CONSENSUS IN NETWORKS OF ANTICIPATORY AGENTS UNDER TRANSMISSION DELAYS

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Zeynep Güven

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Consensus in networks of anticipatory agents under transmission delays
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

CONSENSUS IN NETWORKS OF ANTICIPATORY AGENTS UNDER TRANSMISSION DELAYS

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2024

This thesis examines the dynamics of a coupled system of linear delay differential equations, addressing the normalized consensus problem on undirected and connected graphs of anticipatory agents in the presence of a fixed information transmission delay. The anticipation rule, a first-order linear extrapolation, enables agents to predict the present states of their neighbours using past information, thereby introducing an additional delayed term into the formulation and resulting in a system of delay differential equations with two discrete delays. The main result of this study is the necessary and sufficient condition for the anticipatory consensus protocol under transmission delays to reach consensus. Simulations indicate that the convergence rate of the anticipatory protocol is superior to that of the protocol without anticipatory agents, both under transmission delays. As a natural extension, the findings are applied to the Kuramoto model of coupled phase oscillators to determine the local stability of synchronized states. It is demonstrated that the delay margin for achieving local stability is inversely proportional to the coupling strength between agents. Furthermore, it is shown that the synchronized frequency of the extended model remains the same as that of the original Kuramoto model, contrasting with other extended versions that involve single delays.

Keywords: consensus problem, convergence rate, Laplacian matrix, time-delay systems, stability analysis.

ÖZET

İLETİM GECİKMELERİ VE ÖNGÖRÜLÜ ELEMANLARIN VARLIĞINDA AĞLAR ÜZERİNDE UZLAŞMA PROBLEMİ

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Bu tez iletim gecikmeleri ve öngörülü elemanların varlığında ağlar üzerinde uzlaşma problemini modelleyen bir zaman gecikmeli diferansiyel denklemler sisteminin dinamiklerini incelemektedir. Öngörü kuralı, birinci derece linear ekstrapolasyon, öngörülü elemanların geçmiş bilgiyi kullanarak komşularının mevcut durumlarını tahmin etmelerini sağlar. Elemanların öngörücü davranışı sisteme ikinci bir zaman gecikmeli terim ekleyerek iki zaman gecikmeli terimi olan bir denklemler sistemi ortaya çıkarır. Bu çalışmanın ortaya koyduğu ana sonuç iletim gecikmeleri ve öngörülü elemanların varlığında, sistemin uzlaşmaya ulaşması için gerek ve yeter koşulun elde edilmesidir. Simülasyon sonuçları iletim gecikmeleri varlığında öngörülü elemanların uzlaşma protokolünün yakınsama hızını artırdığını göstermektedir. Doğal bir uzantı olarak, uzlaşma problemi için elde edilen sonuçlar Kuramoto modeline uygulanmış, senkronize çözümlerin yerel kararlılığı için gerek ve yeter koşul elde edilmiş ve senkronize çözümlerin yerel kararlılığı için gerekli ve yeterli olan gecikme marjının ajanlar arasındaki bağlanma gücü ile ters orantılı olduğu gösterilmiştir. Tek bir zaman gecikmeli terim içeren Kuramoto modeli versiyonlarının aksine, iletim gecikmeleri ve öngörülü elemanların varlığında, iki zaman gecikmeli terim içeren Kuramoto modelininin senkronize frekansının orijinal Kuramoto modelinin senkronize frekansı ile aynı kaldığı gösterilmiştir.

Anahtar sözcükler: uzlaşma problemi, yakınsama hızı, Laplace matrisi, zaman-gecikmeli sistemler, kararlılık analizi.

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Chapter 1

Introduction

A cooperative system refers to a collection of dynamical entities sharing information to achieve a single or multiple common objectives; the entities could be mechanical devices, living beings, or abstract mathematical objects exhibiting time-dependent behaviour [1].

Cooperative behaviour in living organisms has been a subject of extensive study by evolutionary biologists and ecologists. Early evolutionary theories, including those by Charles Darwin, viewed cooperation as a challenge to the principle of natural selection, by considering that nature favours the most competitive individuals focused solely on their own success while cooperation could potentially endanger the survival and reproductive success of cooperative actors [2]. On the other hand, self-sacrifice for the success of relatives can enhance the likelihood of gene transmission to future generations through increased reproductive success of relatives, thus cooperation indeed makes evolutionary sense.

Inclusive fitness theory, developed by evolutionary biologist William Hamilton, provides a mathematical model to describe interactions among relatives, showing that species evolve to balance altruism and selfishness, thereby promoting both individual and collective success [3]. Moreover, cooperation extends beyond kin relationships, offering direct benefits to individuals. As ecologist Tim Clutton-Brock discusses, cooperation can enhance group success and individual survival by providing protection from predators and competitive advantages through increased group size [4].

Flocking behaviour of birds is a quintessential example of a coordinated group behaviour emerging from individual interactions. Craig Reynolds pioneered the modeling of this behaviour by identifying three fundamental rules: separation, alignment, and cohesion [5]. Under these simple rules, Reynolds' model allows a complex flocking behaviour to emerge naturally from the local interactions between birds.

A fundamental class of dynamical systems for exploring the cooperative behaviour is the class of consensus problems, which often involves agents trying to coordinate with others through the exchange of information, just as birds in a flock aligning their speed, position, and direction according to the nearby birds. This principle extends beyond natural phenomena and finds applications in engineering fields such as traffic flow modeling. In these models, agents (drivers) interact similarly to birds in a flock, adjusting their speed and position in response to surrounding vehicles.

Consensus problems have a wide range of applications in flocking and swarming theory [6],[7], distributed control [8], distributed computing [9], management science [10], and opinion dynamics [11]. Delayed protocols form a critical subset of consensus protocols, facilitating more realistic models of cooperative systems. Developing realistic models for applications such as traffic flow modeling is crucially important to create reliable models ensuring that undesired outcomes are prevented/minimized. Therefore, these models must consider various real-world complexities, including the time delays inherent in information exchange and processing. Time delays are also commonly used to capture the memory effect of *intelligent agents* [12].

Jeffrey Hawkins argues that the brain creates a predictive model of the world, and that the prediction is an intrinsic property of brain that never stops and serves an essential role in learning [13]. In a discussion about how to create artificial neural networks that have brainlike intelligence, Hawkins highlights the significance of using *auto-associative memory*[1] models, which are able to learn sequences of information due to the delayed feedback mechanism, similar to our brains [14].

In numerous applications, for example in the design of controllers, fast consensus reaching is highly desired [15], [16], [17]. Various protocols with accelerated convergence rates have been studied both for continuous and discrete time consensus problems. Notably, the effect of using delayed feedback to accelerate consensus has been investigated extensively under different settings [18], [19], [20], [21].

Using predictive mechanisms to accelerate consensus reaching was proposed by Zhang et al. [22], [23] for the discrete time problem. Atay and Irofti proposed a predictive mechanism to improve the convergence rate of the continuous time consensus problem through the introduction of *anticipatory agents* [24], where an anticipatory agent refers to an intelligent agent who keeps memory of the past information and predict the future states of its neighbours based on this information.

In this thesis, we study the dynamics of the consensus problem in the presence of a fixed information transmission delay, on networks of anticipatory agents. Following Atay and Irofti [24], who considers first-order linear extrapolation as the anticipation rule, we assume an anticipation rule that is of a first-order estimation by linear extrapolation but agents anticipate the states of their neighbours at present time, not in the future. More precisely, agent i knowing the past states of agent j up to time $t-\tau$, anticipates the current state of agent j:

$$\begin{aligned}\hat{x}_j(t) &= x_j(t-\tau) + \frac{x_j(t-\tau) - x_j(t-2\tau)}{\tau}\tau \\ &= 2x_j(t-\tau) - x_j(t-2\tau).\end{aligned}$$

[1]The term “auto-associative memory” describes a memory system capable of recalling an entire memory from just a fragment of it.

We formulate the consensus problem under transmission delays on networks of anticipatory agents by the system of delay differential equations (DDEs):

$$\begin{aligned}\dot{x}_i(t) &= \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[\hat{x}_j(t) - x_i(t)],\\ &= \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[2x_j(t-\tau) - x_j(t-2\tau) - x_i(t)] \quad i = 1, 2, \ldots, n. \end{aligned} \tag{1.1}$$

Here, $x_i(t)$ is the state of agent i at time t, $a_{ij} = a_{ji} \in \{0, 1\}$ denotes whether i and j influence each other or not, and $d_i = \sum_{j=1}^{n} a_{ij}$ is the number of neighbours of agent i.

In this study, we explore the dynamics of the system (1.1) on undirected and connected graphs and answer the following questions:

1. Under which conditions does the system (1.1) reach consensus?
2. Does anticipation improve the convergence rate in the presence of information transmission delays?
3. How does the network structure affect the consensus dynamics?

The rest of this thesis is organized in six chapters, divided into two parts: Part I (Chapters 2, 3, 4) provides preliminaries, and Part II (Chapters 5, 6, 7) presents the main results of the thesis. In Chapter 2, we review the basics of algebraic graph theory and introduce the linear consensus problem on networks. In Chapter 3, we study time-delay systems with special focus on linear delay differential equations of retarded type, and end the first part with a brief review of convergence conditions of three different delayed consensus protocols in Chapter 4. In Chapter 5, we study the dynamics of the system (1.1), derive an exact condition for reaching consensus, and illustrate by computational examples that anticipation improves the convergence rate of the consensus protocol under transmission delays. As a natural extension, in Chapter 6, we apply our results from Chapter 5 to the Kuramoto model of identical phase oscillators. Finally, we conclude in Chapter 7 with a brief discussion of our results.

Part I

Preliminaries

Chapter 2

Consensus in networks

2.1 Basics of graph theory

We start by reviewing related notions from algebraic and spectral graph theory. The main references for this section are *Algebraic Graph Theory* by C. Godsil & G. F. Royle [25], and *Graph Spectra for Complex Networks* by P. Van Mieghem [26], others will be cited explicitly.

Definition 2.1.1. *A* graph $G = (V, E)$ *consists of a non-empty, finite set of vertices* V *and a set of edges* $E \subseteq V \times V$*.*

The terms *graph* and *network* will be used interchangeably throughout the thesis though they refer to different objects in general; while a network often refers to a real system, a graph refers to its mathematical representation.

Definition 2.1.2. *A graph* $G = (V, E)$ *is said to be* undirected *if* $(i, j) \in E \iff (j, i) \in E$ *for all* $i, j \in V$*. Otherwise, it is called* directed*. In the latter case, edges of the graph are called arcs.*

Definition 2.1.3. *A graph* G *is called* simple *if it contains no self-loops (self-edges) or multiple edges between vertices.*

The following definitions will be given only for simple undirected graphs as we restrict our attention to such graphs in this study.

Definition 2.1.4. *Let $G = (V, E)$ be a graph. Two vertices $i, j \in V$ are called* adjacent (neighbours) *if there exists an edge between them, i.e. $(i, j) \in E$.*

Definition 2.1.5. *A* path *of length r from vertex i to vertex j is a sequence of $r+1$ distinct vertices $\{v_0, \dots, v_r\}$ starting with $v_0 = i$ and ending with $v_r = j$ such that all consecutive pairs of vertices in the sequence are adjacent, i.e. $(v_k, v_{k+1}) \in E \; \forall k \in \{0, 1, \dots, r-1\}$.*

Definition 2.1.6. *A graph $G = (V, E)$ is called* connected *if there exists a path between any two of its vertices.*

Definition 2.1.7. *Let $G = (V, E)$ be a graph on n vertices. The* degree *of vertex $i \in V$ is the number of vertices adjacent to i, denoted by d_i.*

Definition 2.1.8. *A* cycle *is a path for which the first and last vertices defining the path are the same. In other words, a path of length r is called a cycle if $v_0 = v_r$.*

A cycle graph on n vertices, denoted by C_n, is a graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$ such that there exists an edge between vertices v_i and v_j if and only if $j - i \equiv \pm 1 \; (mod \; n)$ for all $i, j \in V$.

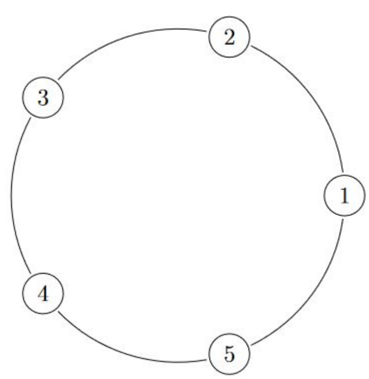


Figure 2.1: Cycle graph on 5 vertices.

2.1.1 The adjacency matrix

Definition 2.1.9. *The* adjacency matrix *of a graph $G = (V, E)$, denoted by $A = A(G)$, is a square matrix with rows and columns indexed by the vertices of G, and with entries a_{ij} given by*

$$a_{ij} = \begin{cases} 1, & \text{if}(j, i) \in E \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix $A = A(G)$ of an undirected graph G is symmetric.

The degrees of vertices can be deduced from the adjacency matrix by the equation $d_i = \sum_{j=1}^{n} a_{ij}$. The degree matrix is the diagonal matrix of degrees, i.e. $D = \text{diag}(d_1, d_2, \ldots, d_n)$.

For the graph C_5, the adjacency matrix A and the degree matrix D are given below.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Definition 2.1.10. *The* volume *of a graph $G = (V, E)$, denoted by* $\text{vol}(G)$, *is defined as the sum of degrees of all vertices $i \in V$, i.e.* $\text{vol}(G) = \sum_{i \in V} d_i$ *[27].*

2.1.2 Transition probability matrix

Definition 2.1.11. *A matrix $B \in \mathbb{R}^{n \times n}$ is said to be* nonnegative *if all of its elements are nonnegative.*

Definition 2.1.12. *A nonnegative matrix $B = [B_{ij}] \in \mathbb{R}^{n \times n}$ satisfying $\sum_{j=1}^{n} B_{ij} = 1$ for all $i = 1, \ldots, n$ is called a* row-stochastic matrix.

Since a row-stochastic matrix $B \in \mathbb{R}^{n\times n}$ has row sums equal to 1, $B\vec{1} = \vec{1}$ always holds, meaning that 1 is always an eigenvalue of B with corresponding right eigenvector $\vec{1} = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^\top$.

Definition 2.1.13. *Let $B = [B_{ij}] \in \mathbb{R}^{n\times n}$. Then* the underlying graph of B *is the graph $G = (V, E)$ with the vertex set $V = \{1, 2, \dots, n\}$ and $(i, j) \in E \iff B_{ij} \neq 0$.*

If $B = B^\top \in \mathbb{R}^{n\times n}$, then the underlying graph of B is undirected, and is directed otherwise.

Definition 2.1.14. *Let $B = B^\top \in \mathbb{R}^{n\times n}$. Then B is called* irreducible *if its underlying graph is connected.*

For nonsymmetric B, the above definition of irreducibility would require the underlying directed graph to be strongly connected, referring to the existence of directed paths from any vertex in the graph to any other vertex.

Relating to Markov processes, an irreducible matrix corresponds to an irreducible Markov chain, i.e. a Markov chain with exactly one communicating class. The reader is referred to [28] for the theory of Markov chains.

Definition 2.1.15. *Let G be a graph on n vertices with all vertices having at least one connection. Then the matrix $P = D^{-1}A$ where A and D respectively denote the adjacency and degree matrices of G, is called* the transition matrix, *or* the one-step transition probability matrix.

Remark 2.1.16. *The transition matrix P characterizes a random walk on G such that $P_{ij} = \frac{a_{ij}}{d_i}$ gives the conditional probability of next vertex being j given the present vertex is i. Since G is assumed not to have any isolated vertices, $d_i > 0\ \forall i$ thus, P_{ij} is defined for all $i, j \in V$.*

Definition 2.1.17. *The* spectral radius *of $B \in \mathbb{R}^{n\times n}$, is defined as*

$$\sigma(B) = \max_{i=1,\dots,n} |\lambda_i(B)|,$$

where $\lambda_i(B)$, $i = 1, \dots, n$, are the eigenvalues of B.

Let $\Delta(a, r) := \{x \in \mathbb{C} : \ |x - a| \leq r\}$ denote the closed disc in the complex plane with radius r, centered at a.

Theorem 2.1.18. *(Gershgorin's theorem) Let $B \in \mathbb{C}^{n \times n}$ whose elements are denoted by B_{ij}, $i, j \in \{1, 2, \ldots, n\}$. Then, all eigenvalues of B belong to the union of discs*

$$\Delta(B_{ii}, \sum_{j=1, j \neq i}^{n} |B_{ij}|).$$

in the complex plane.

Theorem 2.1.19. *(Perron-Frobenius theorem) Let $B \in \mathbb{R}^{n \times n}$ be a non-negative, irreducible matrix. Then, its spectral radius $\sigma(B)$ is a simple and positive eigenvalue with corresponding eigenvector having strictly positive elements [29].*

Remark 2.1.20. *Let $P \in \mathbb{R}^{n \times n}$ be the transition probability matrix of an undirected connected graph G, whose eigenvalues are denoted by μ_k for $k = 1, \ldots, n$.*

(i) P is similar to the symmetric matrix $R = D^{1/2} P D^{-1/2} = D^{-1/2} A D^{-1/2}$ thus, P has real eigenvalues, i.e. $\mu_k \in \mathbb{R}$ $\forall k = 1, \ldots, n$, and the corresponding eigenvectors $\{v_k\}$ form a basis for $\mathbb{R}^n$.

(ii) P is a row stochastic matrix. Thus, 1 is always an eigenvalue of P with associated right eigenvector $v_1 = \vec{1} = \begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}^\top$. The left eigenvector π^1 associated with the eigenvalue 1 is called the steady state vector, which is the row vector given by $\pi^1 = \frac{1}{vol(G)} \begin{pmatrix} d_1 & \ldots & d_n \end{pmatrix}$ satisfying the orthogonality relation $\langle \pi^1, v_1 \rangle = 1$.

(iii) By Gershgorin's theorem 2.1.18, eigenvalues μ_k of P satisfy

$$|\mu_k| \leq 1, \quad k = 1, \ldots, n.$$

(iv) By Perron-Frobenius theorem 2.1.19, the spectral radius $\sigma(P)$ equals 1 and it satisfies $|\mu_k| < \sigma(P)$ for all $k \geq 2$, where $\mu_1 \geq \mu_2 \geq \ldots \mu_n$ denote the eigenvalues of P.

2.1.3 The Laplacian matrix

Laplacian matrices play an important role studying dynamical systems on networks, appearing in various versions depending on the formulations governing the dynamics. The Laplacian matrix can be seen as a discrete (network) form of Laplace operator ∇^2 [1] defined as

$$\nabla^2 = \nabla \cdot \nabla f = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

The Laplace operator appears naturally in the mathematical formulation of diffusion processes which describes the macroscopic behaviour of many particles undergoing Brownian motion[2], mathematically described by second order partial differential equations.

Random walks on networks can be thought of as discrete analogue of the continuous diffusion processes, which gives rise to the Laplacian matrices. There are various definitions for graph Laplacians, given formally in the definition below.

Definition 2.1.21. *Let $G = (V, E)$ be a graph whose adjacency and degree matrices are denoted by A and D, respectively.*

(i) The combinatorial Laplacian matrix is defined as

$$Q = D - A,$$

with entries Q_{ij} given by

$$Q_{ij} = \begin{cases} d_i, & \text{if } i = j \\ -1, & \text{if } i \neq j \text{ and } (i,j) \in E \\ 0, & \text{if } i \neq j \text{ and } (i,j) \notin E. \end{cases}$$

[1]Named after Pierre-Simon de Laplace, the Laplace operator is a second-order differential operator defined on open sets of $\mathbb{R}^n$ which maps C^k functions to C^{k-2} functions for $k \geq 2$.

[2]Named after botanist Robert Brown, Brownian motion refers to the irregular and random motion of particles suspended in a liquid [30].

(ii) The random-walk Laplacian (or normalized Laplacian) is defined as

$$L = I - D^{-1}A$$

with entries L_{ij} given by

$$L_{ij} = \begin{cases} 1, & \text{if } i = j \\ -\frac{1}{d_i}, & \text{if } i \neq j \text{ and } (i,j) \in E \\ 0, & \text{if } i \neq j \text{ and } (i,j) \notin E. \end{cases}$$

(iii) the symmetric Laplacian matrix is defined as

$$S = I - D^{-1/2}AD^{-1/2}$$

with entries S_{ij} given by

$$S_{ij} = \begin{cases} 1, & \text{if } i = j \\ -\frac{1}{\sqrt{d_i d_j}}, & \text{if } i \neq j \text{ and } (i,j) \in E \\ 0, & \text{if } i \neq j \text{ and } (i,j) \notin E. \end{cases}$$

Lemma 2.1.22. *(Chung, 1997) Let G be a graph on n vertices, and S be the symmetric Laplacian matrix of G defined by $S = I - D^{-1/2}AD^{-1/2}$ whose eigenvalues are denoted by λ_k and ordered such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then, the following statements hold [31]:*

(i) $\sum_{k=1}^{n} \lambda_k \leq n$ with equality holding if and only if G has no isolated vertices.

(ii) For $n \geq 2$, $\lambda_2 \leq \frac{n}{n-1}$ with equality holding if and only if G is complete. Also if G has no isolated vertices, then $\lambda_n \geq \frac{n}{n-1}$.

(iii) If G is not complete, then $\lambda_2 \leq 1$.

(iv) If G is connected, then $\lambda_2 > 0$. If $\lambda_1 = \cdots = \lambda_k = 0$ and $\lambda_{k+1} \neq 0$, G has exactly k connected components.

