

**MAINTENANCE AND MARGINAL COST ANALYSIS  
OF A TWO-UNIT COLD STANDBY SYSTEM**

**A THESIS**

**SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL  
ENGINEERING**

**AND THE INSTITUTE OF ENGINEERING AND SCIENCES  
OF BILKENT UNIVERSITY**

**IN PARTIAL FULFILLMENT OF THE REQUIREMENTS  
FOR THE DEGREE OF  
MASTER OF SCIENCE**

**By**

**Chokri Hamdaoui**

**August, 1997**

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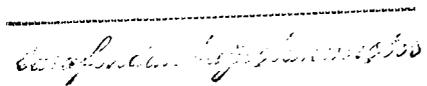
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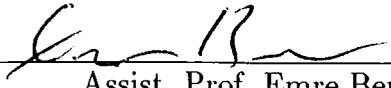
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# ABSTRACT

## MAINTENANCE AND MARGINAL COST ANALYSIS OF A TWO-UNIT COLD STANDBY SYSTEM

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M.S. in Industrial Engineering

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August, 1997

The Marginal Cost Analysis (MCA) of maintenance policies is a concept gaining interest in the recent years. This approach, due to Berg, has been categorized as an Economics Oriented Approach, as different from the classical probability centered approach. The MCA has been successfully applied to the Age Replacement and the Block Replacement policies, and was shown to be flexible enough to permit extensions and generalizations.

In this thesis, we apply the MCA approach to a more complex model. We consider a two-unit cold standby system. Upon failure of the working unit in the time interval  $[0, T)$  the unit is replaced by the standby unit if available. If the standby unit is in repair, the system is down, and a downtime cost is incurred. The item inspected at time  $T$  is in one of two states: “good”, or “critical”. The good unit continues operation, whereas a unit in critical state is sent to repair. The switchover is immediate. We derive and compare the marginal cost function as well as the long-run cost per unit time function.

*Key words:* Maintenance Policies, Age Replacement, Block Replacement, Cold Standby System, Marginal Cost Approach.

## ÖZET

### SOĞUK YEDEKLİ İKİ BİRİMLİ BİR SİSTEMİN BAKIM/ONARIM VE MARJİNAL MALİYET ANALİZİ

Chokri Hamdaoui

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Tez Yöneticisi: Doç. Ülkü Gürler

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Bakım/onarım politikalarının Marjinal Maliyet Analizi (MMA) ilgi kazanmakta olan bir kavramdır. Berg tarafından ortaya atılan ve klasik olasılık yaklaşımından farklı olan bu kavram Ekonomik Bakışaçılı bir yaklaşımdır. MMA yaklaşımı yaşa bağlı ve blok yenileme politikalarında başarılı olarak uygulanmış ve çeşitli yönlerde genelleştirilecek kadar esnek olduğu gösterilmiştir.

Bu tezde MMA daha komplike olan iki-birimli yedekli sistemlere uygulanmıştır. Çalışan birim  $[0, T)$  aralığında bozulduğunda eğer yedek birim hazır durumdaysa onunla değiştirilmektedir. Eğer yedek birim tamirde ise, sistem durmakta ve bir durma maliyeti ortaya çıkmaktadır.  $T$  zamanında kontrol edilen birim “iyi” ya da “kritik” durumda olabilmektedir. “İyi” durumda olan makina çalışmaya devam etmekte, “kritik” ise tamire gitmektedir. Makina değişimi anında olmaktadır. Bu çalışmada ortalama maliyet ve marjinal maliyet fonksiyonları bulunmakta ve karşılaştırılmaktadır.

*Anahtar sözcükler:* Bakım/onarım Politikaları, Yaşa Bağlı Yenileme, Blok Yenileme, Soğuk Yedekli Sistemler, Marjinal Maliyet Analizi.

Dedicated to my Family.

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# Chapter 1

## INTRODUCTION

### 1.1 RELIABILITY AND DESIGN FOR RELIABILITY

The field of reliability analysis dates back to the second world war. It was an outcome of problems with electronic systems designed during the 1940's. The growth in complexity of the electronic systems prompted the US Air Force, Navy and Army to establish committees to investigate the reliability problem. In 1952, the Department of Defense established the Advisory Group on Reliability of Electronic Equipment (AGREE). In the recent years, the field of reliability has become important for systems design and development.

Reliability must be considered early in the design process of a system, where changes can be most easily and economically done. During this design process, the system configuration is chosen. Evidently, the choice of the system configuration has a direct effect on the reliability level. During the design phase, a preliminary reliability analysis should be performed, in addition to the other design factors. The system designer should be familiar with the basic reliability analysis concepts that can be used as performance measures to evaluate the system. It is important that the designer evaluate the reliability

levels, costs and trade-offs to come up with a final decision or choice.

## 1.2 INTEGRATING MAINTENANCE ACTIVITY INTO PMS

Maintenance planning should be integrated in the Production Management System (PMS). In any organization, there are many resources such as workforce, equipment, machinery, work in process, raw materials, etc. However from a management viewpoint, the primary and most valuable resources are time and information. More specifically, timely information is of paramount significance, and is the key to success of today's factory. The reason is obvious. For example, late deliveries of product may cause loss of customers' good will. A plant may have the capacity to produce, but may not have the capacity to produce so many units in a certain period of time. Of course, the issue raised here is Production Activity Control (PAC), and its interface with PMS. We believe that reliability considerations should be integrated in PAC. Such considerations influence PAC in carrying out functions such as operation scheduling and loading, manufacturing order approval and release, capacity and quality control, process evaluation with respect to material and equipment costs and availability.

## 1.3 PERFORMANCE MEASURES AND RELATED CONCEPTS

The concept of a system is needful.

**Definition 1** *A system is a collection of entities (e.g. people, machines, etc.) with a specified interaction to accomplish a goal.*

A system is completely characterized by its state vector, a collection of variables that contain all the information necessary to describe the system.

Grant & Leavenworth [29] suggest the following definitions for reliability.

Reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered.

The reliability (of a system, device, etc.) is the probability that it will give satisfactory performance for a specified period of time under specified operating conditions.

The following definitions are suggested by AGREE.

Failure: the inability of an equipment to perform its required function.

Reliability: the probability of no failure throughout a prescribed operating period.

Similar definitions are due to Smith [54].

Failure: non conformance to some defined performance criterion.

Quality: conformance to specifications.

Reliability is the extension of quality into the time domain, and may be rephrased: the probability of no failure in a given period. Alternatively, quality is the projection of availability onto the space where time is zero.

These definitions are similar. However, there is no consensus as to what constitutes ‘satisfactory performance.’ Naturally, this depends on the use as well as on the operating (or environmental) conditions. For example, a particular chip might have a short life under one set of specifications, and a long life under another set of specifications.

For repairable systems, the concept of minimal repair was first introduced to the literature by Barlow and Hunter [5]. Upon minimal repair, the repair

action returns the system into an operational state in such a way that the system characteristics are the same as they were just before failure, i.e. the system is as good as old. In other terms, the repair restores operability, but makes no improvement. Mathematically, that means that the hazard function following a (minimal) repair is undisturbed, so that the failure process in the process is independent of this and of any other past repair action.

### 1.3.1 What Causes Failure?

There are many causes of failure, known as stresses. Smith [54] classifies stresses into two categories:

a) Environmental:

- Temperature,
- Radiation
- Shock (mechanical, thermal)
- Vibration
- Humidity
- Ingress of foreign body.

b) Self-generated:

- Power dissipation
- Applied voltage and current
- Self-generated vibration
- Wear

### 1.3.2 Preventive Maintenance

These are systems in which the operating units are sent for preventive maintenance according to an age-specific probability distribution. When the

age of an operating unit is in the interval  $(x, x + \Delta)$ , it is sent for preventive maintenance with probability  $a(x)\Delta$ , provided that this action of scheduling the preventive maintenance of the unit does not result in a system failure. Otherwise, the preventive maintenance is either deferred until one of the units under repair becomes available, or the preventive maintenance is cancelled. The concept of preventive maintenance is useful only when the failure rate of a unit is an increasing function of age,  $x$ , and the expected duration of maintenance is not larger than that of the repair time duration.

### 1.3.3 Performance Measures

We present some of the performance measures found in the literature of interest from the system's design and analysis viewpoint.

One of the performance measures is termed *serviceability*, defined as the ease with which a system can be repaired. Clearly, ease of serviceability should be planned at the design stage, as it is a characteristic of the system's design. However, this is not easy to measure on a numerical scale. Usually, it is measured by ranking, or by specifically developed rating procedures.

*Maintainability* is another performance measure, defined as the probability that a failed system can be made operable in a specified interval of downtime, which includes the total time that the system is out of service. Downtime comprises failure detection time, repair time, administration time and the logistics time connected with the repair cycle. The maintainability density function describes probabilistically how long a system remains in a failed state (or down).

A more restricted performance measure is termed repairability.

*Repairability* is the probability that a failed system will be restored to a satisfactory operating condition in a specified interval of active repair time. This performance measure helps quantify the workload of the repair facility, hence it allows a managerial support.

*Operational readiness* is the probability that either a system is operating or can operate satisfactorily when the system is used under stated conditions.

*Availability* is the probability that a system is operating satisfactorily at any point in time. It considers only operating time and downtime, thus it excludes idle time. Availability is a measure of the ratio of the operating time of the system to the operating time plus the downtime. Therefore, it includes both reliability and maintainability.

*Intrinsic availability* refers to the probability that a system is operating in a satisfactory manner at any point in time when used under stated conditions. Hence, time is confined solely to operating and active repair time. Availability is always smaller than or equal to the intrinsic availability. The difference between these two performance measures can be viewed as a measure of administrative and logistics time related to the repair cycle.

## 1.4 PRELIMINARIES

**Definition 2** *Lifetime of a component is the time, usually random, from the beginning of the operation until the occurrence of a failure.*

The reliability, or survival probability, is formally defined.

**Definition 3** *The reliability of a component, with lifetime  $X$  having a distribution  $F(\cdot)$ , is  $\bar{F}(x) = 1 - F(x)$ .*

**Definition 4** *The conditional reliability of a component at age  $t$  is the probability the component will survive  $x$  more time units given that it is operating at age  $t$ . Formally,  $\bar{F}(x | t) = \frac{\bar{F}(t+x)}{\bar{F}(t)}$ , if  $\bar{F}(t) \geq 0$ .*

**Definition 5** *The failure rate  $r(t)$  at time  $t$ , of the random variable  $X$  is,*

$$r(t) = \frac{f(t)}{\bar{F}(t)}, \quad (1)$$

where  $\bar{F}(t) \neq 0$ , and the density  $f(t)$  exists.

A primary role is played by the failure rate function, sometimes called the hazard rate, force of mortality, and intensity rate. Failure rate of a component at time  $t$  is proportional to the probability that it will fail in the next short interval given that it is operational at the start of that time interval.

