\left( 15\pi^{-2} (1 + \log 3z) \right)^{2^r-1}}{z^{r-1}}\tag{2.5}
\]

holds, where the last inequality follows from (1.6). Hence, combining the estimates in (2.1), (2.3)–(2.5), we conclude that
\[ \pi_r(x, a) - \operatorname{li}(x) \sum_{d_i \geq 1} \mu(d_1) \mu(d_2) \cdots \mu(d_t) \frac{\phi([d_1, d_2, \ldots, d_t])}{\phi([d_1, d_2, \ldots, d_t])} \]
\[ \ll \ell z^{r-1}(|a| + x^{1/r}) + \frac{x\ell (1 + \log 3z)^{2r-1}}{z^{r-1}} \]
\[ + \operatorname{li}(x)\ell (15\pi^2 (1 + \log 3z))^{2r-1} z^{r-1} + x^r \exp(-C \sqrt{\log x}). \]

Given \( B > 0 \), we now choose \( A = (B + 2)/(r - 1) \). Then, for sufficiently large \( x \),
\[ \pi_r(x, a) - \operatorname{li}(x) \sum_{d_i \geq 1} \mu(d_1) \cdots \mu(d_t) \frac{\phi([d_1, d_2, \ldots, d_t])}{\phi([d_1, d_2, \ldots, d_t])} \ll \ell (\log x)^{2r-1} (|a| + \frac{\operatorname{li}(x)}{(\log x)^B}, \]
uniformly for \( \ell \leq (\log 2)^{-r} \) \log \log x \) and every sufficiently large \( x \) depending only on \( \varepsilon, A \) and \( r \).

Note that we assumed above that \( |a| \leq x - \varepsilon' \). However, this condition can be removed since for larger \( |a| \), the claim of the theorem becomes trivial.

Now, we assume GRH. For any given modulus \( f \geq 1 \), we have (cf. [15 Corollary 13.8])
\[ \pi(x; f, a) = \frac{\operatorname{li}(x)}{\phi(f)} + O(x^{1/2} \log x). \]

Hence, by (2.1) we obtain
\[ \pi_r(x, a) - \operatorname{li}(x) \sum_{d_i \geq 1} \mu(d_1) \cdots \mu(d_t) \frac{\phi([d_1, d_2, \ldots, d_t])}{\phi([d_1, d_2, \ldots, d_t])} (\frac{\operatorname{li}(x)}{\phi(f)} + O(x^{1/2} \log x)) \]
\[ \ll \ell z^{r-1}(|a| + x^{1/r}) + \frac{x\ell (1 + \log 3z)^{2r-1}}{z^{r-1}}. \]

Using (2.5b) we end up with
\[ \pi_r(x, a) - \operatorname{li}(x) \sum_{d_i \geq 1} \mu(d_1) \cdots \mu(d_t) \frac{\phi([d_1, d_2, \ldots, d_t])}{\phi([d_1, d_2, \ldots, d_t])} \]
\[ \ll \ell z^{r-1}(|a| + x^{1/r}) + \frac{x\ell (\log x)^{2r-1}}{z^{r-1}} \left(1 + \left(\frac{15/\pi^2}{\log x}\right)^{2r-1}\right) + x^{r/2} \log x. \]
We choose
\[ z = \left( x^{1/2} \ell (\log x)^{2\ell - 2} \left( 1 + \frac{(15/\pi^2)2^{\ell - 1}}{\log x} \right) \right)^{\frac{\ell - 1}{\ell}} \]
so as to balance the last two error terms above. Then, we have
\[ \pi_r(x, \mathbf{a}) - \text{li}(x) \sum_{d_i \geq 1 \atop (d_i, a_i) = 1 \atop 1 \leq i \leq \ell} \frac{\mu(d_1) \cdots \mu(d_\ell)}{\phi([d_1, d_2, \ldots, d_\ell])} \]
\[ \ll \ell \left( x^{1/2} \ell (\log x)^{2\ell - 2} \left( 1 + \frac{(15/\pi^2)2^{\ell - 1}}{\log x} \right) \right)^{\frac{\ell - 1}{\ell}} (|a| + x^{1/r}) \]
\[ + \left( \ell (\log x)^{2\ell - 2} \left( 1 + \frac{(15/\pi^2)2^{\ell - 1}}{\log x} \right) \right)^{\frac{\ell - 1}{\ell}} x^{2\ell - 1} \log x. \quad (2.6) \]
The above error terms are easily seen to be \(< x/\log^2 x\) for sufficiently large \(x\) provided that
\[ \ell < \frac{1}{\log 2} \log \left( \frac{1}{(\log(15\pi^{-2}) + \log \log x)} \left( \frac{\log 2}{2} \log x - \log \log x \right) \right), \]
\[ |a| < \frac{x}{\ell x^{\ell - 1} \log^2 x} = \frac{x}{\ell \log^2 x} \left( x^{1/2} \ell (\log x)^{2\ell - 2} \left( 1 + \frac{(15/\pi^2)2^{\ell - 1}}{\log x} \right) \right)^{\frac{\ell - 1}{\ell}}. \quad (2.7) \]
Note that with this bound on \(|a|\), we also have \(|a| \leq x - z^r\), as needed.

We now show that
\[ \Sigma_r(\mathbf{a}) = \prod_p \left( 1 - \frac{\nu_p(\mathbf{a})}{\phi(p^\ell)} \right) = \sum_{d_i \geq 1 \atop (d_i, a_i) = 1 \atop 1 \leq i \leq \ell} \frac{\mu(d_1) \cdots \mu(d_\ell)}{\phi([d_1, d_2, \ldots, d_\ell])}. \quad (2.8) \]
Since
\[ \sum_{d_1 | n, \ldots, d_\ell | n \atop [d_1, \ldots, d_\ell] = n} 1 \leq \tau(n)^\ell, \quad \text{and} \quad \sum_{n=1}^\infty \frac{\tau(n)^\ell}{\phi(n^\ell)} < \infty, \]
we can write
\[ \sum_{d_i \geq 1 \atop (d_i, a_i) = 1 \atop 1 \leq i \leq \ell} \frac{\mu(d_1) \cdots \mu(d_\ell)}{\phi([d_1, d_2, \ldots, d_\ell]^\ell)} = \sum_{n \geq 1} H_\ell(n) \frac{\phi(n^\ell)}{\phi(n^\ell)}. \]
where $H_1(n) = \mu_{a_1}(n)$ and for $\ell > 1$

$$H_{\ell}(n) = \sum_{d_1, \ldots, d_\ell \geq 1 \atop |d_1, \ldots, d_\ell| = n} \mu_{a_1}(d_1) \cdots \mu_{a_\ell}(d_\ell) \rho(d_1, d_2, \ldots, d_\ell).$$

Here,

$$\rho(d_1, \ldots, d_\ell) = \prod_{1 \leq i < j \leq \ell} \psi(d_i, d_j), \quad \psi(d_i, d_j) = \begin{cases} 1 & \text{if } (d_i, d_j)^* | a_i - a_j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mu_{a_i}(d)$ is multiplicative. Since $\psi$ is multiplicative on each of its components, using a method similar to the one used in the proof of Lemma 1.11, one can show that $H_{\ell}(n)$ is multiplicative and is supported only on square-free numbers. This gives

$$\sum_{n \geq 1} \frac{H_{\ell}(n)}{\phi(n')} = \prod_p \left(1 + \frac{H_{\ell}(p)}{\phi(p')}\right).$$

It remains to determine $H_{\ell}(p)$ for each prime $p$. We prove below that $-H_{\ell}(p)$ gives the number of distinct $a_i$'s modulo $p^r$ each of which are coprime to $p$. We can assume that $\ell > 1$.

Since $[d_1, \ldots, d_\ell] = p$, each $d_i \in \{1, p\}$. If $d_i = p$ for some $1 \leq i \leq \ell$, then

$$\mu_{a_1}(d_1) \mu_{a_2}(d_2) \cdots \mu_{a_\ell}(d_\ell) \rho(d_1, d_2, \ldots, d_\ell) \neq 0$$

iff $p \nmid a_i \lor \forall j, d_j = p \Rightarrow a_i \equiv a_j \mod p^r$. Hence, if $p | a_i$ for some $i$, then we must have $d_i = 1$ and these terms have no effect on $H_{\ell}(p)$. This means we can ignore these indices by renaming $a_i$'s if necessary and replacing $\ell$ with a smaller number $\ell'$ and then consider $H_{\ell'}$. Thus, we shall assume below that $p \nmid a_i$ for any $i$.