2.1.4 Spectrum of the Laplacian

We first note that the Laplacian matrices Q, L, and S have zero row sums, therefore zero is always an eigenvalue of each of them with corresponding right eigenvector $\vec{1} = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^\top$ since $Q\vec{1} = L\vec{1} = S\vec{1} = \vec{0}$ always holds.

Remark 2.1.23. *Let G be an undirected connected graph defined on $n \geq 2$ vertices. Then the random-walk Laplacian L is not symmetric in general but is similar to the symmetric real matrix S through the similarity transformation $L = D^{-1/2}SD^{1/2}$, thus L and S have the same spectrum consisting of real eigenvalues. By Chung's lemma (2.1.22), eigenvalues λ_i, $i = 1, 2, \dots, n$, of L satisfy*

(i) $\sum_{i=1}^{n} \lambda_i = n$.

(ii) $\lambda_1 = 0$, *and* $\lambda_i > 0$ *for all* $i \geq 2$.

(iii) If G is complete, $\lambda_1 = 0$ and $\lambda_i = \frac{n}{n-1}$ for all $i \geq 2$. If G is not complete, then $\lambda_2 \leq 1$ and $\lambda_n \geq \frac{n}{n-1}$.

(iv) Gershgorin's thereom (2.1.18) implies that eigenvalues λ_i belongs to the closed disc $\Delta(1,1)$ since $L_{ii} = 1$ and $\sum_{j=1, j\neq i}^{n} |L_{ij}| = 1$ for all i. In other words, $|1 - \lambda| \leq 1$ holds for all eigenvalues λ. Therefore, the eigenvalues of L can be ordered as

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq 2.$$

2.2 Consensus dynamics

2.2.1 The classical consensus protocol

The classical consensus problem on networks in continuous time is defined by the system of differential equations

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}[x_j(t) - x_i(t)], \ \ i = 1, 2, \dots, n. \tag{2.1}$$

Here, $x_i(t)$ denotes the state of agent i at time t and a_{ij} is the strength of influence of agent j on agent i. Letting $x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^\top \in \mathbb{R}^n$ denote the vector of agents' states, the classical consensus problem (2.1) has the vector form

$$\dot{x}(t) = -Qx(t). \tag{2.2}$$

The normalized version of the classical consensus problem is formulated as

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[x_j(t) - x_i(t)], \quad i = 1, 2, \dots, n, \tag{2.3}$$

whose vector form can be derived as follows:

$$\begin{aligned}
\dot{x}_i(t) &= \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[x_j(t) - x_i(t)] \\
&= \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}x_j(t) - \frac{x_i(t)}{d_i}\sum_{j=1}^{n} a_{ij} \\
&= \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}x_j(t) - x_i(t) \\
&= \sum_{j=1}^{n} \frac{a_{ij}}{d_i}x_j(t) - \sum_{j=1}^{n} \delta_{ij}x_j(t) \\
&= -\sum_{j=1}^{n} \left(\delta_{ij} - \frac{a_{ij}}{d_i}\right) x_j(t) \\
&= -\sum_{j=1}^{n} L_{ij}x_j(t)
\end{aligned}$$

where L_{ij} is the $(ij)-$entry of the normalized Laplacian matrix $L = I - D^{-1}A$. Then, the normalized consensus problem (2.3) can be written in vector form as

$$\dot{x}(t) = -Lx(t). \tag{2.4}$$

The vector differential equation (2.4) has the general solution

$$x(t) = e^{-Lt}x(0), \quad t \geq 0, \tag{2.5}$$

where e^{-Lt} is the matrix exponential satisfying $e^{-Lt}v = e^{-\lambda t}v$ for any λ, v satisfying $Lv = \lambda v$.

2.2.2 Reaching consensus

Consensus reaching refers to the case when all agents converge to the same state as $t \to \infty$. More precisely, we give the following definition.

Definition 2.2.1. *The system is said to reach consensus if for any set of initial conditions $\{x_i(0)\}$, there exists $c \in \mathbb{R}$ such that $\lim_{t\to\infty} x_i(t) = c$ for all $i = 1, 2, \ldots, n$. In this case, c is called the consensus value.*

Recall the general solution $x(t) = e^{-Lt}x(0)$ to the normalized consensus problem $\dot{x}(t) = -Lx(t)$.

Considering an undirected and connected graph, since L has a complete set of eigenvectors $\{v_k\}$ that form a basis for $\mathbb{R}^n$, we can express the general solution (2.5) as sums of eigenmode solutions $c_k e^{-\lambda_k t} v_k$

$$x(t) = \sum_{k=1}^{n} c_k e^{-\lambda_k t} v_k,$$

c_k, $k = 1, \ldots, n$ being constants. Since $\lambda_1 = 0$ and $\lambda_k > 0$ for all $k \geq 2$, all eigenmode solutions converge to zero except for the one that corresponds to the zero eigenvalue. This means that solutions converge to the subspace spanned by $v_1 = \vec{1}$ as $t \to \infty$, implying that the system (2.4) reaches consensus under the assumption that L has a simple zero eigenvalue, or equivalently, the underlying network is connected.

We can also compute the consensus value explicitly. Recall that the left eigenvector of $P = D^{-1}A$ corresponding to eigenvalue 1 is the row vector $\pi^1 = \dfrac{1}{vol(G)}\begin{pmatrix} d_1 & \ldots & d_n \end{pmatrix}$, which is also the left eigenvector of $L = I - P$ corresponding to the zero eigenvalue. Let Λ denote the diagonal matrix of eigenvalues of L and V denote the matrix whose columns are the right eigenvectors $\{v_i\}$ of L. Then, V^{-1} is the matrix whose rows, denoted by $\{\pi^i\}$, are the left eigenvectors, which constitutes a basis for the dual space of V. Then,

$$\begin{aligned} LV &= V\Lambda \\ L &= V\Lambda V^{-1} \end{aligned}$$

and

$$e^{-Lt} = Ve^{-\Lambda t}V^{-1}.$$

Then the general solution (2.5) can be written as

$$x(t) = Ve^{-\Lambda t}V^{-1}x(0).$$

Multiplying both sides by V^{-1} and applying the change of variables $y(t) = V^{-1}x(t)$, we get

$$\begin{aligned} y(t) &= e^{-\Lambda t}y(0) \\ &= \begin{pmatrix} e^{-\lambda_1 t} & & \\ & \ddots & \\ & & e^{-\lambda_n t} \end{pmatrix} \begin{pmatrix} y_1(0) \\ \vdots \\ y_n(0) \end{pmatrix} = \begin{pmatrix} e^{-\lambda_1 t}y_1(0) \\ \vdots \\ e^{-\lambda_n t}y_n(0) \end{pmatrix}. \end{aligned}$$

That is, $y_k(t) = e^{-\lambda_k t}y_k(0)$. Since $\lambda_k > 0 \; \forall k \geq 2$, $e^{-\lambda_k t}y_k(0) \to 0$ as $t \to \infty$ for all $k \geq 2$. Hence, as $t \to \infty$

$$y(t) \to \begin{pmatrix} y_1(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.6}$$

Since $x(t) = Vy(t)$,

$$x(t) \to V \begin{pmatrix} \sum_{k=1}^{n} V_{1k}^{-1}x_k(0) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{k=1}^{n} V_{1k}^{-1}x_k(0) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

The first row of V^{-1} is the row vector π^1, therefore

$$x(t) \to \langle \pi^1, x(0) \rangle \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

This proves that all agents' states converge to the same point, $\langle \pi^1, x(0) \rangle$, on the subspace spanned by $\vec{1}$. The consensus value is $c = \langle \pi^1, x(0) \rangle$ since the first row of the matrix V^{-1} is the left eigenvector π^1, that is, the consensus value is a weighted average of initial states of agents with weights proportional to degrees of vertices, normalized so that the sum of weights is equal to 1.

The graph being connected is shown to be a sufficient condition for reaching consensus, yet is also necessary. If zero eigenvalue has algebraic multiplicity $k > 1$, then consensus cannot be achieved in both of the following scenarios [32]:

(i) if geometric multiplicity of $\lambda = 0$ equals k: In this case, $\lambda_1, \lambda_2, \dots, \lambda_k = 0$ have corresponding eigenvectors $v_1, \dots, v_k$ that are linearly independent of each other, meaning that connected components of the graph, let be denoted by $C_1, C_2, \dots, C_k$, separately converge to points on the subspaces spanned by $v_1, v_2, \dots, v_k$, respectively. That is, there is no global consensus among the whole graph, but each disjoint connected component separately reaches consensus.

(ii) If the geometric multiplicity of $\lambda = 0$ is less than k: This never happens in undirected graphs since L always has a complete eigenbasis. Suppose that this is not the case and that the geometric multiplicity of $\lambda = 0$ is less than k. In this case, L is not diagonalizable, so is the matrix exponential e^{-Lt}. Using the Jordan normal form, however, we can show that e^{-Lt} diverges. For some invertible P, $P^{-1}(-L)P = J$ so that $e^{-Lt} = Pe^{Jt}P^{-1}$. Here, J is an $n \times n$ Jordan matrix

$$J = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \dots & J_m \end{pmatrix}$$

consisting of m Jordan blocks J_i each has the form

$$J_i = \begin{pmatrix} -\lambda_i & 1 & 0 & \dots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & -\lambda_i & 1 \\ 0 & \dots & 0 & -\lambda_i \end{pmatrix},$$

where λ_i are the eigenvalues of L. Finally, since the exponential of Jordan block of the form J_i is given by

$$e^{J_i t} = e^{-\lambda_i t} \begin{pmatrix} 1 & t & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & 1 & t \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

$e^{J_1 t}$ diverges since $\lambda_1 = 0$ and

$$e^{J_1 t} = \begin{pmatrix} 1 & t & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & 1 & t \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

which implies the divergence of e^{-Lt}.

The convergence of solutions $x(t) = e^{-Qt}x(0)$ of the classical consensus problem (2.2) can be shown in a similar way; the only difference is due to the left eigenvectors of L and Q, which affects the consensus value. Since Q is symmetric, the normalized eigenvectors u_i of Q form an orthogonal matrix U such that $u_i^\top u_j = \delta_{ij}$, which implies that the left eigenvector corresponding to zero eigenvalue, let be denoted by $u^1 = u_1^\top$, is given by

$$u^1 = \frac{1}{n}\begin{pmatrix} 1 & \dots & 1 \end{pmatrix}.$$

Then,

$$c = \langle u^1, x(0) \rangle = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$$

gives the consensus value, which is the average of agents' initial states.

Using normalized Laplacian L changes the dynamics in favour of vertices with higher degrees as the initial states of agents affect the consensus value with a weight proportional to their degrees, unlike the Laplacian Q.

2.2.3 Convergence rate of the classical consensus protocol

The second smallest eigenvalue of Laplacian L, which is also called the algebraic connectivity of the graph, gives an indication how difficult it is to disconnect the graph by removing edges.

Considering the general solution $x(t) = e^{-Lt}x(0) = \sum_{k=1}^{n}\langle c_k, x(0)\rangle e^{-\lambda_k t} v_k$, the slowest eigenmode solution converges to zero at the rate $e^{-\lambda_2 t}$ since $\lambda_2 \leq \lambda_k$ $\forall k \geq 2$. This shows that the convergence rate of the consensus protocol and the connectivity structure of the underlying graph are closely related, and λ_2 can be seen as a measure of the speed of convergence to consensus [33].

Chapter 3

Time-delay systems

At the IVth International Congress of Mathematicians in 1908, Picard made the following statements, translated from French to English and paraphrased as [34], [35]: "The differential equations of classical mechanics assume that the motion is determined by the simple knowledge of positions and velocities, or, by the state at a given instant and at the instants that are infinitely close, hence they are applicable to model the systems where the motions are of a reversible nature and do not involve hereditary effects. We can envision functional equations that are more complicated than the classical equations as they will also contain integrals taken between a very distant past time and the current time, thereby incorporating the effect of heredity."

Delay differential equations (DDEs) belong to the class of functional differential equations (FDEs) and attract attention in various fields of science and engineering, particularly in control engineering [36].

In this chapter, we review related concepts from delay differential equations of retarded type. The main references for this section are [37] and [38], other sources are cited explicitly.

3.1 Initial value problem

Let $\tau \in \mathbb{R}$ be nonnegative. We denote by $(\mathcal{C}([-\tau, 0], \mathbb{R}^n), ||.||)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence, i.e. $||\phi|| = \sup_{-\tau \leq \theta \leq 0} ||\phi(\theta)||$ for $\phi \in C([-\tau, 0], \mathbb{R}^n)$. Let $t \in \mathbb{R}$ denote the time, $\alpha \geq 0$ be a real number, and $x \in \mathcal{C}([t-\tau, t+\alpha], \mathbb{R}^n)$. Then, we let $x_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ be defined by

$$x_t(\theta) = x(t+\theta), \quad -\tau \leq t \leq 0.$$

Let Ω be a subset of $\mathbb{R} \times \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ and $f : \Omega \to \mathbb{R}^n$ be a given function. Then,

$$\dot{x}(t) = f(t, x_t) \tag{3.1}$$

is called a retarded functional differential equation on Ω, where $\dot{x}$ denotes the right-hand derivative of x.

Let $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ be a given function. The Cauchy problem, also called the initial value problem (IVP), of retarded functional differential equation $\dot{x}(t) = f(t, x_t)$ with initial condition ϕ is defined by (3.2)-(3.3).

$$\dot{x} = f(t, x_t), \quad t \geq t_0 \tag{3.2}$$

$$x(t) = \phi(t), \quad t \in [t_0 - \tau, t_0]. \tag{3.3}$$

Definition 3.1.1. *A function x is said to be a solution of (3.2) on $[t_0 - \tau, t_0 + \alpha)$ if there are $t_0 \in \mathbb{R}$ and $\alpha > 0$ such that $x \in \mathcal{C}([t_0 - \tau, t_0 + \alpha), \mathbb{R}^n)$, $(t, x_t) \in \Omega$ and x satisfies (3.2) for $t \in [t_0, t_0 + \alpha)$. For given $t_0 \in \mathbb{R}$, $\phi \in \mathcal{C}([t_0 - \tau, t_0 + \alpha))$, we say $x(t_0, \phi, f)$ is a solution of IVP, or a solution of (3.2) passing through (t_0, ϕ), if there exists $\alpha > 0$ such that $x(t_0, \phi, f)$ satisfies (3.2) on $[t_0 - \tau, t_0 + \alpha)$ and $x_{t_0}(t_0, \phi, f) = \phi$.*

3.2 Existence, uniqueness and continuous dependence

Lemma 3.2.1. *If $t_0 \in \mathbb{R}$, $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ are given and $f(t, \phi)$ is continuous, then finding a solution of IVP (3.2)-(3.3) is equivalent to solving the integral equation*

$$\begin{aligned} x(t) &= \phi(t_0) + \int_{t_0}^{t} f(s, x_s) ds, \quad t \geq t_0. \\ x_{t_0} &= \phi \end{aligned} \tag{3.4}$$

Definition 3.2.2. *Let $D \subset \mathbb{R} \times \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. If g is a function on D satisfying*

$$||g(t, \psi_1) - g(t, \psi_2)|| \leq K||\psi_1 - \psi_2||$$

for all (t, ψ_1) and (t, ψ_2) in D, then g is said to satisfy the Lipschitz condition with respect to the second variable, with a Lipschitz constant K. We call such a g Lipschitzian in the second variable.

Theorem 3.2.3. *(Existence and Uniqueness) Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$ is continuous, and $f(t, \phi)$ is Lipschitzian in the second variable in each compact set of Ω. If $(t_0, \phi) \in \Omega$, then there exists a unique solution of (3.2) passing through (t_0, ϕ).*

Theorem 3.2.4. *(Continuous dependence) Suppose $\Omega \subseteq \mathbb{R} \times C$ is open, $(t_0, \phi) \in \Omega$, $f \in C(\Omega, \mathbb{R}^n)$, and x is the unique solution of (3.2) passing through (t_0, ϕ), defined on $[t_0 - \tau, \sigma]$, $\sigma > t_0 - \tau$. Let $W \subseteq \Omega$ be the compact set defined by*

$$W = \{(t, x_t) : t \in [t_0, \sigma]\},$$

and let V be a neighbourhood of W such that f is bounded on V. If (t_0^k, ϕ^k, f^k), $k = 1, 2, \ldots$, satisfies $t_0^k \to t_0$, $\phi^k \to \phi$, and $|f_k - f|_V \to 0$ as $k \to \infty$, then there exists K such that for $k \geq K$, each solution $x^k = x^k(t_0^k, \phi^k, f^k)$ through (t_0^k, ϕ^k) of

$$\dot{x}(t) = f^k(t, x_t)$$

exists on $[t_0^k - \tau, \sigma]$ and $x^k \to x$ uniformly on $[t_0 - \tau, \sigma]$.

Definition 3.2.5. *Let x be a solution of (3.2) on the interval $[t_0, a)$, $a > t_0$, and f be continuous. We say that $\overline{x}$ is a continuation of x if there is $b > a$ such that $\overline{x}$ is defined on $[t_0 - \sigma, b)$, coincides with x on $[t_0 - \sigma, a)$ and x satisfies (3.2) for $t \in [t_0, b)$. If there is no such continuation exists, x is said to be a non-continuable solution.*

The following theorem states that the solutions of (3.2) either exist for all $t \geq t_0$ or diverge in finite time.

Theorem 3.2.6. *Let Ω be an open subset of $\mathbb{R} \times C$ and $f \in C(\Omega, \mathbb{R}^n)$. If x is a non-continuable solution of (3.2) on $[t_0 - \sigma, b)$, then for any compact set W in Ω, there exists t_W such that $(t, x_t) \notin W$ for $t_W \leq t < b$.*

3.3 Equilibrium solutions and stability

Definition 3.3.1. *A constant solution of (3.1) satisfying $x(t) = x^*$ for all t is called an equilibrium solution.*

The equilibrium points of the delay differential equation given by $x' = f(x_t)$ are the roots of the algebraic equation $f(x^*) = 0$. Therefore, the equilibria of DDEs are identical to those of the corresponding delay-free equations. However, the stability of these equilibrium points is highly dependent on the time delays.

Definition 3.3.2. *An equilibrium point x^* of the system $\dot{x}(t) = g(x_t)$ is said to be hyperbolic if $\mathrm{Re}(\lambda) \neq 0$ for every eigenvalue λ of the Jacobian of g evaluated at x^*, i.e. if $Dg(x^*)$ has no eigenvalues on the imaginary axis.*

The stability of hyperbolic equilibrium points is determined by studying the linearized equation.

3.4 Linear time-delay systems

Consider the linear time-delay system with multiple discrete delays, formulated by the system of DDEs

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i), \tag{3.5}$$

where $x(t) \in \mathbb{R}^n$ is the state variable at time t, $A_i \in \mathbb{R}^{n\times n}$ for $i = 0, 1, 2, \dots, m$, and $0 < \tau_1 < \cdots < \tau_m = \tau$ represent the discrete time delays. The initial condition for (3.5) is given by a continuous function ϕ mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$, i.e. $\phi \in C([-\tau, 0], \mathbb{R}^n)$.

The mapping $f : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ defined by

$$f(\phi) := A_0\phi(0) + \sum_{i=1}^{m} A_i \phi(-\tau_i)$$

is linear, thus, the existence and uniqueness are guaranteed for all initial conditions. Given an initial function $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, the unique solution can be established explicitly by *the method of steps.* Considering the initial value problem formulated as

$$\begin{aligned} &\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i), \quad t \geq 0 \\ &x(t) = \phi(t), \quad t = [-\tau, 0], \end{aligned}$$

the first step consists of solving the initial value problem given by

$$\begin{aligned} &\dot{x}_1(t) = A_0 x_1(t) + \sum_{i=1}^{m} A_i \phi(t-\tau_i), \quad t \in [0, \tau_1], \\ &x_1(0) = x(0) = \phi(0). \end{aligned} \tag{3.6}$$

Since $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, the value of $\phi(t-\tau_i)$ is known for all $t \in [0, \tau_1]$ and for all $i = 1, \dots, m$. Therefore, equation (3.6) is a linear system of ordinary differential equations. We note that since ϕ is continuous, x_1 is also continuous. Having found the solution x_1 on $[0, \tau_1]$, the next step consists of solving the below initial value problem to construct the continuation of x_1 on the interval $[\tau_1, 2\tau_1]$.

$$\dot{x}_2(t) = A_0 x_2(t) + \sum_{i=1}^{m} A_i x_1(t - \tau_i), \quad \tau \in [\tau_1, 2\tau_1]$$
$$x_2(\tau_1) = x(\tau_1) = x_1(\tau_1).$$

Similarly, one can construct x_k using x_{k-1} as the initial condition at $t = k\tau_1$ for any positive integer k. This process yields a unique forward solution defined for all $t \geq 0$, and the unique solution x is given by the collection of function segments x_k, each defined on $[(k-1)\tau_1, k\tau_1]$ for $k = \{1, 2, \dots\}$.

It is significant to note that the solution x becomes smoother as t increases. In other words, the function segments x_k become smoother as k increases. This also implies that to construct the backward continuation of x on $[-T, -\tau]$, $T > \tau$, further smoothness conditions on ϕ are required.