The failure rate has a particular usefulness in reliability analysis. The probability that a component will fail in the next  $\Delta$  time units (for infinitesimal  $\Delta$ ) given that the component is surviving now is equal to  $r(t)\Delta$ . If  $r(t)$  is an increasing function of  $t$ , then the lifetime distribution  $F(t)$  is an Increasing Failure Rate (IFR) distribution. This describes a class of distributions corresponding to adverse aging. Note that IFR class can alternatively be characterized by decreasing conditional reliability,  $\bar{F}(x | t)$ , as a function of  $t$ , for each  $x \geq 0$ . If  $r(t)$  is a decreasing function of  $t$ , then the lifetime distribution  $F(t)$  is a Decreasing Failure Rate (DFR) distribution. In this case, aging is beneficial. A component whose lifetime distribution is in the DFR class has increasing survival probability, or equivalently, it has increasing conditional reliability,  $\bar{F}(x | t)$ , as a function of  $t$ , for each  $x \geq 0$ . The failure rate function  $r(t)$  at age  $t$  is derived from the conditional reliability at age  $t$ ,

$$\begin{aligned}
 r(t) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} [1 - \bar{F}(\Delta | t)] \\
 &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left[ \frac{\bar{F}(t) - \bar{F}(t + \Delta)}{\bar{F}(t)} \right] \\
 &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left[ \frac{F(t + \Delta) - F(t)}{\bar{F}(t)} \right] \\
 &= \lim_{\Delta \rightarrow 0} \frac{1}{\bar{F}(t)} \left[ \frac{F(t + \Delta) - F(t)}{\Delta} \right] \\
 &= \frac{f(t)}{\bar{F}(t)}.
 \end{aligned}$$

Some useful identities are straightforwardly obtained from the above definitions. The reliability function and the failure rate function are intimately related to each other. Multiplying equation (1) by  $-1$ , then exponentiating,

we have

$$-r(t) = -\frac{f(t)}{\bar{F}(t)}. \quad (2)$$

Upon integrating both sides of equation (2) with respect to  $t$ ,

$$\begin{aligned} -\int_0^x r(t)dt &= \log \bar{F}(x) \\ \bar{F}(x) &= e^{-\int_0^x r(t)dt} \end{aligned}$$

**Definition 6** *The hazard function is the cumulative failure rate up to age  $x$ . Formally, it is*

$$R(x) = \int_0^x r(t)dt \quad (3)$$

The reliability function and the hazard function are related to each other, as follows,

$$\bar{F}(x) = e^{-R(x)} \quad (4)$$

**Definition 7** *Age Replacement Policy (ARP) is a policy where the component is replaced upon failure or at age  $T$ , whichever comes first. If  $c_1$  is the replacement cost at failure and  $c_2$ , ( $0 < c_2 < c_1$ ), is the replacement cost at age  $T$ , then the average long-run cost per unit time is given by,*

$$C(T) = \frac{c_1 F(T) + c_2 \bar{F}(T)}{\int_0^T \bar{F}(t)dt} \quad (5)$$

**Definition 8** *Block Replacement Policy (BRP) is a policy where the component is replaced upon failure and at times  $T, 2T, 3T, \dots$ . The expected cost per unit time, in the long run, is given by,*

$$C(T) = \frac{c_1 m(T) + c_2}{T}, \quad (6)$$

where  $m(T)$  is the renewal function of failures, that is the expected number of failures in the interval  $[0, T)$ .

**Definition 9** *The Failure Replacement Policy (FRP) is a policy where no preventive replacements are made at all. FRP is a non-preventive policy, where action (replacement) is taken only after the failure has occurred.*

Under FRP, the long-run average maintenance cost per component per unit time is  $\frac{c}{\tau}$ , where  $c$  is the replacement cost, and  $\tau$  is the expected life of a component. The result follows from the renewal theory. The interested reader is referred to Barlow & Proschan [7].

## 1.5 SCOPE OF THE THESIS

In this thesis, we study and perform Marginal Cost Analysis of a two-unit cold redundant standby system, supported by one repair facility. Our objective is to determine the optimal preventive maintenance time that minimizes the expected cost per unit time in the long run.

In Chapter 2, we review the related literature. Chapter 3 provides a complete description of the model. Two maintenance policies are studied. Their Marginal Cost Functions are derived. In Chapter 4, we present a general maintenance model and derive its objective and Marginal Cost Functions. Numerical results for this model are provided in Chapter 5. Finally, in Chapter 6, we conclude and suggest some future research directions.

## Chapter 2

# LITERATURE REVIEW

The earlier work on reliability theory and replacement problems is the book, Barlow & Proschan [6]. The subsequent version of this book, Barlow & Proschan [7], is a good reference. It treats a wide range of topics, including coherent structure theory, the theory of extreme value distributions, fault tree analysis, availability theory, shock models, the Marshall-Olkin multivariate exponential distribution, as well as dependence of various kinds among random variables. This book is intended for reliability analysts, operations researchers, and statisticians.

Many survey papers covering the literature in maintenance and reliability theory have been published, including McCall [41], Pierskella & Voelker, Sherif & Smith [52], El-Newehi & Proschan [21], Valdez-Flores & Feldman [57] and Cho & Parlar [17]. Review papers of standby systems include Osaki & Nakagawa [45], Kumar & Agarwal [38], and Yearout *et al.* [60]. Abouammoh & Quamber [1] trace the development and application of the statistical theory in modeling and solving problems in reliability engineering. They review criteria of aging; partial ordering; closure properties; discrete aging and shock model; and testing of exponentiality versus aging.

We start by single unit maintenance systems and policies, then we review some multi-unit systems. Finally, we review the two-unit redundant standby

systems.

Berg & Epstein [12] describe and compare three failure replacement policies: the age replacement policy; the block replacement policy; and the failure replacement policy. They propose a rule to choose the best (least costly) policy under some conditions specified on model parameters. Langberg [40] studies and compares age replacement policy (ARP) and block replacement policy (BRP), based on number of failures and removals. The number of renewals in the time interval  $[0, s]$  under ARP is shown to be stochastically smaller than the number of renewals in the same interval under the corresponding BRP. The author introduces the concepts New Better Than Used (NBU) in sequence and New Worse Than Used (NWU) in sequence. Under the assumption of NBU in sequence, the author shows that the number of failures in  $[0, s]$  under ARP is stochastically smaller than the number of failures in  $[0, s]$  when no replacement is applied.

The age replacement policy is studied by Berg [9]. Using renewal theory, and continuous Markov decision processes, he proves that it is an optimal decision rule amongst all reasonable replacement policies. Block *et al.* [15] propose a general age replacement model with minimal repair. It is assumed that the cost of a minimal repair is dependent on the number of minimal repairs done so far. Two kinds of replacement are considered: planned (at age  $T$ ) and unplanned replacement (upon failure, with probability  $p(y)$ , where  $y$  is the age). They also consider a shock model, where the system is replaced at a fixed cost at periods  $T, 2T, 3T$ , etc. The authors derive the total expected long-run cost per unit time, and provide optimal results (replacement age,  $T^*$ ) for finite and infinite horizon. A similar study is done by Block *et al.* [14]. They investigate a maintenance policy where upon failure a maintenance action is undertaken. However, the maintenance action is a complete repair with probability  $p(t)$ , and is a minimal repair with probability  $q(t) = 1 - p(t)$ , where  $t$  is the age at failure. Repairs are assumed to take negligible time. The authors establish that the successive complete repairs are a renewal process, and derive its interarrival distribution.

Another variation of the age replacement model is the paper by Tilquin & Cl eroux [56]. They investigate periodic replacement policies with minimal repair and discrete adjustment costs. The adjustment costs are those costs incurred to keep the system operating. They suggest the replacement age that minimizes the expected maintenance cost per unit time over an infinite horizon.

Kamien & Schwartz [33] study the problem of optimal maintenance policy for a machine subject to failure. It is assumed that the natural probability of machine failure is increasing with age, whereas the value of the machine's output is independent of age. There is the option of selling the machine at any time, at a salvage value decreasing with time. The authors use optimal control theory, and distinguish between the machine's natural failure rate and actual failure rate resulting from the application of a preventive maintenance policy, including when to sell the machine.

Cl eroux *et al.* [18] treat the age replacement problem with minimal repair and random repair costs. They assume instant repair and replacement. At failure, the unit is replaced or repaired depending on the random cost of repair. Under this policy, at failure, the equipment is replaced if the repair cost is greater than  $\delta c_1$ , where  $c_1$  is the replacement cost and  $\delta$  is a predetermined fraction (parameter). They provide a method to determine the optimal replacement age that minimizes the long-run expected total maintenance cost per unit time.

Barlow & Hunter [5] define and investigate two preventive maintenance policies, one applicable to simple equipment and one applicable to large, complex systems. Policy I performs preventive maintenance after  $t_0$  hours of continuous operation. Policy II performs preventive maintenance on the system when its total number of working hours reaches  $t^*$ , even if operation has been interrupted because of failure. Failures are assumed to be repaired minimally. The system is assumed to restore its as good as new state after a preventive maintenance. The optimum policy is the policy which maximizes the fraction of up-time, over long time intervals. The policies are compared under various restrictions on  $T_e$ ,  $T_s$ , and  $T_m$ , defined as the expected time

to perform emergency maintenance, the expected time to perform scheduled maintenance, and the expected time to perform minimal repair, respectively.

A maintenance policy accounting for the accumulated damage is studied by Zuckerman [62]. It is assumed that upon failure, a system is replaced by a new identical one, and that the replacement cycles are repeated indefinitely. The system is subject to shocks occurring in a Poisson stream. The author establishes an optimal control limit policy, a policy in which the system is replaced either upon failure, or when the accumulated damage first exceeds a critical control level  $\xi^*$ . He provides the optimal replacement policy that maximizes the total long-run average net income per unit time, and the policy that maximizes the total long-run expected discounted net income.

Most maintenance and replacement models consider deterministic time horizon. Banerjee & Kabadi [4] study optimal replacement policies under random horizon. Upon failure of a functional part, the part is replaced immediately to keep the system in operation. There is the option to replace the part by two brands having different unit costs and life distributions. The authors determine a time-dependent replacement policy that minimizes the expected operational cost. They establish conditions that secure the existence of the optimal policy.

Technological advances and learning effects have also been considered in modeling some maintenance and replacement problems. Goldstein *et al.* [26] analyze a machine replacement model with an expected technological breakthrough. They assume a stationary environment of the time at which a new technology may appear, and devise a dynamic programming method to determine the optimal replacement age of an operating machine that minimizes the expected cost in the long run. Chand *et al.* [16] investigate the single machine replacement problem with learning. They assume that there is a setup cost to replace a machine, and that due to learning this cost is a nonincreasing function of the number of setups made so far. They develop a forward dynamic programming algorithm to determine how frequently replacements should be done over a finite horizon. They also obtain optimal results for what is

effectively infinite-horizon problem, while only using data over a finite period of time. Nakagawa [43] study the effect of imperfect preventive maintenance. After repair, the unit is as good as new (i.e. repair is perfect). Preventive maintenance (PM) is imperfect in the sense that with probability  $p$  after PM the unit is the same as just before PM, and with probability  $\bar{p} = 1 - p$  it is as good as new. Repair and PM are assumed to take negligible time. The author studies three models: Model I the unit is repaired upon failure; Model II the unit undergoes minimal repair at failure; and Model III the failed unit is detected only by perfect PM. Expected cost rate is derived for each model, and is minimized and the optimal policy is determined.

After reviewing some well known maintenance policies, let us now focus on multi-state maintenance systems, with many units. Lam & Yeh [39] study optimal replacement policies of multistate deteriorating systems. They classify the deterioration of a system into a finite number of states ( $n + 2$ ), and characterize it by a semi-Markov process. Furthermore, the failure rate function of the sojourn time distribution in each state is assumed to be independent of time. It is assumed that the system can be replaced at any point of time. Cost parameters, replacement and sojourn time distributions are state dependent. The authors investigate optimal state-age-dependent replacement policies that minimize the expected long-run cost rate. They show that under some regularity conditions the optimal policies have monotonic properties.

Kander [34] investigates inspection policies for deteriorating equipment. He deals with systems in which  $N + 1$  quality levels can be diagnosed, characterizing them by a semi-markov process. He assumes no aging during the sojourn in any state. He develops optimal inspection policies yielding minimal loss. The optimal policy is a sequence of checking (inspection) times minimizing the loss per life cycle. He also deals with models of pure checking, truncated checking, and checking followed by monitoring.