The sets

$$V_{\ell}(k) = \{i : 1 \leq i \leq \ell, a_i \equiv k \mod p^r\} \quad (0 \leq k < p^r)$$

provide a partition of the set of indices $\{1, \ldots, \ell\}$, and we can consider the terms of $H_{\ell}(p)$ coming from each non-empty set $V_{\ell}(k)$. Now, assume $V_{\ell}(k)$ is non-empty for some $k$ (necessarily coprime to $p$), and put $v = \#V_{\ell}(k)$. For each $1 \leq s \leq v$, choosing $s$ $d_i$'s equal to $p$ and the rest equal to $1$ gives a contribution of $(-1)^s$. Since there are $\binom{v}{s}$ such choices for each $s$, each non-empty $V_{\ell}(k)$ contributes

$$\sum_{s=1}^{v} \binom{v}{s} (-1)^s = (1 - 1)^v - 1 = -1.$$ 

Hence, $-H_{\ell}(p)$ gives the number of non-empty sets $V_{\ell}(k)$ for $0 \leq k < p^r$, as claimed above and in Theorem 1.1.

We next construct a permissible $\ell$-tuple and obtain the aforementioned lower bound in Remark 1.2.
Let $\ell \geq 2$ be given. If $\ell < \phi(2^r)$, then any $\ell$-tuple $\mathbf{a}$ works since $\nu_p(\mathbf{a}) \leq \ell < \phi(2^r) \leq \phi(p^r)$ for every prime $p \geq 2$. Otherwise, choose $n$ as the largest integer satisfying

$$\phi(p_n^r) \leq \ell < \phi(p_{n+1}^r),$$

where $p_n$ stands for the $n$th prime. Pick $\mathbf{a} = (k_1, k_2, \ldots, k_\ell)$, where $k_i = 1 + i(p_1p_2 \cdots p_n)^r$ for $i = 1, 2, \ldots, \ell$. It now follows that for such a tuple

$$1 = \nu_{p_j}(\mathbf{a}) < 2 < \phi(p_j) \quad j = 1, 2, \ldots, n$$

and $\nu_p(\mathbf{a}) \leq \ell < \phi(p^r)$ for $p \geq p_{n+1}$.

As for the lower bound, we consider a permissible $\ell$-tuple $\mathbf{a}$, where $\ell$ is a large number. Let $p_n$ be the largest prime such that

$$\phi(p_n^r) \leq \ell < \phi(p_{n+1}^r).$$

We now determine the size of $n$ in terms of $\ell$. As a consequence of Chebyshev’s inequalities (see [21]),

$$p_n \approx n \log 2n$$

for every $n \geq 1$. Actually cooperating with [13, Theorem 3] together with some numerical computation with PARI/GP, one has

$$0.3n \log(2n) \leq p_n \leq 3n \log(2n) \quad (2.9)$$

for every $n \geq 1$. Therefore,

$$\phi(p_n^r) = p_n^r - p_n^{r-1} \approx (n \log 2n)^r$$

for every $n \geq 1$, where the implied constants depend only on $r$. Hence,

$$(n \log 2n)^r \approx \phi(p_n^r) \leq \ell < \phi(p_{n+1}^r) \approx ((n + 1) \log 2(n + 1))^r.$$

This yields

$$\ell \approx p_n^r \approx (n \log 2n)^r, \quad (2.10)$$

where the implied constants depend only on $r$. Let $m > n + 1$ be an integer to be determined in terms of $n$. Then as $m \to \infty$

$$\prod_{p \leq p_m} \left(1 - \frac{\nu_p(\mathbf{a})}{\phi(p^r)}\right) \geq \frac{1}{\phi(2^r \cdots p_m^r)} \geq \frac{1}{2^r \cdots p_m^r} = e^{-r \sum_{p \leq p_m} \log p} = e^{-r p_m(1 + o(1))}, \quad (2.11)$$

where in the last step we used the prime number theorem in the form

$$\sum_{p \leq p_m} \log p = p_m(1 + o(1)), \quad (m \to \infty).$$

We next write

$$\prod_{p > p_m} \left(1 - \frac{\nu_p(\mathbf{a})}{\phi(p^r)}\right) \geq \prod_{p > p_m} \left(1 - \frac{\ell}{\phi(p^r)}\right).$$
Let $p > p_m$. Then, since $\ell < \phi(p_m^{r+1})$, it follows by (2.9) that
\[
\ell / \phi(p^r) < (p_{m+1}/p_m)^r \leq 10^r ((n+1) \log(2n+2)/(m \log 2m))^r
\]
for any $m > n + 1$. Choose $m = \lceil (3/2)^{1/20} \rceil n$ so that
\[
\left( \frac{10(n+1) \log(2n+2)}{m \log(2m)} \right)^r \leq \frac{2}{3}
\]
holds. This choice of $m$ yields $p_m > p_n$ for every $n \geq 1$ as well.

This gives rise to the fact that for all $p > p_m$, the inequality $\ell / \phi(p^r) < 2/3$ holds. We next use the lower bound $\log(1 - x) > -\frac{\log 3}{2} x$ being valid for $0 < x < 2/3$. Therefore, we have
\[
\sum_{p > p_m} \log \left( 1 - \frac{\ell}{\phi(p^r)} \right) \geq -\frac{3 \log 3}{2} \sum_{p > p_m} \frac{\ell}{\phi(p^r)} \\
\geq -\frac{3 \log 3}{2} \sum_{k > p_m} \frac{\ell}{k^{r-1}(k - 1)} > -\frac{\ell C}{(p_m)^{r-1}},
\]
for some positive constant $C$, where, in the last inequality, we used the fact that
\[
\sum_{k > K} \frac{1}{k^r} < \frac{1}{K^{r-1}},
\]
which is valid for each integer $K \geq 1$. Furthermore, we can write
\[
\frac{\ell}{(p_m)^{r-1}} = \frac{\ell p_m^{-1} - 1}{p_m^{r-1} p_m^{-1}} \geq \frac{\ell}{p_m^{r-1}} \geq \ell / r,
\]
where in the last two steps we used (2.10) together with the observation that $p_m > p_n$. Furthermore,
\[
\prod_{p > p_m} \left( 1 - \frac{\ell}{\phi(p^r)} \right) = e^{\sum_{p > p_m} \log(1 - \frac{\ell}{\phi(p^r)})} \geq e^{-C \ell / r}
\]
for some $C > 0$. Also with this choice of $m$ and (2.10), the left-hand side of (2.11) is easily seen to be $\geq e^{-Cr^1/r}$ for some positive constant $C$ not necessarily the same as above. Therefore the lower bound in (1.2) now follows.

2.2. Proof of Theorem 1.3

Using (2.6) and (2.8), we have for every $z, H \geq 1$
\[
\sum_{|a| \leq H} \mathcal{S}_r(a) - \sum_{d_1, \ldots, d_r \leq z} \mu(d_1) \mu(d_2) \cdots \mu(d_r) \phi((d_1, d_2, \ldots, d_r)^r) \sum_{0 < |a| \leq H} \frac{1}{(d_1, a_1)^1 \cdots (d_r, a_r)^1} \bigg| a_i - a_j \bigg| 1 \leq i, j \leq r
\]
\[
< \frac{\ell (2H)^r (15 \pi^{-2} (1 + \log 3z))^2}{2^{r-1} x^{r-1}}.
\]
Recall that the first sum above runs over \( \ell \)-tuples \( a \) with non-zero entries. Writing

\[
\sum_{0 < |a_i| \leq H \atop (d_i, a_i) = 1 \atop 1 \leq i \leq \ell} 1 = \sum_{m_i \equiv a_i \mod d_i} 1 \sum_{0 < |a_i| \leq H \atop 1 \leq i \leq \ell} 1,  
\tag{2.13}
\]

and using the estimate

\[
\left| \sum_{n \equiv a \mod d} 1 - \frac{H}{d} \right| \leq 1, 
\]

being valid for all \( H \geq 1 \) and \( 1 \leq a \leq d \), we get

\[
\sum_{0 < |a_i| \leq H \atop a_i \equiv m_i \mod d_i} 1 = \prod_{i=1}^{\ell} \left( \frac{2H}{d_i^r} + 2\theta_i \right) = \frac{(2H)^{\ell}}{(d_1d_2 \cdots d_{\ell})^r} + O(2^\ell E(d_1, d_2, \ldots, d_{\ell})),  
\tag{2.14}
\]

where \( |\theta_i| \leq 1 \) and

\[
E(d_1, d_2, \ldots, d_{\ell}) = 1 + \sum_{i=1}^{\ell} \frac{H}{d_i} + \sum_{1 \leq i, j \leq \ell, i \neq j} \frac{H^2}{(d_i d_j)^r} + \cdots + \sum_{i=1}^{\ell} \frac{H^{\ell-1}}{(d_1 d_2 \cdots d_{i-1} d_{i+1} \cdots d_{\ell})^r}.
\]

