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ON THE RANK AND EXPONENT OF THE FIXED POINTS OF COPRIME ACTIONS

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Abstract. Let A be a group acting on a p-group P coprimely. We show that if A centralizes some specified abelian subgroups of P, then A acts trivially on P. As a consequence of this, we obtain that the special rank of $C_P(A)$ is strictly less than that of P unless the action of A on P is trivial. Secondly, we prove that if A acts on a group G coprimely and $[G,A]=G$, then the exponent of $C_G(A)/(C_G(A))'$ divides $|G:C_G(A)|$.

1. Introduction

All groups considered in this paper are finite. Notation and terminology are standard as in [2]. Let A be a group acting on a group G via automorphisms. In the case that $(|A|,|G|)=1$, we say that A acts coprimely on G. Now let p be a prime and A be a group acting on a p-group P coprimely. Recall that $\Omega_i(P)$ is defined to be $\langle x \in P \mid x^{p^i}=1\rangle$ for each positive integer i and we simply write $\Omega(P)$ when $i=1$. The following result in coprime action is quite useful and well known:

Theorem. *Assume $[\Omega(P),A]=1$ in the case that p is odd and assume $[\Omega_2(P),A]=1$ in the case that $p=2$. Then A acts trivially on P.*

Isaacs and Navarro improved this result for $p=2$ (see [3, Theorem B]). A simpler (character free) proof of [3, Theorem B] is given in [4] by the author. We shall improve [3, Theorem B] by refining the proof of [4, Theorem B] in our first main result. We also note that there is a small gap (a missed subcase) in the proof of [4, Corollary C]. We shall remedy this by obtaining a generalization of the result in Corollary B. Before stating our main results, we note some of our conventions:

Recall that an element x of a group G is called a *real element* if there exists $y \in G$ such that $x^y = x^{-1}$. We say that x is a *special real element* if

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x is of order 4 and the inverting element y can be chosen of order 4. Let P be a p-group, and let $\mathcal{E}(P)$ be the set of all maximal elementary abelian subgroups of P. Now consider the set of all abelian subgroups of P having a generating set whose each element is of order p or is a special real element. We define $\mathcal{E}^*(P)$ to be the set of all maximal members of these subgroups under inclusion. Note that $\mathcal{E}^*(P) = \mathcal{E}(P)$ when p is odd.

A p-group P is said to be of *rank* n if $p^n = \max\{|E| \mid E \in \mathcal{E}(P)\}$. Similarly, we define the *special rank* of P to be the natural number n where $p^n = \max\{|E| \mid E \in \mathcal{E}^*(P)\}$. We denote the rank and the special rank of P by $\mathrm{rk}(P)$ and $\mathrm{srk}(P)$, respectively. Note that $\mathrm{rk}(P) = \mathrm{srk}(P)$ when p is odd, however, the rank and the special rank of a 2-group can be different. For example, if P is a quaternion group, then $\mathrm{rk}(P) = 1$, but $\mathrm{srk}(P) = 2$.

THEOREM A. *Let A be a group acting coprimely on a p-group P and $E \in \mathcal{E}^*(P)$. If $[E, A] = 1$, then A acts trivially on P.*

In the case that p is odd, the above result can be obtained by the main theorem of [1], and so the main contribution of the above theorem is for $p = 2$.

Let Γ be the nontrivial semidirect product of $\mathbb{Z}_4$ by $\mathbb{Z}_4$. We say that P is of *odd type* if P has no subgroup isomorphic to Q_8 and Γ. Notice that P is always of odd type when p is odd.

COROLLARY B. *Let A be a group acting coprimely on a p-group P and $E \in \mathcal{E}(P)$. If P is of odd type and $[E, A] = 1$, then A acts trivially on P.*

By Corollary B, we see that there are only two possible "bad subgroups", namely, "Q_8 and Γ". On the absence of these subgroups, the $p = 2$ case behaves like odd primes. The assumption of being of odd type is weaker than that of being quaternion free (see Example 4.1). On the other hand, we only suppose $[E, A] = 1$ instead of $[\Omega(P), A]{=}1$, and so this generalizes [4, Corollary C] in two different directions. The following corollary is immediate from Theorem A and Corollary B.

COROLLARY C. *Let A be a group acting on a p-group P nontrivially and coprimely. Then the inequality $\mathrm{srk}(C_P(A)) < \mathrm{srk}(P)$ holds. Moreover, $\mathrm{rk}(C_P(A)) < \mathrm{rk}(P)$ if P is of odd type.*

REMARK 1.1. The above corollary can be quite useful especially when P is of "small rank". For example it implies that if $\mathrm{srk}(P) = 2$, then $C_P(A)$ is cyclic unless A acts trivially on P.