Sharma *et al.* [51] study the transient behavior of multiple-unit reliability systems, characterizing a system by a Markovian process, representing the number of operable units at any point in time. They model the problem as a

birth and death process. The repair time and lifetime distributions of a unit are assumed to be exponential. For the system to be operable, there must be at least  $k$  ( $k \leq n$ ) operable units. The authors derive the reliability and the availability functions of the system for the transient state.

Wilken & Langford [59] study  $n$  identical systems assumed to operate independently, each characterized by an exponential time to failure with mean  $1/\gamma$ . Upon failure of a system, it is immediately replaced by a spare (if there are any.) The authors start by writing the probability of exactly  $i$  failures in a time period of length  $t$  in the case where no spares are provisioned. Then, they establish a recurrence relation between this probability and the conditional probability of exactly  $i$  failures in time  $t$ , given that the first failure of any system is at time  $t'$ . Finally, by induction, they derive the expression for the probability of exactly  $i$  failures in a time period of length  $t$ , for the general case when  $s$  spares are provisioned.

Van Der Duyn Schouten & Vanneste [48] propose two control policies for a multicomponent system, consisting of  $M$  identical machines in series, with the objective of minimizing the long-run average cost of the system. Each component has four possible states: good (0), doubtful (1), bad (2), and down (3). The sojourn times in states 0 and 1 are taken to be exponentially distributed. The two policies are Policy A: a complete replacement is carried out if and only if a single component enters state 2 or 3 and the number of components in state 1 is greater than or equal to  $K$ . Policy B: a complete system replacement is carried out at the first time epoch at which a component enters state 2 or 3 after the first moment at which the number of doubtful components has reached the level  $K$ . Average cost analysis of these policies is done, and the four states are identified with age intervals. Furthermore, the authors derive approximations for some performance measures, such as time to system replacement, and validate their results with simulation. Goel *al.* [22] present a cost-benefit analysis of a complex system comprised of two subsystems A and B connected in series. Subsystem B has one unit whereas subsystem A has two identical units. Failure and repair times for each unit are exponentially distributed. Switchover is instantaneous. The system fails

either because of a failure of subsystem A or subsystem B, or because of a failure of subsystem B and one of the two units of subsystem A. The authors use the regenerative point technique, and derive some performance measures: mean time to system failure; availability measures; repairman busy period fraction. The authors perform profit analysis, and conclude that a high positive correlation between failure and repair times leads to a high mean time to system failure.

Drinkwater & Hastings [20] study an economic replacement model of vehicles. They propose a theory of repair limits. They suggest two methods to determine the optimum repair limits, and show that these lead to financial savings. Doğrusöz and Karabakal [19] investigate a replacement problem of garbage collection trucks, with increasing operating costs and decreasing availability. Use the discounted value of all cash flows, they develop a measure, the average cost per unit quantity of service provided per truck load garbage collected throughout the service life of the equipment. They determine the optimal replacement age.

Baxter [8] derives some reliability measures for a two-state system. An alternating renewal process is used to model the operation and repair periods. The author defines an indicator variable  $I_k(t)$  which is equal to 1 if the system is operating at time  $t$ , and 0 otherwise, where  $k = 0$  if the system is initially down, and 1 otherwise. The concept of interval of a two-state system is defined as  $R_k(x, t) = p\{I_k(u) = 1 \forall u \in [t, t + x]\}$  for  $k = 0, 1$ . The author introduces a new density function representing the density of the asymptotic recurrence time of a renewal process whose renewal is according to the distribution  $F$ , as  $\psi(\cdot) = \bar{F}(\cdot)/\mu_1$ .  $\mu_1$  and  $\mu_2$  designate the expectations of  $F$  and  $G$  which are the distribution functions of the failure times and the repair times, respectively. Then the author establishes that the limiting interval reliability  $R(x) = \frac{\mu_1}{\mu_1 + \mu_2}[1 - \Psi(x)]$ , where  $\Psi$  is the distribution function with density  $\psi$ . The author also derives expressions for the joint availability and for the forward recurrence times to break down at time  $t$  given that  $I_k(t) = 1, k = 0, 1$ .

We now focus on redundant cold standby systems. Khalil [37] derives an

asymptotic distribution for the time to system failure, for the 1-out-of-  $(n + 1)$  redundant cold standby system. There is one repair facility which starts repair only when the queue length becomes  $k$ . The repair time distribution is exponential, whereas the failure time distribution of the active unit is arbitrary. The author notes that in reality the time between failures of the active unit is considerably larger than the repair time. Using this observation, he shows that asymptotic distribution of the time to system failure is exponential. He takes the Laplace-Stieltjes transform of the system's time to failure distribution, and shows that it converges to  $(1 + s)^{-1}$ , where  $s$  is the number of spare units.

Shao & Lamberson [50] examine an  $n$ -unit cold standby system with built-in test equipment (BIT), which can detect system failure, distinguish and report failure mode and occurrence time, and record all the information to a data base. If the BIT cannot detect a failed active unit, a system failure occurs. BIT itself has a failure rate. Switching is instantaneous. Time to failure of a unit is exponentially distributed. Units are non identical. It is assumed that the switch is under the command of the BIT, and that the successful switching probability is  $k$ . By properly defining the states of the system, and treating it as a discrete-state, continuous-time homogeneous Markov process, the author establishes a set of first-order linear differential equations. These are solved by taking Laplace Transforms. The system reliability is obtained. A numerical example is considered for three identical units.

The two-unit cold standby systems have a particular importance. Osaki & Asakura [44] study a two-unit standby redundant system with repair and preventive maintenance. Using Laplace-Stieltjes transform (L.S.T.), the authors provide the distribution of time to the first system down, and derive the mean time. They assume arbitrary time distributions for failure, repair, inspection and preventive maintenance. Switchover times are instantaneous. After repair, the unit can operate perfectly. At age  $t$ , the operating unit is inspected. To avoid system downtime, an inspection is not performed when the other unit is not in standby. Failure time distribution of a functioning unit has IFR. They define four states, which are time instance of the system. Then they establish the distributions and their L.S.T. of the time between

these states. They provide an expression for the mean time to system down. In addition, they prove that repair is effective, under the condition that the failure rate of the failure time distribution is strictly increasing.

Gopalan & D'Souza [27] investigate the two-unit system with a cold standby, with a single repair facility that performs repair and preventive maintenance, the lengths of which (i.e. service times) are governed by exponential distributions. Initially, one unit is in standby and one unit starts functioning. The times at which an operating unit is sent for preventive maintenance or repair are assumed to be exponential distributions. The authors derive the Laplace Transforms for the mean downtime of the system, and for the mean time to system failure, and obtain the availability and the reliability functions of the system.

Van Der Duyn Schouten & Wartenhorst [49] study a two-unit cold standby system with one repairman. After inspecting a failed unit, the repairman has the option to choose either a slow or a fast repair rate. The amount of work is known before repair is started. Additionally, the repairman can switch to the fast repair rate when the system breaks down. However, switching back to slow rate during a fast repair is not allowed. Based on the breakdown cost, the authors investigate long-run average costs. Using the theory of optimal control, and semi-Markov decision processes, they present some useful performance measures, including long-run average costs, all moments of system up- and downtimes, and system availability.

Wang [58] analyzes the steady state behavior of a cold standby system with  $R$  repairmen. The repair follows the FIFO policy. At most  $M$  machines can be operating simultaneously, while  $S$  machines are cold standby spares. Two failure modes are allowed, and they have equal probability of repair. Failure times and repair times are exponentially distributed. The failure rates depend on the failure mode: that is, there are two failure rates which are exponentially distributed. The author develops a profit model to determine the optimal number of standbys and the optimal number of repairmen simultaneously, constrained by system availability. He uses a direct search method, which

terminates after obtaining a global optimum value  $(S^*, R^*)$ . The author uses birth and death balance equations, and describes the system by the number of failed machines in mode 1 and mode 2. A numerical illustration is provided. The profit function is unimodal, resulting in a unique optimal global solution. The author conjectures that the profit function is generally so, though not necessarily convex.

Gupta & Chaudhary [30] study a two-unit standby system with Rayleigh downtime and Gamma failure time distributions. The system consists of two units: one priority and the other ordinary. The priority unit has three distinct failure modes: normal, partial failure, and total failure. The ordinary unit has 2 modes: normal and total failure. Allowed downtime is  $T_d$ . System failure occurs if the repair of the priority unit is completed during the downtime  $T_d$ , which is assumed to have a Rayleigh distribution. The author identifies 6 states, depending on the state of each unit, and on whether in standby or repair. Using the technique of regenerative points, the author derives system characteristics, such as system reliability, transition probabilities, sojourn times, mean time to system failure, and the distribution of time to system recovery and its mean. Laplace and Laplace-Stieltjes transforms are used.

Agnihotri *et al.* [2] examine a two-unit cold standby system with 2 types of failure. The operating unit fails due to machinery defects as well as due to random shocks. The two units are non identical. Repair and unit life distributions are general. Failure rate of an operative unit after it undergoes the first shock is assumed to increase. The authors identify 8 up states and 2 down states. Using the regenerative point method, and Markov chains, the authors derive some characteristics of concern: transient and steady state transition probabilities; mean sojourn times in regenerative states; distribution of time to system failure; pointwise availability and steady state; availability of the system; expected busy period of the repairman and the expected number of visits he makes in the time interval  $(0, t]$ .

Gupta *et al.* [32] study a two-unit cold standby system, subject to random shocks and degradation. There is a single repairman. The two units are

one priority and the other ordinary. The priority unit gets preference over the ordinary unit for both operation and repair. The priority unit has three operative stages: excellent, good and satisfactory. Shocks occur randomly, and may only affect the priority unit. Only a unit in a failure mode is repaired. Shock time and repair time are exponentially distributed. After repair, a unit is as good as new. Using the regenerative point technique, the authors perform profit analysis, and derive measures of system effectiveness, such as: reliability of the system, and mean time to system failure; the expected up time due to each unit and the expected number of repairs in the time interval  $(0, t]$ .

Agnihotri *et al.* [3] investigate a two-unit warm standby system supported by a single repair facility. The units are identical. There are two types of failure: failure due to the machine effect, and failure due to critical human error. These have fixed probabilities. The warm standby unit can only fail due to machine defect. A failed unit is sent for fault detection to see the cause of the failure. After defining the states of the system, the authors use the regenerative point technique in the Markov process, and derive the transient probability matrix. Using Laplace Transforms, they determine some measures of system effectiveness, such as mean time to system failure, availability measures, expected number of visits by the repairman.

Goel & Shrivastava [24] investigate a two-unit cold standby system, with imperfect switch. One unit is priority and one is ordinary. The switch may be available with probability  $\theta$ , and it is subject to failure. It is repaired before taking up a failed repair for repair. The failure and repair times for both units follow exponential distributions. Preventive maintenance is provided for the priority units at random epochs. The effect of correlated failure and repair times on system performance is studied. Profit analysis is performed, and various system effectiveness measures are derived: mean time to system failure; point availability and expected up-time; busy period of the repairman; expected frequency of preventive maintenance. The authors employ the regenerative point technique. They conclude that the preventive maintenance should be allowed as long as its cost is small compared to the cost of actual repair; otherwise it may be uneconomical. They also conclude that, other parameters

held constant, a high positive correlation between failure and repair times is desirable to increase profits.

Gupta & Chaudhary [31] examine a two-unit cold standby system, supported by a repair facility. The units are identical. Each unit has 3 modes: normal, partial failure, and total failure. A unit cannot fail totally directly without passing through a partial failure. The time to failure is exponentially distributed. The time to failure of a partially failed unit is exponentially distributed while the time to repair a totally failed units has a gamma distribution. After repair, the unit is as good as new. Using the regenerative point technique and Laplace Transforms, the authors perform profit analysis, and derive some useful measures: the reliability and the mean time to system failure; expected up-time of the system; and expected busy period of the repairman.