Note here that the implied constant in the \( O \)-term in (2.14) is independent of \( m_i \)'s. Inserting (2.14) into (2.13) and using the fact that

\[
\sum_{m_i \equiv a_i \mod d_i \atop (m_i, d_i) = 1 \atop 1 \leq i \leq \ell} 1 \equiv \phi([d_1, \ldots, d_{\ell}]^r),
\]

which follows by the Chinese Remainder Theorem, we obtain

\[
\sum_{0 < |a_i| \leq H \atop (d_i, a_i) = 1 \atop 1 \leq i \leq \ell} 1 = \sum_{m_i \equiv a_i \mod d_i \atop (m_i, d_i) = 1 \atop 1 \leq i \leq \ell} \left( \frac{(2H)^{\ell}}{(d_1 d_2 \cdots d_{\ell})^r} + O(2^\ell E(d_1, d_2, \ldots, d_{\ell})) \right)
\]

\[
= \phi([d_1, \ldots, d_{\ell}]^r) \left( \frac{(2H)^{\ell}}{(d_1 d_2 \cdots d_{\ell})^r} + O(2^\ell E(d_1, d_2, \ldots, d_{\ell})) \right).  
\tag{2.15}
\]
We have
\[ \sum_{d_1, \ldots, d_\ell \leq z} E(d_1, \ldots, d_\ell) \leq \sum_{i=0}^{\ell} z^{\ell-i} \binom{\ell}{i} H^i \zeta(r)^i - H^\ell \zeta(r)^\ell \]
\[ = (H \zeta(r) + z)^\ell - (H \zeta(r))^\ell \leq \ell \ell (H \zeta(r) + z)^{\ell-1}, \]
(2.16)

where the last inequality follows by the mean value theorem. Inserting (2.15) and (2.16) into (2.12), we conclude that
\[ \sum |a| \leq H S(r(a)) - (2H) ^\ell \left( \sum_{d \leq z} \frac{\mu(d)}{d^r} \right)^\ell \ll \ell \ell z^{\ell} (H \zeta(r) + z)^{\ell-1} \]
\[ + \ell (2H) ^\ell \left( 15 \pi^{-2} (1 + \log 3z) \right)^{2^{\ell-1}} z^{\ell-1}. \]

By the mean value theorem and the inequality
\[ \left| \zeta(r)^{-1} - \sum_{d \leq z} \frac{\mu(d)}{d^r} \right| \leq \frac{1}{(r-1) [z]^{r-1}}, \]
we have
\[ \left( \sum_{d \leq z} \frac{\mu(d)}{d^r} \right)^\ell - \zeta(r)^{-\ell} \ll \frac{\ell \zeta(r)^{\ell-1}}{(r-1) z^{r-1}}. \]

Hence, assuming \( 1 \leq z < H/(3e) \) it follows that
\[ \sum_{|a| \leq H} \mathcal{G}_r(a) - \left( \frac{2H}{\zeta(r)} \right)^\ell \ll z^{\ell} 2^{\ell} H^\ell (\zeta(r) + 1)^{\ell-1} + \frac{\ell (2H) ^\ell \zeta(r)^{\ell-1}}{(r-1) z^{r-1}} \]
\[ + \ell (2H) ^\ell \left( 15 \pi^{-2} (\log H) \right)^{2^{\ell-1}} z^{\ell-1}. \]

We can omit the second term on the right when \( H \) is large as it is dominated by the third term. Thus, choosing
\[ z = \left( \frac{H \left( 15 \pi^{-2} (\log H) \right)^{2^{\ell-1}}}{(1 + \zeta(r))^{\ell-1}} \right)^{1/r} \]

to balance the two remaining terms, we arrive at the claimed result in (1.1) since for \( 1 \leq \ell \leq (1/\log 2 - \varepsilon) \log \log H \) our assumption above that \( 1 \leq z < H/(3e) \) is satisfied for \( H \) sufficiently large.
2.3. Proof of Theorem 1.4

Let \( A \) be a fixed real number with \( A > 1/(r-1) \). Assume that \( \log^A x \leq H < x \). By (2.1) we can write

\[
\sum_{|a| \leq H} \pi_r(x, a) - \sum_{p \leq x} \sum_{d_1, \ldots, d_\ell \leq z} \mu(d_1) \mu(d_2) \cdots \mu(d_\ell) \sum_{0 < |a| \leq H} 1
\]

\[
\sum_{a_1 \equiv -p \mod d_1, \ldots, a_\ell \equiv -p \mod d_\ell} \mu(d_1) \mu(d_2) \cdots \mu(d_\ell)
\]

\[
\sum_{|a| \leq H} \pi_r(x, a) - \sum_{p \leq x} \sum_{d_1, \ldots, d_\ell \leq z} \mu(d_1) \mu(d_2) \cdots \mu(d_\ell) \sum_{0 < |a| \leq H} 1
\]

\[
\sum_{a_1 \equiv -p \mod d_1, \ldots, a_\ell \equiv -p \mod d_\ell} \mu(d_1) \mu(d_2) \cdots \mu(d_\ell)
\]

\[
\ll \ell z^{\ell-1}(2H)^\ell ((2H) + x^{1/r}) + \frac{x\ell(2H)^\ell (1 + \log 3z)^{2\ell-1}}{z\ell^{\ell-1}} \quad (2.17)
\]

for every \( 1 \leq z \leq (x - H)^{1/r} \). For \( p \leq x \), we obtain by (2.14) and (2.16) that

\[
\sum_{d_1, \ldots, d_\ell \leq z \atop p \nmid d_1} \mu(d_1) \mu(d_2) \cdots \mu(d_\ell) \sum_{0 < |a| \leq H} \frac{1}{a_1 \equiv -p \mod d_1, \ldots, a_\ell \equiv -p \mod d_\ell} \sum_{0 < |a| \leq H} 1
\]

\[
\sum_{d_1, \ldots, d_\ell \leq z \atop p \nmid d_1} \mu(d_1) \mu(d_2) \cdots \mu(d_\ell) \sum_{0 < |a| \leq H} \frac{1}{a_1 \equiv -p \mod d_1, \ldots, a_\ell \equiv -p \mod d_\ell}
\]

\[
\left( 2H \sum_{d \leq z \atop p \nmid d} \frac{\mu(d)}{d^r} \right)^\ell + O(z\ell^2 \ell^\ell (H \zeta(r) + z)^{\ell-1}). \quad (2.18)
\]

By the mean value theorem and the inequality

\[
\left| \sum_{d \leq z \atop p \nmid d} \frac{\mu(d)}{d^r} - \zeta(r)^{-1} \left( 1 - \frac{1}{p^r} \right)^{-1} \right| \leq \frac{1}{(r-1)|z|^{r-1}},
\]