DEFINITION 1.2. Let P be a p-group. In the case that $\mathcal{E}(P)$ has a member of order p^m, we define $\omega_m(P)$ to be the subgroup generated by all members of $\mathcal{E}(P)$ of order p^m. When m is the smallest possible, we shall simply write $\omega(P)$. Similarly, we define $\omega^*_m(P)$ and $\omega^*(P)$ by replacing $\mathcal{E}(P)$ with $\mathcal{E}^*(P)$.

Clearly, both $\omega_m(P)$ and $\omega_m^*(P)$ are characteristic subgroups of P that are contained in $\Omega(P)$ and $\Omega_2(P)$ respectively for all possible m. The containment might be proper (see Example 4.2).

Corollary D. *Let A be a group acting coprimely on a p-group P. Assume one of the following*:

(a) $[A, \omega_m^*(P)] = 1$ *for some m.*

(b) $[A, \omega_m(P)] = 1$ *for some m and P is of odd type.*

Then A acts trivially on P.

The transfer homomorphism generally produces useful results when the target group is a Sylow or a Hall subgroup. Surprisingly, we obtain the following result via the transfer map $v \colon G \to C_G(A)/(C_G(A))'$. The proof uses a basic observation Lemma 3.2, which simply says that the transfer map is invariant under the action of automorphisms.

Theorem E. *Let A be a group acting on a group G coprimely with $[G, A] = G$. Then the exponent of $C_G(A)/(C_G(A))'$ divides $|G : C_G(A)|$.*

The following corollary shows that Theorem E can be used to bound the exponent of the fixed point subgroup in a p-group, although possibly the bound is not the best possible for large n.

Corollary F. *Let A be a group acting on a p-group P of order p^n coprimely where $n \geq 5$ and $[P, A] = P$. Then $C_P(A)$ is of exponent at most p^{n-3}.*

2. Proof of Theorem A

The following lemma is possibly noted somewhere but we could not find a reference.

Lemma 2.1. *Let A be a group acting on a p-group P coprimely, and write $C = C_P(A)$. If $C_P(C) \leq C$, then $C = P$, that is, A acts trivially on P.*

Proof. Assume that $C < P$ and set $N = N_P(C)$. Then $C < N$ as P is a p-group. Moreover, N is an A-invariant subgroup of P.

Now we have $[A, C, N] = [C, N, A] = 1$, which yields that $[N, A, C] = 1$ by three subgroups lemma. Thus, $[N, A] \leq C_P(C) \leq C$. Then $[N, A, A] = 1$, and so $[N, A] = 1$ due to the coprime action. It follows that $N = N_P(C) \leq C$. This contradiction completes the proof. $\square$

Corollary 2.2. *Let A be a group acting on a p-group P coprimely and let $D \leq P$. If $[P, A] \neq 1$, then $[DC_P(D), A] \neq 1$.*

Proof. Assume $[DC_P(D), A] = 1$. Then $DC_P(D) \leq C_P(A)$, and so $C_P(C_P(A)) \leq C_P(D) \leq C_P(A)$. It follows that $[P, A] = 1$ by Lemma 2.1. $\square$

Remark 2.3. It seems that Corollary 2.2 is equivalent to the $P \times Q$ lemma of Thompson (see [2, Theorem 4.31]).

Lemma 2.4. *Let P be a p-group and $E \in \mathcal{E}^*(P)$. Then $\Omega(E) \in \mathcal{E}^*(C_P(E))$.*

Proof. Write $C = C_P(E)$. Assume that $c \in C$ is a special real element in C. Then c is also a special real element in P, and so $c \in E$ as $[E, c] = 1$ and $E \in \mathcal{E}^*(P)$. However, $c \in E \leq Z(C)$, which leads that c can not be a special real element in C. Thus, C does not have any special real elements. Now let $d \in C$ of order p. Since $[d, E] = 1$, we obtain that $d \in E$, and so $d \in \Omega(E)$. The absence of a special real element in C leads to $\Omega(E)$ $\in \mathcal{E}^*(C) = \mathcal{E}^*(C_P(E))$. $\square$

Lemma 2.5 [4, Lemma D]. *Let G be a group and $x, y \in G$ be of order 4 such that $x^2 = y^2$ and $[x, y]$ is an involution lying in $Z(G)$. Then xy is a special real element.*

Remark 2.6. The following proof can be shortened by appealing the known results in that type that are mentioned in the introduction. However, we would like to present a proof which covers the previous known results as well at once. Beside some of the well known reductions, we use some of the ideas from the proof of [4, Theorem B].