Goel & Shrivastava [23] examine a two-unit cold standby system with two identical units. Each unit has three modes: normal, partial failure, and total failure. A unit cannot fail from normal to complete failure without passing through partial failure. Failure and repair times follow exponential distributions. The authors use the regenerative point technique, and study the effect of correlation between failure and repair times on the system effectiveness measures, such as the mean time to system failure and availability. They show that the increased correlation between failure and repair times tend to increase the mean time to system failure and the steady state availability.

Bhat & Gururajan [13] investigate a two-unit cold standby redundant system with imperfect repair, and one repair facility. Switchover is instantaneous. The two units are dissimilar. Each unit is assumed to have a random excessive available period. The repaired unit will not behave like a new one. After repair of a unit, the failure time distribution of the unit is different and its expected life decreases after each failure. A unit can be repaired at most  $k$  times beyond which it cannot be repaired any more. The regenerative point technique and Laplace transforms are used, and availability and reliability analysis is done.

Goel & Tyagi [25] study a two-unit cold standby redundant system with allowed downtime  $T_d$ , i.e., time beyond which the system enters into the failed state, where one unit is under repair and the other is waiting for repair. Each unit has three modes: normal, partial failure, and total failure. A unit passes from normal to total failure through partial failure. Switching is perfect and instantaneous. The single repair facility repair a partially or totally failed unit. A unit in the partial failure mode can be repaired while operating. A partial failure supersedingly gets repair when a totally failed unit is under repair. Repair times from partial failure to normal mode and from total failure to normal mode are exponentially distributed. After repair, the unit is as good as new, and is taken to operation. It is assumed that repair and failure times are correlated, and that their joint distribution is bivariate exponential. The authors employ the regenerative point technique, and perform reliability and profit analysis, and derive the distribution of time to system failure and its mean; availability of the system; mean up time and down time, expected busy period and idle period of the repairman and expected profit during the time interval  $(0, t]$ . A particular case when no downtime is allowed ( $T_d = 0$ ) is studied. It is observed that more positive correlations lead to higher mean time to system failure.

Singh *et al.* [53] examine a two-unit cold standby redundant system. The units are identical, and have three modes: normal, partial failure, and total failure. Failure time is exponentially distributed. Repair time is arbitrary. After repair, the unit is as good as new. When an operating unit fails partially, its operation is stopped and the standby unit goes for operation. The method of regenerative point is employed, and system effectiveness measure are derived: distribution of time to system failure and its mean; pointwise and steady state availability of the system. A numerical example is provided for illustration.

Mohamoud & Mohie El-Din [42] study a two-unit cold standby system. Each unit has two modes: normal and total failure. The switch is imperfect. The probability that the switch is good when used is equal to  $p$  and the probability that it is bad is equal to  $q = 1 - p$ . The repair consists of two stages. A unit on repair goes to the second phase of repair, called post repair. At age

$t$  an operating unit undergoes preventive maintenance provided that there is a unit in standby. All distributions are arbitrary. Using the regenerative point technique, the authors perform profit analysis, and derive system performance measures: distribution of time to system failure and its mean; availability of the system; busy period; expected number of visits by the repairman and the expected profit in  $(0, t]$  and steady state.

Kapur & Kapoor [35] study a two-unit cold standby redundant system. They study 4 general models, each consisting of two identical units, and perfect switchovers. Model I is an intermittently available redundant system; Model II is an intermittently used system; Model III is a redundant system with repair and post repair; Model IV is a repair limit suspension model. The authors use the Markov renewal process technique, and derive system effectiveness measures, including the distribution to system failure, availability and expected number of system failures in  $(0, t]$ . Kapur & Kapoor [36] investigate the stochastic behavior of a two-unit standby system. Three models are studied. Model I is a warm standby redundant system with preventive maintenance; Model II is a cold standby redundant system with preventive maintenance; Model III is a warm standby redundant system with delay (set up or waiting for server.) For models I and II, the server is available intermittently. The authors use the Markov renewal processes and Laplace-Stieltjes transforms to derive such performance measures as the distribution of time to system failure and its mean; pointwise unavailability and the expected number of failures in  $(0, t]$ .

Yeh [61] studies a maintenance model for two-unit redundant standby system with one repairman. After repair unit 2 is as good as new, while unit 1 is not. He establishes differential equations and their Laplace transforms, and derives a formula for the rate of occurrence of failures as a function of time  $t$ . He suggests solving these equations numerically. Gopalan & Simhan [28] consider a two-unit cold standby redundant system, propose and analyze 4 monitoring policies: continuous monitoring of the switch and the units; continuous monitoring of the switch; continuous monitoring of the units; no monitoring at all. Continuous monitoring enables to detect failure instantaneously. All failure

time distributions are exponential. Repair time distributions are arbitrary. For each of the monitoring policy, the authors derive net gains (revenues - costs), and conclude that suitable levels of monitoring depend on the monitoring cost as well as on the quantitative effect of monitoring on the system performance. Srinivasan [55] investigates the effect of cold standby redundancy on two models. In one of the models, demand pattern of the system is ignored; in the other model, intermittent demand of the system is considered. The author uses Laplace transforms, and derives the distribution of the system's failure for each model, as well as the expected time to system failure. He also measures the gain resulting from introducing standby redundancy.

Our study is different from the above studies, since it is the first study that applies the marginal cost approach to the two-unit cold standby system, which is a relatively complex system. Furthermore, the general maintenance policy we study is different from the ones studied before. Our model resembles the one studied by Osaki & Asakura [44]. However, their objective function and their solution method are different from ours. In addition, the maintenance policies we deal with are quite realistic and practical.

# Chapter 3

## THE MODEL AND POLICIES

### 3.1 THE MODEL DESCRIPTION

We consider a two-unit redundant cold standby system, with a single repair facility, having exponential service time with rate  $\mu$ . The lifetime distribution is general. The switchover is immediate. Our aim is to find the optimal replacement age  $T^*$ , the time interval that minimizes the total costs incurred per unit time, in the long run.

Figure 3.1 depicts the system at hand. Upon failure of the working unit at age  $x$  in the time interval  $[0, T)$ , the unit is replaced by the standby unit if available, at a cost  $C_f(x)$ . Unit repair cost is  $C_r$ . Preventive maintenance cost of a unit of age  $x$  is  $C_p(x)$ . Denote the service cdf and pdf by  $H(\cdot)$  and  $h(\cdot)$ , respectively, which is exponentially distributed with parameter  $\mu$ .

Policy 1: At age  $T$ , replace the operating unit, even if the other unit is in repair.

Policy 2: At age  $T$ , replace the operating unit by the other unit if available; if not available, put off replacement for a while until the other unit returns from repair.

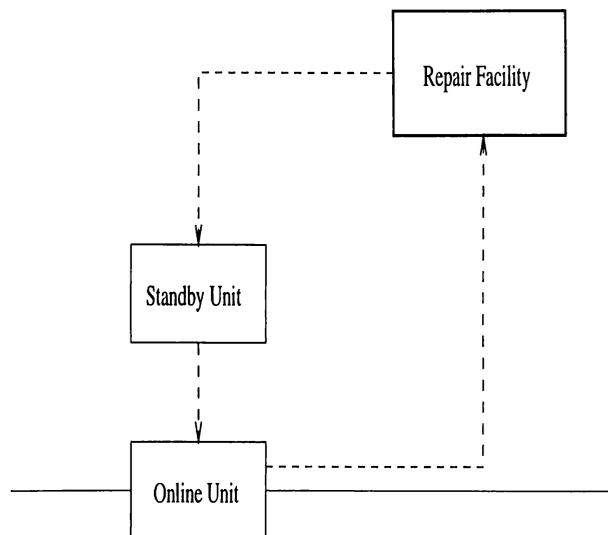


Figure 3.1: Sketch of the System

Let  $\{Z(t), t \geq 0\}$  and  $\{R(t), t \geq 0\}$  denote, respectively, the number of units available and the number of units in the repair facility, at time  $t$ . These are stochastic processes, with state spaces  $S_Z = \{0, 1, 2\}$  and  $S_R = \{0, 1, 2\}$ . At any time  $t \geq 0$ ,  $Z(t) + R(t) = 2$ . This is trivial, since upon failure, the unit is carried to the repair facility in a negligible amount of time; the same is true upon repair completion.

A typical system behavior is illustrated in Figure 3.2.

## 3.2 THE MARGINAL COST APPROACH AND ITS MOTIVATION

The Marginal Cost Analysis (MCA) of maintenance policies is a concept gaining more interest in the recent years. This approach has been categorized as an Economics Oriented approach, as different from the classical probability centered approach. The MCA has been successfully applied to the Age Replacement and the Block Replacement policies, and was shown to be flexible enough to permit extensions and generalizations. The MCA concept was introduced by Berg [10] for the solution of maintenance problems. Denote

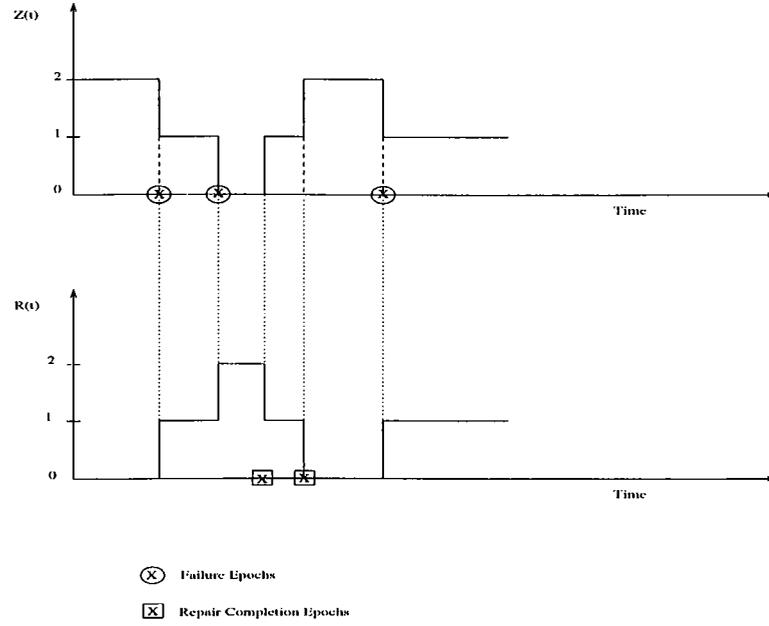


Figure 3.2: A Typical Realization of the Processes  $Z(t)$  and  $R(t)$

by  $C(T)$  the long-run expected cost per unit of time incurred when preventive maintenance is performed at time intervals of length  $T$ . Let  $V_1(x)$  be the costs incurred if the preventive maintenance is performed at age  $x$ . Let  $V_2(x, \Delta)$  stand for the expected costs incurred in  $(x, x + \Delta]$  if the preventive maintenance is deferred to age  $x + \Delta$  (for an infinitesimal  $\Delta$ .)