we obtain

\[
\left( \sum_{d \leq z \atop p \nmid d} \frac{\mu(d)}{d^r} \right)^\ell = \left( 1 - \frac{1}{p^r} \right)^{-\ell} \zeta(r)^{-\ell} + O \left( \frac{\ell(2\zeta(r))^{\ell-1}}{z^{r-1}} \right). \quad (2.19)
\]
Therefore, combining (2.18) and (2.19) and inserting the result into (2.17) gives
\[
\sum_{0 < |a| \leq H} \pi_r(x, a) - \left( \frac{2H}{\zeta(r)} \right) \sum_{p \leq x} \left( 1 - \frac{1}{p^r} \right)^{-\ell} \ll \ell z^{\ell-1}(2H)^\ell((2H) + x^{1/r}) + \frac{x\ell(2H)^\ell(1 + \log 3z)^{2^\ell - 1}}{z^{r-1}} \\
+ \frac{x z \ell((2H)^\ell-1(\zeta(r) + 1)^{\ell-1})}{\log x} + \frac{x \ell(2H)^\ell(2\zeta(r))^{\ell-1}}{z^{r-1} \log x}
\]
for every \(2 \leq z \leq \min\{(x - H)^{1/r}, H/(3\varepsilon)\} \). Finally noting that
\[
\sum_{p \leq x} \left( 1 - \frac{1}{p^r} \right)^{-\ell} = \pi(x) + O(\ell^2),
\]
where the implied constant depends only on \(r \geq 2\), we conclude that
\[
\sum_{0 < |a| \leq H} \pi_r(x, a) - \left( \frac{2H}{\zeta(r)} \right) \pi(x) \ll \ell z^{\ell-1}(2H)^\ell((2H) + x^{1/r}) \\
+ \frac{x \ell(2H)^\ell(\log H)^{2^\ell - 1}}{z^{r-1}} + \frac{x z \ell((2H)^\ell-1(\zeta(r) + 1)^{\ell-1})}{\log x}
\]
for every \(2 \leq z \leq \min\{(x - H)^{1/r}, H/(3\varepsilon)\} \). We choose
\[
z = \left( \frac{2H (\log H)^{2^\ell - 1} \log x}{(1 + \zeta(r))^{\ell-1}} \right)^{1/r}
\]
so that the last two terms are
\[
\ll \frac{x \ell(2H)^\ell-1+\frac{\ell}{2} (\log H)^{2^\ell - 1} (\zeta(r) + 1)^{(\ell-1)(r-1)}}{\log^{1-1/r} x}.
\]
The proof follows after a quick check that \(z \leq \min\{(x - H)^{1/r}, H/(3\varepsilon)\} \) holds whenever (1.2) is satisfied.

### 2.4. Proof of Theorem 1.6

We shall make use of the circle method. Throughout the proof, we fix \(k \in \mathbb{Z}\setminus\{0\}\) and \(r \geq 2\). For \(\alpha \in \mathbb{R}\), we define the following auxiliary function:
\[
f_{r,k}(\alpha, N) = \sum_{p \in \mathcal{P}_{r,k}(N)} (\log p)e(\alpha p).
\]
Set \(R_s(N) = \#\{(p_1, p_2, \ldots, p_s) \in (\mathcal{P}_{r,k})^s : N = p_1 + p_2 + \cdots + p_s\} \). Then
\[
R_s(N) = \int_0^1 (f_{r,k}(\alpha, N))^s e(-\alpha N)d\alpha.
\]
Let \( \mathcal{L} = \log^A N \), where \( A > 0 \) is to be determined. Then we set

\[
\mathfrak{M} = \bigcup_{1 \leq q \leq \mathcal{L}} \mathfrak{M}(a, q),
\]

where \( \mathfrak{M}(a, q) = \left[ \frac{a}{q}, \frac{a}{q} + \frac{6}{qN} \right] \) and we define the minor arc as \( m = [0, 1) \mod \mathfrak{M} \).

As the integrand is of period 1, we can think of \( \mathfrak{M}(1, 1) \) as the interval \( \left[ -\frac{6}{N}, \frac{6}{N} \right] \). We first do the minor arc analysis.

**Lemma 2.1.** Suppose that \( |a - a/q| < 2/N \) with \( q \geq 1 \) and \( (a, q) = 1 \). Then,

\[
f_r, k(\alpha, N) \ll 2^{a(q)} q^{-1/2} N \log^3 N + N^{(r+1)/2} q^{(r-1)/2r} \log^{(3r-2)/r} N
\]

\[
+ \begin{cases} 
N^{5/6} \log^{8/3} N & \text{if } r = 2, \\
N^{4/5} \log^3 N & \text{if } r > 2.
\end{cases}
\]

The implied constant depends at most on \( k \) and \( r \).

**Proof.** We prove only the case \( r > 2 \) since the case \( r = 2 \) follows similarly. We may assume that \( q \leq N/\log^4 N \), otherwise the above inequality is worse than the trivial estimate of \( f_r, k(\alpha, N) \).

By Lemma 1.15, with the choices \( Q = 1 \) and the arithmetic function \( \frac{\log p}{\log N} e(\alpha p) \), one has

\[
f_r, k(\alpha, N) = \sum_{d \leq z} \mu(d) \sum_{p \leq N, p \equiv -k (\text{mod } d')} (\log p) e(\alpha p) + O \left( \frac{N \log N}{z^{r-1}} \right)
\]

for any \( 1 < z \ll_k N^{1/r} \), where the implied constant depends on \( k \) and \( r \). By Lemma 1.13, one has

\[
f_r, k(\alpha, N) \ll \sum_{d \leq z} \mu^2(d) \left( \frac{(q, d')}d \frac{q^{1/2} N^{1/2} + N^{4/5} d^{2r/5}}{(q, d')^{1/2}} \right) \log^3 N + \frac{N \log N}{z^{r-1}}.
\]

Here, the trivial upper bound

\[
\sum_{d \leq z} \frac{1}{(d', q)^{1/2}} \leq \sum_{d \leq z} 1 \leq z
\]

and the less trivial upper bound

\[
\sum_{d \leq z} \frac{\mu^2(d)(d', q)} d \leq \prod_{p \leq z} \left( 1 + \frac{(p^r, q)} p \right) \ll \prod_{p \leq z} \left( 1 + \frac{(p^r, q)} p \right) \ll 2^{a(q)}
\]
yield
\[ f_{r,k}(\alpha, N) \ll \frac{2^{\omega(q)}N}{q^{1/2}} \log^3 N + \left( q^{1/2} N^{1/2} \log^3 N \right) z + N^{4/5} \log^3 N + \frac{N \log N}{z^{r-1}}. \]

Appropriate choice of \(1 \leq z \ll k N^{1/r}\) via \([7, \text{Lemma 2.4}]\) together with the initial assumption \(q \leq N/\log^4 N\) gives the desired result.

Take \(\alpha \in \mathbb{m}\). By Dirichlet’s approximation theorem, there are co-prime integers \(a\) and \(q\) with \(1 \leq a \leq q \leq N/\mathcal{L}\) such that
\[
|\alpha - \frac{a}{q}| < \frac{\mathcal{L}}{q N}.
\]
Since \(\alpha \in \mathbb{m}\), we have \(\mathcal{L} < q\), yielding by Lemma 2.1 and the divisor bounds \(2^{\omega(q)} \ll \tau(q) \ll q^\varepsilon\),
\[
\sup_{\alpha \in \mathbb{m}} |f_{r,k}(\alpha, N)| \ll \frac{N}{\log^{A_r} N},
\]
where for \(r \geq 2\),
\[
A_r = \frac{(r-1)}{2r} A - \frac{(3r-2)}{r}
\]
if \(A > 4\) and \(N\) is large. It follows now by a standard argument
\[
\int_{\mathbb{m}} (f_{r,k}(\alpha, N))^* e(-\alpha N) \ll \frac{N^{s-1}}{(\log N)^{(s-2)A_r}}.
\]
This estimate is nontrivial if \((s-2)A_r > 0\) and this forces
\[
A > 6 + 2/(r-1) > 8.
\]

**Lemma 2.2.** Let \(\alpha \in \mathbb{M}(a, q)\) and let \(\beta = \alpha - \frac{a}{q}\). Then,
\[
f_{r,k}(\alpha, N) = \sum_{l \mod q} e\left(\frac{al}{q}\right) \sum_{d \mid \mid k+l} \frac{\mu_k(d)}{\phi([q, d^r]k+l)} \sum_{n \leq N} e(\beta n) \ll \frac{N}{\log^B N}
\]
for any fixed \(B > 0\), where the implied constant depends only on \(A, B, k\) and \(r\). Furthermore,
\[
\sum_{l \mod q} e\left(\frac{al}{q}\right) \sum_{d \mid \mid k+l} \frac{\mu_k(d)}{\phi([q, d^r]k+l)} \ll \frac{2^{\omega(q)}}{\phi(q)}
\]
where the constant in \(2.24\) is independent of \(k\).