Taking the Laplace transform of (3.5) with initial condition $x(t) = \phi(t), \quad t \in [-\tau, 0]$, $\phi \in \mathcal{C}([-\tau, 0])$, we obtain the characteristic equation

$$\det \triangle(s) = 0,$$

where

$$\triangle(s) = sI - A_0 - \sum_{i=1}^{m} e^{-\tau_i s} A_i$$

is the characteristic matrix of (3.5) and $\det \triangle(s)$ is called the characteristic function, whose roots are called the characteristic roots. Due to the presence of time delays τ_i, the characteristic function is not a polynomial as in the case of ODEs. Instead, it is a transcendental equation, called a quasipolynomial, which has infinitely many roots.

Proposition 3.4.1. *If there exists a sequence $\{s_k\}_{k\geq 1}$ of characteristic roots of (3.5) such that $\lim_{k\to\infty} |s_k| \to \infty$, then $\lim_{k\to\infty} \mathrm{Re}(s_k) \to -\infty$.*

Proposition 3.4.2. *The following statements hold.*
(i) There are only a finite number of characteristic roots in any vertical strip of the complex plane given by $\{s \in \mathbb{C} : a < \mathrm{Re}(s) < b\}$ with $a, b \in \mathbb{R}$ and $a < b$.
(ii) There exists $\gamma \in \mathbb{R}$ such that $\mathrm{Re}(s) < \gamma$ holds for all characteristic roots s.

To emphasize the dependence of characteristic roots on parameters, we can write $\triangle(s; \vec{\tau}, A_0, \ldots, A_n)$ instead of $\triangle(s)$ where $\vec{\tau} = [\tau_1, \ldots, \tau_m]^T$.

Proposition 3.4.3. *Let s_0 be a characteristic root of (3.5) with multiplicity k. Then, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon < \bar{\varepsilon}$, $\exists \delta > 0$ such that*

$$\triangle(s; \vec{\tau} + \delta\vec{\tau}, A_0 + \delta A_0, \ldots, A_n + \delta A_n),$$

has exactly k zeros in the disc $\{s \in \mathbb{C} : |s - s_0| < \varepsilon\}$, where $\delta\vec{\tau} \in \mathbb{R}^n$, $||\delta\vec{\tau}|| < \delta$, $\vec{\tau} + \delta\vec{\tau} \geq 0$, $\delta A_k \in \mathbb{R}^{n \times n}$, $||\delta A_k||_2 < \delta$, $k = 0, 1, \ldots, n$.

Let *spectral abscissa function* of system (3.5) be denoted by η which is defined as

$$\eta = \eta(\vec{\tau}, A_0, \ldots, A_n) = \sup\{\mathrm{Re}(s) : \det \triangle(s) = 0\}.$$

η always exists and is finite by (ii) of Proposition (3.4.2). In fact, the supremum can be replaced with a maximum, since the rightmost characteristic root, denoted by s^*, always exists such that $\sigma = \mathrm{Re}(s^*)$.

Proposition 3.4.4. *The spectral abscissa function $\eta : (\vec{\tau}, A_0, \ldots, A_n) \to \eta(\vec{\tau}, A_0, \ldots, A_n)$ is continuous.*

Theorem 3.4.5. *If the matrices $A_0, \ldots, A_n$ and the delays $\tau_1, \ldots, \tau_m$ vary continuously, then the exponential stability of the zero solution of (3.5) is lost only if a characteristic root appears on or crosses the imaginary axis.*

This theorem leads to well-known methods for determining the stability regions of delay differential equations. For instance, the *D-Subdivision method* assumes fixed time delays and identifies stability crossing curves by finding purely imaginary roots of the characteristic equation (see [38] and [39] for details).

Our analysis in Chapter 5 also relies on the continuity argument, with the delay parameter τ as the sole system parameter. Therefore, our approach can be viewed as an application of the $\tau-$decomposition method, which is used to determine the stability regions for delay equations only with respect to the delay parameter τ [38].

3.4.1 Stability of the origin

The zero solution $x(t) = 0$ is always an equilibrium solution for (3.5).

Definition 3.4.6. *The zero solution of (3.5) is asymptotically stable if and only if* $\forall \varepsilon > 0$, $\exists \delta > 0$ *s.t.* $\forall \phi \in C([-\tau, 0], \mathbb{R}^n)$, $||\phi|| < \delta \implies ||x_t(\phi)|| < \varepsilon$ $\forall t \geq 0$ *and* $\forall \phi \in C([-\tau, 0], \mathbb{R}^n)$, $\lim_{t\to\infty} x(\phi)(t) = 0$ *holds.*

Definition 3.4.7. *The zero solution of (3.5) is exponentially stable if and only if there exists* $c > 0$, $\gamma > 0$ *such that*

$$||x_t(\phi)|| \leq ce^{-\gamma t}||\phi|| \quad \forall \phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n).$$

In the case of linear DDEs, exponential stability and asymptotic stability are equivalent.

Proposition 3.4.8. *The zero solution of (3.5) is asymptotically stable if and only if all characteristic roots of (3.5) have negative real parts.*

Consider the linear scalar DDE given by

$$\dot{x}(t) = a_0 x(t) + \sum_{j=1}^{m} a_j x(t - \tau_j), \quad a_j \in \mathbb{R},\ j = 0, \ldots, m. \tag{3.7}$$

whose characteristic equation is given by

$$\phi(s) = s - a_0 - \sum_{j=1}^{n} a_j e^{-s\tau_j}. \tag{3.8}$$

A characteristic root s of (3.8) has multiplicity k if $\phi(s) = \phi'(s) = \cdots = \phi^{(k-1)}(s) = 0$, $\phi^{(k)}(s) \neq 0$.

Theorem 3.4.9. *Let s be a characteristic root of (3.8) with multiplicity m. Then, for $k = 0, 1, \ldots, m-1$, $t^k e^{st}$ is a solution of the linear DDE (3.7). Moreover, any finite linear combination of solutions of the form $t^k e^{st}$ is also a solution.*

A solution of (3.7) corresponding to a characteristic root s with multiplicity k can be expressed as $x(t) = p(s)e^{st}$ where $p(s)$ is a polynomial of degree $k - 1$.

Proposition 3.4.10. *The zero solution of (3.7) is asymptotically stable if and only if all roots of ϕ have strictly negative real parts.*

3.4.2 Linear DDEs with single delay

Consider the linear constant coefficient scalar delay differential equation

$$\dot{x}(t) = a_1 x(t) + a_2 x(t - \tau)$$

with single delay τ, which has the characteristic equation

$$\psi(s) = s - a_1 - a_2 e^{-s\tau} = 0. \tag{3.9}$$

3.4.2.1 Exact stability region for single-delay equations

The exact stability region for single delay linear constant coefficient differential equations was established by Hayes [40] who obtained the necessary and sufficient condition for all roots of the equation

$$se^s - a_1 e^s - a_2 = 0 \tag{3.10}$$

to be on the open left half-plane, which is

$$a_1 < 1, \quad a_1 < -a_2 < \sqrt{v^2 + a_1^2}$$

where v is the root of $v \cot v = a_1$ such that $0 < v < \pi$. For the equation (3.10) to have a single root on the imaginary axis and all other roots have negative real parts, the necessary and sufficient condition is

$$a_1 \leq 1, \quad -a_2 = a_1$$

whilst for (3.10) to have two roots on the imaginary axis while all other roots having real parts, the condition is given by

$$a_1 \leq 1, \quad -a_2 = \sqrt{v^2 - a_1^2}.$$

Hayes' result is directly applicable to determine the exact stability regions of consensus problems under single fixed delays, which will be the subject of the next chapter.

3.4.2.2 Lambert W function

The Lambert W function is defined to be the multi-valued inverse of the mapping $s \to se^s$, $s \in \mathbb{C}$. The function has infinite branches W_k, $k = 0, \pm 1, \pm 2, \ldots$ where each $W_k : \mathbb{C} \to \mathbb{C}$ is a single-valued function. W_0 is called the principal branch, and it has been used to numerically determine the rightmost characteristic root of delay equations with single discrete delay due to the following lemma.

Lemma 3.4.11. *For arbitrary $z \in \mathbb{C}$,*

$$\max\{\mathrm{Re}(W_k(z))|0, \pm 1, \pm 2, \ldots\} = \mathrm{Re}(W_0(z))$$

holds [41].

Letting $z = s\tau$, we can rewrite (3.9) as

$$\begin{aligned} \frac{z}{\tau} - a - be^{-z} &= 0 \\ z - a\tau &= b\tau e^{-z} \\ (z - a\tau)e^{z-a\tau} &= b\tau e^{-a\tau}. \end{aligned} \tag{3.11}$$

Then,

$$\begin{aligned} z - a\tau &= W_k(b\tau e^{\tau}) \\ z &= a\tau + W_k(b\tau e^{\tau}) \end{aligned}$$

Substituting $s = \frac{z}{\tau}$ back,

$$s = a + \frac{1}{\tau} W_k(b\tau e^{\tau}).$$

Then, by Lemma (3.4.11) the rightmost characteristic root is given by

$$s = a + \frac{1}{\tau} W_0(b\tau e^{\tau}).$$

3.4.3 Linear DDEs with two delays

This section can be seen as an introduction to our analysis in Chapter 5.

Consider the linear constant coefficient differential equation with two discrete delays

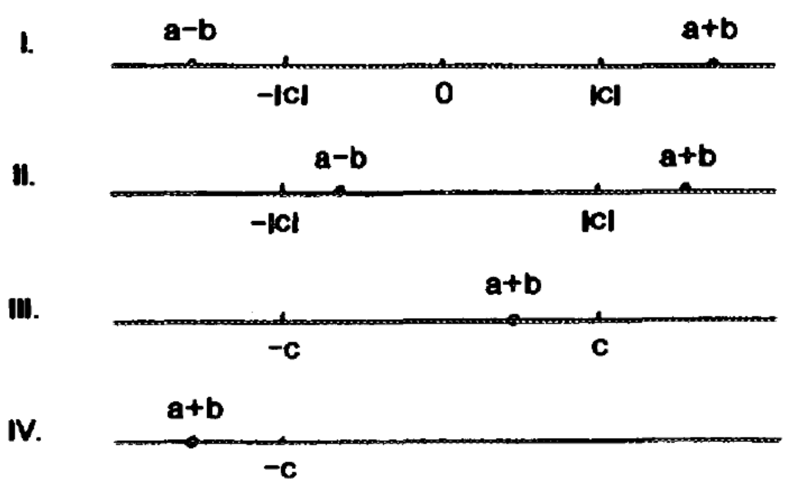
$$\dot{x}(t) + ax(t) + bx(t-\tau) + cx(t-\gamma) = 0, \quad \gamma, \tau \geq 0, \ t \geq 0, \tag{3.12}$$

whose characteristic equation is given by the transcendental polynomial

$$\psi(s, \tau, \gamma) = s + a + be^{-s\tau} + ce^{-s\gamma}. \tag{3.13}$$

The stability region of (3.12) has been extensively studied in simplified settings; for example, the exact stability region is determined by Braddock and van den Driessche [42] when $b = c$. Mahaffy considered the case of rationally dependent when $a, b, c < 0$ [43], Belair studied the problem when $a \leq 0$, $bc \neq 0$ [44], and the case $a = 0$, $b, c > 0$ is studied by Belair and Campbell [45].

Hale and Huang [46] gives a complete geometrical description of the stable region for (3.12) in $\tau - \sigma$ plane[1]. In their study, they provide a complete analysis by dividing the relation of the coefficients a, b, c into four main cases according to their positions in the real line, which is illustrated below.



[1] Hale and Huang defines the stable region as the maximal connected set $D \subset [0, \infty) \times [(0, \infty)$ which contains the origin [46].

Assuming $b > 0$ without loss of generality, the four cases are enumerated below:

I. $b \pm a \geq |c|$

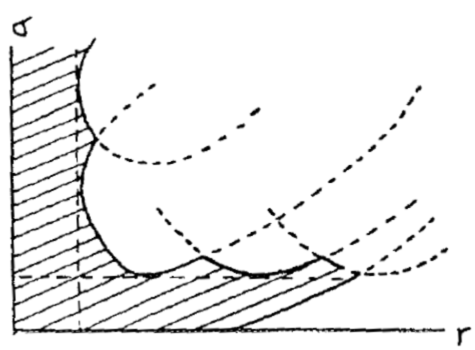
II. $b + a \geq |c| > b - a$

III. $c > a + b \geq -c$, $c > 0$

IV. $a + b + c < 0$

In Chapter 5, we will study a coupled system of DDEs of the form (3.12) with $\sigma = 2\tau$, $a = 1$, $b = -2(1 - \lambda_k)$, $c = (1 - \lambda_k)$, where λ_k, $k = 1, \ldots, n$, are the eigenvalues of the Laplacian L. Our stability analysis in Chapter 5 will be simpler than of Hale and Huang's due to the simplifications coming with the assumption that $\sigma = 2\tau$. Nevertheless, we derive inspiration from their approach and results.

Consider the case when $\lambda_k > \frac{4}{3}$ so that $a + b = 2\lambda_k - 1 > |c|$ and $a - b = 3 - 2\lambda_k < |c|$ where $|c| = \lambda_k - 1$, which corresponds to $a + b > |c| > a - b$, $c < 0$, which is covered under the case IIB in Hale & Huang's analysis [46]. Their illustration of the stability region for this case is provided below.



When $\sigma = 2\tau$, the above drawing directly implies the existence of $\tau^* > 0$ such that the zero solution of (3.12) is asymptotically stable if and only if $\tau < \tau^*$, where τ^* is the point where the line $\sigma = 2\tau$ intersects with the stability switching curves shown in the figure above.

When $0 < \lambda < \frac{2}{3}$, $b = -2(1 - \lambda_k) < 0$, $c > 0$, therefore we switch τ and σ and write

$$\dot{x}(t) + ax(t) + cx(t - \tau) + bx(t - \sigma) = 0,$$

where $a = 1$, $c = -2(1 - \lambda_k) < 0$, $b = (1 - \lambda_k) > 0$ and $\tau = 2\sigma$. Switching τ and σ is not needed, it is just to be able to use Hale and Huang's results directly as they assume $b > 0$ and divide the real plane into four regions accordingly, without having any constraints on τ and σ. In this case, $a + b = 2 - \lambda_k$, $|c| = 2(1 - \lambda_k)$ and again we have $a + b > |c| > a - b$, which is covered under the case IIB in Hale and Huang's analysis, but with $\tau = 2\sigma$. The stability region is again of the form $[0, \tau^*)$ where τ^* is the point the line $\tau = 2\sigma$ intersects with the stability switching curves drawn in the figure above.

Under the assumption that $\sigma = 2\tau$, (3.12) rewrites as

$$\dot{x}(t) + ax(t) + bx(t - \tau) + cx(t - 2\tau) = 0, \tag{3.14}$$

whose characteristic equation is given by

$$f(s, \tau) = s + a + be^{-s\tau} + ce^{-2s\tau}. \tag{3.15}$$

Hata and Matsunaga established conditions for the asymtotic stability of the zero solution of (3.14), which clearly requires $a + b + c > 0$.

When $a + b + c > 0$, it is possible to find a sufficient condition for all roots of f to have negative real parts directly by the continuity of roots with respect to time delay τ for which we search for imaginary crossings $s = i\omega$, due to the following proposition.

Proposition 3.4.12. *As τ varies continuously, the number of roots (counting multiplicity) of (3.15) on $\mathbb{C}_+$ can change only if a root appears on or crosses the imaginary axis.*

Since $\tau = 0$ yields $f(s) = s + a + b + c$, Proposition 3.4.12 implies when $a + b + c > 0$ that f has roots with negative real parts either for all $\tau \geq 0$ or there exists $D \subset \mathbb{R}$ and $\tau^* > 0$ such that $[0, \tau^*) \subseteq D$ and when $\tau \in D$, f has roots with negative real parts.

However, when $a + b + c = 0$, $s = 0$ is a root that is invariant of τ. Thus, additional work is required to be able to use the continuity argument to prove the existence of τ^* such that when $\tau < \tau^*$, $s = 0$ is a simple root and all other roots have negative real parts.

Our solution method in Chapter 5 will be very similar to that of Hata and Matsunaga at some parts, therefore we will not cover their analysis here.

Chapter 4

Delayed consensus protocols in networks

In this chapter, we review three delayed consensus protocols, each with a time delay arising from a different cause. Specifically, in Section 4.1, we study the consensus problem under processing delays, followed by the consensus problem under transmission delays in Section 4.2. In Section 4.3, we review the anticipatory consensus protocol, where the delayed argument arises from the predictive mechanisms of the agents unlike the first two protocols where the time delay is inherent to the system.

4.1 Information processing delays

Consider a system of agents that require a certain amount of time to process the information they receive from their neighbours. Assuming a fixed information processing delay $\tau > 0$ for all agents, the formulation of the consensus problem takes the form:

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}[x_j(t-\tau) - x_i(t-\tau)], \quad i = 1, 2, \ldots, n. \tag{4.1}$$

These system of equations can be written in vector form as

$$\dot{x}(t) = -Qx(t-\tau), \tag{4.2}$$

where $Q = D - A$ is the combinatorial Laplacian matrix. The conditions for the system (4.2) defined on undirected and connected graphs to reach consensus is established by Olfati-Saber and Murray [47], given in Theorem 4.1.1.

Theorem 4.1.1. *(Olfati-Saber & Murray, 2004) The system (4.2) defined on a connected, undirected graph reaches consensus from arbitrary initial conditions if and only if* $\tau < \dfrac{\pi}{2\lambda_n}$ *where* λ_n *is the largest eigenvalue of the Laplacian* Q*.*

The normalized version of (4.1) is given by

$$\dot{x}(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[x_j(t-\tau) - x_i(t-\tau)], \quad i = 1, 2, \ldots, n, \tag{4.3}$$

which has the vector form

$$\dot{x}(t) = -Lx(t-\tau). \tag{4.4}$$

Then, we can express the conditions for the convergence of the system (4.4) as a corollary of theorem (4.1.1).

Corollary 4.1.2. *The system (4.4) defined on a connected, undirected graph reaches consensus from arbitrary initial conditions if and only if* $\tau < \dfrac{\pi}{2\lambda_n}$ *where* λ_n *is the largest eigenvalue of the normalized Laplacian* L*.*

The convergence rate of the systems (4.2)-(4.4) was shown to be superior to the convergence rates of the delay-free systems $\dot{x}(t) = -Qx(t)$ and $\dot{x}(t) = -Lx(t)$ respectively under certain conditions on τ [48].

If the processing delay is not fixed for all agents but distributed according to a probability density function f, then the formulation (4.3) becomes

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}\left[\int_{-\tau}^{0} (x_j(t+\theta) - x_i(t+\theta))f(\theta)d\theta\right], \quad i = 1, \ldots, n. \tag{4.5}$$

We can write (4.5) as

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}\mathcal{L}(x_j^t - x_i^t) \tag{4.6}$$

where $x_j^t(\theta) = x_j(t+\theta)$ for all $\theta \in [-\tau, 0]$ and $\mathcal{L}$ is a linear operator whose action on $\psi \in \mathcal{C} = C([-\tau, 0], \mathbb{R})$ can be represented as a Stieltjes integral

$$\mathcal{L}\psi = \int_{-\tau}^{0} \psi(\theta) d\eta(\theta). \tag{4.7}$$

Here, η is a function of bounded variation defined on the compact interval $[-\tau, 0]$ satisfying $\int_{-\tau}^{0} d\eta(\theta) = 1$ and is non-decreasing on $[-\tau, 0]$. The operator $\mathcal{L}$, known as the delay operator, facilitates further generalizations of time delays [49].

For example, (4.6) can be reduced to (4.3) if η is chosen to be the Heaviside step function at $-\tau$, or one obtains (4.5) if η is chosen to be the probability density function f.

4.2 Information transmission delays

Network consensus under delayed information transmission has been studied by Seuret et al. [50], and Moreau [51]. The necessary and sufficient conditions for reaching consensus, as well as the exact consensus value, were established by Atay [49] under more general settings. We start with the most general setting, which involves distributed transmission delays on directed networks.

Atay studied the consensus problem of the form

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}(\mathcal{L}x_j^t - x_i(t)), \quad i = 1, 2, \ldots, n \tag{4.8}$$

with $\mathcal{L}$ as in (4.7) and x_j^t is the function defined by $x_j^t(\theta) = x_j(t+\theta)$ for all $\theta \in [-\tau, 0]$, that is, $\mathcal{L}x_j^t = \int_{-\tau}^{0} x_j(t+\theta)d\eta(\theta)$.

Using (4.7), we can write (4.8) in vector form as

$$\dot{x}(t) = -x(t) + D^{-1}A\int_{-\tau}^{\theta} x(t+\theta)d\eta(\theta). \tag{4.9}$$

The following theorem not only provides the necessary and sufficient condition for reaching consensus but also the consensus value explicitly.

Theorem 4.2.1. *(Atay, 2014) The system (4.9) reaches consensus if and only if zero is a simple eigenvalue of the Laplacian L. Moreover, the consensus value is given by*

$$c = \frac{1}{1+\overline{\tau}}\langle \pi^*, x(0) - \int_{-\tau}^{0}\int_{0}^{\theta} x(\xi)d\xi d\eta(\theta)\rangle$$

where π^* is the left eigenvector associated with the zero eigenvalue and $\overline{\tau} = \int_{-\tau}^{0} \theta d\eta(\theta)$ is the mean of distributed delay [49].