Proof of Theorem A. Let P be a minimal counter example to the theorem. Then $[P, A] \neq 1$ but $[E, A] = 1$ where E is a member of $\mathcal{E}^*(P)$. Note that $E \leq C_P(E)$ as E is abelian and clearly $C_P(E)$ is A-invariant. Then we see that $[C_P(E), A] \neq 1$ by Corollary 2.2. Now assume that $C_P(E) < P$. Since $\Omega(E) \in \mathcal{E}^*(C_P(E))$ by Lemma 2.4 and $[\Omega(E), A] = 1$, we get that $[C_P(E), A] = 1$ by the minimality of P. This contradiction shows that $P = C_P(E)$, that is, $E \leq Z(P)$. In particular, we see that $\Omega(P) = E \leq Z(P)$ and there is no special real element lying in P. We also see that P is nonabelian by [2, Corollary 4.35].

Now let D be an A-invariant proper subgroup of P. Note that $E \cap D = \Omega(D) \in \mathcal{E}^*(D)$ and $[\Omega(D), A] = 1$. Thus we obtain that $[A, D] = 1$ by the inductive argument. If $C_P(D)$ is also proper in P, then we have $[C_P(D), A] = 1$, and so $[DC_P(D), A] = 1$, which is not possible by Corollary 2.2. Thus, $C_P(D) = P$, that is, every A-invariant proper subgroup of P is contained in $Z(P)$. In particular, we have $P' \leq \Phi(P) \leq Z(P)$.

Note that $[P, A]$ is an A-invariant subgroup of P and if $[P, A] < P$, then $[P, A, A] = 1$. The coprime action yields that $[P, A] = 1$, which is not the case. Thus, $[P, A] = P$, and so $[P/P', A] = P/P'$ which yields that $C_{P/P'}(A) = 1$ by [2, Theorem 4.34]. Consequently, we have $Z(P) = \Phi(P) = P'$ as $Z(P)/P'$ $\leq C_{P/P'}(A) = 1$.

We have $1 = [x^p, y] = [x, y]^p$ as $\Phi(P) \leq Z(P)$, and so P' is of exponent p. Now assume that p is odd. Then we have $(xy)^p = x^p y^p [x, y]^{p(p-1)/2} = x^p y^p$

for all $x, y \in P$ as P is of class 2 and P' is of exponent p. Then

$$[x,a]^p = (x^{-1}x^a)^p = x^{-p}(x^p)^a = [x^p,a] = 1$$

for all $a \in A$ and $x \in P$. That leads that $[P,A] \leq \Omega(P) = E < P$, which is a contradiction. Thus, $p = 2$.

Let x be an element of order 4 in P and $a \in A$ such that $[x,x^a] \neq 1$. Set $y = x^a$. Note that $(x^2)^a = x^2$ as $x^2 \in E$. Then we see that $x^2 = y^2$. Since P' is of exponent $p = 2$, we obtain that $1 \neq [x,y]$ is an involution lying in the center of P. It follows that xy is a special real element by Lemma 2.5, which is not the case. This contradiction shows that $[x,x^a] = 1$, and so $\langle x^A \rangle$ is an abelian group. Note that $\langle x^A \rangle$ is a proper A-invariant subgroup as P is nonabelian. Then we have $\langle x^A \rangle \leq Z(P)$. Consequently, each element of order 4 lies in $Z(P)$. However, $P' = Z(P)$ is of exponent 2, and so there is no element of order 4 in P, that is, $E = P$. This contradiction completes the proof. $\square$

Proof of Corollary B. Let x be a special real element in P. Then we have $y \in P$ of order 4 such that $x^y = x^{-1}$ by the definition of a special real element. In the case that $\langle x \rangle \cap \langle y \rangle = 1$, we have $\langle x \rangle \langle y \rangle \cong \Gamma$. If $|\langle x \rangle \cap \langle y \rangle| = 2$, then $x^2 = y^2$, which leads $\langle x,y \rangle \cong Q_8$. Both cases are impossible as P is of odd type. Hence, there is no special real element in P, which leads to $\mathcal{E}^*(P) = \mathcal{E}(P)$.