**Proof of Lemma 2.2** To prove \(2.24\), we swap the summations and use \(1.3\) to get
\[
\sum_{1 \leq l \leq q} e\left(\frac{al}{q}\right) \ll 1,
\]
which then yields
\[
\sum_{l \equiv a \pmod{q}} e\left(\frac{al}{q}\right) \sum_{d=1}^{\infty} \frac{\mu_k(d)}{\phi([q, d^2])} \lesssim \frac{1}{\phi(q)} \prod_{p \nmid q} \left(1 + \frac{\phi\left((q, p^r)\right)}{\phi(p^r)}\right) \prod_{p | q} \left(1 + \frac{1}{\phi(p^r)}\right) \leq \frac{2^{\omega(q)}}{\phi(q)}
\]

Next, we write
\[
f_{r,k}(\alpha, N) = \sum_{p \in \mathcal{P}_{r,k}(N)} \sum_{p \equiv l \pmod{q}} e(\alpha p) \log p + O\left(\sum_{p \nmid q} \log p\right)
\]
\[
= \sum_{1 \leq \ell \leq q} \sum_{p \in \mathcal{P}_{r,k}(N)} e(\ell p) \log p + O(\omega(q) \log q).
\]

Given any $B' > A$, by Lemma 1.16 and partial integration we obtain
\[
\sum_{p \in \mathcal{P}_{r,k}(N)} e(\ell p) \log p = \sum_{d \geq 1} \frac{\mu_k(d)}{\phi([q, d^2])} \int_{\sqrt{N}}^{N} e(\beta x) dx
\]
\[
\quad + \int_{\sqrt{N}}^{N} e(\beta x) d\mathfrak{e}(x; l, q) + O(\sqrt{N})
\]
\[
= \sum_{d \geq 1} \frac{\mu_k(d)}{\phi([q, d^2])} \left(\sum_{n \leq N} e(\beta n) + O(\mathcal{L}/q + \sqrt{N})\right)
\]
\[
\quad + O \left(\frac{N \mathcal{L}}{q \log B'}\right)
\]
for sufficiently large $N$, since
\[
\sum_{1 \leq n \leq N} e(\beta n) = \int_{\sqrt{N}}^{N} e(\beta x) dx + 2\pi i \beta \int_{0}^{N} e(\beta x) [x] dx + O(\sqrt{N}).
\]
Hence, choosing $B' = B + A$ gives (2.23).

We next establish the asymptotic formula in (1.3) briefly. We write
\[
R_\alpha(N) = \int_{0}^{1} (f_{r,k}(\alpha, N))^* e(-\alpha N) da = \left(\int_{\mathfrak{m}} + \int_{\mathfrak{m}}\right) (f_{r,k}(\alpha, N))^* e(-\alpha N) da,
\]
where $\mathfrak{m}$ and $\mathfrak{m}$ are defined in (2.20). Since we have (2.22), we shall look at only the major arc analysis; that is, we shall determine the asymptotic behavior of
\[
\int_{\mathfrak{m}} (f_{r,k}(\alpha, N))^* da = \sum_{q \leq \mathcal{L}} \sum_{\alpha \equiv 1 \pmod{q}} \int_{\mathfrak{m}(\alpha, q)} (f_{r,k}(\alpha, N))^* e(-\alpha N) da.
\]
By Lemma 2.2 together with the upper bound in (2.24), it follows that for \( \alpha \in \mathcal{M}(a, q) \)

\[
(f_{r,k}(\alpha, N))^s = \left( \sum_{l \equiv q \mod q \atop (l,q)=1} e\left(\frac{al}{q}\right) \sum_{d=1}^{\infty} \frac{\mu_k(d)}{\phi([q, d^r])} \sum_{n \leq N} e(\beta n) + O\left(\frac{N}{\log B N}\right) \right)^s
\]

\[
= \left( \sum_{l \equiv q \mod q \atop (l,q)=1} e\left(\frac{al}{q}\right) \sum_{d=1}^{\infty} \frac{\mu_k(d)}{\phi([q, d^r])} \sum_{n \leq N} e(\beta n) \right)^s + O\left(\frac{N^s}{\log B N}\right).
\]

This, after an obvious change of variable, yields

\[
\sum_{q \leq L} \sum_{a \equiv q \mod q \atop (a,q)=1} \int_{\mathcal{M}(a,q)} (f_{r,k}(\alpha, N))^s e(-\alpha N) d\alpha
\]

\[
= \sum_{q \leq L} \sum_{a \equiv q \mod q \atop (a,q)=1} \left( \sum_{l \equiv q \mod q \atop (l,q)=1} e\left(\frac{al}{q}\right) \sum_{d=1}^{\infty} \frac{\mu_k(d)}{\phi([q, d^r])} \right)^s e\left(-\frac{\alpha N}{q}\right) J(N, \ell/qN)
\]

\[
+ O\left(\frac{N^{s-1}}{\log B^{2s-1} N}\right),
\]

where

\[
J(N, \ell) = \int_{-t}^{t} \left( \sum_{n \leq N} e(\beta n) \right)^s e(-\beta N) d\beta
\]

for any \( 0 < t \leq 1/2 \). Recalling the simple upper bound

\[
\sum_{n \leq x} e(\beta n) \ll \min \left\{ x, \frac{1}{||\beta||} \right\},
\]

it follows that

\[
\int_{-1/2}^{t} \left( \sum_{n \leq N} e(\beta n) \right)^s e(-\beta N) d\beta + \int_{t}^{1/2} \left( \sum_{n \leq N} e(\beta n) \right)^s e(-\beta N) d\beta \ll \frac{1}{t^{s-1}}.
\]

Therefore,

\[
J(N, \ell/qN) - J(N, 1/2) \ll \frac{N^{s-1} N^{s-1}}{\ell^{s-1}}.
\]
Replacing $\mathcal{J}(N, \mathcal{L}/qN)$ by $\mathcal{J}(N, 1/2)$ in (2.25) using the upper bound (2.24) introduces an error

$$\ll \frac{N^{s-1}}{L^{s-1}} \sum_{q \leq L} \frac{2^{s}\omega(q)q^{s-1}}{\phi^{s-1}(q)} \ll \frac{N^{s-1}}{L^{s-1}} \sum_{q \leq L} 2^{s}\omega(q) \left( \prod_{p \mid q} p^{1-1}\right)^{s-1}$$

$$\ll \frac{N^{s-1}}{L^{s-1}} \sum_{q \leq L} \tau(q)2^{s-1} \ll \frac{N^{s-1}}{L^{s-1}} L(\log L)^{2^{s-1}-1},$$

where we used

$$\sum_{n \leq x} \tau(n)^k \ll x(\log x)^{2k-1}.$$

Therefore, combining (2.25) and this estimate we derive

$$\int_{\mathbb{R}} (f_{r,k}(\alpha, N))^s \, d\alpha - \mathcal{G}_{r,k}(N, \mathcal{L}) \mathcal{J}(N, 1/2)$$

$$\ll \frac{N^{s-1}}{\log^{2\gamma-2} \mathcal{L} N} + \frac{N^{s-1}}{L^{s-2}} (\log L)^{2^{s-1}-1},$$

where

$$\mathcal{G}_{r,k}(N, \mathcal{L}) = \sum_{q \leq L} \sum_{a \mod q} \left( \sum_{\chi \mod q} \frac{\mu_k(d)}{\phi(d)} \sum_{d=1}^{\infty} \frac{\phi([q, d^r])}{\phi(q)} \right)^s x \left( \frac{aN}{q} \right).$$

Now, using the upper bound in (2.24) and Rankin’s trick, one has

$$\mathcal{G}_{r,k}(N, \mathcal{L}) - \mathcal{G}_{r,k}(N) \ll \sum_{q > L} \frac{2^{s}\omega(q)q^{s'}}{\phi^{s-1}(q)} \ll \frac{1}{\log^{s'} \mathcal{L}} \sum_{q=1}^{\infty} \frac{2^{s}\omega(q)q^{s'}}{\phi^{s-1}(q)}$$

for any $s' < s-2$, where the singular series $\mathcal{G}_{r,k}(N)$ is given by (1.4).