To model the consensus problem under a fixed information transmission delay of τ, we define η as the heaviside step function at $-\tau$ so that (4.7) gives $\mathcal{L}x_j^t = x_j(t-\tau)$ and (4.9) becomes

$$\dot{x}(t) = -x(t) + D^{-1}Ax(t-\tau). \tag{4.10}$$

Corollary 4.2.2. *(Atay, 2014) For the system (4.10) with single discrete transmission delay τ, the consensus value is*

$$c = \frac{1}{1+\tau}\langle \pi^*, x(0) + \int_{-\tau}^{0} x(\xi)d\xi\rangle.$$

Proof. Taking η as the heaviside step function at $-\tau$, the proof follows from Theorem (4.2.1). □

Reaching consensus is independent of the presence of transmission delays. However, the rate of convergence of the consensus protocol is adversely affected by the presence of a fixed information transmission delay $\tau > 0$ on undirected graphs, except when the graph is complete [52].

Theorem 4.2.3. *(Alhassan, 2020) For an undirected graph G, small delays improve the convergence rate of (4.10) if and only if G is a complete graph [52].*

4.3 Consensus in networks of anticipatory agents

Another delayed consensus protocol, studied by Atay and Irofti [24], is the anticipatory consensus protocol, which assumes intelligent agents that retain past information and predict the future states of their neighbours using that information following an anticipation rule based on a first-order estimation by linear extrapolation.

Under a delay-free setting, Atay and Irofti formulated the anticipatory consensus problem as

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[\hat{x}_j(t+\tau) - x_i(t)], \quad i = 1, 2, \ldots, n,$$

where $\hat{x}_j(t+\tau)$ is the anticipated state of agent j at future time $t+\tau$, given by

$$\begin{aligned}\hat{x}_j(t+\tau) &= x_j(t) + \frac{x_j(t) - x_j(t-\tau)}{\tau}\tau \\ &= 2x_j(t) - x_j(t-\tau),\end{aligned}$$

depicted in the figure below.

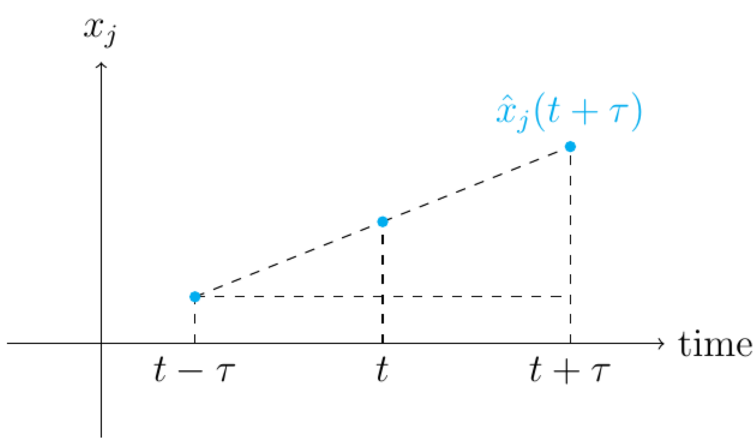


Figure 4.1: Anticipation rule to predict $\hat{x}_j(t+\tau)$.

Linear extrapolation introduces a delayed term into the system, and the formulation of the consensus problem on networks of anticipatory agents becomes

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[2x_j(t) - x_j(t-\tau) - x_i(t)], \quad i = 1, 2, \ldots, n \tag{4.11}$$

which can be written in vector form as

$$\dot{x}(t) = -x(t) + D^{-1}A(2x(t) - x(t-\tau)). \tag{4.12}$$

Then, the characteristic equation for the system (4.12) is given by

$$\Psi(s) = \prod_{k=1}^{n} \psi_k(s) = 0$$

where

$$\psi_k(s) = s + 1 - 2(1-\lambda_k) + (1-\lambda_k)e^{-s\tau}. \tag{4.13}$$

Theorem 4.3.1. *(Atay & Irofti, 2016) The system (4.12) defined on undirected and connected graphs reaches consensus from arbitrary initial conditions if and only if* $\tau < 1$ *[24].*

Atay and Irofti also compute the dominant transverse root of the anticipatory consensus protocol by using Lambert W function.

Proposition 4.3.2. *The root of the characteristic factor (4.13) having the largest real part is given by*

$$s^* = \frac{1}{\tau} W_0(\tau b_k e^{-a_k \tau}) + a_k$$

where W_0 *is the principal branch of the Lambert* W *function and* $a_k = 2(1-\lambda_k) - 1$, $b_k = -(1-\lambda_k)$.

The preceding proposition allows one to determine whether for a given $\tau > 0$ the anticipatory protocol reaches consensus faster than the classical consensus protocol by comparing the dominant roots.

The findings of Alhassan [52] and Atay & Irofti [24] raise the following question: How anticipation affects the consensus dynamics in the presence of transmission delays?

In the next chapter, we will examine the consensus problem under a fixed transmission delay of τ in networks of anticipatory agents, under the assumption that the agents predict the current states of their neighbours by a first-order linear extrapolation. The anticipatory behaviour of agents introduces another delayed term into the system, resulting in a linear system of delay differential equations with two discrete delays. We will derive an exact condition for reaching consensus, and present computational examples suggesting that anticipation improves the convergence rate of the consensus protocol under transmission delays.

Part II

Main Results

Chapter 5

Consensus in networks of anticipatory agents under transmission delays

In this chapter, we introduce anticipatory agents into the linear normalized consensus problem under transmission delays. First, we define the model and derive the characteristic equation Ψ (Section 5.1). We then demonstrate that when the time delay is sufficiently small, Ψ has a simple root at zero, and all other roots have negative real parts (Section 5.2). To determine the stability switching curves, we fully characterize the purely imaginary roots of Ψ and the corresponding values of the time delay τ (Section 5.2). Next, we present the main result of this thesis in Theorem 5.3.3, which provides the necessary and sufficient condition for the consensus protocol under transmission delays on networks of anticipatory agents to to reach consensus (Section 5.3). Finally, we present computational examples to further elucidate the findings of Theorem 5.3.3 and compare the convergence rates of consensus protocols with and without anticipation in the presence of transmission delays (Section 5.4).

5.1 Consensus protocol under transmission delays and anticipatory agents

The consensus problem under a fixed information transmission delay of τ is given in (4.10), formulated by the system of DDEs with a single delay $\tau > 0$:

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[x_j(t-\tau) - x_i(t)], \quad i = 1, 2, \ldots, n.$$

Assuming that the underlying network consists of anticipatory agents, we replace the delayed term $x_j(t-\tau)$ by $\hat{x}_j(t)$, the predicted state of agent j at present time. This leads to the model:

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[\hat{x}_j(t) - x_i(t)] \quad i = 1, 2, \ldots, n, \tag{5.1}$$

where $\hat{x}_j(t)$ is given by

$$\begin{aligned}\hat{x}_j(t) &= x_j(t-\tau) + \frac{x_j(t-\tau) - x_j(t-2\tau)}{\tau}\tau \\ &= 2x_j(t-\tau) - x_j(t-2\tau).\end{aligned}$$

Following [24], we assume an anticipation rule based on a first-order estimation by linear extrapolation using the points $x_j(t-\tau)$ and $x_j(t-2\tau)$ to predict $\hat{x}_j(t)$, which is illustrated in the figure below.

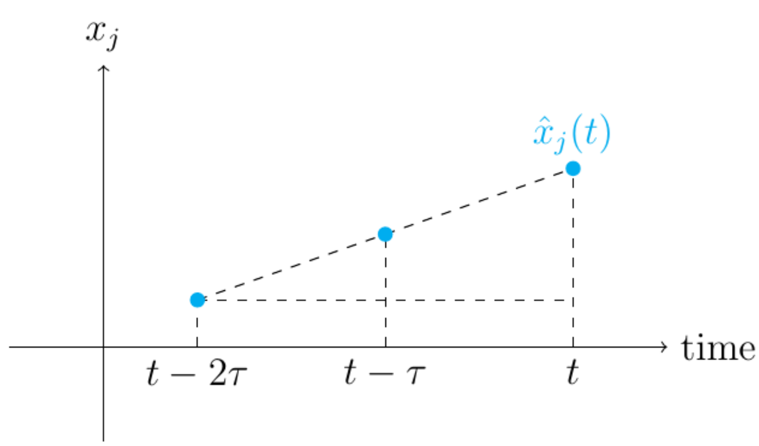


Figure 5.1: Anticipation rule to predict $\hat{x}_j(t)$.

We choose equal length for the history and future windows for linear extrapolation. The reason for this, not mentioning intuitiveness, will be clear when we study the stability of the system (5.1); the assumption of $\sigma = 2\tau$ brings simplifications, which allow us to establish explicit conditions for reaching consensus.

Substitution of $\hat{x}_j(t)$ into (5.1) gives

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[2x_j(t-\tau) - x_j(t-2\tau) - x_i(t)], \quad i = 1, 2, \ldots, n, \tag{5.2}$$

which can be written in vector form as

$$\dot{x}(t) = -x(t) + D^{-1}A[2x(t-\tau) - x(t-2\tau)]. \tag{5.3}$$

Expressing $x(t)$ in the eigenbasis of L as $x(t) = \sum_{k=1}^{n} \alpha_k(t) v_k$ yields

$$\begin{aligned}
\sum_{k=1}^{n} \dot{\alpha}_k(t) v_k &= -\sum_{k=1}^{n} \alpha_k(t) v_k + D^{-1}A\sum_{k=1}^{n}(2\alpha_k(t-\tau) - \alpha_k(t-2\tau))v_k \\
\sum_{k=1}^{n} \dot{\alpha}_k(t) v_k &= -\sum_{k=1}^{n} \alpha_k(t) v_k + \sum_{k=1}^{n}(1-\lambda_k)(2\alpha_k(t-\tau) - \alpha_k(t-2\tau))v_k \\
\sum_{k=1}^{n}[\dot{\alpha}_k(t) &+ \alpha_k(t) - (1-\lambda_k)(2\alpha_k(t-\tau) - \alpha_k(t-2\tau))]v_k = 0
\end{aligned}$$

where $\{v_1, \ldots, v_n\}$ denote the set of right eigenvectors of L. Then we have $(I - L)v_k = D^{-1}Av_k = (1-\lambda_k)v_k$ since v_k satisfies $Lv_k = \lambda_k v_k$ and $v_k \neq \vec{0}$ for all $k = 1, \ldots, n$. This gives n decoupled scalar DDEs:

$$\dot{\alpha}_k(t) = -\alpha_k(t) + (1-\lambda_k)[2\alpha_k(t-\tau) - \alpha_k(t-2\tau)], \quad k = 1, \ldots, n. \tag{5.4}$$

The ansatz $\alpha_k(t) = ce^{st}$ yields the characteristic factor corresponding to eigenmode k

$$\psi_k(s) = s + 1 - 2(1-\lambda_k)e^{-s\tau} + (1-\lambda_k)e^{-2s\tau}. \tag{5.5}$$

Then the characteristic equation for the whole system is

$$\Psi(s) = \prod_{k=1}^{n} \psi_k(s) = 0.$$

5.2 On the roots of Ψ

The first characteristic factor ψ_1, which corresponds to the zero eigenvalue $\lambda_1 = 0$, is given by

$$\psi_1(s) = s + 1 - 2e^{-s\tau} + e^{-2s\tau}. \tag{5.6}$$

Lemma 5.2.1. *If $0 \leq \tau < \frac{\pi}{6}$, then ψ_1 has a simple root at zero and all its other roots have negative real parts.*

Proof. First, note that if $s = \mu + i\omega$ is a root of ψ_1 then the imaginary part of $\psi_1(s)$ is given by

$$\omega + 2e^{-\mu\tau}\sin(\omega\tau) - e^{-2\mu\tau}\sin(2\omega\tau) = 0. \tag{5.7}$$

Then,

$$\begin{aligned} |\omega| &= |-2e^{-\mu\tau}\sin(\omega\tau) + e^{-2\mu\tau}\sin(2\omega\tau)| \\ &\leq 2e^{-\mu\tau}|\sin(\omega\tau)| + e^{-2\mu\tau}|\sin(2\omega\tau)|, \end{aligned}$$

where the inequality follows by the triangle inequality. If $\mu \geq 0$, then

$$|\omega| \leq 3 \tag{5.8}$$

since $e^{-\mu\tau} \leq 1$ for all $\mu \geq 0$, and $|\sin(x)| \leq 1$ for all x.

Clearly, $s = 0$ is a root of ψ_1. Moreover, it is a simple root since

$$\psi_1'(0) = 1 + 2\tau e^0 - 2\tau e^0 = 1 \neq 0.$$

We prove by contradiction that all other roots have negative real parts. Suppose $s = \mu + i\omega$ is a root of ψ_1 with $\mu \geq 0$. We can rewrite (5.7) as

$$\omega = 2e^{-\mu\tau}\sin(\omega\tau)[e^{-\mu\tau}\cos(\omega\tau) - 1]. \tag{5.9}$$

Note that $-\frac{\pi}{2} < \omega\tau < \frac{\pi}{2}$ is always satisfied since $|\omega| \leq 3$ by (5.8) and $0 \leq \tau < \frac{\pi}{6}$. Clearly, $\omega = 0$ is the only solution of (5.9) when $\tau = 0$. In fact, $\omega = 0$ satisfies (5.9) for any $\tau \geq 0$. Observing that ω satisfies (5.9) if and only $-\omega$ satisfies (5.9), we assume $\omega > 0$ in the following without loss of generality.

If $\tau > 0$, we have

$$0 < \cos(\omega\tau) < 1, \quad 0 < \sin(\omega\tau) < 1, \text{and } 0 < e^{-\mu\tau} \leq 1$$

since $\mu \geq 0$. Therefore,

$$2e^{-\mu\tau}\sin(\omega\tau) > 0 \text{ and } e^{-\mu\tau}\cos(\omega\tau) - 1 < 0,$$

meaning that the right hand side of (5.9) is negative while the left hand side is positive. We conclude that ω cannot be positive (or negative), so it must be equal to zero for all $\tau \geq 0$, which implies that if there exists a characteristic root with non-negative real part, then this root must be real. To find the real roots on the right half-plane, we substitute $\omega = 0$ and rewrite $\psi_1(s)$ as

$$\phi(\mu) = \mu + 1 - 2e^{-\mu\tau} + e^{-2\mu\tau}, \tag{5.10}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(s) = \psi_1(s)$ when $s \in \mathbb{R}$. Note that ϕ is strictly increasing in μ for all $\mu \geq 0$ since

$$\begin{aligned} \phi'(\mu) &= 1 + 2\tau e^{-\mu\tau} - 2\tau e^{-2\mu\tau} \\ &= 1 + 2\tau e^{-\mu\tau}(1 - e^{-\mu\tau}) > 0 \quad \forall \mu, \tau \geq 0 \end{aligned}$$

and $\mu = 0$ is a root of (5.10). Thus, a positive root of (5.10) cannot exist, which completes the proof.

□

Remark 5.2.2. *When $\tau = 0$, (5.5) reduces to $\psi_k(s) = s + \lambda_k$, which implies $s = -\lambda_k < 0$ for all $k \geq 2$. By the continuity of roots, for sufficiently small $\tau > 0$, the roots cannot cross the imaginary axis and enter the right half-plane, thus all roots of ψ_k, $k \geq 2$, are on the left half-plane for sufficiently small values of τ. Together with Lemma 5.2.1, this implies that Ψ has a simple root at zero and all its other roots have negative real parts when $\tau > 0$ is sufficiently small.*

The following lemma will be useful for the characterization of roots $s = i\omega$, as we will show afterwards that $s = i\omega$ is a root if and only if ω^2 satisfies a certain quadratic equation.

Lemma 5.2.3. *Let $\lambda \in \mathbb{R}$ satisfy $|1-\lambda| \leq 1$. Then the real roots of quadratic equation*

$$r^2 + (2 - 6(1-\lambda)^2)r + \lambda^3(4 - 3\lambda) = 0$$

are classified according to their signs as follows:

(i) If $\lambda = 0$, then there is the zero root and a positive root.

(ii) If $0 < \lambda < \frac{1}{3}$, there are two distinct positive roots.

(iii) If $\lambda = \frac{1}{3}$, then there is double positive roots.

(iv) If $\frac{1}{3} < \lambda < \frac{4}{3}$, then the quadratic equation has either no real roots, or all the roots are negative.

(v) If $\lambda = \frac{4}{3}$, then there is the zero root and a negative root.

(vi) If $\lambda > \frac{4}{3}$, then there is one positive and one negative root.

Proof. The proof is straightforward, a more general version can be found in [53]. □

Lemma 5.2.4. *Let $0 \leq \lambda \leq 2$, $\tau \geq 0$, $s \in \mathbb{C}$ and $\psi(s, \lambda, \tau)$ be a complex-valued function defined by*

$$\psi(s, \lambda, \tau) = s + 1 - 2(1-\lambda)e^{-s\tau} + (1-\lambda)e^{-2s\tau}. \tag{5.11}$$

Then $\psi(\pm i\omega, \lambda, \tau) = 0$ if and only if $r = \omega^2$ is a positive root of the quadratic equation

$$r^2 + (2 - 6(1-\lambda)^2)r + \lambda^3(4 - 3\lambda) = 0 \tag{5.12}$$

and

$$\tau = \begin{cases} \dfrac{1}{\omega}\left(2\pi - \arccos\left(\frac{2\lambda(1-\lambda)}{\lambda(2-\lambda)+\omega^2}\right) + 2m\pi\right), & \lambda \in [0, \frac{1}{3}] \\[2em] \dfrac{1}{\omega}\left(\arccos\left(\frac{2\lambda(1-\lambda)}{\lambda(2-\lambda)+\omega^2}\right) + 2m\pi\right), & \lambda \in (\frac{4}{3}, 2] \\[2em] +\infty, & \text{otherwise} \end{cases} \tag{5.13}$$

for some $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Proof. We note that $\psi(0, \lambda, \tau) = \lambda$, which means that $s = 0$ cannot be a root of ψ for $\lambda > 0$. Together with the fact that $s = 0$ is a simple root of ψ when $\lambda = 0$ regardless of τ, which we have shown in Lemma 5.2.1, we assume $\omega \neq 0$ without loss of generality.

Multiplying (5.11) by $e^{s\tau}$, we obtain

$$\psi(s, \lambda, \tau)e^{s\tau} = (s + 1)e^{s\tau} - 2(1 - \lambda) + (1 - \lambda)e^{-s\tau} = 0.$$

Substituting $s = i\omega$, and equating real and imaginary parts yields

$$\begin{aligned}(2 - \lambda)\cos(\omega\tau) - \omega\sin(\omega\tau) &= 2(1 - \lambda)\\ \omega\cos(\omega\tau) + \lambda\sin(\omega\tau) &= 0,\end{aligned}$$

which can be written as $Tx = b$ where

$$T = \begin{bmatrix} 2-\lambda & -\omega \\ \omega & \lambda \end{bmatrix}, \quad x = \begin{bmatrix} \cos(\omega\tau) \\ \sin(\omega\tau) \end{bmatrix}, \quad b = \begin{bmatrix} 2(1-\lambda) \\ 0 \end{bmatrix}.$$

T is invertible since $\det T = (2 - \lambda)\lambda + \omega^2 > 0$ for all $\lambda \in [0, 2]$ and $\omega > 0$. Applying $T^{-1} = \dfrac{1}{(2-\lambda)\lambda + \omega^2}\begin{bmatrix} \lambda & \omega \\ -\omega & 2-\lambda \end{bmatrix}$ to b, we get

$$\begin{bmatrix} \cos(\omega\tau) \\ \sin(\omega\tau) \end{bmatrix} = \begin{bmatrix} \dfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda + \omega^2} \\ \dfrac{-2\omega(1-\lambda)}{(2-\lambda)\lambda + \omega^2} \end{bmatrix}. \tag{5.14}$$

Noting that ω satisfies (5.14) if and only $-\omega$ does so, we assume $\omega > 0$ in the following without loss of generality.

($\implies$) If $s = i\omega$ is a root of (5.11), ω satisfies (5.14). Then, $\cos^2(\omega\tau) + \sin^2(\omega\tau) = 1$ gives

$$\omega^4 + (2 - 6(1-\lambda)^2)\omega^2 + \lambda^3(4 - 3\lambda) = 0.$$

Substitution of $r = \omega^2$ yields the quadratic equation (5.12).