**Definition 10** *The Marginal Cost Function (MCF),  $\eta(x)$ , is the cost incurred if the preventive maintenance action is put off by an infinitesimal amount of time  $\Delta$ . That is,*

$$\eta(x) = \lim_{\Delta \rightarrow 0} \left( \frac{V_2(x, \Delta) - V_1(x)}{\Delta} \right) \quad (1)$$

In order to find  $C(T)$ , a renewal process has to be appropriately chosen. Let the renewal interval have a cdf and a pdf denoted by  $G(\cdot)$  and  $g(\cdot)$ , respectively. Furthermore, let  $D(T)$  and  $U(T)$  stand for the expected cost during a renewal interval (cycle) and the expected cycle length, respectively. Using the Renewal-Reward Theorem (see e.g. Ross [47]), the long-run expected cost per unit of time,  $C(T)$ , is obtained by dividing the expected cost during a cycle,  $D(T)$ , by the expected cycle length,  $U(T)$ ,

$$C(T) = \frac{D(T)}{U(T)} \quad (2)$$

The aim is to determine  $T^*$  that minimizes  $C(T)$ , that is, the problem is

$$\min_T C(T).$$

There is a theorem from Mathematical Economics, Theorem 1, that makes the MCF a useful and a powerful tool for optimization. Here, we give the theorem, without proof.

**Theorem 1 (Berg [10], [11])** *The optimal value  $T^*$  in the problem  $\min_T C(T)$  is a solution to the equation*

$$C(T) = \eta(T) \quad (3)$$

As shown in Figure 3.3, the  $C(T)$  curve is flat in the neighborhood of  $T^*$ , whereas the  $\eta(T)$  curve is not. This improves the efficiency of algorithms to solve for  $T^*$ , if the MCA is used. In addition to that, for the age replacement policy, the MCA has a particular usefulness. Extensions and generalizations are handled by slightly modifying the MCF. The interested reader is referred to Berg [11].

### 3.3 POLICY 1

Under this policy, at age  $T$ , the operating unit is replaced, even if the other unit is in repair. Hence the system is down. In the following derivations, failure cost and preventive maintenance cost are assumed to be independent of time, and are denoted by  $c_f$  and  $c_p$ , respectively. In the marginal cost derivations, however, we allow them to depend on age.

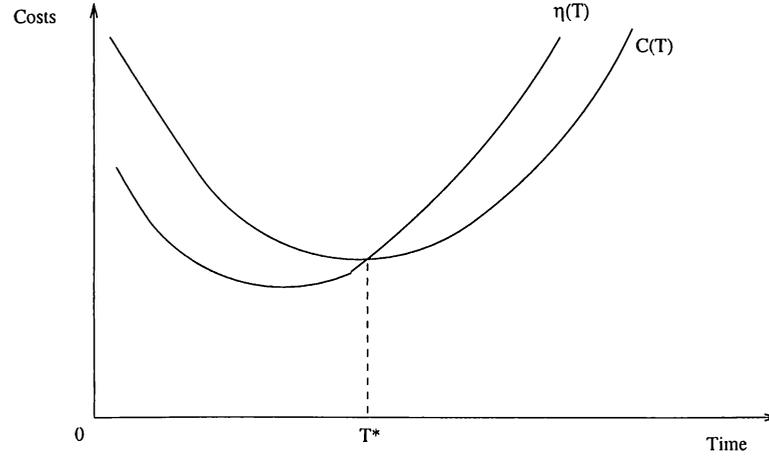


Figure 3.3: Total costs  $C(T)$ , and marginal cost,  $\eta(T)$ , functions versus time  $T$ .

### 3.3.1 Cycle Length

Figure 3.4 depicts the cycle length for this maintenance policy.

We have the following:

$$T_1 = \begin{cases} X & \text{if } \{X < \min\{T, Y\}\} \\ T & \text{if } \{T < X < Y\} \text{ or } \{T < Y < X\} \\ T + T_2 & \text{if } \{Y < T < X\} \\ X + T_2 & \text{if } \{Y < X < T\} \end{cases}$$

We write the corresponding probabilities to the above events,

$$T_1 = \begin{cases} X & \text{wp } \int_0^T e^{-\mu x} dF(x) \\ T & \text{wp } \int_T^\infty e^{-\mu x} dF(x) + \int_T^\infty (e^{-\mu T} - e^{-\mu x}) dF(x) \\ T + T_2 & \text{wp } (1 - e^{-\mu T}) \bar{F}(T) \\ X + T_2 & \text{wp } \int_0^T (1 - e^{-\mu x}) dF(x) \end{cases}$$

Let  $I(A)$  denote the indicator function of the event  $A$ .

$$\begin{aligned} T_1 &= XI(X < T < Y) + XI(X < Y < T) + XI(Y < X < T) \\ &+ Te^{-\mu T} \bar{F}(T) \\ &+ T_2I(Y < T < X) + T_2I(Y < X < T) \end{aligned}$$

Upon taking the expectation, we obtain

$$\begin{aligned}
E[T_1] &= \int_0^T xf(x)dx\bar{H}(T) + \int_{x=0}^T \int_{y=x}^T xf(x)h(y)dydx + \int_{x=0}^T \int_{y=0}^x xf(x)h(y)dydx \\
&\quad + T\bar{F}(T) + E[T_1] \left( e^{-\mu T}\bar{F}(T) + \int_0^T e^{-\mu x}dF(x) \right) \\
E[T_1] &= \int_0^T xf(x)dx\bar{H}(T) + \int_0^T [H(T) - H(x)]xf(x)dx + \int_0^T xf(x)H(x)dx \\
&\quad + T\bar{F}(T) + E[T_1] \left( e^{-\mu T}\bar{F}(T) + \int_0^T e^{-\mu x}dF(x) \right) \\
E[T_1] &= \int_0^T xf(x)dx[\bar{H}(T) + H(T)] - \int_0^T xf(x)(1 - e^{-\mu x})dx + \int_0^T xf(x)(1 - e^{-\mu x})dx \\
&\quad + T\bar{F}(T) + E[T_1] \left( e^{-\mu T}\bar{F}(T) + \int_0^T e^{-\mu x}dF(x) \right)
\end{aligned}$$

After doing some algebra, we obtain

$$E[T_1] = \frac{\int_0^T xdF(x) + T\bar{F}(T)}{e^{-\mu T}\bar{F}(T) + \int_0^T e^{-\mu x}dF(x)} \quad (4)$$

Finally, the expected value of the cycle length is

$$E[CL] = E[Y] + E[T_1]. \quad (5)$$

### 3.3.2 Cycle Cost

In a similar way, we derive  $E(CC)$ , the expected cost incurred during a cycle.

Let  $C_i$  denote the cost incurred during  $T_i, i = 1, 2$ .

$$C_1 = \begin{cases} c_f & \text{wp } \int_0^T e^{-\mu x}dF(x) \\ c_p & \text{wp } \int_T^\infty e^{-\mu x}dF(x) + \int_T^\infty (e^{-\mu T} - e^{-\mu x})dF(x) \\ c_r + c_p + C_2 & \text{wp } (1 - e^{-\mu T})\bar{F}(T) \\ c_f + c_r + C_2 & \text{wp } \int_0^T (1 - e^{-\mu x})dF(x) \end{cases}$$

Upon taking the expectation, we obtain

$$\begin{aligned}
E(C_1) &= c_f \int_0^T e^{-\mu x} dF(x) + c_p e^{-\mu T} \bar{F}(T) + c_p (1 - e^{-\mu T}) \bar{F}(T) \\
&+ E(C_1) (1 - e^{-\mu T}) \bar{F}(T) + c_f \int_0^T (1 - e^{-\mu x}) dF(x) \\
&+ c_r (1 - e^{-\mu T}) \bar{F}(T) + c_r \int_0^T (1 - e^{-\mu x}) dF(x) \\
&+ E(C_1) \int_0^T (1 - e^{-\mu x}) dF(x)
\end{aligned}$$

$$E(C_1) = \frac{c_f F(T) + c_p \bar{F}(T) + c_r (1 - e^{-\mu T}) \bar{F}(T) + c_r \int_0^T (1 - e^{-\mu x}) dF(x)}{e^{-\mu T} \bar{F}(T) + \int_0^T e^{-\mu x} dF(x)} \quad (6)$$

Finally, the expected value of the cycle cost is

$$E[CC] = c_d + c_r + E[C_1]. \quad (7)$$

Our objective function is the expected total cost per unit time, in the long run, which is, using the Renewal-Reward Theorem,

$$C(T) = \frac{E(CC)}{E(CL)} \quad (8)$$

## 3.4 POLICY 2

Recall that under this maintenance policy, at age  $T$ , the operating unit is replaced by the standby one if available; if not available, replacement is put off for a while until the other unit returns from repair. This policy, in contrast with Policy 1, tries to avoid system failure.

### 3.4.1 Cycle Length

Figure 3.5 depicts the cycle length for this maintenance policy.

We have the following:

$$T_1 = \begin{cases} X & \text{if } \{X < \min\{T, Y\}\} \text{ or } \{T < X < Y\} \\ Y + T_2 & \text{if } \{T < Y < X\} \\ T + T_2 & \text{if } \{Y < T < X\} \\ X + T_2 & \text{if } \{Y < X < T\} \end{cases}$$

We write the corresponding probabilities to the above events,

$$T_1 = \begin{cases} X & \text{wp } \int_0^T e^{-\mu x} dF(x) + \int_T^\infty e^{-\mu x} dF(x) \\ Y + T_2 & \text{wp } \int_T^\infty (e^{-\mu T} - e^{-\mu x}) dF(x) \\ T + T_2 & \text{wp } (1 - e^{-\mu T}) \bar{F}(T) \\ X + T_2 & \text{wp } \int_0^T (1 - e^{-\mu x}) dF(x) \end{cases}$$

Let  $I(A)$  denote the indicator function of the event  $A$ . Upon taking the expectation, we obtain

$$\begin{aligned} T_1 &= XI(X < \min\{T, Y\}) + XI(T < X < Y) + XI(Y < X < T) \\ &+ YI(T < Y < X) + TI(Y < TX) \\ &+ T_2I(T < Y < X) + T_2I(Y < T < X) + T_2I(Y < X < T) \end{aligned}$$

$$\begin{aligned} E[T_1] &= \int_0^T xe^{-\mu T} dF(x) + \int_0^T x(e^{-\mu x} - e^{-\mu T}) dF(x) + \int_{x=0}^T \int_{y=0}^x xf(x)h(y)dydx \\ &+ \int_T^\infty xe^{-\mu x} f(x)dx + \int_{x=y}^\infty \int_{y=T}^\infty yh(y)f(x)dx dy \\ &+ T(1 - e^{-\mu T}) \bar{F}(T) \\ &+ E[T_1] \left( \int_T^\infty (e^{-\mu T} - e^{-\mu x}) dF(x) + (1 - e^{-\mu T}) \bar{F}(T) + \int_0^T (1 - e^{-\mu x}) dF(x) \right) \\ E[T_1] &= \int_0^T xe^{-\mu x} dF(x) + \int_0^T xf(x)H(x)dx + \int_T^\infty xe^{-\mu x} f(x)dx + \int_T^\infty x\bar{F}(x)h(x)dx \\ &+ T(1 - e^{-\mu T}) \bar{F}(T) \\ &+ E[T_1] \left( \int_T^\infty (e^{-\mu T} - e^{-\mu x}) dF(x) + (1 - e^{-\mu T}) \bar{F}(T) + \int_0^T (1 - e^{-\mu x}) dF(x) \right) \\ E[T_1] &= \int_0^T x dF(x) + \int_T^\infty xe^{-\mu x} f(x)dx + \int_T^\infty x\mu e^{-\mu x} \bar{F}(x)dx + T(1 - e^{-\mu T}) \bar{F}(T) \end{aligned}$$

$$+ E[T_1] \left( \int_T^\infty (e^{-\mu T} - e^{-\mu x}) dF(x) + (1 - e^{-\mu T}) \bar{F}(T) + \int_0^T (1 - e^{-\mu x}) dF(x) \right)$$

We obtain the following result

$$E[T_1] = \frac{\int_0^T x dF(x) + \int_T^\infty x e^{-\mu x} f(x) dx + \int_T^\infty x \mu e^{-\mu x} \bar{F}(x) dx + T(1 - e^{-\mu T}) \bar{F}(T)}{\int_0^\infty e^{-\mu x} dF(x)} \quad (9)$$

Finally, the expected value of the cycle length is

$$E[CL] = E[Y] + E[T_1]. \quad (10)$$

### 3.4.2 Cycle Cost

In a similar way, we derive  $E(CC)$ , the expected cost incurred during a cycle.