Using $\phi(q) = q \prod_{p \mid q} (1 - 1/p)$, we see that

$$\sum_{q=1}^{\infty} \frac{2^{s}\omega(q)q^{s'}}{\phi^{s-1}(q)} \leq \sum_{q=1}^{\infty} \frac{2^{s}\omega(q)q^{s'}}{q^{s-1}} \ll \prod_{p} \left( 1 + \frac{4^{s}}{p^{s-s'-1}} \right)$$

$$< \zeta(s-s')^{-1} 4^{s'} < \left( 1 + \frac{1}{s-s'-2} \right)^{4^{s'}}.$$

Choosing $s' = s-2 - \frac{1}{\log L}$, it follows that

$$\mathcal{G}_{r,k}(N, \mathcal{L}) - \mathcal{G}_{r,k}(N) \ll \frac{(\log L)^{4^{s'}}}{L^{s-2}}.$$

Since $\mathcal{J}(N, 1/2)$ counts the number of ways to express $N$ as the sum of $s$ positive integers, we have

$$\mathcal{J}(N, 1/2) = \frac{(N-1)!}{(N-s)!(s-1)!} = \frac{N^{s-1}}{(s-1)!} + O(N^{s-2}),$$
where the constant in the error term certainly depends on $s$. Finally, noting
\[
\mathcal{S}_{r,k}(N) \ll \sum_{q=1}^{\infty} \frac{q^{s-1}(q)}{\phi^{s-1}(q)} \ll 1,
\]
and using \([2.22]\) we arrive at
\[
R_s(N) - \frac{\mathcal{S}_{r,k}(N)N^{s-1}}{(s-1)!} \ll N^{s-2} + \frac{N^{s-1}(\log L)^{2s-1}}{L^{s-2}} + \frac{N^{s-1}}{\log B^{-2A}N} + \frac{N^{s-1}}{(\log N)^{(s-2)A}},
\]
where $A_r$ is defined in \([2.21]\). This proves \((1.3)\) in Theorem 1.6. We next prove the rest of Theorem 1.6.

**Proposition 2.3.** As a function of $q$, the following function is multiplicative in $q$:
\[
F(q,l) = \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-1} \sum_{d=1}^{\infty} \frac{\mu_k(d)}{\phi(q,d^r)};
\]
that is, for every integer $l$ and non-zero integer $k$, $F(q_1,q_2,l) = F(q_1,l)F(q_2,l)$ whenever $(q_1,q_2) = 1$.

**Proof of Proposition 2.3** We have
\[
F(q,l) = \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-1} \sum_{d=1}^{\infty} \frac{\mu_k(d)}{\phi(q,d^r)}
\]
\[
= \frac{1}{\phi(q)} \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-1} \prod_p \left(1 + \frac{\mu_k(p)^\theta(q,p^r)\phi(q,p^r)}{\phi(p^r)}\right),
\]
where $\theta(q,d) = 1$ if $(q,d) | k + l$ and 0 otherwise. Hence,
\[
F(q,l) = \frac{1}{\phi(q)} \prod_{p|q} \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right)^{-1} \prod_{p|q} \left(1 + \frac{\mu_k(p)^\theta(q,p^r)\phi(q,p^r)}{\phi(p^r)}\right).
\]
Since $\theta$ is multiplicative in $q$ and $d$, the result follows. \(\square\)

**Lemma 2.4 (Non-vanishing of the Singular Series).** For any non-zero integer $k$ and any integer $s \geq 3$, the singular series $\mathcal{S}_{r,k}(N)$ given by \([2.4]\) satisfies $\mathcal{S}_{r,k}(N) > C$ for some $C > 0$ independent of $N$, provided that the parities of $N$...
and $s$ are the same and in the case $k$ is odd and $r = 2$, one also needs $4 \mid ks + N$ when $N$ is odd, and $4 \nmid ks + N$ when $N$ is even.

**Proof.** Define

$$b_{r,N}(q) = \prod_p \left( 1 + \frac{\mu_k(p)}{\varphi(p^r)} \right)^{-s} \sum_{0 < a \leq q \atop (a, q) = 1} \left( \sum_{1 \leq l \leq q \atop (l, q) = 1} e \left( \frac{al}{q} \right) \left( \prod_{d \mid (q, d^r)} \frac{\mu_k(d)}{\varphi([q, d^r])} \right)^s \right) \times \left( -aN \over q \right).$$

We shall show that, for fixed $r$ and $N$, $b$ is a multiplicative function of $q$, and then investigate properties of $b$ at prime powers. First, we write

$$b_{r,N}(q) = \sum_{0 < a \leq q \atop (a, q) = 1} \left( \sum_{0 < l \leq q \atop (l, q) = 1} e \left( \frac{al}{q} \right) F(q, l) \right) e \left( \frac{-aN}{q} \right),$$

where $F(q, l)$ is given by (2.26). Assume that $q = q_1q_2$, where $(q_1, q_2) = 1$. Then,

$$b_{r,N}(q_1q_2) = \sum_{0 < a \leq q_1q_2 \atop (a, q_1q_2) = 1} \left( \sum_{0 < l \leq q_1q_2 \atop (l, q_1q_2) = 1} e \left( \frac{al}{q_1q_2} \right) F(q_1q_2, l) \right) e \left( \frac{-aN}{q_1q_2} \right).$$

Given $l \equiv q_1q_2$ and $(l, q_1q_2) = 1$, there are unique $l_1 \mod q_1$ and $l_2 \mod q_2$ with $(l_1, q_1) = 1$ and $(l_2, q_2) = 1$ such that

$$l \equiv l_1q_2 + l_2q_1 \mod q_1q_2.$$

Therefore, by Proposition 2.3,

$$\sum_{0 < l \leq q_1q_2 \atop (l, q_1q_2) = 1} e \left( \frac{al}{q_1q_2} \right) F(q_1q_2, l) = \sum_{0 < l_1 \leq q_1 \atop (l_1, q_1) = 1} \sum_{0 < l_2 \leq q_2 \atop (l_2, q_2) = 1} e \left( \frac{al_1q_2}{q_1q_2} \right) F(q_1, l_1q_2) F(q_2, l_2q_1)$$

$$\times F(q_1, l_1q_2) F(q_2, l_2q_1)$$

$$= \sum_{0 < l_1 \leq q_1 \atop (l_1, q_1) = 1} e \left( \frac{al_1q_2}{q_1} \right) F(q_1, l_1q_2) \sum_{0 < l_2 \leq q_2 \atop (l_2, q_2) = 1} e \left( \frac{al_2q_1}{q_2} \right) F(q_2, l_2q_1).$$
Similarly, picking \(a_i \mod q_i\) with \((a_i, q_i) = 1\) such that \(a \equiv a_1q_2 + a_2q_1 \mod q_1q_2\), and replacing \(l_1q_2\) by \(l_1\) and \(l_2q_1\) by \(l_2\), it follows that

\[
b_{r,N}(q_1q_2) = \sum_{0 < a_1 \leq q_1} \left( \sum_{0 < l_1 \leq q_1} e\left( \frac{al_1}{q_1} \right) F(q_1, l_1) \right)^s e\left( \frac{-a_1N}{q_1} \right)
\]

\[
\cdot \sum_{0 < a_2 \leq q_2} \left( \sum_{0 < l_2 \leq q_2} e\left( \frac{al_2}{q_2} \right) F(q_2, l_2) \right)^s e\left( \frac{-a_2N}{q_2} \right)
\]

\[= b_{r,N}(q_1)b_{r,N}(q_2).\]

This proves the desired multiplicativity of \(b_{r,N}\). We shall now investigate \(b_{r,N}\) at prime powers. We write

\[
F(p^m, l) = \prod_{q \text{ prime}} \left( 1 + \frac{\mu_k(q)}{\phi(q^r)} \right)^{-1} \sum_{d \mid p^m} \frac{\mu_k(d)}{\phi([p^m, d^r])}
\]

\[= \frac{1}{\phi(p^m)} \left( 1 + \frac{\mu_k(p^m)}{\phi(p^r)} \right)^{-1} \left( 1 + \frac{\mu_k(p)\phi(p^m, p^r)\rho((p^m, p^r), l)}{\phi(p^r)} \right),\]

(2.28)

where \(\rho(v, l) = 1\) if \(v \mid (l + k)\) and equals 0 otherwise. Note that for \(p \mid k\), we have \(F(p^m, l) = \frac{1}{\phi(p^m)}\), in which case Lemma 1.10 gives

\[
b_{r,N}(p^m) = \frac{1}{\phi^s(p^m)} \sum_{0 < a \leq p^m \atop (a, p) = 1} \left( \sum_{0 < l \leq p^m \atop \text{lcm}(l, p^m)} e\left( \frac{al}{p^m} \right) \right)^s e\left( \frac{-aN}{p^m} \right) = \frac{\mu^s(p^m)c_{p^m}(N)}{\phi^s(p^m)}.
\]