We denote the roots of (5.12) by $r_+ = \omega_+^2$ and $r_- = \omega_-^2$ with $r_+ \geq r_-$, which are given by

$$\begin{aligned} r_+ = \omega_+^2 &= -1 + 3(1-\lambda)^2 + 2\sqrt{(1-\lambda)^3(1-3\lambda)} \\ r_- = \omega_-^2 &= -1 + 3(1-\lambda)^2 - 2\sqrt{(1-\lambda)^3(1-3\lambda)}, \end{aligned}$$

and thus

$$\begin{aligned} \omega_+ &= \sqrt{-1 + 3(1-\lambda)^2 + 2\sqrt{(1-\lambda)^3(1-3\lambda)}} \\ \omega_- &= \sqrt{-1 + 3(1-\lambda)^2 - 2\sqrt{(1-\lambda)^3(1-3\lambda)}}. \end{aligned}$$

Note that ω_+ and ω_- are continuous functions of λ whenever they are defined; and ω_+ is well-defined for $\lambda \in [0, \frac{1}{3}] \cup [\frac{4}{3}, 2]$ whereas ω_- is well-defined when $\lambda \in [0, \frac{1}{3}]$, consistent with our findings in Lemma (5.2.3). However, when $\lambda = 0$, $\omega_-(\lambda) = 0$ and when $\lambda = \frac{4}{3}$, $\omega_+(\lambda) = 0$ but we assumed $\omega > 0$. Therefore, we exclude those points of λ from the domains of definitions of ω_+ and ω_-, and write

$$\omega_+(\lambda) = \sqrt{-1 + 3(1-\lambda)^2 + 2\sqrt{(1-\lambda)^3(1-3\lambda)}}, \quad \lambda \in [0, \tfrac{1}{3}] \cup (\tfrac{4}{3}, 2] \tag{5.15}$$

$$\omega_-(\lambda) = \sqrt{-1 + 3(1-\lambda)^2 - 2\sqrt{(1-\lambda)^3(1-3\lambda)}}, \quad \lambda \in (0, \tfrac{1}{3}]. \tag{5.16}$$

Now, to find the value(s) of τ satisfying (5.14), we solve

$$\omega\tau = \arccos\left(\frac{2\lambda(1-\lambda)}{(2-\lambda)\lambda + \omega^2}\right), \quad \omega\tau = \arcsin\left(\frac{-2\omega(1-\lambda)}{(2-\lambda)\lambda + \omega^2}\right). \tag{5.17}$$

For both equations in (5.17) to be satisfied, one needs to make adjustments using the trigonometric identities

$$\begin{aligned} \cos(x) = \cos(2\pi - x) = \cos(x + 2\pi) = \cos(-x) \\ \sin(x) = \sin(\pi - x) = \sin(x + 2\pi) = -\sin(-x) \end{aligned}$$

so that both equations will give the same value for $\omega\tau > 0$. The need for such adjustments is due to the fact that cosine and sine functions are not one-to-one on the unit circle thus, their domains are restricted to smaller intervals so that the inverse functions, arccos and arcsin, are defined. We use the standard definitions where the domains and ranges of inverse functions are given by

$$\arcsin : [-1,1] \to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \quad \arccos : [-1,1] \to [0,\pi].$$

When $\lambda < 1$, we have

$$\cos(\omega\tau) \geq 0 \text{ and } \sin(\omega\tau) < 0,$$

meaning that $\omega\tau$ must be in the fourth region of the unit circle, that is, $\omega\tau \in [\frac{3\pi}{2} + 2m\pi, 2(m+1)\pi]$ for some $m \in \mathbb{N}_0$. In this case,

$$\omega\tau = 2\pi - \arccos\left(\frac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega^2}\right) + 2m\pi = \arcsin\left(\frac{-2\omega(1-\lambda)}{(2-\lambda)\lambda+\omega^2}\right) + 2(m+1)\pi.$$

When $\lambda > 1$, we have

$$\cos(\omega\tau) < 0 \text{ and } \sin(\omega\tau) > 0,$$

meaning that $\omega\tau$ must be in the second region of the unit circle, that is, $\omega\tau \in [\frac{\pi}{2} + 2m\pi, \pi + 2m\pi]$ for some $m \in \mathbb{N}_0$. In this case,

$$\omega\tau = \arccos\left(\frac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega^2}\right) + 2m\pi = \pi - \arcsin\left(\frac{-2\omega(1-\lambda)}{(2-\lambda)\lambda+\omega^2}\right) + 2m\pi.$$

Hence, we can write $\omega\tau$ in terms of arccos function as

$$\omega\tau = \begin{cases} 2\pi - \arccos\left(\dfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega^2}\right) + 2m\pi, & \lambda < 1 \\[2ex] \arccos\left(\dfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega^2}\right) + 2m\pi, & \lambda > 1 \end{cases}$$

where $m \in \mathbb{N}_0$.

For fixed m, denoting the corresponding value(s) of τ by τ^m, we write

$$\tau^m = \begin{cases} \dfrac{1}{\omega}\left(2(m+1)\pi - \arccos\left(\dfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega^2}\right)\right), & \lambda < 1, \\[2ex] \dfrac{1}{\omega}\left(2m\pi + \arccos\left(\dfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega^2}\right)\right), & \lambda > 1. \end{cases}$$

Here, m is restricted to take values in $\mathbb{N}_0$ instead of $\mathbb{Z}$ because negative values of τ does not make sense as τ is the time delay in the system. Another note is on the fact that τ being a continuous function of λ whenever it is well-defined since ω is a continuous function of λ as well. Therefore, we can write

$$\tau^m(\lambda) = \begin{cases} \dfrac{1}{\omega(\lambda)}\left(2(m+1)\pi - \arccos\left(\dfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega(\lambda)^2}\right)\right), & \lambda < 1, \\ \\ \dfrac{1}{\omega(\lambda)}\left(2m\pi + \arccos\left(\dfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega(\lambda)^2}\right)\right), & \lambda > 1. \end{cases}$$

Finally, letting $\omega = \omega_+$ and $\omega = \omega_-$, and denoting the corresponding values of τ by τ_+^m and τ_-^m for fixed $m \in \mathbb{N}_0$ respectively, we get

$$\tau_+^m = \begin{cases} \dfrac{1}{\omega_+}\left(2(m+1)\pi - \arccos\left(\dfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right)\right), & \lambda \in [0, \frac{1}{3}], m \in \mathbb{N}_0 \\ \\ \dfrac{1}{\omega_+}\left(2m\pi + \arccos\left(\dfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right)\right), & \lambda \in (\frac{4}{3}, 2], m \in \mathbb{N}_0 \end{cases} \tag{5.18}$$

$$\tau_-^m = \frac{1}{\omega_-}\left(2(m+1)\pi - \arccos\left(\frac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_-^2}\right)\right), \quad \lambda \in (0, \tfrac{1}{3}], m \in \mathbb{N}_0 \tag{5.19}$$

which gives

$$\tau = \begin{cases} \dfrac{1}{\omega}\left(2\pi - \arccos\left(\frac{2\lambda(1-\lambda)}{\lambda(2-\lambda)+\omega^2}\right) + 2m\pi\right), & \lambda \in [0, \frac{1}{3}], m \in \mathbb{N}_0 \\ \\ \dfrac{1}{\omega}\left(\arccos\left(\frac{2\lambda(1-\lambda)}{\lambda(2-\lambda)+\omega^2}\right) + 2m\pi\right), & \lambda \in (\frac{4}{3}, 2], m \in \mathbb{N}_0 \\ \\ +\infty, & \lambda \in (\frac{1}{3}, \frac{4}{3}] \end{cases}$$

where we write $\tau = +\infty$ for $\lambda \in (\frac{1}{3}, \frac{4}{3}]$ by convention.

($\Longleftarrow$) Conversely, suppose that $r = \omega^2$ is a positive root of quadratic equation (5.12) and τ is given by (5.13). In the first part of the proof, we have shown that if $s = \pm i\omega$ are purely imaginary roots of (5.11), then $r = \omega^2$ is a positive root of (5.12). By statements (iv)-(v) of Lemma (5.2.3), the quadratic equation (5.12) has no positive roots when $\lambda \in (\frac{1}{3}, \frac{4}{3}]$. Hence, by the law of contraposition, there cannot be any purely imaginary roots of (5.11) when $\lambda \in (\frac{1}{3}, \frac{4}{3}]$.

It suffices to show that $\omega_+\tau_+$ and $\omega_-\tau_-$ satisfy (5.14) whenever defined.

$$\omega_+\tau_+ = \begin{cases} 2(m+1)\pi - \arccos\left(\frac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right), & \lambda \in [0, \frac{1}{3}],\ m \in \mathbb{N}_0 \\ 2m\pi + \arccos\left(\frac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right), & \lambda \in (\frac{4}{3}, 2],\ m \in \mathbb{N}_0 \end{cases}$$

$$\omega_-\tau_- = 2(m+1)\pi - \arccos\left(\frac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_-^2}\right), \quad \lambda \in (0, \frac{1}{3}],\ m \in \mathbb{N}_0$$

Similarly, in terms of arcsin function

$$\omega_+\tau_+ = \begin{cases} 2m\pi + \arcsin\left(\frac{-2\omega_+(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right), & \lambda \in [0, \frac{1}{3}],\ m \in \mathbb{N}_0 \\ (2m+1)\pi - \arcsin\left(\frac{-2\omega_+(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right), & \lambda \in (\frac{4}{3}, 2],\ m \in \mathbb{N}_0 \end{cases}$$

$$\omega_-\tau_- = 2m\pi + \arcsin\left(\frac{-2\omega_-(1-\lambda)}{(2-\lambda)\lambda+\omega_-^2}\right), \quad \lambda \in (0, \frac{1}{3}],\ m \in \mathbb{N}_0$$

Hence, we have

$$\cos(\omega_+\tau_+) = \frac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}, \quad \lambda \in [0, \tfrac{1}{3}] \cup (\tfrac{4}{3}, 2]$$
$$\sin(\omega_+\tau_+) = \frac{-2\omega_+(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}, \quad \lambda \in [0, \tfrac{1}{3}] \cup (\tfrac{4}{3}, 2]$$

and

$$\cos(\omega_-\tau_-) = \frac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_-^2}, \quad \lambda \in (0, \tfrac{1}{3}]$$
$$\sin(\omega_-\tau_-) = \frac{-2\omega_-(1-\lambda)}{(2-\lambda)\lambda+\omega_-^2}, \quad \lambda \in (0, \tfrac{1}{3}]$$

Thus, both $\omega_+\tau_+$ and $\omega_-\tau_-$ satisfy (5.14), which completes the proof. □

Clearly, the smallest positive values of τ_+^m and τ_-^m are τ_+^0 and τ_-^0, which we will denote by τ_+ and τ_- from now on, for brevity.

5.3 Convergence of the anticipatory protocol under transmission delays

We will present the main result of this thesis in Theorem 5.3.3, which establishes the necessary and sufficient condition for the system

$$\dot{x}(t) = -x(t) + D^{-1}A[2x(t-\tau) - x(t-2\tau)]$$

to achieve consensus on undirected and connected graphs.

Initially, we will determine the minimum value of τ that yields purely imaginary roots of Ψ, denoted by τ^*. The following two lemmas will aid in obtaining the exact value of τ^*.

Lemma 5.3.1. *$\tau_+(\lambda) \leq \tau_-(\lambda)$ for all $\lambda \in (0, \frac{1}{3}]$.*

Proof. Since $\omega_+(\lambda) \geq \omega_-(\lambda) > 0$ holds for all $\lambda \in (0, \frac{1}{3}]$, it follows that

$$\tfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_-^2} \geq \tfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}.$$

Then,

$$2\pi - \arccos\left(\tfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_-^2}\right) \geq 2\pi - \arccos\left(\tfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right)$$

holds for all $\lambda \in (0, \frac{1}{3}]$ since arccos is decreasing. Therefore,

$$\tau_+(\lambda) = \frac{2\pi - \arccos\left(\tfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right)}{\omega_+} \leq \frac{2\pi - \arccos\left(\tfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_-^2}\right)}{\omega_-} = \tau_-(\lambda),$$

which completes the proof. $\square$

Substituting

$$\omega_+(\lambda) = \sqrt{-1 + 3(1-\lambda)^2 + 2\sqrt{(1-\lambda)^3(1-3\lambda)}}$$

into

$$\tau_+ = \begin{cases} \dfrac{2\pi - \arccos\left(\tfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right)}{\omega_+}, & \lambda \in [0, \frac{1}{3}] \\ \dfrac{\arccos\left(\tfrac{2\lambda(1-\lambda)}{(2-\lambda)\lambda+\omega_+^2}\right)}{\omega_+}, & \lambda \in (\frac{4}{3}, 2], \end{cases}$$

we obtain

$$\tau_+(\lambda) = \begin{cases} \dfrac{2\pi - \arccos\left(\frac{2\lambda(1-\lambda)}{2(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}\right)}{\sqrt{-1+3(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}}, & \lambda \in [0, \frac{1}{3}] \\ \dfrac{\arccos\left(\frac{2\lambda(1-\lambda)}{2(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}\right)}{\sqrt{-1+3(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}}, & \lambda \in (\frac{4}{3}, 2]. \end{cases}$$

Lemma 5.3.2. *$\tau_+(\lambda)$ is increasing in λ for $\lambda \in [0, \frac{1}{3}]$ and decreasing in λ for $\lambda \in (\frac{4}{3}, 2]$.*

Proof. Define the functions

$$f(\lambda) = \frac{1}{\sqrt{-1+3(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}}, \quad \lambda \in [0, \tfrac{1}{3}] \cup (\tfrac{4}{3}, 2]$$

$$g(\lambda) = \arccos\left(\frac{2\lambda(1-\lambda)}{2(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}\right), \quad \lambda \in [0, \tfrac{1}{3}] \cup (\tfrac{4}{3}, 2].$$

Then,

$$\tau_+(\lambda) = \begin{cases} (2\pi - g(\lambda))f(\lambda), & \lambda \leq \frac{1}{3} \\ g(\lambda)f(\lambda), & \lambda > \frac{4}{3}. \end{cases}$$

It is enough to prove the following two items:

1. f is increasing for $\lambda \in [0, \frac{1}{3}]$ and decreasing for $\lambda \in (\frac{4}{3}, 2]$.

   We take the derivative with respect to λ

   $$\frac{df}{d\lambda} = \frac{6(1-\lambda)+\frac{3(1-\lambda)^2(2-4\lambda)}{\sqrt{(1-\lambda)^3(1-3\lambda)}}}{2\left(-1+3(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}\right)^{\frac{3}{2}}}.$$

   $\frac{df}{d\lambda} > 0$ when $\lambda < \frac{1}{3}$ and $\frac{df}{d\lambda} < 0$ when $\lambda > 1$, which proves the first claim.

2. g is decreasing for $\lambda \in [0, \frac{1}{3}) \cup (\frac{4}{3}, 2]$.

   Let $p(\lambda) = \frac{2\lambda(1-\lambda)}{2(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}$, so $g(\lambda) = \arccos(p(\lambda))$. It is enough to show that $p(\lambda)$ is an increasing function of $\lambda \in [0, \frac{1}{3}] \cup (\frac{4}{3}, 2]$ since arccos is decreasing. Taking the derivative of p with respect to λ,

   $$\frac{dp}{d\lambda} = \frac{4(1-\lambda)^2[1+\frac{4(1-\lambda)(1-2\lambda)}{\sqrt{(1-\lambda)^3(1-3\lambda)}}]}{\left(2(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}\right)^2}.$$

Letting $q(\lambda) = 1+\frac{4(1-\lambda)(1-2\lambda)}{\sqrt{(1-\lambda)^3(1-3\lambda)}}$, which is well-defined when $\lambda \in [0, \frac{1}{3}) \cup (1, 2]$, we have $q(0) = 5$, $q(2) = 1 + \frac{12}{\sqrt{5}}$ and $q'(\lambda) = \frac{4\lambda(1-\lambda)^3}{((1-\lambda)^3(1-3\lambda))^{\frac{3}{2}}}$, so $q'(\lambda) > 0$ when $\lambda < 1$ and $q'(\lambda) < 0$ when $\lambda > 1$, meaning that $q(\lambda) > 0$ for any $\lambda \in [0, \frac{1}{3}] \cup (\frac{4}{3}, 2]$ and therefore $p'(\lambda) > 0$ everywhere it is defined. Since arccos is decreasing and p is increasing whenever defined, $g = \arccos(p(\lambda))$.

The proof is complete. See the illustration of the continuous function τ_+ wherever well-defined in the figure below.

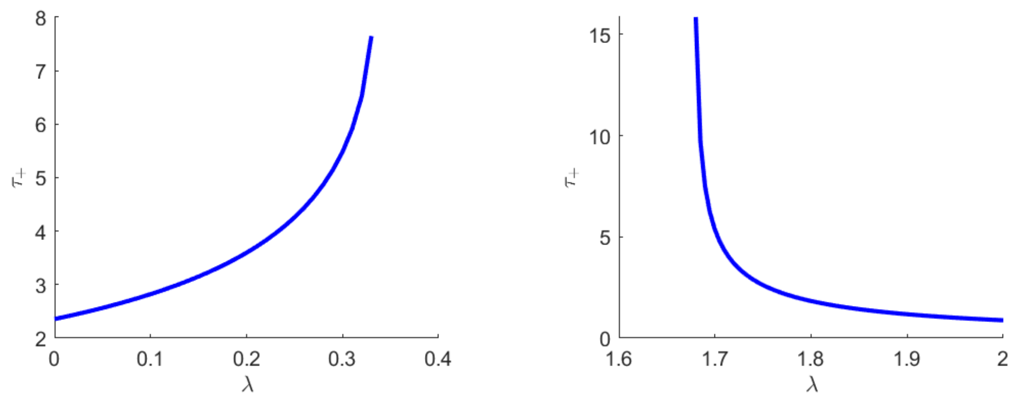


Figure 5.2: Illustration of τ_+: $\lambda \in [0, \frac{1}{3}]$ on the left, $\lambda \in (\frac{4}{3}, 2]$ on the right.

$\square$

Let τ^* be defined by $\tau^* = \min\{\tau_+(\lambda_1), \tau_+(\lambda_n)\}$. Now, we are ready to derive an exact condition for the system (5.3) formulated as

$$\dot{x}(t) = -x(t) + D^{-1}A[2x(t-\tau) - x(t-2\tau)]. \tag{5.20}$$

Theorem 5.3.3. *The system (5.20) defined on a connected and undirected graph reaches consensus from arbitrary initial conditions if and only if $\tau < \tau^*$.*

Proof. We first show that $\tau < \tau^*$ implies that all roots of Ψ have negative real parts except for the simple root at zero. Since the network is connected, $\lambda_1 = 0$ is a simple eigenvalue of the normalized Laplacian matrix, and by lemma (5.2.1), when $0 \leq \tau < \frac{\pi}{6}$, ψ_1 has a simple root at zero and all other roots satisfy $Re(s) < 0$. Furthermore, the zero root of ψ_1 is independent of the value of time delay, i.e. invariant with respect to τ since

$$\left.\frac{\partial s}{\partial \tau}\right|_{s=0} = \frac{-2se^{-s\tau(1-e^{-s\tau})}}{1+2\tau e^{-s\tau}(1-e^{-s\tau})} = 0.$$

For $k \geq 2$, all roots of ψ_k have negative real parts when $\tau = 0$ since $\psi_k(s) = s + 1 - 2(1-\lambda_k) + (1-\lambda_k) = s + \lambda_k = 0$ implies $s = -\lambda_k < 0$.

By the continuity of roots with respect to τ, the roots on the left half-plane cannot enter the right half-plane without crossing the imaginary axis as τ increases from zero. Thus, to find a condition for the stability of the system, knowing that the roots are located on the left half-plane when $\tau = 0$, we search for roots crossing the imaginary axis as τ increases from zero. Therefore, it suffices to show that when $\tau < \tau^*$, no imaginary crossings happen.

The preceding lemmas (5.3.1) and (5.3.2) imply that $\tau^* = \min\{\tau_+(\lambda_1), \tau_+(\lambda_n)\}$ is the minimum value of τ leading to a pair of purely imaginary roots of Ψ. We will show that $\tau^* \leq \tau_+(\lambda_k)$ for all λ_k, $k = 1, 2, \ldots, n$ to prove this claim.

(i) If $1 < \lambda_n \leq \frac{4}{3}$

In this case, $\tau_+(\lambda_n)$ is not defined and we write $\tau_+(\lambda_n) = \infty$ by convention. Thus,

$$\tau^* = \min\{\tau_+(0), \tau_+(\lambda_n)\} = \tau_+(0) \leq \tau_+(\lambda_k) \quad \forall k = 1, 2, \ldots, n$$

since $\tau_+(0) \leq \tau_+(\lambda)$ for all $\lambda \in (0, \frac{1}{3}]$ by Lemma (5.3.2) and clearly for any $\lambda > \frac{1}{3}$ we have $\tau_+(\lambda) = \infty$ since $\lambda_n \leq \frac{4}{3}$, and therefore $\tau_+(0) \leq \tau_+(\lambda)$ is satisfied.

(ii) If $\lambda_n > \frac{4}{3}$

In this case, by Lemma (5.3.2), $\tau_+(\lambda_n) \leq \tau_+(\lambda)$ for all $\lambda \in (\frac{4}{3}, \lambda_n]$ and $\tau_+(0) = \tau_+(\lambda_1) \leq \tau_+(\lambda)$ for $\lambda \in [0, \frac{1}{3}]$. Thus,

$$\tau^* = \min\{\tau_+(\lambda_1), \tau_+(\lambda_n)\} \leq \tau_+(\lambda_k) \quad \forall k = 1, 2, \ldots, n.$$

This proves the claim that when $\tau < \tau^*$ there is no imaginary crossings, which implies that all roots of Ψ have negative real parts except for the simple zero root.

Conversely, assume that the system (5.20) reaches consensus from arbitrary initial conditions. To show that $\tau < \tau^*$, we will prove the contrapositive.

When $\tau = \tau^*$, $\tau = \tau_+(\lambda_1)$ or $\tau = \tau_+(\lambda_n)$ meaning that either ψ_1 or ψ_n has a pair of purely imaginary roots. Thus, Ψ has a pair of purely imaginary roots, hence the system can at best converge to a periodic solution.