Let  $C_i$  denote the cost incurred during  $T_i, i = 1, 2, \dots$

$$C_1 = \begin{cases} c_f & \text{if } \{X < \min\{T, Y\}\} \\ c_f + c_p N_X(T) & \text{if } \{T < X < Y\} \\ c_r + c_p N_Y(T) + C_2 & \text{if } \{T < Y < X\} \\ c_p + c_r + C_2 & \text{if } \{Y < T < X\} \\ c_f + c_r + C_2 & \text{if } \{Y < X < T\} \end{cases}$$

$$C_1 = \begin{cases} c_f & \text{wp } \int_0^T e^{-\mu x} dF(x) \\ c_f + c_p N_X(T) & \text{wp } \int_T^\infty e^{-\mu x} dF(x) \\ c_r + c_p N_Y(T) + C_2 & \text{wp } \int_T^\infty (e^{-\mu T} - e^{-\mu x}) dF(x) \\ c_p + c_r + C_2 & \text{wp } (1 - e^{-\mu T}) \bar{F}(T) \\ c_f + c_r + C_2 & \text{wp } \int_0^T (1 - e^{-\mu x}) dF(x) \end{cases}$$

$$\begin{aligned} C_1 &= c_f I(X < \min\{T, Y\}) \\ &+ (c_f + c_p N_X(T)) I(T < X < Y) \end{aligned}$$

$$\begin{aligned}
& + (c_r + c_p N_Y(T) + C_2)I(T < Y < X) \\
& + (c_p + c_r + C_2)I(Y < T < X) \\
& + (c_f + c_r + C_2)I(Y < X < T)
\end{aligned}$$

Upon taking the expectation, we obtain

$$\begin{aligned}
E(C_1) & = c_f \left( \int_0^T e^{-\mu x} dF(x) + \int_T^\infty e^{-\mu x} dF(x) + \int_0^T (1 - e^{-\mu x}) dF(x) \right) \\
& + c_r \left( \int_T^\infty (e^{-\mu T} - e^{-\mu x}) dF(x) + (1 - e^{-\mu T}) \bar{F}(T) + \int_0^T (1 - e^{-\mu x}) dF(x) \right) \\
& + c_p (1 - e^{-\mu T}) \bar{F}(T) \\
& + c_p \left( 1.Pr\{T < X < 2T, X < Y\} + 2.Pr\{2T < X < 3T, X < Y\} + \dots \right) \\
& + c_p \left( 1.Pr\{T < Y < 2T, Y < X\} + 2.Pr\{2T < Y < 3T, Y < X\} + \dots \right) \\
& + E[C_1] \left( \int_T^\infty (e^{-\mu T} - e^{-\mu x}) dF(x) + (1 - e^{-\mu T}) \bar{F}(T) + \int_0^T (1 - e^{-\mu x}) dF(x) \right)
\end{aligned}$$

After simplifying and arranging terms, we obtain

$$E[C_1] = \frac{U}{\int_0^\infty e^{-\mu x} dF(x)}$$

$$\begin{aligned}
U & = c_f \left( F(T) + \int_T^\infty e^{-\mu x} dF(x) \right) + c_r \left( 1 - \int_0^\infty e^{-\mu x} dF(x) \right) \\
& + c_p (1 - e^{-\mu T}) \bar{F}(T) \\
& + c_p \left( \sum_{k=1}^\infty k.Pr\{kT < X < (k+1)T\} \right)
\end{aligned}$$

Finally, the expected value of the cycle cost is

$$E[CC] = c_d + c_r + E[C_1]. \quad (11)$$

$C(T)$ , our objective function is the expected long-run cost per unit time,

$$C(T) = \frac{E(CC)}{E(CL)} \quad (12)$$

### 3.5 POLICY COMPARISON

Let us compare the main policy (Policy 1) and the altered policy (Policy 2). We have two policies: Policy 1: take a machine off line immediately when it reaches the replacement age; Policy 2: wait a while if the repair facility is busy, until it becomes idle. Before any analysis, one is inclined to think that Policy 2 is superior to Policy 1, since it tries to avoid system failure, which is very costly. In order to be able to compare, we experimented with the two policies under the same experimental conditions, i.e., parameters ( $c_p=0.1$ ;  $c_f=3$ ;  $c_r=0.5$ ;  $c_d=5$ ;  $(\alpha, \beta) = (5, 2)$ ;  $\mu = 0.25$ ). These parameters are reasonably chosen, as described in Section 5.1. Policy 1 achieves a reduction of about %8.48 whereas Policy 2 achieves a reduction of about %8.37, which are comparable. Furthermore, if no preventive maintenance is performed (i.e.  $T \rightarrow \infty$ ), both policies incur 0.908, the same cost. The unit is monetary unit per unit time, in the long run. Actually, this result is intuitive, since there is no human intervention to the physical system. The numerical results confirm that policy 2 is superior to policy1, as shown in Figure 3.6.

### 3.6 MARGINAL COSTS

#### 3.6.1 Policy1

We now apply the Marginal Cost Approach to the two maintenance policies introduced in Section 3.1. Note that here we allow the failure cost and the preventive maintenance cost to depend on age  $x$ .

**Lemma 1** *The cost incurred if an inspection is carried out at age  $x$ , is given by*

$$V_1(x) = C_p(x) + \bar{H}(x)C_d. \quad (13)$$

*Proof:* The cost incurred includes replacement cost,  $C_p(x)$ . In addition, if the other unit is still in repair, downtime cost  $C_d$  is incurred. The probability of the latter event is  $\bar{H}(x)$ .

**Lemma 2** *If the inspection is deferred to age  $x + \Delta$ , then the expected cost incurred is given by*

$$\begin{aligned} V_2(x, \Delta) &= \bar{H}(x)r(x)\Delta \left( \bar{H}(\Delta)C_d + H(\Delta)C_r \right) + r(x)\Delta C_f(x) \\ &+ [1 - r(x)\Delta] \left( C_p(x + \Delta) + \bar{H}(x + \Delta)C_d \right) \end{aligned}$$

*Proof:* Below, we provide a sketch of the proof.

- a) In the next  $\Delta$  time units (for infinitesimal  $\Delta$ ), the probability that a failure occurs, and the standby unit is not available is equal to  $\bar{H}(x)[r(x)\Delta]$ .
  - The probability that there will be no repair completion in  $(x, x + \Delta]$  is equal to  $\bar{H}(\Delta)$ . A downtime cost is incurred,  $C_d$ .
  - The probability that there will be repair completion in  $(x, x + \Delta]$  is equal to  $H(\Delta)$ . Repair cost,  $C_r$ , is incurred.
- b) If a failure occurs in  $(x, x + \Delta]$ , a failure cost  $C_f(x)$  is incurred. This event has a probability equal to  $r(x)\Delta$ .
- c) The probability that the (operating) unit will not fail in  $(x, x + \Delta]$  is equal to  $[1 - r(x)\Delta]$ .
  - At the end of the interval  $(x, x + \Delta]$  replacement cost is equal to  $C_p(x + \Delta)$ .
  - In addition, the probability that the other unit is still in repair is equal to  $\bar{H}(x + \Delta)$ . A downtime cost,  $C_d$ , is incurred.

Now recalling Equation 1, we write

$$\eta_{\Theta}(x) = \lim_{\Delta \rightarrow 0} \left( \frac{V_2(x, \Delta) - V_1(x)}{\Delta} \right) \quad (14)$$

Using the results of Lemma 1 and Lemma 2, we derive the following,

**Theorem 2** *The MCF for Policy 1 is given by,*

$$\eta_{\Theta}(x) = r(x) \left( C_f(x) - C_p(x) \right) + C'_p(x) - h(x)C_d \quad (15)$$

where the operator ' stands for the derivative, and  $r(\cdot)$  denotes the failure rate function of the lifetime of a unit.

*Proof*

$$\begin{aligned} \eta(x) &= \lim_{\Delta \rightarrow 0} \left( \frac{V_2(x, \Delta) - V_1(x)}{\Delta} \right) \\ \eta(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[\bar{H}(\Delta)C_d + H(\Delta)C_r] + r(x)\Delta C_f(x) \right. \\ &\quad \left. + [1 - r(x)\Delta][C_p(x + \Delta) + \bar{H}(x + \Delta)C_d] - C_p(x) - \bar{H}(x)C_d \right) \\ \eta(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[C_d - H(\Delta)C_d + H(\Delta)C_r] + r(x)\Delta C_f(x) \right. \\ &\quad \left. + C_p(x + \Delta) + \bar{H}(x + \Delta)C_d - r(x)\Delta C_p(x + \Delta) - r(x)\Delta \bar{H}C_d - C_p(x) - \bar{H}(x)C_d \right) \\ \eta(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[C_d - H(\Delta)C_d + H(\Delta)C_r] + C_d - H(x + \Delta)C_d \right. \\ &\quad \left. - C_d + H(x)C_d - r(x)\Delta \bar{H}(x + \Delta)C_d \right) + r(x)C_f(x) + C'_p(x) - r(x)C_p(x), \\ \eta(x) &= \bar{H}(x)r(x)C_d + r(x)C_f(x) + C'_p(x) - r(x)C_p(x) - h(x)C_d - r(x)\bar{H}(x)C_d \\ \eta(x) &= r(x) \left( C_f(x) - C_p(x) \right) + C'_p(x) - h(x)C_d \end{aligned}$$

### 3.6.2 Policy2

After applying it to Policy 1, we now apply the Marginal Cost Approach to Policy 2, described in Section 3.1.

**Lemma 3** *The cost,  $V_1(x)$ , incurred if an inspection is carried out at age  $x$ , is given by*

$$V_1(x) = C_p(x) \quad (16)$$

*Proof:* The cost incurred includes inspection cost,  $C_p(x)$ . In addition, if the other unit is still in repair, no cost is incurred. The probability of the latter event is  $\bar{H}(x)$ .

**Lemma 4** *If the inspection is deferred to age  $x + \Delta$ , then the expected cost,  $V_2(x, \Delta)$ , incurred is given by*

$$\begin{aligned} V_2(x, \Delta) &= \bar{H}(x) r(x) \Delta \left( \bar{H}(\Delta) C_d + H(\Delta) C_r \right) + r(x) \Delta C_f(x) \\ &+ [1 - r(x) \Delta] C_p(x + \Delta) \end{aligned}$$

*Proof:*

Below, we provide a sketch of the proof.

- a) In the next  $\Delta$  time units (for infinitesimal  $\Delta$ ), the probability that a failure occurs, and the standby unit is not available is equal to  $\bar{H}(x)[r(x)\Delta]$ .
  - The probability that there will be no repair completion in  $(x, x + \Delta]$  is equal to  $\bar{H}(\Delta)$ . A downtime cost,  $C_d$ , is incurred.
  - The probability that there will be repair completion in  $(x, x + \Delta]$  is equal to  $H(\Delta)$ . Repair cost,  $C_r$ , is incurred.
- b) If a failure occurs in  $(x, x + \Delta]$ , failure cost  $C_f(x)$  is incurred. This occurs with probability  $r(x)\Delta$ .
- c) The probability that the (operating) unit will not fail in  $(x, x + \Delta]$  is equal to  $[1 - r(x)\Delta]$ .

- At the end of the interval  $(x, x + \Delta]$  replacement cost is equal to  $C_p(x + \Delta)$ .
- In addition, the probability that the other unit is still in repair is equal to  $\bar{H}(x + \Delta)$ . The unit will continue operation, so no cost (downtime cost) is incurred.