By absolute convergence we can therefore write

\[
\prod_p \left( 1 + \frac{\mu_k(p)}{\phi(p^r)} \right)^{-s} \theta_{r,k}(N) = \sum_{q \geq 1} b_{r,N}(q)
\]

\[= \prod_{p \mid k} \left( 1 + \frac{(-1)^s c_p(N)}{\phi^s(p)} \right) \prod_{p\notmid k} (1 + b_{r,N}(p) + b_{r,N}(p^2) + \cdots).\]
For any $m \geq 1$ and $p \nmid k$, we obtain using (2.27) and (2.28) that

$$b_{r,N}(p^m) = \frac{1}{\phi^s(p^r)} \left( 1 - \frac{1}{\phi(p^r)} \right)^{-s} \times \sum_{0 < a \leq p^m \atop (a,p^r) = 1} \left( \sum_{0 < l \leq p^m \atop (l,p^r) = 1} e \left( \frac{al}{p^m} \right) \left( 1 - \frac{p^{\min(m,r)\rho(p^{\min(m,r)})}}{p^r} \right) \right)^s \times \left( -\frac{aN}{p^m} \right).$$

For $m \geq 2$,

$$\sum_{0 < l \leq p^m \atop (l,p^r) = 1} e \left( \frac{al}{p^m} \right) = \mu(p^m) = 0.$$

Therefore, for $2 \leq m \leq r$ and $p \nmid k$,

$$b_{r,N}(p^m) = \frac{(-1)^s}{\phi^s(p^r)} \left( 1 - \frac{1}{\phi(p^r)} \right)^{-s} \times \sum_{0 < a \leq p^m \atop (a,p^r) = 1} \left( \sum_{0 < l \leq p^m \atop (l,p^r) = 1} e \left( \frac{al}{p^m} \right) \right)^s e \left( \frac{-aN}{p^m} \right).$$

Note that since the sum over $l$ reduces to one term only and $p^{s(r-m)} \phi^s(p^m) = \phi^s(p^r)$, we get

$$b_{r,N}(p^m) = \frac{(-1)^s}{\phi^s(p^r)} \left( 1 - \frac{1}{\phi(p^r)} \right)^{-s} \sum_{0 < a \leq p^m \atop (a,p^r) = 1} e \left( \frac{-aks - aN}{p^m} \right) = \frac{(-1)^s}{\phi^s(p^r)} \left( 1 - \frac{1}{\phi(p^r)} \right)^{-s} c_{p^m}(ks + N).$$

If $m > r$, then by (1.5)

$$\sum_{0 < l \leq p^m \atop (l,p^r) = 1} e \left( \frac{al}{p^m} \right) = 0.$$
since \((p^{m-r}, p') \neq 1\). Hence, we get \(b_{r,N}(p^m) = 0\) when \(m > r\) and \(p \nmid k\). It remains to treat the case \(m = 1\) and \(p \nmid k\). This time we have

\[
b_{r,N}(p) = \frac{1}{\phi^*(p)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \sum_{0 < a \leq p} \left(\sum_{0 < l \leq p} e\left(\frac{al}{p}\right) \left(1 - \frac{\rho(p, l)}{p^{r-1}}\right)\right)^s \left(-\frac{aN}{p}\right).
\]

Here,

\[
\sum_{0 < l \leq p} e\left(\frac{al}{p}\right) \left(1 - \frac{\rho(p, l)}{p^{r-1}}\right) = -1 - \frac{1}{p^{r-1}} \sum_{0 < l \leq p} e\left(\frac{al}{p}\right)
\]

\[= -1 - \frac{1}{p^{r-1}} e\left(-\frac{ak}{p}\right).
\]

It follows by the binomial theorem that

\[
b_{r,N}(p) = \frac{(-1)^s}{\phi^*(p)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \sum_{0 < a \leq p} \left(\sum_{i=0}^{s} \binom{s}{i} \frac{1}{p^{i(r-1)}} e\left(-\frac{a(ki + N)}{p}\right)\right)
\]

\[
\times \left(-\frac{aN}{p}\right).
\]

This yields

\[
b_{r,N}(p) = \frac{(-1)^s}{\phi^*(p)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \sum_{i=0}^{s} \binom{s}{i} \frac{1}{p^{i(r-1)}} \sum_{0 < a \leq p} \left(-\frac{a(ki + N)}{p}\right).
\]

To wrap up, one has

\[
\mathcal{G}_{r,k}(N) = \prod_p \left(1 + \frac{\mu_k(p)}{\phi(p^r)}\right) \prod_{p | k} \left(1 + \frac{(-1)^s c_p(N)}{\phi^*(p)}\right)
\]

\[
\times \prod_{p | k} \left(1 + \frac{(-1)^s}{\phi^*(p)} \left(1 - \frac{1}{\phi(p^r)}\right)^{-s} \left(\sum_{i=0}^{s} \binom{s}{i} \frac{1}{p^{i(r-1)}} c_p(ki + N)\right)+ \frac{1}{p^{r-1}} \left(c_p^r(ks + N) + \cdots + c_p(ks + N)\right)\right).
\]

One important observation here is

\[
\lim_{r \to \infty} \mathcal{G}_{r,k}(N) = \prod_p \left(1 + \frac{(-1)^s c_p(N)}{\phi^*(p)}\right).
\]
From [5, §26], we have the identity
\[ \sum_{d|q} c_d(N) = \begin{cases} q & \text{if } q | N, \\ 0 & \text{if } q \nmid N. \end{cases} \]

Therefore, we have
\[ c_{p^r}(ks + N) + \cdots + c_{p^r}(ks + N) = p^r \rho(p^r, k(s-1)+N) - c_p(ks+N) - 1. \]

For \( p > 2 \), we note that
\[ \left| 1 + \frac{(-1)^s c_p(ks+N)}{\phi^s(p)} \right| \geq 1 - (p-1)^{1-s} \geq 3/4. \]

Therefore the factors in (2.29) corresponding to odd \( p | k \) do not vanish. On the other hand, for \( p = 2 \), this factor vanishes if and only if \( s \) and \( N \) have different parities.

Assume now that \( p \nmid k \). Since \( c_p(ik+N) \leq p - 1 \) for all \( i \), the expression
\[ \left( \frac{1}{\phi^s(p^r)} \right)^s \left( p^{s(r-1)} \sum_{i=0}^{s} \frac{1}{p^{s(r-1)}} c_p(ks+N) \right) \]
\[ + (c_{p^r}(ks+N) + \cdots + c_{p^r}(ks+N)) \] (2.30)

has absolute value
\[ \leq h(p,r,s) := \frac{(p-1)(1+p^{r-1})^s + p^r - p}{(p^r - p^r-1)^s}. \]

Since \( s \geq 3 \) and \( r \geq 2 \) are fixed, \( h(p,r,s) \ll p^{-s+1} \) for every large \( p \). Hence, the factors in \( \mathcal{G}_{r,k}(N) \) corresponding to large \( p \nmid k \) do not vanish. Therefore, it is enough to show that \( h(p,r,s) < 1 \) for every \( s \geq 3, r \geq 2 \) and small primes \( p \).

When \( p \geq 3 \) and \( r \geq 2 \) are fixed, then \( h \) is decreasing as a function of \( s \), since for \( p \geq 3 \) and \( r \geq 2 \)
\[ \frac{p^{r-1} + 1}{p^r - p^{r-1} - 1} < 1. \]

Hence, \( h(p,r,s) \leq h(p,r,3) \). Define
\[ f(r) = \frac{1 + p^{r-1}}{p^r - p^{r-1} - 1}, \quad \text{and} \quad g(r) = \frac{p^r - p}{(p^r - p^{r-1} - 1)^3}. \]

Since
\[ f'(r) = -\frac{p^r \log p}{(p^r - p^{r-1} - 1)^2} < 0, \]
\[ g'(r) = -\frac{p^r \log p (2p^r - 2p^{r-1} - 3p + 4)}{(p^r - p^{r-1} - 1)^4} < 0 \]

for \( r \geq 2 \) and each fixed \( p \geq 2 \), we conclude that \( h(p,r,3) \) is decreasing as a function of \( r \). For \( r = 2 \), we see that
\[ h(p,2,3) = \frac{p^4 + 2p^3 + p^2 - 3p - 1}{p^6 - 3p^5 + 5p^4 - 3p - 1} < 1. \]
for all $p > 3$. This shows $h(p, r, s) < h(p, 2, 3) < 1$ for all $r \geq 2, s \geq 3$, and $p > 3$.

Furthermore, we note that $h(3, 3, 3) < 1$. Thus, $h(3, r, s) < 1$ for all $r, s \geq 3$.

It remains to investigate (2.30) in the following cases:

(1) $p, s = 3, r = 2$ and $3 \nmid k$,
(2) $p = 2, s \geq 3, r \geq 2$ and $2 \nmid k$.