We will compute $\frac{\partial Re(s)}{\partial \tau}$ to understand what happens to the real parts of roots appearing on the imaginary axis (when τ hits τ^*), as τ increases from τ^*. Differentiating ψ_k, which is given by

$$\psi_k(s) = s + 1 - 2(1 - \lambda_k)e^{-s\tau} + (1 - \lambda_k)e^{-2s\tau}, \tag{5.21}$$

with respect to τ, we obtain

$$\begin{aligned} \frac{\partial s}{\partial \tau} &= \frac{-2s(1-\lambda_k)e^{-s\tau}(1-e^{-s\tau})}{1 + 2\tau(1-\lambda_k)e^{-s\tau}(1-e^{-s\tau})} \\ &= \frac{-2s^2 - 2s + 2s(1-\lambda_k)e^{-s\tau}}{1 + 2s\tau + 2\tau - 2\tau(1-\lambda_k)e^{-s\tau}}. \end{aligned} \tag{5.22}$$

The second equality is obtained by substituting $s + 1 - (1 - \lambda_k)e^{-s\tau} = (1 - \lambda_k)e^{-s\tau}(1 - e^{-s\tau})$, which always holds by (5.21). Substituting $s = i\omega$ into (5.22), then separating real and imaginary parts, we obtain

$$Re\left(\frac{\partial s}{\partial \tau}\right)\Big|_{s=i\omega} = \frac{2\omega^2 + 2\omega(1-\lambda_k)\sin(\omega\tau)}{(1 + 2\tau - 2\tau(1-\lambda_k)\cos(\omega\tau))^2 + (2\tau\omega + 2\tau(1-\lambda_k)\sin(\omega\tau))^2}$$

Substituting $\sin(\omega\tau) = \dfrac{-2\omega(1-\lambda_k)}{(2-\lambda_k)\lambda_k + \omega^2}$ which we know by (5.14),

$$Re\left(\frac{\partial s}{\partial \tau}\right)\Big|_{s=i\omega} = \frac{2\omega^2[1 + \omega^2 - 3(1-\lambda_k)^2]/[(2-\lambda_k)\lambda_k + \omega^2]}{[(1 + 2\tau - 2\tau(1-\lambda_k)\cos(\omega\tau))^2 + (2\tau\omega + 2\tau(1-\lambda_k)\sin(\omega\tau))^2]}$$

The denominator is always positive, so is the term $2\omega^2$ in the numerator, thus

$$\begin{aligned} Re\left(\frac{\partial s}{\partial \tau}\right)\Big|_{s=i\omega} &> 0 \iff 1+\omega^2-3(1-\lambda_k)^2 > 0,\\ &= 0 \iff 1+\omega^2-3(1-\lambda_k)^2 = 0,\\ &< 0 \iff 1+\omega^2-3(1-\lambda_k)^2 < 0. \end{aligned}$$

By equations (5.15)-(5.16), we have

$$\begin{aligned} &\omega_+^2+1-3(1-\lambda_k)^2 = 2\sqrt{(1-\lambda_k)^3(1-3\lambda_k)} > 0, \quad \lambda_k \in [0,\tfrac{1}{3}] \cup (\tfrac{4}{3},2],\\ &\omega_-^2+1-3(1-\lambda_k)^2 = -2\sqrt{(1-\lambda_k)^3(1-3\lambda_k)} < 0, \quad \lambda_k \in (0,\tfrac{1}{3}],\\ &\omega_+^2+1-3(1-\lambda_k)^2 = \omega_-^2+1-3(1-\lambda_k)^2 = 0, \quad \lambda_k = \tfrac{1}{3}. \end{aligned}$$

Hence,

$$\begin{aligned} Re\left(\frac{\partial s}{\partial \tau}\right)\Big|_{s=i\omega} = \frac{\partial Re(s)}{\partial \tau}\Big|_{s=i\omega} &> 0 \iff \omega=\omega_+ \text{ and } \lambda_k \neq \tfrac{1}{3}\\ &= 0 \iff \lambda_k = \tfrac{1}{3},\\ &< 0 \iff \omega=\omega_- \text{ and } \lambda_k \neq \tfrac{1}{3}. \end{aligned}$$

This means that roots crossing the imaginary axis at $s = \pm i\omega_+(\lambda_k)$ enters the right half-plane as τ increases from $\tau_+(\lambda_k)$, and roots crossing the imaginary axis at $s = \pm i\omega_-(\lambda_k)$ enters the left half-plane as τ increases from $\tau_-(\lambda_k)$. However, we have already proved that $\omega_-(\lambda_k)$ exists only for $\lambda_k \in (0, \frac{1}{3}]$ in lemma 5.2.4 and $\tau^* = \min\{\tau_+(\lambda_1), \tau_+(\lambda_n)\}$ where $\lambda_1 = 0$ and $\lambda_n > 1$ meaning that neither for λ_1 nor for λ_n there exists such ω_-. This implies that the roots crossing the imaginary axis when τ increases from τ^* cannot cross the imaginary axis again and turn back to the left half-plane, which proves that when $\tau > \tau^*$, Ψ has at least two roots on $\mathbb{C}_+$ thus, consensus cannot be achieved. $\square$

Corollary 5.3.4. *There exists λ^* such that if all eigenvalues λ_k of L satisfy $\lambda_k \leq \lambda^*$, then $\tau^* = \frac{3\pi}{4}$.*

Proof. $\tau_+ : [0, \frac{1}{3}] \cup (\frac{4}{3}, 2] \to \mathbb{R}_+$ is a continuous function and $\tau_+(\frac{3}{2}) > \frac{3\pi}{4}$, $\tau_+(2) < \frac{3\pi}{4}$, thus by intermediate value theorem, there exists $\lambda^* \in (\frac{3}{2}, 2)$ such that $\tau_+(\lambda^*) =$

$\frac{3\pi}{4}$. Since τ_+ is decreasing in λ when $\lambda > \frac{4}{3}$ by Lemma 5.3.2, if $\lambda_n \leq \lambda^*$, then $\tau_+(\lambda_n) \geq \tau_+(\lambda^*) = \frac{3\pi}{4}$. Therefore, $\tau^* = \min\{\tau_+(0), \tau_+(\lambda_n)\} = \tau_+(0) = \frac{3\pi}{4}$. $\square$

Corollary 5.3.5. *If $\tau < \tau_+(2) \approx 0.8793$, the system (5.3) defined on a connected and undirected network reaches consensus.*

Proof. Since the largest eigenvalue of Laplacian L, denoted by λ_n, satisfies $1 < \lambda_n \leq 2$ and since $\tau_+(2) \leq \tau_+(\lambda_n)$ by Lemma 5.3.2, it follows that

$$\tau_+(2) \leq \tau^* = \min\{\tau_+(0), \tau_+(\lambda_n)\} \leq \min\{\tau_+(0), \tau_+(2)\} = \tau_+(2).$$

Thus, $\tau < \tau^*$ serves as a sufficient condition for reaching consensus for any network that is undirected and connected. $\square$

The following corollary will be needed to apply our results to the Kuramoto model in the next chapter.

Corollary 5.3.6. *The linear consensus problem formulated as*

$$\dot{x}_i(t) = \frac{K}{d_i} \sum_{j=1}^{n} a_{ij}[2x_j(t-\tau) - x_j(t-2\tau) - x_i(t)], \quad K > 0 \tag{5.23}$$

defined on an undirected and connected graph reaches consensus from arbitrary initial conditions if and only if $\tau < \dfrac{\tau^*}{K}$.

Proof. The characteristic equation for this system is given by $\Phi(z,\tau) = \prod_{k=1}^{n} \phi_k(z,\tau) = 0$ where

$$\phi_k(z,\tau) = z + K(1 - 2(1-\lambda_k)e^{-z\tau} + (1-\lambda_k)e^{-2z\tau}). \tag{5.24}$$

Let $z \in \mathbb{C}$ be a root of ϕ, i.e. $\phi_k(z,\tau) = 0$ and let $s = \dfrac{z}{K}$ and $\varsigma = K\tau$, we can rewrite (5.24) as

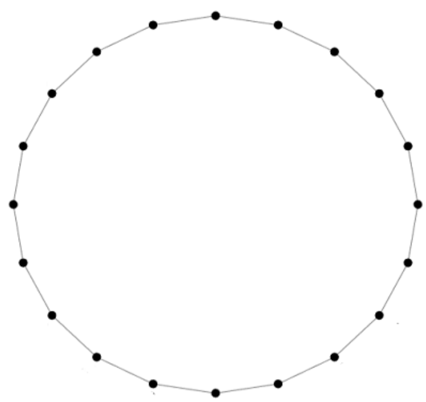
$$\begin{aligned}
\phi_k(Ks, \tfrac{\varsigma}{K}) &= Ks + K(1 - 2(1-\lambda_k)e^{-s\varsigma} + (1-\lambda)e^{-2s\varsigma}) \\
&= K(s + 1 - 2(1-\lambda_k)e^{-s\varsigma} + (1-\lambda_k)e^{-2s\varsigma}) \\
&= K\psi_k(s,\varsigma)
\end{aligned}$$

where $\psi_k(s,\varsigma) = s + 1 - 2(1-\lambda_k)e^{-s\varsigma} + (1-\lambda_k)e^{-2s\varsigma}$. The proof is complete by Theorem 5.3.3. $\square$

5.4 Computational examples

Figures 5.3, 5.5, 5.7 illustrate the findings of Theorem 5.3.3 on various networks.

Network 1. A cycle graph on 20 vertices, for which $\lambda_n = 2$ and $\tau^* = \min\{\tau_+(0), \tau_+(2)\} \approx 0.8793$.



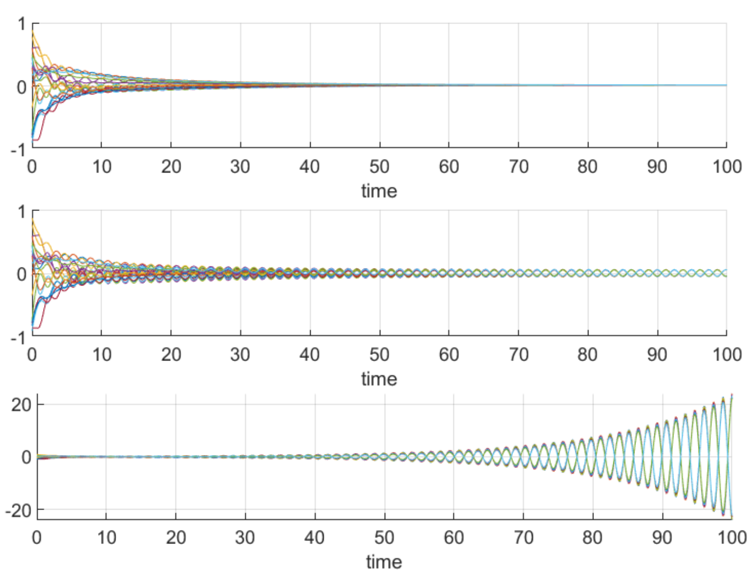


Figure 5.3: Evolution of agents' states on network 1 under anticipatory consensus algorithm with $\tau = 0.7793$, $\tau = 0.8793$, and $\tau = 0.9793$ from top to bottom, respectively.

Network 2. An Erdős–Rényi random graph, denoted by $\mathrm{ER}(n, p)$ where n is the number of vertices and p is the probability that two randomly selected vertices are connected. For the network illustrated below, $n = 20$, $p = 0.3$ and $\lambda_n = 1.7048$ gives $\tau^* = \min\{\tau_+(0), \tau_+(1.7048)\} \approx 1.3865$.

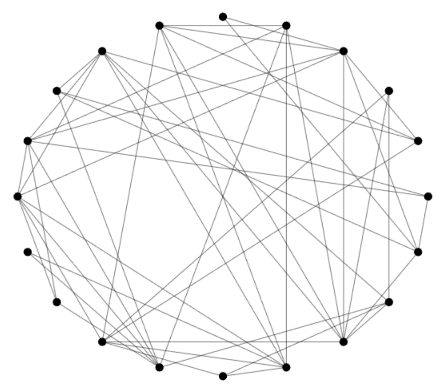


Figure 5.4: An Erdős–Rényi random graph with $n = 20$ and $p = 0.3$

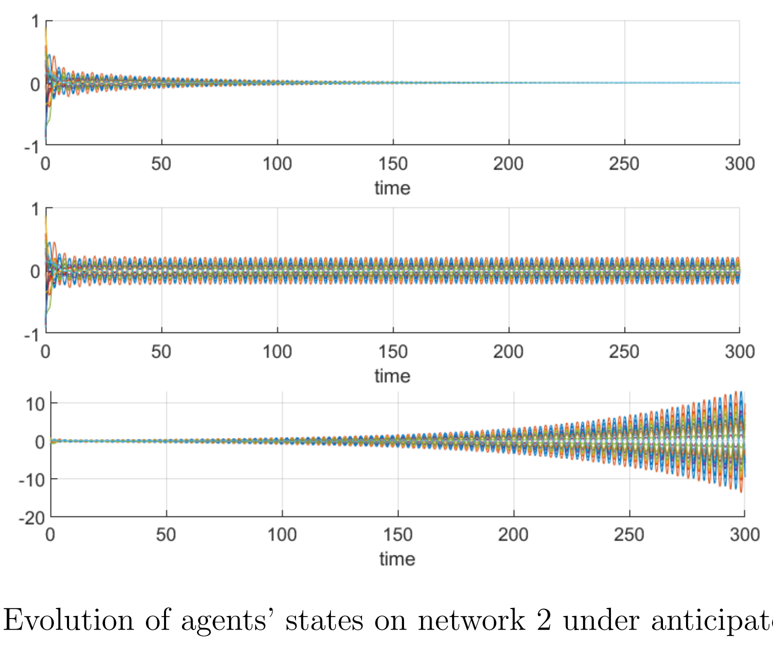


Figure 5.5: Evolution of agents' states on network 2 under anticipatory consensus algorithm with $\tau = 1.2865$, $\tau = 1.3865$, and $\tau = 1.4865$ from top to bottom, respectively.

Network 3. A Watts-Strogatz random graph, denoted by WS(N, K, β) where N is the number of vertices of the $K-$regular graph, K is an even integer such that each node has exactly $K/2$ neighbours on each side, and β represents the random rewiring probability. For the network illustrated below, $N = 20$, $K = 4$, $\beta = 0.3$ and $\lambda_n = 1.6667$ gives $\tau^* = \min\{\tau_+(0), \tau_+(1.6667)\} \approx 1.5033$.

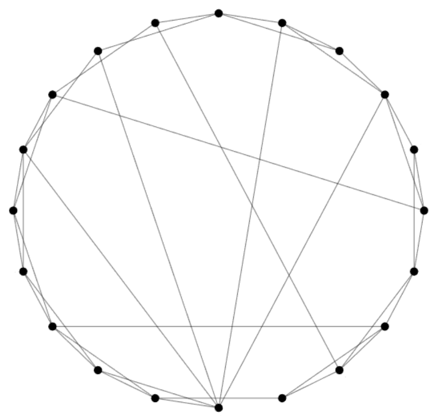


Figure 5.6: A Watts-Strogatz random graph with $N = 20$, $K = 4$ and $\beta = 0.3$

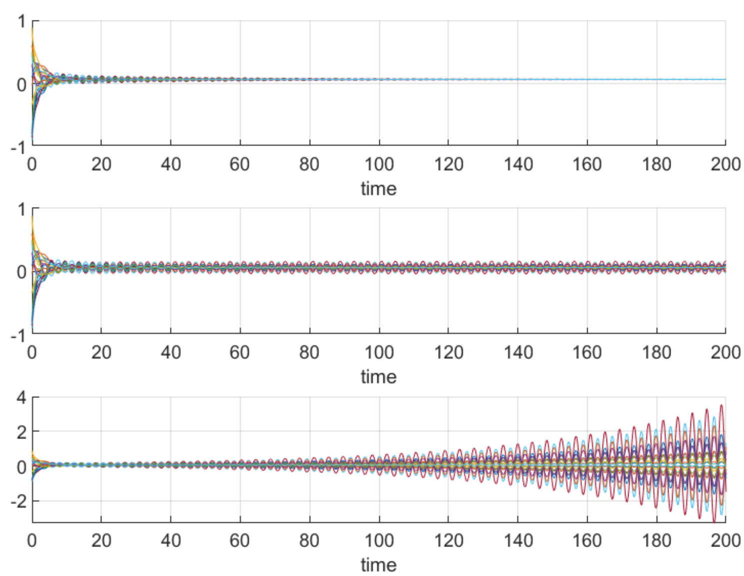


Figure 5.7: Evolution of agents' states on network 3 under anticipatory consensus algorithm with $\tau = 1.4033$, $\tau = 1.5033$, and $\tau = 1.6033$ from top to bottom, respectively.

5.4.1 Comparison of convergence rates with and without anticipation

Figures 5.8, 5.9, 5.10 show the improved convergence rate of the consensus protocol under transmission delays when agents employ anticipation.

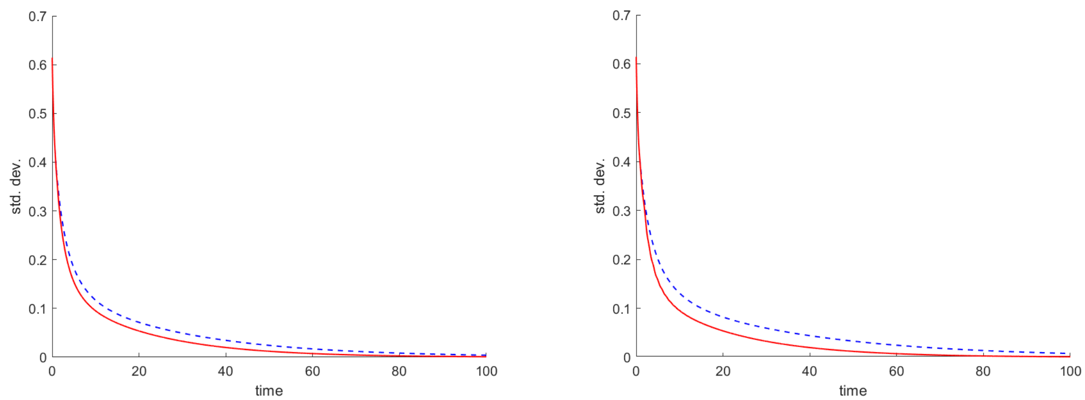


Figure 5.8: Convergence of consensus protocols in network 1 under transmission delays with $\tau = 0.4$ on the left and $\tau = 0.7$ on the right, with anticipation (red lines) and without anticipation (blue dashed lines).

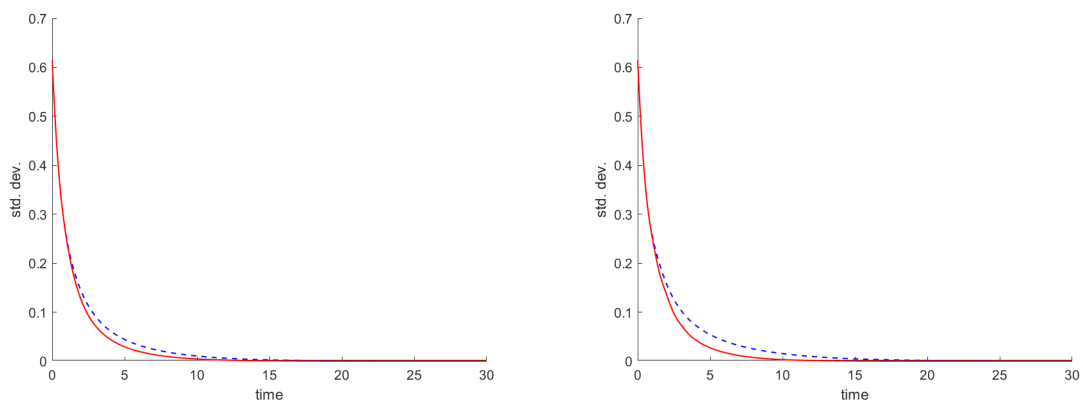


Figure 5.9: Convergence of consensus protocols in network 2 under transmission delays with $\tau = 0.4$ on the left and $\tau = 0.7$ on the right, with anticipation (red lines) and without anticipation (blue dashed lines).

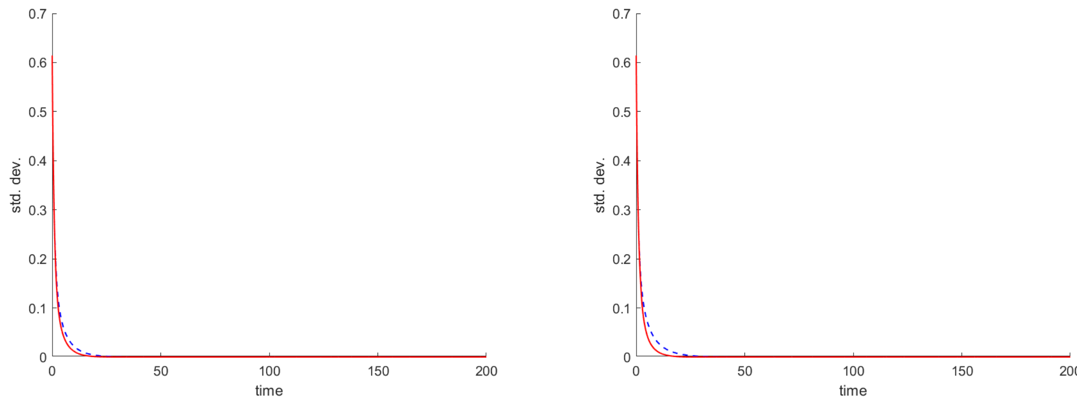


Figure 5.10: Convergence of consensus protocols in network 3 under transmission delays with $\tau = 0.4$ on the left and $\tau = 0.7$ on the right, with anticipation (red lines) and without anticipation (blue dashed lines).

Chapter 6

Application to Kuramoto model of phase oscillators

This chapter applies the findings from Chapter 5 to the Kuramoto model, a well-known synchronization model of coupled phase oscillators.

We begin with a brief review of coupled oscillators and synchronization phenomenon, which will be followed by an examination of the original Kuramoto model, Kuramoto's analysis, and two extended versions of the model, each involving a single delayed term.