By hypothesis we have $E \in \mathcal{E}(P)$ such that $[A,E] = 1$. We see that $E \in \mathcal{E}^*(P)$ by the previous paragraph, and so we obtain that $[P,A] = 1$ by Theorem A. $\square$

3. Proof of Theorem E

Definition 3.1. Let A be a group acting on groups G and H via automorphisms. A homomorphism ϕ from G to H is called *A-invariant* if $\phi(g)^a = \phi(g^a)$ for all $g \in G$ and $a \in A$.

If ϕ is an A-invariant homomorphism from G to H, then one can observe that $\phi(G)$ and $\operatorname{Ker}(\phi)$ are A-invariant subgroups of H and G, respectively. Moreover, the action of A on $\phi(G)$ is equivalent to the action of A on $G/\operatorname{Ker}(\phi)$.

Lemma 3.2. *Let A be a group acting on a group G via automorphisms and H be an A-invariant subgroup of G. The transfer map v from G to H/H' is A-invariant.*

Proof. Let $T = \{t_1, t_2, \ldots, t_n\}$ be a transversal set for the right cosets of H in G. Consider the usual right action of G on T. Pick $g \in G$, $a \in A$. Then $t_i.g = t_{k_i}$ where $t_{k_i} \in T$ and $Ht_ig = Ht_{k_i}$. Hence, we get $(Ht_ig)^a = (Ht_{k_i})^a$.

It follows that $Ht_i^a g^a = Ht_{k_i}^a$. The previous equality holds as H is A-invariant. Consequently, we can say that

$$(t_i.g)^a = t_{k_i}^a = t_i^a.(g^a). \tag{1}$$

Note that both t_i^a and $t_{k_i}^a$ are elements of the set $T^a = \{t^a \mid t \in T\}$, which is also a right transversal for H in G as a permutes the cosets of H in G.

Now let V be a pretransfer map from G to H constructed by using T. Then $V(g) = \prod_{t_i \in T} t_i g (t_i.g)^{-1}$. It follows that

$$V(g)^a = \Big(\prod_{t_i \in T} t_i g (t_i.g)^{-1}\Big)^a = \prod_{t_i \in T} (t_i g)^a ((t_i.g)^a)^{-1}$$
$$= \prod_{t_i \in T} t_i^a g^a (t_i^a.g^a)^{-1} = \prod_{t_i^a \in T^a} t_i^a g^a (t_i^a.g^a)^{-1} \equiv V(g^a) \bmod H'$$

by using (1) in the third equality. Thus, $V(g)^a \equiv V(g^a) \bmod H'$. The congruence relation holds as the transfer homomorphism is independent of the choice of transversal set (see [2, Theorem 5.1]). Hence, $v(g)^a = v(g^a)$ for all $g \in G$ and $a \in A$ as required. $\square$

Corollary 3.3. *Let N be a normal subgroup of a group G and v be a transfer map from G to N/N'. Then $v(G) \leq Z(G/N')$. In particular, if N is abelian, then $v(G)$ is central in G.*

Proof. As both G and N are G-invariant under the conjugation action of G, the transfer map $v\colon G \to N/N'$ is also G-invariant by Lemma 3.2. Note that G acts trivially on $G/\operatorname{Ker}(v)$ as $G' \leq \operatorname{Ker}(v)$. Thus, G acts trivially on $v(G)$. Hence, $v(G) \leq Z(G/N')$. Then the result follows. $\square$

Proof of Theorem E. Let V be a transfer map from G to $C_G(A)$ and v be the corresponding transfer map from G to $C_G(A)/(C_G(A))'$. Note that the transfer map $v\colon G \to C_G(A)/(C_G(A))'$ is A-invariant by Lemma 3.2. Thus, the action of A on $v(G)$ is equivalent to the action of A on $G/\operatorname{Ker}(v)$. As A acts trivially on $C_G(A)/(C_G(A))'$, it also acts trivially on $G/\operatorname{Ker}(v)$, and so $G = [G, A] \leq \operatorname{Ker}(v)$. As a result v is the trivial map.

Let x be an element of $C_G(A)$ such that the order of the image of x in the quotient group $C_G(A)/(C_G(A))'$ is equal to the exponent e of $C_G(A)/(C_G(A))'$. We have

$$V(x) \equiv \prod_{t_i \in T_0} t_i x^{n_{t_i}} t_i^{-1} \bmod (C_G(A))'$$

where n_{t_i} is a positive integer for each $t_i \in T_0$ such that $\sum_{t_i \in T_0} n_{t_i} = |G : C_G(A)|$ (see [2, Lemma 5.5(b) and (c)]). Note also that $t_i x^{n_{t_i}} t_i^{-1}$ and

$x^{n_{t_i}}$ are elements of $C_G(A)$ (see [2, Lemma 5.5(a)]). Then we see that they are also conjugate via an element of $C_G(A)$ by [2, Theorem 3.26]. Now set

$$t_i x^{n_{t_i}} t_i^{-1} = c_i x^{n_{t_i}} c_i^{-1}$$

for some $c_i \in C_G(A)$.