In the first case, (2.30) becomes

$$\frac{1}{53} \left( \sum_{i=0}^{3} \binom{3}{i} 3^{3-i} c_3(ki + N) + c_9(3k + N) + c_3(3k + N) \right).$$

An easy computation reveals that the maximum value of this expression has absolute value 37/125 for $1 \leq k \leq 9$ with $(k, 9) = 1$ and $1 \leq N \leq 9$.

We can now work on the case $p = 2$. Assume now $2 \nmid k$ and $N$ is odd. In this case, $c_2(ki + N) = (-1)^{s+1}$. Therefore,

$$\sum_{i=0}^{s} \binom{s}{i} \frac{1}{2^{s(r-1)}} c_2(ki + N) = - \left(1 - \frac{1}{2^{r-1}}\right)^s.$$

Inserting this in (2.30), we get

$$\frac{(-1)^s}{(2^{r-1} - 1)^s} \left( -2^{s(r-1)} \left(1 - \frac{1}{2^{r-1}}\right)^s + 2^r \rho(2^r, k(s-1) + N) + (-1)^s - 1 \right)$$

$$= (-1)^s \left( -1 + \frac{2^r \rho(2^r, k(s-1) + N) + (-1)^s - 1}{(2^{r-1} - 1)^s} \right).$$

This expression equals $-1$, and hence the corresponding local factor in (2.20) vanishes, only if $s$ is even, or if $s$ is odd, and $r = 2$ and $2 \nmid (ks + N)$.

If $N$ is even, $c_2(ki + N) = (-1)^i$. This time (2.30) becomes

$$(-1)^s \left( 1 + \frac{2^r \rho(2^r, k(s-1) + N) + (-1)^{s+1} - 1}{(2^{r-1} - 1)^s} \right).$$

Therefore, the corresponding factor in (2.20) vanishes only if $s$ is odd or if $s$ is even, and $r = 2$ and $2 \nmid (ks + N)$. This completes the proof.

2.5. Proof of Theorem 1.7

We borrow the notation $f_{r,k}(\alpha, N)$ from the proof of Theorem 1.6. Let $\alpha > 0$ be an irrational number. For the first claim, it will be sufficient to show that

$$f_{r,k}(\alpha, N) = o(N),$$

as this will imply by partial summation that

$$\sum_{p \in \mathcal{P}_{r,k}(N)} e(h(\alpha p + \beta)) = o(|\mathcal{P}_{r,k}(N)|)$$

(2.32)
for any fixed non-zero integer \( h \) and any real number \( \beta \). This shows by Weyl’s criterion (see [11, Chap. 1, Theorem 2.1]) that the fractional parts \( \{\alpha p + \beta\} \) with \( p \in \mathcal{P}_{r,k} \) are uniformly distributed modulo 1.

To prove (2.31), we assume, shifting by an appropriate integer if needed, that \( \alpha \in (0,1) \). Let \( A \) be a sufficiently large number. By Dirichlet’s approximation theorem, we find a reduced rational number \( a/q \) satisfying

\[
|\alpha - \frac{a}{q}| < \frac{\log^A N}{qN}
\]

with \( 1 \leq q \leq N/\log^A N \). Note that \( q \to \infty \) as \( N \to \infty \) since \( \alpha \) is irrational. If \( q \geq \log^A N \), then the result follows by Lemma (2.1). Otherwise, by Lemma (2.2) it follows that

\[
f_{r,k}(\alpha, N) \ll \frac{N^{2\omega(q)}}{\phi(q)} + \frac{N}{\log N} \ll \frac{N^{2\omega(q)} \log \log q}{q} + \frac{N}{\log N},
\]

yielding \( \limsup_{N \to \infty} |f_{r,k}(\alpha, N)|/N = 0 \). This proves the first assertion in Theorem 1.

To prove the second assertion, we assume that \( \alpha > 0 \) is of finite type \( r \geq 1 \). Hence, for a given \( \varepsilon > 0 \), there exists a constant \( C = C(\varepsilon, r) > 0 \) such that \( \|\alpha n\| \geq C n^{-(r+\varepsilon)} \) for every positive integer \( n \).

Let \( h \in [1, 2(CN)^{-\overline{w}}] \) be an integer. By Dirichlet’s approximation theorem, there exist co-prime integers \( a, q \) with \( 1 \leq q \leq N/Q \) such that \( |\alpha h - a/q| < Q/(qN) \), where \( Q \) is to be chosen in \( [1, (CN)^{-1/2} (2/h)^{1/2}] \). Since

\[
C(hq)^{-(r+\varepsilon)} \leq \|\alpha h q\| \leq |\alpha h q - a| < Q/N,
\]

we find

\[
q > h^{-1}(CN/Q)^{1 r} \geq Q/2,
\]

last inequality by our choice of the upper limit of \( Q \). Hence, it follows that \( |\alpha h - a/q| < 2/N \) and we can use Lemma (2.1) to get

\[
f_{r,k}(\alpha, N) N^{-\varepsilon} \ll h^+ Q^{1\over 2} N^{1\over r} + N Q^{-1\over r} + \begin{cases} N^{1\over 2} & \text{if } r = 2, \\ N^{1\over 3} & \text{if } r > 2, \end{cases}
\]

where we used the elementary inequality \( 2^{\omega(q)} \leq \tau(q) \ll (N/Q)^{1/2} \).

Using [7, Lemma 2.4] to choose \( Q \) optimally, we obtain

\[
f_{r,k}(\alpha h, N) N^{-\varepsilon} \ll h^+ N^{1\over r} + h^{(r-1)(1/2)} N^{1\over r} + \begin{cases} N^{1\over 2} & \text{if } r = 2, \\ N^{1\over 3} & \text{if } r > 2, \end{cases}
\]

\[
\ll N((h^{r+\varepsilon}/N)^{1\over r} + (h^{r+\varepsilon}/N)^{1\over 2(r+1\varepsilon)})
\]

\[
+ (h^{r+\varepsilon}/N)^{1\over 2(r+1\varepsilon)} + \begin{cases} N^{1\over 2} & \text{if } r = 2, \\ N^{1\over 3} & \text{if } r > 2. \end{cases}
\]
We can assume that 
\[ h \leq N^{1/(r+\varepsilon)/2} \] 
by taking a smaller \( C \) if necessary. Hence, the second term dominates the first and the third terms, and we can write
\[ f_{r,k}(\alpha h, N) N^{-\varepsilon} \ll \frac{h^{(r-1)/(r+1)}}{x^{(r-1)/(r+1)}} N^{1-(r-1)/(r+1)} + \begin{cases} \frac{N^\varepsilon}{x^{r}} & \text{if } r = 2, \\ \frac{N^\varepsilon}{x^{r}} & \text{if } r > 2. \end{cases} \]
Note that if \( h \gg N^{1/(r+\varepsilon)} \), then the above inequality is trivially satisfied. Therefore, the inequality holds for all non-zero integers \( h \).

Finally, by using Lemma 1.14 subsequent to an application of partial summation, it follows that
\[
\# \{ p \leq x : p \in \mathcal{P}_{r,k} \text{ and } a \leq \{ \alpha p + \beta \} \leq b \} - (b - a) \# \mathcal{P}_{r,k}(x) \ll \frac{x}{H \log x} + \frac{x^\varepsilon}{\log x} \left( H^{(r-1)/(r+1)} x^{-1} + \log H \left( \frac{x^\varepsilon}{x^{r}} \text{ if } r = 2, \frac{x^\varepsilon}{x^{r}} \text{ if } r > 2 \right) \right).
\]
Using [7, Lemma 2.4] with \( H \in [1,x] \) gives the error stated in Theorem 1.7.

**Proof of Corollary** 1.8 We shall show that
\[
\sum_{p \in (\mathcal{P}_{r,k} \cap \mathcal{B}_a, \beta)(x)} \frac{1}{\alpha} \approx 1.
\]
It is not hard to show that if \( \alpha > 1 \) irrational, then
\[ p \in \mathcal{P}_{r,k} \cap \mathcal{B}_{a,\beta} \iff p \in \mathcal{P}_{r,k} \text{ and } 0 < \left( \frac{p + 1 - \beta}{\alpha} \right) \leq \frac{1}{\alpha}, \]
where equality can hold for at most one prime \( p \). Hence, the proof of the corollary is apparent from Theorem 1.7.

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**References**