Next, we define the Kuramoto model under transmission delays with anticipatory agents and we study the local stability of synchronized states of this model through the linearized equation, where our results from chapter 5 are directly applicable.

6.1 Coupled oscillators and synchronization

An oscillator is any object or system that exhibits periodic behaviour, which is characterized by the period T which satisfies $x(t) = x(t+T)$ for all $t \in \mathbb{R}$, where

$x(t)$ denotes the state of oscillator at time t.

Oscillatory systems are everywhere in the natural or engineered systems, ranging from the circadian rhythms governing sleep-wake cycle in humans and animals to the motion of a swinging pendulum in a frictionless environment.

A set of interacting oscillators connected by an underlying graph is referred to as coupled oscillators. The study of synchronization of coupled oscillators dates back to 17th century, when Christiaan Huygens[1] observed that two pendulums next to each other were working in perfect synchrony if they were close enough [55]. Coupled oscillators are ubiquitous in nature; examples include pacemaker cells in the heart, insulin-secreting cells in the pancreas, neural networks in the brain, crickets chirping in unison, and synchronously flashing fireflies [55]. When two or more oscillators are coupled, they exhibit a much richer variety of behaviours than of a single oscillator, such as phase and frequency synchronization[2], cluster synchronization[3], and chimera states[4].

An oscillator whose state at time t is determined solely by its phase $\theta(t)$ is called a phase oscillator. A phase oscillator model can be formulated by the set of differential equations

$$\dot{\theta}_i(t) = \omega_i + \sum_{j=1}^{N} a_{ij} g(\theta_j(t) - \theta_i(t)), \quad i = 1, 2, \ldots, N,$$

where N is the number of oscillators in the system, $\theta_i(t)$ denotes the phase of oscillator i at time t, a_{ij} is the $ij-$ entry of the adjacency matrix of the underlying graph representing relations between oscillators, and $g : S^1 \to \mathbb{R}$ is a function that is monotone increasing, differentiable such that $g(0) = 0$.

Synchronization can be defined in various ways to describe different coherent patterns of coupled phase oscillators. Here we will focus on the notion of full

[1]Christiaan Huygens (1629 - 1695) was a Dutch mathematician, physicist, engineer, astronomer and inventor, who invented the pendulum clocks [54].

[2]Phase synchronization occurs when $\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0$, which implies frequency synchronization, i.e. $\lim_{t\to\infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0$

[3]When oscillators split into groups where each group synchronizes within the group but not with other groups.

[4]When both synchronous and asynchronous behaviour exist at the same time.

phase synchronization.

Definition 6.1.1. *A finite set of oscillators $i = 1, 2, \ldots, N$ is said to achieve full phase synchronization if*

$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = 0 \ \forall i, j \in \{1, \ldots, N\}.$$

6.2 The Kuramoto model

A classical model for the synchronization of coupled phase oscillators is the Kuramoto model, named after Yoshiki Kuramoto [56], [57]. The Kuramoto model assumes N coupled oscillators, all-to-all coupled with coupling constant $K > 0$, whose dynamics is governed by the system of differential equations

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t)), \quad i = 1, \ldots, N, \tag{6.1}$$

where $\omega_i \in \mathbb{R}$ denotes the natural frequency of oscillator i. Kuramoto assumes that the distribution of ω_i is given by the probability distribution function $g(\omega)$, which is unimodal and symmetric about its mean frequency Ω, that is, $g(\Omega+\omega) = g(\Omega - \omega)$ for all ω [58].

Kuramoto's analysis of the model (6.1) is based on an order parameter r which satisfies

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \tag{6.2}$$

where $re^{i\psi}$ gives the centroid of all oscillators when represented as points on the unit circle S^1, ψ representing the average phase of oscillators and r can be seen as a measure of phase coherence such that $r = 1$ represents the case of full phase synchronization [59].

Multiplying both sides of (6.2) by $e^{-i\theta_i}$ gives

$$re^{i(\psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}.$$

Equating imaginary parts, we get

$$r \sin(\psi - \theta_i) = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i).$$

Substitution of $r\sin(\psi - \theta_i)$ into the model (6.1) then yields

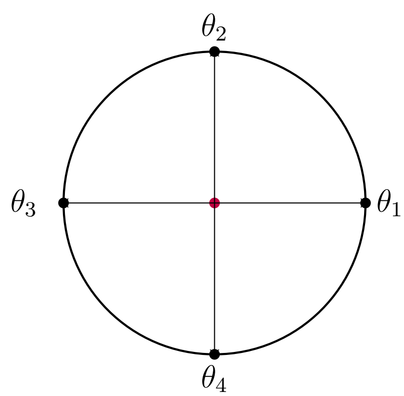
$$\dot{\theta}_i = \omega_i + Kr \sin(\psi - \theta_i), \quad i = 1, 2, \ldots, N. \tag{6.3}$$

Expressing the synchronized state as $\theta_i(t) = \Omega t + c$, we can set the mean frequency Ω to 0 by letting $\theta_i \to \theta_i + \Omega t$. Then setting $\omega_i \to \omega_i - \Omega$ gives

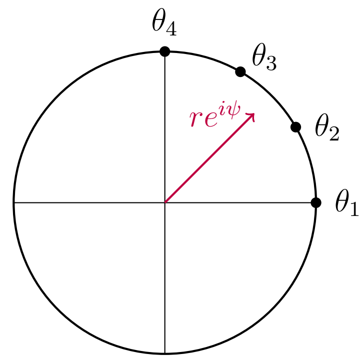
$$\dot{\theta}_i = \omega_i - Kr \sin(\theta_i), \quad i = 1, \ldots, N. \tag{6.4}$$

The long-term behaviour of solutions of (6.4) can be analyzed in two cases, depending on the relative size of $|\omega_i|$ compared to Kr. The oscillators whose natural frequency satisfies $|\omega_i| \leq Kr$ approach a stable fixed point given by $\omega_i = Kr\sin(\theta_i)$ where $|\theta_i| \leq \frac{\pi}{2}$. On the other hand, oscillators with $|\omega_i| > Kr$ rotate around the circle in a non-uniform way; the oscillators with inherently larger frequencies tend to catch the locked oscillators, while the inherently slower oscillators tend to be caught by the locked oscillators in time [58].

With the representation (6.3), each oscillator seems to move independently of other oscillators although they are coupled, but only through the mean-field quantities r and ψ [58]. The constant Kr is sometimes called the effective coupling, which increases further as oscillators tend to synchronize or form clusters, thereby it accelerates oscillators' phases θ_i to approach ψ.



$r = 0$



$\psi = \frac{\pi}{4}$, $r = \frac{3+\sqrt{3}}{4\sqrt{2}}$

For the drawing on the left, $r = 0$ since

$$
\begin{aligned}
re^{i\psi} &= \tfrac{1}{4}\sum_{j=1}^{4} e^{i\theta_j} \\
&= \tfrac{1}{4}(e^0 + e^{i\frac{\pi}{2}} + e^{i\pi} + e^{i\frac{3\pi}{2}}) \\
&= \tfrac{1}{4}(1 + i - 1 - i) = 0.
\end{aligned}
$$

Similarly, r and ψ is obtained for the drawing on the right as follows:

$$
\begin{aligned}
re^{i\psi} &= \tfrac{1}{4}\sum_{j=1}^{4} e^{i\theta_j} \\
&= \tfrac{1}{4}(e^0 + e^{i\frac{\pi}{6}} + e^{i\frac{\pi}{3}} + e^{i\frac{\pi}{2}}) \\
&= \tfrac{1}{4}(1 + \tfrac{\sqrt{3}}{2} + i\tfrac{1}{2} + \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2} + i) \\
&= \tfrac{1}{4}\tfrac{(3+\sqrt{3})}{2}(1+i) \\
&= \tfrac{3+\sqrt{3}}{4\sqrt{2}}^{i\frac{\pi}{4}}
\end{aligned}
$$

which gives $r = \frac{3+\sqrt{3}}{4\sqrt{2}}$ and $\psi = \frac{\pi}{4}$.

Though Kuramoto originally considered a coupled system of oscillators where each oscillator connected to each other with the same strength, in most of the physical systems, not all pairs of oscillators are coupled but there is an underlying network representing the interactions/coupling between them.

The extended Kuramoto model with underlying graph G has the form

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin(\theta_j(t) - \theta_i(t)), \quad i = 1, 2, \ldots, N,$$

where a_{ij} is the $ij-$ entry of the adjacency matrix of the underlying graph G, $d_i = \sum_{j=1}^{n} a_{ij}$ is the degree (in-degree) of oscillator i if G is undirected (directed).

When all oscillators have the same natural frequency, i.e. $\omega_i = \omega \; \forall i = 1, 2, \ldots, N$, then the formulation reduces to

$$\dot{\theta}_i(t) = \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin(\theta_j(t) - \theta_i(t)), \quad i = 1, 2, \ldots, N. \tag{6.5}$$

The phase synchronization in this model ($\theta_i = \theta_j$ for all i, j) implies that $\dot{\theta}_i(t) = \omega$ for all i, which then implies that the synchronized state θ_i must satisfy $\theta_i(t) = \omega t + c$ for all $i = 1, 2, \ldots, N$, where c is a constant phase shift.

6.3 Synchronization of Kuramoto model under transmission delays

The Kuramoto model under discrete delayed interactions between oscillators is first studied by Schuster and Wagner [60] in the setting of two coupled oscillators:

$$\begin{aligned}
\dot{\theta}_1(t) &= \omega_1 + K\sin(\theta_2(t-\tau) - \theta_1(t)) \\
\dot{\theta}_2(t) &= \omega_2 + K\sin(\theta_1(t-\tau) - \theta_2(t)).
\end{aligned}$$

They have shown that delayed interactions lead to a range of synchronized states [60]. Yeung and Strogatz [61] generalized the work of Schuster and Wagner to N coupled oscillators with noisy, randomly distributed intrinsic frequencies; they study the dynamics of the system

$$\dot{\theta}_i(t) = \omega_i + \xi_i(t) + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j(t-\tau) - \theta_i(t) - \alpha), \quad i = 1, 2, \ldots, N,$$

where $\xi_i(t)$ represents the frequency fluctuations and α is the phase shift parameter [61]. For the special case of identical oscillators, i.e. $\omega_i = \omega$, they explicitly define the stability boundaries of the incoherent and synchronized states as a function of time delay [61].

Later, Earl and Strogatz [62] studied the model under a generalized coupling function f on $k-$regular graphs (graphs where each vertex has degree k). Under this setting, the formulation takes the form

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{k}\sum_{j=1}^{N} a_{ij} f(\theta_j(t-\tau) - \theta_i(t)).$$

Letting f to be the sinusoidal coupling function as in the Kuramoto model, and relaxing the $k-$regularity assumption, the model reduces to the Kuramoto model with an underlying graph under delayed interactions, formulated as

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij} \sin(\theta_j(t-\tau) - \theta_i(t)), \quad i = 1, \ldots, N.$$

If oscillators are identical ($\omega_i = \omega \ \forall i$), then

$$\dot{\theta}_i(t) = \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij} \sin(\theta_j(t-\tau) - \theta_i(t)), \quad i = 1, \ldots, N. \tag{6.6}$$

A synchronized state $\phi_i(t) = \Omega t + c$ of (6.6) must satisfy

$$\begin{aligned} \Omega &= \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij} \sin(\Omega(t-\tau) + c - \Omega t - c) \\ \Omega &= \omega - K\sin(\Omega\tau). \end{aligned} \tag{6.7}$$

Then, τ must satisfy

$$\tau = \frac{\arcsin(\frac{\omega-\Omega}{K})}{\Omega}. \tag{6.8}$$

For fixed $K > 0$ and ω, the synchronized frequency Ω is a root of (6.8).

Theorem 6.3.1. *The synchronized state $\theta_i(t) = \Omega t + c$ of (6.6) is linearly stable if and only if $K\cos(\Omega\tau) > 0$. [62].*

6.4 Synchronization of Kuramoto model with anticipatory agents

The anticipatory agents introduced into the Kuramoto model by Dönmez and Atay [63], [64]. Assuming an undirected and connected graph G on N identical oscillators, they study the model

$$\dot{\theta}_i(t) = \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin(\hat{\theta}_j(t+\delta) - \theta_i(t)), \quad i = 1, 2, \ldots, N.$$

where

$$\hat{\theta}_j(t+\delta) = (1+\alpha)\theta_j(t) - \alpha\theta_j(t-\tau), \quad \alpha = \frac{\delta}{\tau}.$$

Here, $\hat{\theta}_j(t+\delta)$ denotes the anticipated phase of oscillator i at future time $t+\delta$. Substituting $\hat{\theta}_j(t+\delta)$ into the model,

$$\dot{\theta}_i(t) = \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin((1+\alpha)\theta_j(t) - \alpha\theta_j(t-\tau) - \theta_i(t)), \quad i = 1, 2, \ldots, N. \tag{6.9}$$

The synchronized states $\theta_i = \Omega t + \phi$ of (6.9) satisfies $\Omega = \omega + K\sin(\Omega\delta)$ where $\delta = \alpha\tau$.

Theorem 6.4.1. *Consider the system (6.9) defined on an undirected and connected graph G. The synchronized state $\phi_i = \Omega t + \phi$ where Ω satisfies $\Omega = \omega + K\sin(\Omega\delta)$ is locally exponentially stable if and only if $0 < \delta K\cos(\Omega\delta) < 1$ [64].*

6.5 Synchronization of Kuramoto model with anticipatory agents under transmission delays

We have reviewed in sections 6.3 and 6.4 the Kuramoto model under transmission delays and the Kuramoto model with anticipatory agents, respectively. The formulations of those models involve a single delayed term, and in both models the time delay leads to a synchronized frequency $\Omega \neq \omega$ in general.

We consider an undirected and connected graph G of N anticipatory oscillators with identical natural frequency ω. Assuming delayed interactions between oscillators, we study the model

$$\dot{\theta}_i(t) = \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin(\hat{\theta}_j(t) - \theta_i(t)), i = 1, \ldots, N \tag{6.10}$$

where $\hat{\theta}_j(t)$ represents the anticipated phase of agent j at present time. We study the same rule as in Chapter 5 for anticipation, that is, we assume $\hat{\theta}_j(t) = 2\theta_j(t-\tau) - \theta_j(t-2\tau)$. Substituting into (6.10), we get

$$\dot{\theta}_i(t) = \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin(2\theta_j(t-\tau) - \theta_j(t-2\tau) - \theta_i(t)), \quad i = 1, 2, \ldots, N. \tag{6.11}$$

Searching for synchronized states of the form $\theta_i(t) = \Omega t + \phi$, we substitute $\theta_i = \Omega t + \phi$ into (6.11) to obtain

$$\begin{aligned}\Omega &= \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin(2\Omega(t-\tau) + 2\phi - \Omega(t-2\tau) - \phi - \Omega t - \phi)\\ &= \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin(0)\\ &= \omega.\end{aligned}$$

Let τ^* be defined as in Chapter 5. That is, $\tau^* = \min\{\tau_+(\lambda_1), \tau_+(\lambda_n)\}$ where $\lambda_1 = 0$ and λ_n are the smallest and the largest eigenvalues of Laplacian L and

$\tau_+ : [0, \frac{1}{3}] \cup (\frac{4}{3}, 2] \to \mathbb{R}_+$ is the continuous function defined by

$$\tau_+(\lambda) = \begin{cases} \dfrac{2\pi - \arccos\left(\frac{2\lambda(1-\lambda)}{2(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}\right)}{\sqrt{-1+3(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}}, & \lambda \in [0, \frac{1}{3}] \\ \dfrac{\arccos\left(\frac{2\lambda(1-\lambda)}{2(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}\right)}{\sqrt{-1+3(1-\lambda)^2+2\sqrt{(1-\lambda)^3(1-3\lambda)}}}, & \lambda \in (\frac{4}{3}, 2] \\ +\infty, & \lambda \in (\frac{1}{3}, \frac{4}{3}]. \end{cases}$$

Theorem 6.5.1. *The synchronized state $\theta_i(t) = \omega t + \phi$ of the system (6.11) defined on a connected and undirected graph is locally exponentially stable if and only if $\tau < \dfrac{\tau^*}{K}$.*

Proof. Let u_i denote the small perturbation, applied to the synchronized state $\theta_i(t) = \omega t + \phi + u_i(t)$. Substitution of perturbed solution into (6.11) gives

$$\begin{aligned} \omega + \dot{u}_i(t) &= \omega + \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin[2\omega(t-\tau) + 2\phi + 2u_j(t-\tau) - \omega(t-2\tau) - \phi \\ &\qquad - u_j(t-2\tau) - \omega t - \phi - u_i(t)] \\ \dot{u}_i(t) &= \frac{K}{d_i}\sum_{j=1}^{N} a_{ij}\sin(2u_j(t-\tau) - u_j(t-2\tau) - u_i(t)). \end{aligned}$$

Linearization around 0 yields

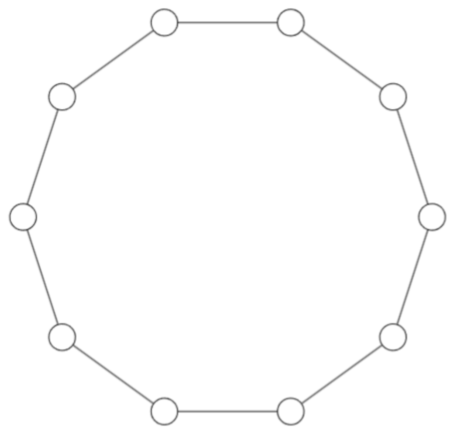
$$\begin{aligned} \dot{u}_i(t) &= K\sin(0) + \frac{K\cos(0)}{d_i}\sum_{j=1}^{n} a_{ij}[2u_j(t-\tau) - u_j(t-2\tau) - u_i(t)] \\ \dot{u}_i(t) &= \frac{K}{d_i}\sum_{j=1}^{n} a_{ij}[2u_j(t-\tau) - u_j(t-2\tau) - u_i(t)], \end{aligned}$$

which is equivalent to the linear consensus problem studied in Chapter 5. It follows from Corollary 5.3.6 that there exists u^* such that $\lim_{t\to\infty} u_i(t) = u^*$ for all $i = 1, \ldots, n$, which completes the proof. □

Remark 6.5.2. *The delay margin for the local stability of the synchronized state $\theta_i(t) = \omega t + \phi$, given by $\dfrac{\tau^*}{K}$, is inversely proportional to the coupling constant between oscillators.*

6.6 Computational examples

The simulations in this section are performed on a cycle graph on 10 vertices, illustrated below, for which $\lambda_n = 2$ and $\tau^* \approx 0.8793$.



I. $K = 1$, $\tau^* \approx 0.8793$

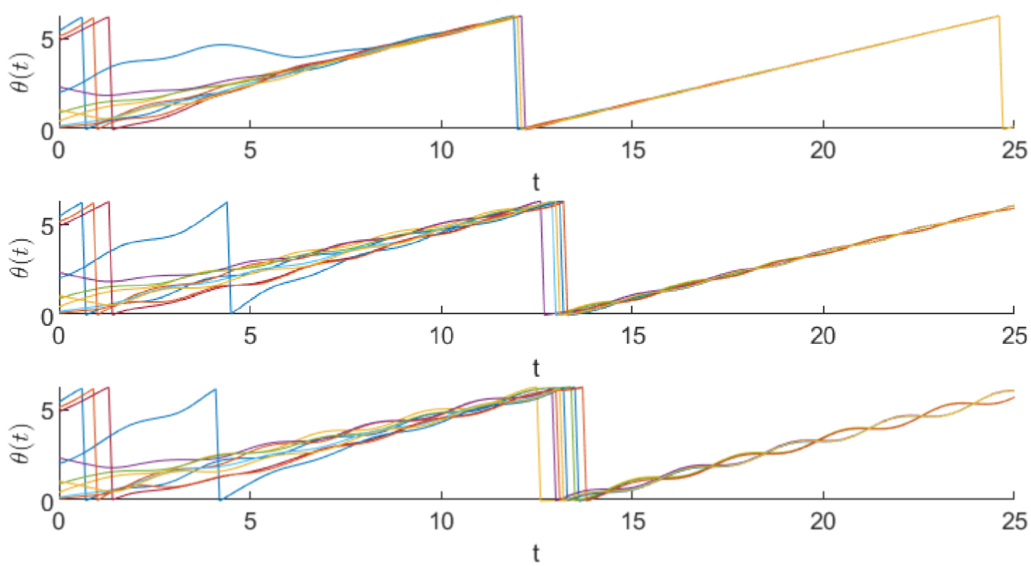


Figure 6.1: Evolution of phases of oscillators with $\tau = 0.7793$, $\tau = 0.8793$, $\tau = 0.9793$, from top to bottom respectively.