Then we have

$$V(x) \equiv \prod_{t_i \in T_0} c_i x^{n_{t_i}} c_i^{-1} \equiv \prod_{t_i \in T_0} x^{n_{t_i}} \bmod (C_G(A))'.$$

Thus, $V(x) \equiv x^{|G:C_G(A)|} \bmod (C_G(A))'$ by [2, Lemma 5.5(c)]. Note that $V(x) \in (C_G(A))'$ as v is the trivial map. Then $x^{|G:C_G(A)|} \in (C_G(A))'$. It follows that e divides $|G : C_G(A)|$ as desired. $\square$

Proof of Corollary F. It is enough to show that A does not centralize any element of order p^{n-2}. Assume the contrary and let $[A, u] = 1$ for some $u \in P$ of order p^{n-2}. If $C_P(A)$ is abelian, then p^{n-2} is a divisor of $|P : C_P(A)| \in \{p, p^2\}$ by Theorem E, which is impossible as $n \geq 5$. Thus, $C_P(A)$ is nonabelian. In particular we have $\langle u\rangle < C_P(A) < P$, which forces that $|C_P(A)| = p^{n-1}$. Now set $C = C_P(C_P(A))$. Note that $C \nsubseteq C_P(A)$ by Lemma 2.1, and so $C_P(A)C = P$ as $C_P(A)$ is a maximal subgroup of P. Since C is also A-invariant, $[P, A] = [C_P(A)C, A] = [C, A] = P$, that is, $C = P$. It follows that $C_P(A) \leq Z(P)$, and so $C_P(A)$ is abelian. This contradiction completes the proof. $\square$

4. Examples

Note that $\Gamma = \langle x, y \mid x^4 = y^4 = 1, y^{-1}xy = x^{-1}\rangle$, and so $\Gamma/\langle x^2y^{-2}\rangle \cong Q_8$. Thus, if a p-group P is quaternion free, then it has no subgroup isomorphic to Q_8 or Γ, that is, P is of odd type. The following example shows there are 2-groups of odd type which are not quaternion free, and so the assumption of being of odd type is weaker than that of being quaternion free.

Example 4.1. Let $P = \langle a, b \mid a^8 = b^8 = 1, b^{-1}ab = a^{-1}\rangle$. Then it is easy to check that the elements of the form $a^n b^m$ where m or n is odd are of order 8. Thus, all elements of order 4 are of the form $a^{2n}b^{2m}$ for some suitable m and n. It follows that $\Omega_2(P)$ is abelian, and so P has no subgroup isomorphic to Q_8 or Γ, that is, P is of odd type. Let $Z = \langle a^4, b^4, a^2b^{-2}\rangle$. Then $P/Z = \langle x, y \mid x^4 = y^4 = 1,\, x^2 = y^2,\, y^{-1}xy = x^{-1}\rangle$, and so $P/Z \cong Q_8$. Thus, P is not quaternion free.

In the following example, we see that in general $\omega(P) \neq \Omega(P)$ and $\omega^*(P) \neq \Omega_2(P)$.

Example 4.2. We shall appeal to GAP[5] for the first example. Consider the group $P = \mathrm{SmallGroup}(81, 7)$, which is isomorphic to $\mathbb{Z}_3 \wr \mathbb{Z}_3$ and a Sylow 3-subgroup of S_9. Note that up to conjugacy $\mathcal{E}(P)$ involves two groups whose orders are 9 and 27. Both $\omega(P) = \omega_2(P)$ and $\omega_3(P)$ are groups of order 27. They are proper in $\Omega(P)$ as $\Omega(P) = P$.

Now let $P = Q_8 \times \mathbb{Z}_4$. Then any special real element u of P is in the form of (x, y) where $x \in \{i, j, k\}$ and $y \in \{0, 2\}$. Thus, each $E \in \mathcal{E}^*(P)$ has order 8 and $\omega^*(P) = \omega_3^*(P) = Q_8 \times \{0, 2\} < P = \Omega_2(P)$.

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