Using the results of Lemma 3 and Lemma 4, we derive the following,

**Theorem 3** *The MCF for Policy 2 is given by,*

$$\eta_{\Theta}(x) = r(x) \left( \bar{H}(x)C_d + C_f(x) - C_p(x) \right) + C'_p(x) \quad (17)$$

where the operator ' stands for the derivative, and  $r(\cdot)$  denotes the failure rate function of the lifetime of a unit.

*Proof*

$$\begin{aligned} \eta(x) &= \lim_{\Delta \rightarrow 0} \left( \frac{V_2(x, \Delta) - V_1(x)}{\Delta} \right) \\ \eta(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[\bar{H}(\Delta)C_d + H(\Delta)C_r] + r(x)\Delta C_f \right. \\ &\quad \left. + C_p(x + \Delta) - C_p(x) - r(x)\Delta C_p(x + \Delta) \right) \\ \eta(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[C_d - H(\Delta)C_d + H(\Delta)C_r] \right) \\ &\quad + r(x)C_f(x) + C'_p(x) - r(x)C_p(x) \\ \eta(x) &= \bar{H}(x)r(x)C_d + r(x)C_f(x) + c'_p(x) - r(x)C_p(x) \\ \eta(x) &= r(x) \left( \bar{H}(x)C_d + C_f(x) - C_p(x) \right) + C'_p(x) \end{aligned}$$

### 3.7 EXPERIMENTATION

After deriving the cost objective function, and the marginal cost functions, we experiment with the model, using the math computer software Maple v.

### 3.7.1 Policy 1

For this policy, results are summarized in Table 3.1, Table 3.2 and Figure 3.7. As shown in Figure 3.7, the minimum of the objective function occurs slightly to the left of the intersection of the curves  $C(T)$  and  $\eta(T)$ . Table 3.1 shows how flat the  $C(T)$  is. Table 3.2 shows the cost rates  $C'(T)$ . Note that the optimal replacement age is  $T^* = 4.2$ , as can be seen in Table 3.2.

### 3.7.2 Policy 2

For Policy 2, the results are summarized in Table 3.3 and Figure 3.8. As shown in Figure 3.8, the two curves  $C(T)$  and  $\eta(T)$  intersect at a point located slightly to the left of the optimal replacement age. Table 3.3 demonstrates how flat the  $C(T)$  curve is.

$\alpha = 5, \beta = 2, \mu = 0.25$			
T	ECL(T)	ECC(T)	C(T) ( $10^{-3}$ )
0	4.000	5.600	1400
1	5.263	5.913	1124
2	7.041	6.648	944
3	9.152	7.843	857
3.5	10.219	8.568	838
3.6	10.426	8.719	836
3.7	10.632	8.871	834
3.8	10.834	9.024	833
3.9	11.032	9.178	832
4.0	11.227	9.331	831
4.1	11.418	9.485	831
4.2	11.604	9.638	831*
4.3	11.787	9.791	831
4.4	11.964	9.942	831
4.5	12.137	10.092	832
4.6	12.305	10.241	832
4.7	12.467	10.388	833
4.8	12.624	10.532	834
4.9	12.777	10.675	835
5.0	12.923	10.814	837
5.1	13.065	10.952	838
5.2	13.201	11.086	840
5.3	13.332	11.218	841
5.4	13.457	11.346	843
5.5	13.577	11.471	845
6	14.102	12.044	854
7	14.819	12.926	872
8	15.212	13.489	887
9	15.409	13.815	896
10	15.500	13.988	902
11	15.540	14.072	906
12	15.555	14.110	907
13	15.563	14.125	908
14	15.563	14.131	908
15	15.564	14.133	908
16	15.564	14.134	908
17	15.564	14.134	908
18	15.564	14.134	908

Table 3.1: Numerical Results, Policy 1

$\alpha = 5, \beta = 2, \mu = 0.25$			
T	$C'(T)$	T	$C'(T)$
0	-0.312	3.8	-0.012
0.1	-0.307	4.0	-0.006
0.2	-0.301	4.1	-0.003
0.3	-0.294	4.2	0*
0.4	-0.287	4.3	0.002
0.5	-0.279	4.4	0.004
0.6	-0.270	4.5	0.006
0.7	-0.261	4.6	0.008
0.8	-0.251	5	0.014
0.9	-0.241	6	0.019
1.0	-0.231	7	0.017
1.2	-0.211	8	0.012
1.4	-0.190	9	0.008
1.6	-0.169	10	0.004
1.8	-0.148	11	0.002
2.0	-0.129	12	0.001
2.2	-0.111	13	0
2.4	-0.094	14	0
2.6	-0.078	15	0
2.8	-0.064	16	0
3.0	-0.051	17	0
3.2	-0.039	18	0
3.4	-0.029	19	0
3.6	-0.020	20	0

Table 3.2: Numerical Results, Policy 1: Cost Rates

$\alpha = 5, \beta = 2, \mu = 0.25$			
T	ECL(T)	ECC(T)	C(T) ( $10^{-3}$ )
0.5	10.517	10.502	998
1.0	10.734	9.926	925
1.5	11.062	9.832	889
2.0	11.470	9.912	864
2.5	11.923	10.105	847
3.0	12.405	10.383	837
3.1	12.501	10.446	836
3.2	12.597	10.512	834
3.3	12.692	10.580	834
3.4	12.786	10.650	833
3.5	12.880	10.722	832
3.6	12.973	10.796	832
3.7	13.065	10.871	832*
3.8	13.155	10.947	832
3.9	13.244	11.024	832
4.0	13.332	11.103	833
4.5	13.746	11.502	837
5.0	14.114	11.899	843
5.5	14.430	12.279	851
6.0	14.696	12.629	859
7	15.084	13.206	876
8	15.319	13.610	888
9	15.448	13.862	897
10	15.514	14.005	903
11	15.544	14.078	906
12	15.558	14.111	907
13	15.562	14.126	908
14	15.564	14.131	908
15	15.564	14.133	908
16	15.564	14.134	908
17	15.564	14.134	908
18	15.564	14.134	908

Table 3.3: Numerical Results, Policy 2

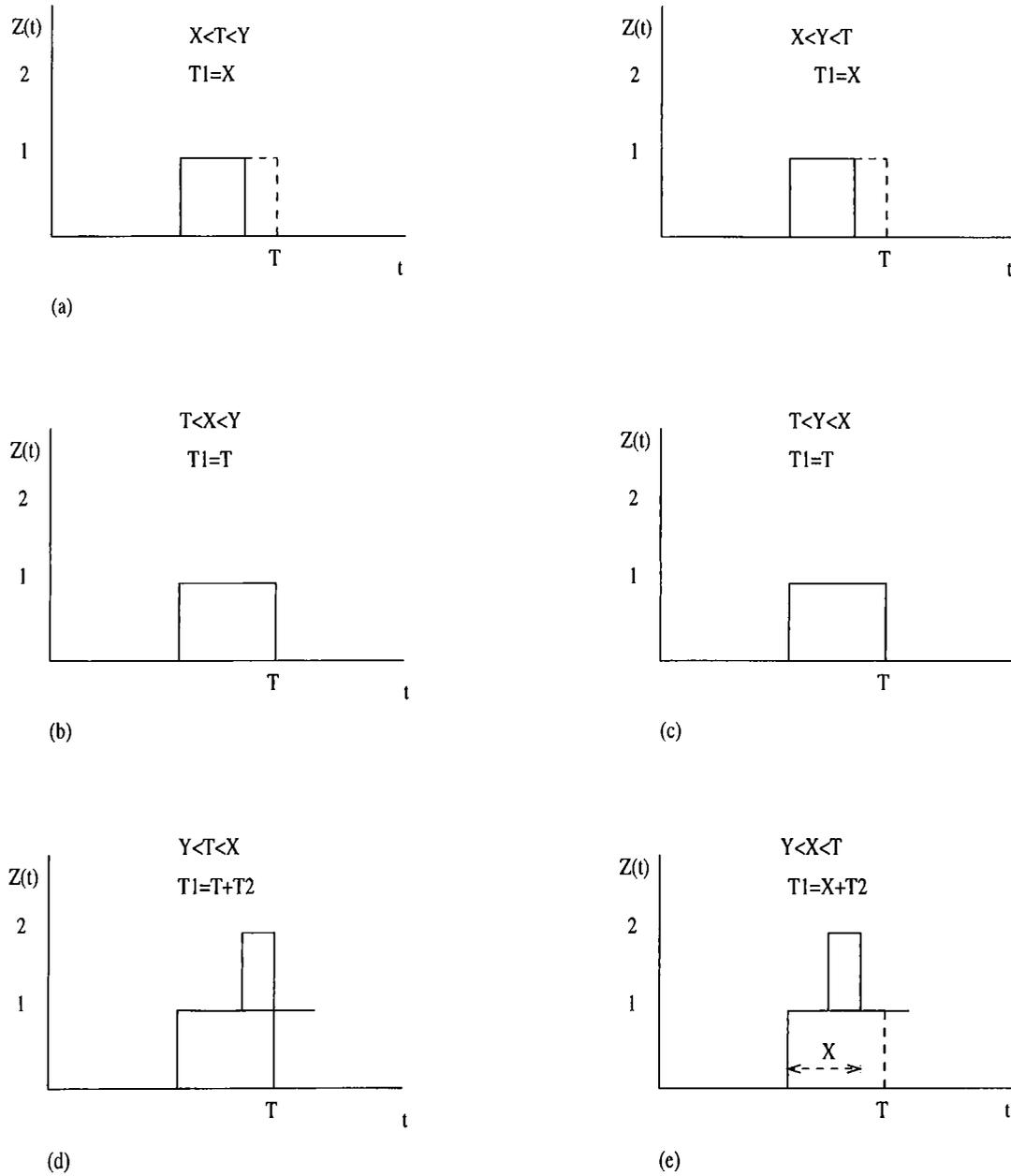


Figure 3.4: Cycle length, for Policy 1, with preventive maintenance carried out at intervals of length  $T$ : (a)  $X < \min\{T, Y\}$ ; (b)  $T < X < Y$ ; (c)  $T < Y < X$ ; (d)  $Y < T < X$ ; (e)  $Y < X < T$ .