II. $K = 3, \dfrac{\tau^*}{K} \approx 0.2931$

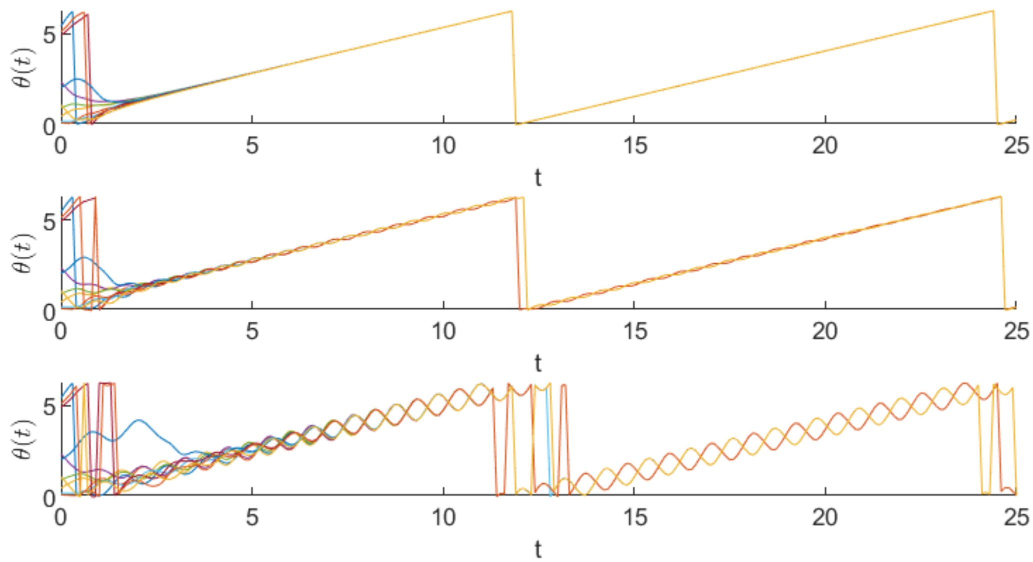


Figure 6.2: Evolution of phases of oscillators with $\tau = 0.1931$, $\tau = 0.2931$, $\tau = 0.3931$, from top to bottom respectively.

III. $K = 5, \dfrac{\tau^*}{5} \approx 0.1758$

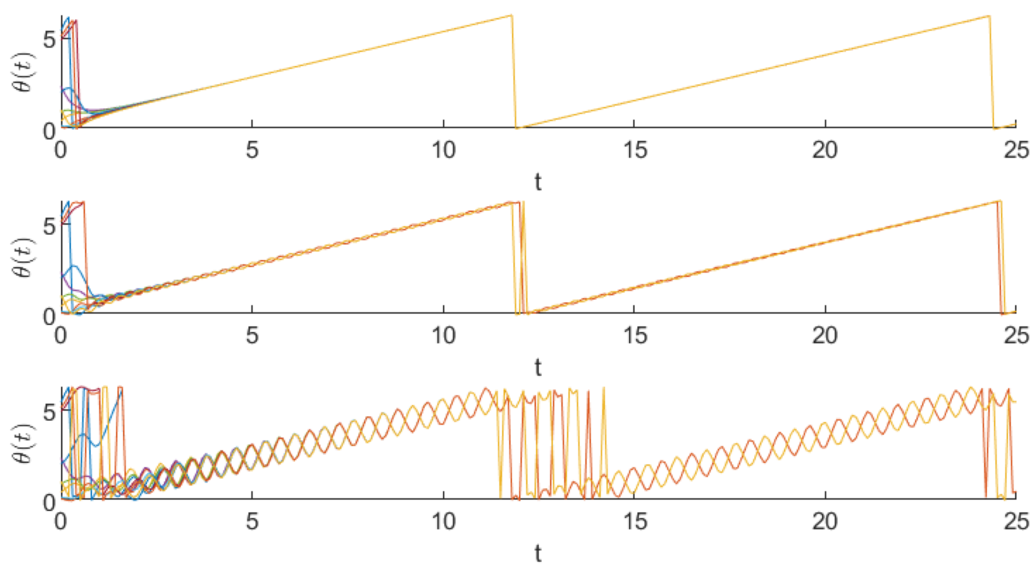
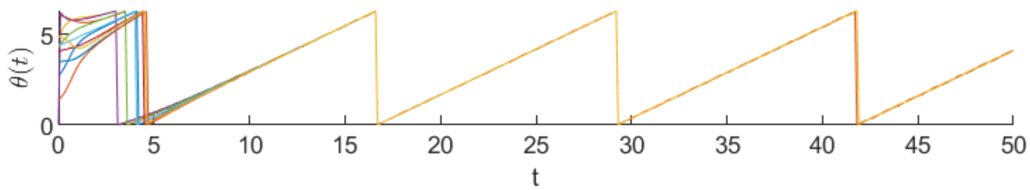
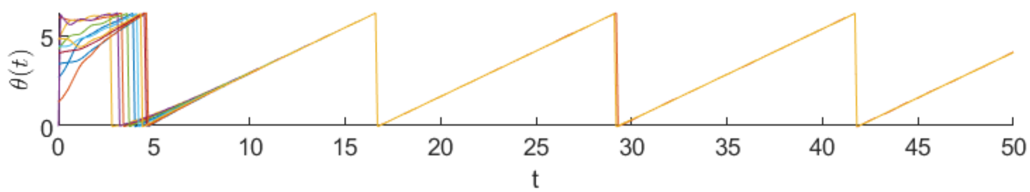


Figure 6.3: Evolution of phases of oscillators with $\tau = 0.0758$, $\tau = 0.1758$, $\tau = 0.2758$, from top to bottom respectively.

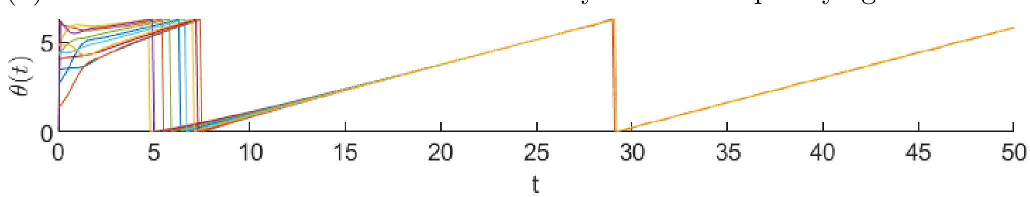
Another note is on the fact that the synchronized frequency Ω of the extended model under transmission delays with anticipatory agents is the same as of the original Kuramoto model, which is given by $\Omega = \omega$, where ω is the natural frequency of oscillators. This is in contrast with the other extended versions involving a single delayed argument, where the synchronized frequency $\Omega \neq \omega$ in general. The Figure 6.4 below illustrates this result.



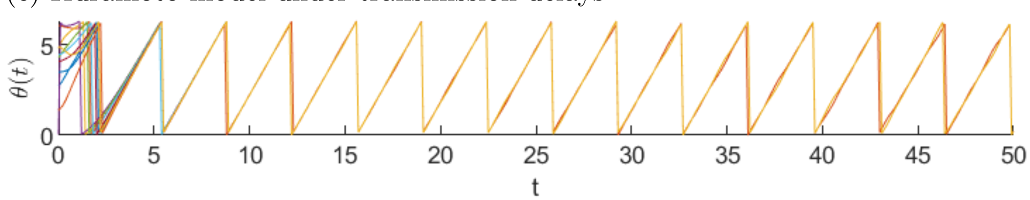
(a) Classical Kuramoto model



(b) Kuramoto model under transmission delays with anticipatory agents



(c) Kuramoto model under transmission delays



(d) Kuramoto model with anticipatory agents

Figure 6.4: Evolution of oscillators' phases with $K = 2$, $\omega = 0.5$ and $\tau = 0.4$.

Chapter 7

Discussion & Conclusion

In this thesis, we have explored the dynamics of the linear normalized consensus problem on networks of anticipatory agents under a fixed transmission delay of $\tau > 0$. Specifically, we examined the consensus problem defined on an undirected and connected graph G on n vertices, whose dynamics is governed by the system of DDEs given by

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}[2x_j(t-\tau) - x_j(t-2\tau) - x_i(t)], \quad i = 1, \dots, n. \tag{7.1}$$

In Chapter 5, we proved the existence of τ^* such that the system (7.1) defined on undirected connected graphs reaches consensus from arbitrary initial conditions if and only if $\tau < \tau^*$. Furthermore, we determined the exact value of τ^*, which depends solely on the network topology through the smallest and the largest eigenvalues of Laplacian L.

As natural consequences of our main result, Theorem 5.3.3, we have established the following:

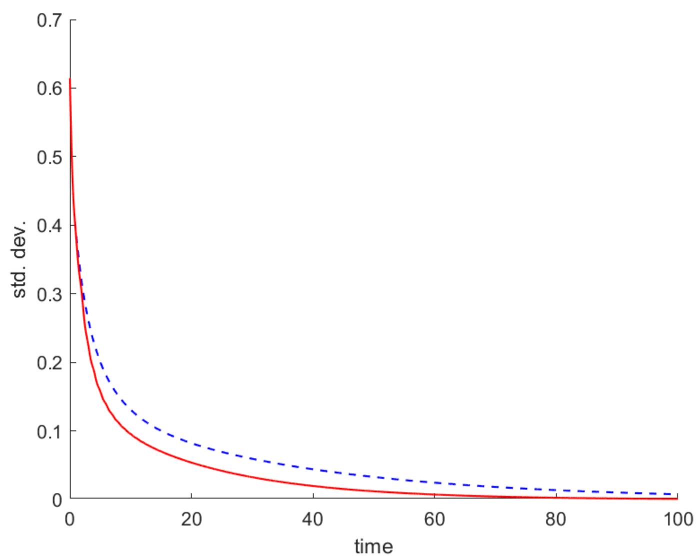
(i) A sufficient condition for the system (7.1) to reach consensus from arbitrary initial conditions is given by

$$\tau < \tau_+(2) \approx 0.8793,$$

which is independent of the network topology, given that the network is connected and undirected.

(ii) There exists $\lambda^* \in (1.5, 2)$ such that if $\lambda_n \leq \lambda^*$, then $\tau^* = \frac{3\pi}{4}$. This result directly applies to complete graphs with $n \geq 3$ since $\lambda_n = \frac{n}{n-1} \leq \frac{3}{2} < \lambda^*$.

Besides the convergence conditions, we observed the improving effect of anticipation on the convergence rate of the consensus protocol under transmission delays through simulations. As an example, see the below figure which shows the superior performance of the consensus protocol with anticipation in terms of convergence rate, on cycle graph of 20 vertices.



We have also examined the Kuramoto model of identical oscillators under transmission delays with anticipatory agents, where our results for the corresponding linear consensus problem were applicable for studying the local stability of the synchronized solutions. We obtained the following:

(i) Unlike the extended versions of the Kuramoto model involving a single delayed argument, our model, which involves two discrete delays τ and 2τ, the synchronized frequency Ω is equal to the natural frequency of oscillators, i.e. $\Omega = \omega$.

(ii) The necessary and sufficient condition for the local stability of the synchronized solution of the form $\theta_i(t) = \Omega t + c$ is given by

$$\tau < \frac{\tau^*}{K}.$$

This means that the delay margin for the local stability of the synchronized solution $\theta_i(t) = \Omega t + c$ is inversely proportional to the coupling strength $K > 0$.

Consequently, we have examined a coupled system of linear differential equations with two discrete delays, addressing the consensus problem on networks of anticipatory agents under transmission delays. Furthermore, we have extended our findings to a nonlinear system by analyzing its corresponding linearized equation.

Our computational results reveal that employing anticipation can be an effective strategy to achieve rapid consensus in linear multi-agent systems incorporating a single time delay due to information transmission. Additionally, our model and results may inspire and motivate further research into consensus/synchronization problems under various settings, as well as into anticipation rules.

Bibliography

[1] R. Murphey, "An Introduction to Collective and Cooperative Systems," *Cooperative Control and Optimization*, pp. 171–197, 2002.

[2] A. Gardner, A. S. Griffin, and S. A. West, "Theory of Cooperation," *eLS*, pp. 1–8, 2016.

[3] W. D. Hamilton, "The Genetical Evolution of Social Behaviour. I," *Journal of Theoretical Biology*, vol. 7, no. 1, pp. 1–16, 1964.

[4] T. Clutton-Brock, "Breeding Together: Kin Selection and Mutualism in Cooperative Vertebrates," *Science*, vol. 296, no. 5565, pp. 69–72, 2002.

[5] C. W. Reynolds, "Flocks, herds and schools: A distributed behavioral model," in *Proceedings of the 14th Annual Conference on Computer Graphics and Interactive Techniques*, pp. 25–34, 1987.

[6] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Physical Review Letters*, vol. 75, no. 6, p. 1226, 1995.

[7] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Transactions on Automatic Control*, vol. 51, no. 3, pp. 401–420, 2006.

[8] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, 2004.

[9] N. A. Lynch, *Distributed algorithms.* Elsevier, 1996.

[10] M. H. DeGroot, “Reaching a consensus,” *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118–121, 1974.

[11] A. Nugent, S. N. Gomes, and M.-T. Wolfram, “Steering opinion dynamics through control of social networks,” *arXiv preprint arXiv:2404.09849*, 2024.

[12] R. Sipahi, F. M. Atay, and S.-I. Niculescu, “Stability of traffic flow behavior with distributed delays modeling the memory effects of the drivers,” *SIAM Journal on Applied Mathematics*, vol. 68, no. 3, pp. 738–759, 2008.

[13] J. Hawkins, *A thousand brains: A new theory of intelligence.* Basic Books, 2021.

[14] J. Hawkins and S. Blakeslee, *On intelligence.* Macmillan, 2004.

[15] M. Koh, A. Ramírez, and R. Sipahi, “Single-delay proportional-retarded (PR) protocols for fast consensus in a multi-agent system,” *IFAC-PapersOnLine*, vol. 51, no. 14, pp. 31–36, 2018.

[16] A. Ramírez and R. Sipahi, “Single-delay and multiple-delay proportional-retarded (PR) protocols for fast consensus in a large-scale network,” *IEEE Transactions on Automatic Control*, vol. 64, no. 5, pp. 2142–2149, 2018.

[17] H. Fan, A. Ramírez, S. Mondié, and R. Sipahi, “Proportional-retarded controller design for single-integrator dynamics against unintentional delays,” *IEEE Control Systems Letters*, vol. 7, pp. 514–519, 2022.

[18] T. C. Aysal, B. N. Oreshkin, and M. J. Coates, “Accelerated distributed average consensus via localized node state prediction,” *IEEE Transactions on Signal Processing*, vol. 57, no. 4, pp. 1563–1576, 2008.

[19] Y. Cao and W. Ren, “Multi-agent consensus using both current and outdated states with fixed and undirected interaction,” *Journal of Intelligent and Robotic Systems*, vol. 58, no. 1, pp. 95–106, 2010.

[20] A. Ramírez and R. Sipahi, "Multiple Intentional Delays Can Facilitate Fast Consensus and Noise Reduction in a Multiagent System," *IEEE Transactions on Cybernetics*, vol. 49, no. 4, pp. 1224–1235, 2018.

[21] H. Moradian and S. S. Kia, "A study on accelerating average consensus algorithms using delayed feedback," *IEEE Transactions on Control of Network Systems*, vol. 10, no. 1, pp. 157–168, 2022.

[22] H.-T. Zhang, M. Z. Chen, G.-B. Stan, T. Zhou, and J. M. Maciejowski, "Collective behavior coordination with predictive mechanisms," *IEEE Circuits and Systems Magazine*, vol. 8, no. 3, pp. 67–85, 2008.

[23] H.-T. Zhang, M. Z. Chen, T. Zhou, and G.-B. Stan, "Ultrafast consensus via predictive mechanisms," *Europhysics Letters*, vol. 83, no. 4, p. 40003, 2008.

[24] F. M. Atay and D. Irofti, "A delayed consensus algorithm in networks of anticipatory agents," in *European Control Conference*, pp. 1880–1885, IEEE, 2016.

[25] C. Godsil and G. F. Royle, *Algebraic Graph Theory*, vol. 207. Springer Science & Business Media, 2001.

[26] P. Van Mieghem, *Graph Spectra for Complex Networks.* Cambridge University Press, 2023.

[27] F. Atay, "MATH 586 Lecture Notes," May 2021.

[28] E. Cinlar, *Introduction to Stochastic Processes.* Courier Corporation, 2013.

[29] O. Axelsson, "Reducible and Irreducible Matrices and the Perron-Frobenius Theory for Nonnegative Matrices," *Iterative Solution Methods*, pp. 122–157, 1994.

[30] M. H. Holmes, *Introduction to the Foundations of Applied Mathematics.* Springer, 2009.

[31] F. R. Chung, *Spectral Graph Theory.* American Mathematical Soc., 1997.

[32] F. M. Atay, "On the duality between consensus problems and markov processes, with application to delay systems," *Markov Processes And Related Fields*, vol. 22, pp. 537–553, 2014.

[33] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[34] J. Hale, "History of Delay Equations," *Delay Differential Equations and Applications*, pp. 1–28, 2006.

[35] M. E. Picard, "La mathématique dans ses rapports avec la physique," *Il Nuovo Cimento*, vol. 16, no. 1, pp. 165–183, 1908.

[36] J.-P. Richard, "Time-delay Systems: An Overview of Some Recent Advances and Open Problems," *Automatica*, vol. 39, no. 10, pp. 1667–1694, 2003.

[37] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, vol. 99. Springer Science & Business Media, 2013.

[38] W. Michiels and S.-I. Niculescu, *Stability, control, and computation for time-delay systems: an eigenvalue-based approach.* SIAM, 2014.

[39] O. Diekmann, S. A. Van Gils, S. M. Lunel, and H.-O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, vol. 110. Springer Science & Business Media, 2012.

[40] N. Hayes, "Roots of the transcendental equation associated with a certain difference-differential equation," *Journal of the London Mathematical Society*, vol. 1, no. 3, pp. 226–232, 1950.

[41] H. Shinozaki and T. Mori, "Robust Stability Analysis of Linear Time-Delay Systems by Lambert W Function: Some Extreme Point Results," *Automatica*, vol. 42, no. 10, pp. 1791–1799, 2006.

[42] R. Braddock and P. Van den Driessche, "On a two lag differential delay equation," *J. Austral. Math. Soc. Ser. B*, vol. 24, pp. 292–317, 1983.

[43] J. M. Mahaffy and T. C. Busken, "Regions of stability for a linear differential equation with two rationally dependent delays," *DYNAMICAL SYSTEMS*, vol. 35, no. 10, pp. 4955–4986, 2015.

[44] J. Bélair, "Stability of a differential-delay equation with two time lags," in *Oscillation, Bifurcation and Chaos: Proceedings of the 1986 Annual Seminar Held July 13-25, 1986*, vol. 8, p. 305, American Mathematical Soc., 1987.

[45] J. Bélair and S. A. Campbell, "Stability and bifurcations of equilibria in a multiple-delayed differential equation," *SIAM Journal on Applied Mathematics*, vol. 54, no. 5, pp. 1402–1424, 1994.

[46] J. K. Hale and W. Huang, "Global geometry of the stable regions for two delay differential equations," *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 344–362, 1993.

[47] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[48] H. Moradian and S. S. Kia, "On the positive effect of delay on the rate of convergence of a class of linear time-delayed systems," *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4832–4839, 2019.

[49] F. M. Atay, "Network consensus under distributed transmission delays," in *2014 European Control Conference (ECC)*, pp. 1110–1114, IEEE, 2014.

[50] A. Seuret, D. V. Dimarogonas, and K. H. Johansson, "Consensus under communication delays," in *2008 47th IEEE Conference on Decision and Control (CDC)*, pp. 4922–4927, IEEE, 2008.

[51] L. Moreau, "Stability of continuous-time distributed consensus algorithms," in *2004 43rd IEEE Conference on Decision and Control (CDC)*, vol. 4, pp. 3998–4003, IEEE, 2004.

[52] M. K. Alhassan, "Effect of time delays on the convergence speed of consensus dynamics," MS Thesis, Bilkent University, 2020.

[53] Y. Hata and H. Matsunaga, "Stability switches in a linear differential equation with two delays," *Opuscula Mathematica*, vol. 42, no. 5, pp. 673–690, 2022.

[54] J. Herivel, "Christiaan Huygens. Dutch scientist and mathematician." Available at https://www.britannica.com/biography/Christiaan-Huygens.

[55] S. H. Strogatz and I. Stewart, "Coupled Oscillators and Biological Synchronization," *Scientific American*, vol. 269, no. 6, pp. 102–109, 1993.

[56] Y. Kuramoto, "Self-entrainment of a population of coupled nonlinear oscillators," in *Int. Symp. on Math. Prob. in Theor. Phys.*, Springer, 1975.

[57] Y. Kuramoto, *Chemical turbulence*. Springer, 1984.

[58] S. H. Strogatz, "From Kuramoto to Crawford: Exploring the Onset of Synchronization in Populations of Coupled Oscillators," *Physica D: Nonlinear Phenomena*, vol. 143, no. 1-4, pp. 1–20, 2000.

[59] F. Dörfler and F. Bullo, "On the critical coupling for kuramoto oscillators," *SIAM Journal on Applied Dynamical Systems*, vol. 10, no. 3, pp. 1070–1099, 2011.

[60] H. G. Schuster and P. Wagner, "Mutual entrainment of two limit cycle oscillators with time delayed coupling," *Progress of Theoretical Physics*, vol. 81, no. 5, pp. 939–945, 1989.

[61] M. S. Yeung and S. H. Strogatz, "Time delay in the Kuramoto model of coupled oscillators," *Physical Review Letters*, vol. 82, no. 3, p. 648, 1999.

[62] M. G. Earl and S. H. Strogatz, "Synchronization in oscillator networks with delayed coupling: A stability criterion," *Physical Review E*, vol. 67, no. 3, p. 036204, 2003.

[63] B. Dönmez, "Synchronization of Kuramoto model with anticipatory agents," MS Thesis, Bilkent University, 2023.

[64] B. Dönmez and F. M. Atay, "Synchronization in Networks of Anticipatory Agents," in *2023 European Control Conference (ECC)*, pp. 1–5, IEEE, 2023.