HAMILTONIAN STRUCTURE OF THE LOTKA–VOLTERRA EQUATIONS

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The Lotka–Volterra equations governing predator–prey relations are shown to admit Hamiltonian structure with respect to a generalized Poisson bracket. These equations provide an example of a system for which the naive criterion for the existence of Hamiltonian structure fails. We show further that there is a three-component generalization of the Lotka–Volterra equations which is a bi-Hamiltonian system.

The conditions for a dynamical system

$$\dot{x}^k = X^k, \quad k=1,2,\ldots,2n, \tag{1}$$

to admit Hamiltonian structure are naively given by [1]

$$X^k{}_{,k} = 0. \tag{2}$$

Gonzalez-Gascon [2] has noted that this criterion is valid only when the variables x^i are chosen such that the symplectic two-form ω is cast into the canonical form

$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^{n+1} + \mathrm{d}x^2 \wedge \mathrm{d}x^{n+2} + \ldots + \mathrm{d}x^n \wedge \mathrm{d}x^{2n} \tag{3}$$

according to Darboux's theorem. When the original variables defining the dynamical system are not of this form, the criterion (2) is too restrictive. In fact Gonzalez-Gascon has given an example of a Hamiltonian system where this condition is violated. Gonzalez-Gascon's counter-example does not represent a familiar dynamical system. We shall show that the predator–prey equations of Lotka and Volterra provide another example of a Hamiltonian system for which the criterion (2) fails. It is surprising that the Hamiltonian structure of such a well-known system as the Lotka–Volterra equations has not been noted earlier.

The Lotka–Volterra equations are given by

$$\dot{x} = (A-By)x, \quad \dot{y} = (Cx-D)y, \tag{4}$$

where A, B, C and D are constants. Since the vector field

$$X = (A-By)x\frac{\partial}{\partial x} + (Cx-D)y\frac{\partial}{\partial y} \tag{5}$$

is not divergence free, the naive criterion for the existence of Hamiltonian structure fails. On the other hand we may consider the following ansatz for the symplectic two-form,

$$\omega = f(x,y)\,\mathrm{d}x \wedge \mathrm{d}y, \tag{6}$$

which is always closed in two dimensions. Hamilton's equations require that

$$\omega \rfloor X = \mathrm{d}H, \tag{7}$$

where the Hamiltonian function H is a zero-form. The integrability conditions of eqs. (7) are obtained by applying the exterior derivative. Thus we find that ω given by eq. (6) will be symplectic provided f satisfies

$$[(A-By)xf]_x + [(Cx-D)yf]_y = 0. \tag{8}$$

This first-order equation has the solution

$$f = \frac{1}{xy} \tag{9}$$

plus an arbitrary function of its characteristic,

$$H = A\ln y + D\ln x - Cx - By, \tag{10}$$

which also plays the role of the Hamiltonian function. Eq. (10) is well-known as the Liapunov function for the Lotka–Volterra equations.

In the dual representation [3] eqs. (4) can be written as

$$\dot{x}^i = J^{ik}\nabla_k H, \tag{11}$$

with $x^1 = x$, $x^2 = y$, where

$$J = \begin{pmatrix} 0 & xy \\ -xy & 0 \end{pmatrix} \tag{12}$$

are the structure functions. The Jacobi identities

$$J^{k[m}\nabla_k J^{np]} = 0 \tag{13}$$

are satisfied automatically because we are in two dimensions.

The Lotka–Volterra equations are a Hamiltonian system with respect to the generalized Poisson bracket defined in terms of eq. (12). They do not admit a second Hamiltonian structure as an examination of the above general solution of eq. (8) reveals. There exist several generalizations of the Lotka–Volterra equations [4,5] which are going to admit a similar Hamiltonian structure and we shall now consider a three-component generalization which is a bi-Hamiltonian system.

Grammaticos et al. [5] have discussed the system

$$\dot{x} = x(cy + z + \lambda), \quad \dot{y} = y(x + az + \mu),$$
$$\dot{z} = z(bx + y + \nu), \tag{14}$$

where some of the constants appearing in these equations can be related to those in eqs. (4) by scaling the dynamical variables and time. It was pointed out in ref. [5] that subject to the conditions

$$abc = -1, \quad \nu = \mu b - \lambda ab \tag{15}$$

eqs. (14) admit two conserved quantities,

$$H_1 = ab \ln x - b \ln y + \ln z,$$
$$H_2 = abx + y - az + \nu \ln y - \mu \ln z. \tag{16}$$

This particular case is a bi-Hamiltonian system.

It can be readily verified that eqs. (14) can be written as Hamilton's equations in two distinct ways,

$$\dot{x}^i = J_1^{ik}\nabla_k H_2 = J_2^{ik}\nabla_k H_1, \tag{17}$$

where the components of x^i are given by x, y, z respectively and

$$J_1 = \begin{pmatrix} 0 & cxy & bcxz \\ -cxy & 0 & -yz \\ -bcxz & yz & 0 \end{pmatrix}, \tag{18}$$

$$J_2 = \begin{pmatrix} 0 & cxy(az+\mu) & cxz(y+\nu) \\ -cxy(az+\mu) & 0 & xyz \\ -cxz(y+\nu) & -xyz & 0 \end{pmatrix}. \tag{19}$$

In three dimensions the Jacobi identities (13) reduce to a single equation which is satisfied by any linear combination of J_1 and J_2 with constant coefficients. Thus they are compatible. No new conserved Hamiltonians are generated from the recursion relation (17) because

$$J_1\nabla H_1 = 0, \quad J_2 \nabla H_2 = 0, \tag{20}$$

that is, H_1 and H_2 are Casimirs of J_1 and J_2 respectively.

The multi-Hamiltonian structure of Lotka–Volterra equations is evidently a rich subject as the above examples indicate.

References

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