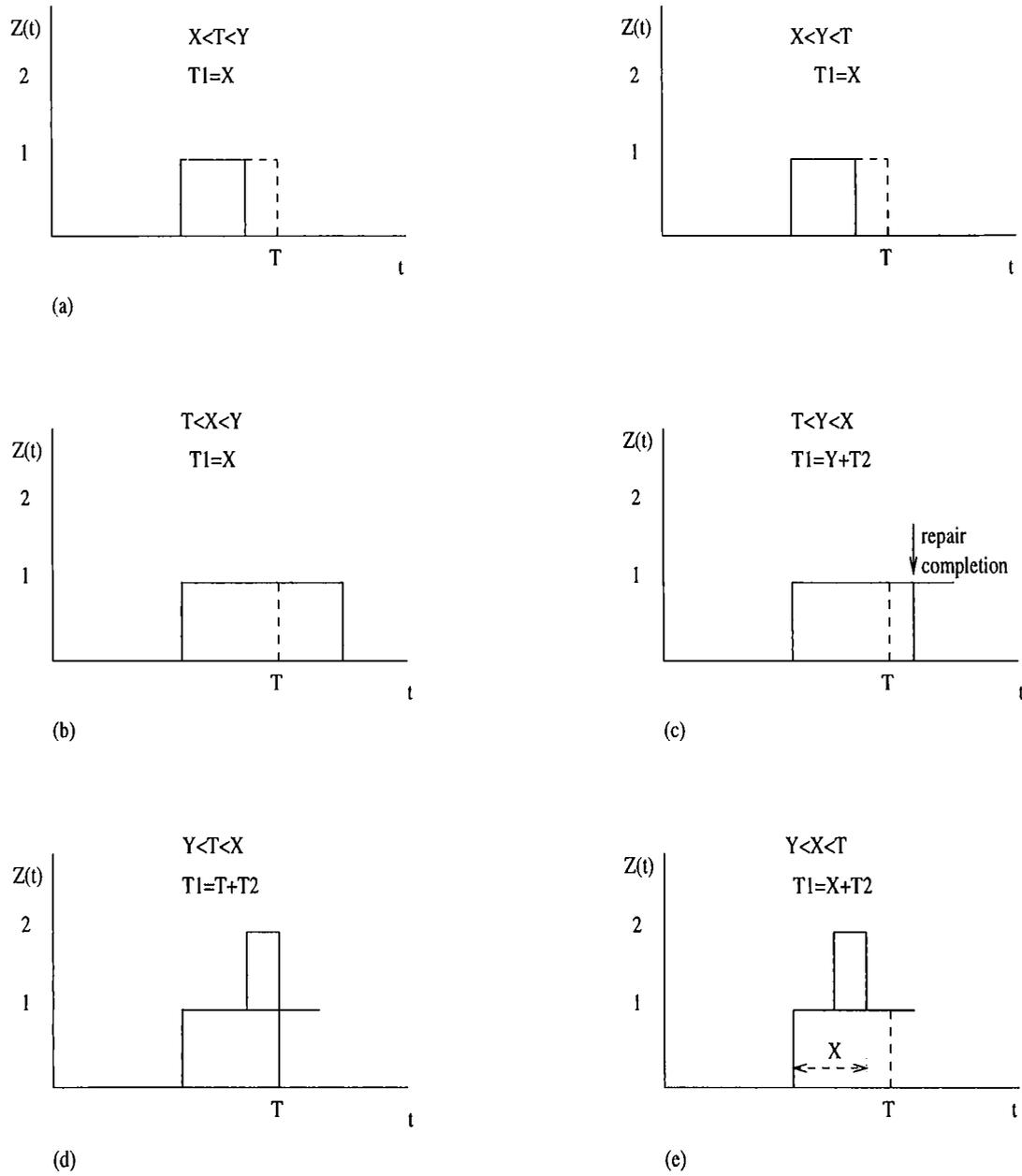


Figure 3.5: Cycle length, for Policy 2, with preventive maintenance carried out at intervals of length  $T$ : (a)  $X < \min\{T, Y\}$ ; (b)  $T < X < Y$ ; (c)  $T < Y < X$ ; (d)  $Y < T < X$ ; (e)  $Y < X < T$ .

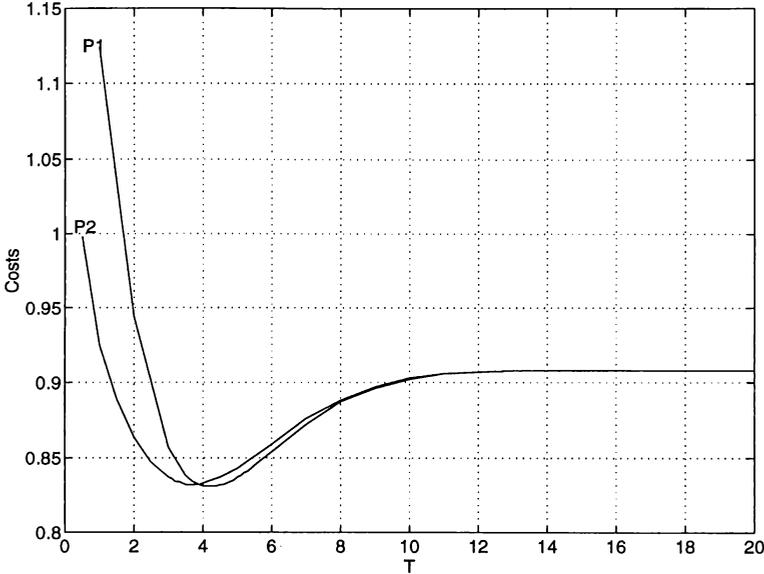


Figure 3.6: Cost curves,  $C(T)$ : Comparison of Policy 1 (P1) and Policy 2 (P2)

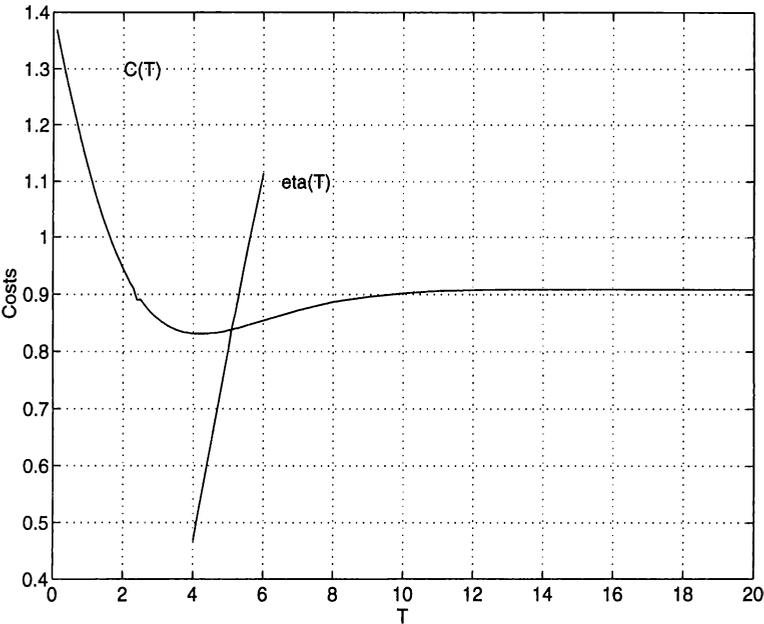


Figure 3.7: Cost curve,  $C(T)$ , and marginal cost function,  $\eta(T)$ , versus replacement age  $T$ , for the simple Policy 1

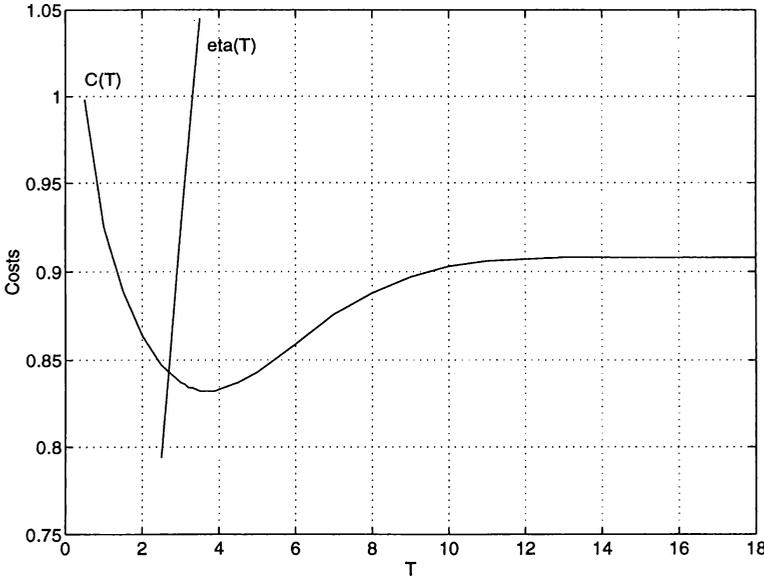


Figure 3.8: Cost curve,  $C(T)$ , and marginal cost function,  $\eta(T)$ , versus replacement age  $T$ , for the simple Policy 2

# Chapter 4

## GENERAL MODEL

### 4.1 GENERAL MODEL

Policy 1, as described in Chapter 3, considers replacement upon failure or at interval of length  $T$ . Suppose inspection is performed at intervals of length  $T$ . We consider a two-unit redundant cold standby system, with a single repair facility, having exponential service time with rate  $\mu$ . The lifetime distribution is general. The switchover is immediate. Our aim is to find the optimal inspection time  $T^*$ , the time interval that minimizes the total costs incurred per unit time, in the long run. Figure 3.1 depicts the system at hand. Upon failure of the working unit at age  $x$  in the time interval  $[0, T)$ , the unit is replaced by the standby unit if available, at a cost  $C_f(x)$ . Unit repair cost is  $C_r$ . Inspection cost of a unit of age  $x$  is  $C_p(x)$ . The inspected item is in one of two states: Good, or Critical. The good unit continues operation, whereas a unit in critical state is sent to repair. Let  $\Theta(x)$  denote the probability that an inspected unit at age  $x$  is in a critical state. Denote the service cdf and pdf by  $H(\cdot)$  and  $h(\cdot)$ , respectively, which is exponentially distributed with parameter  $\mu$ .

Recall that  $\{Z(t), t \geq 0\}$  denotes the number of units available at time  $t$ . The process  $\{Z(t), t \geq 0\}$  is a continuous time regenerative process, with state space  $S_Z = \{0, 1, 2\}$ . Let us define the renewal interval as follows.

**Definition 11** *The renewal interval of the process  $\{Z(t), t \geq 0\}$  is the time between two consecutive epochs at which the process enters state 0.*

An illustration of a cycle is given in Figure 4.1.

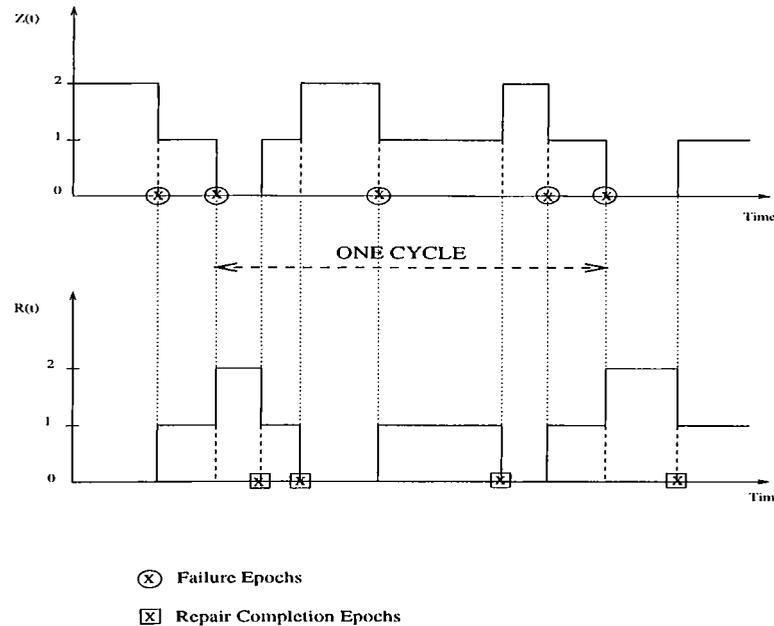


Figure 4.1: Illustration of a Typical Cycle

### 4.1.1 Cycle Length - Without Inspection

Denote by the random variables  $X$  and  $Y$  the lifetime and repair time, respectively. Let  $\rho \equiv P\{X < Y\} = \int_0^\infty e^{-\mu x} dF(x)$  be the probability that the lifetime of a unit is shorter than the repair time of a unit. We recall the repair time is exponentially distributed with parameter  $\mu$ .

$CL$  denotes the cycle length, which is a random variable. As shown in Figure 4.2, the cycle length varies, depending on the number of times (if any) the process  $Z(t)$  enters state 2 from state 1.

$$CL = \begin{cases} Y + X & \text{with probability } \rho \\ Y + X + T_1 & \text{with probability } (1 - \rho) \end{cases}$$

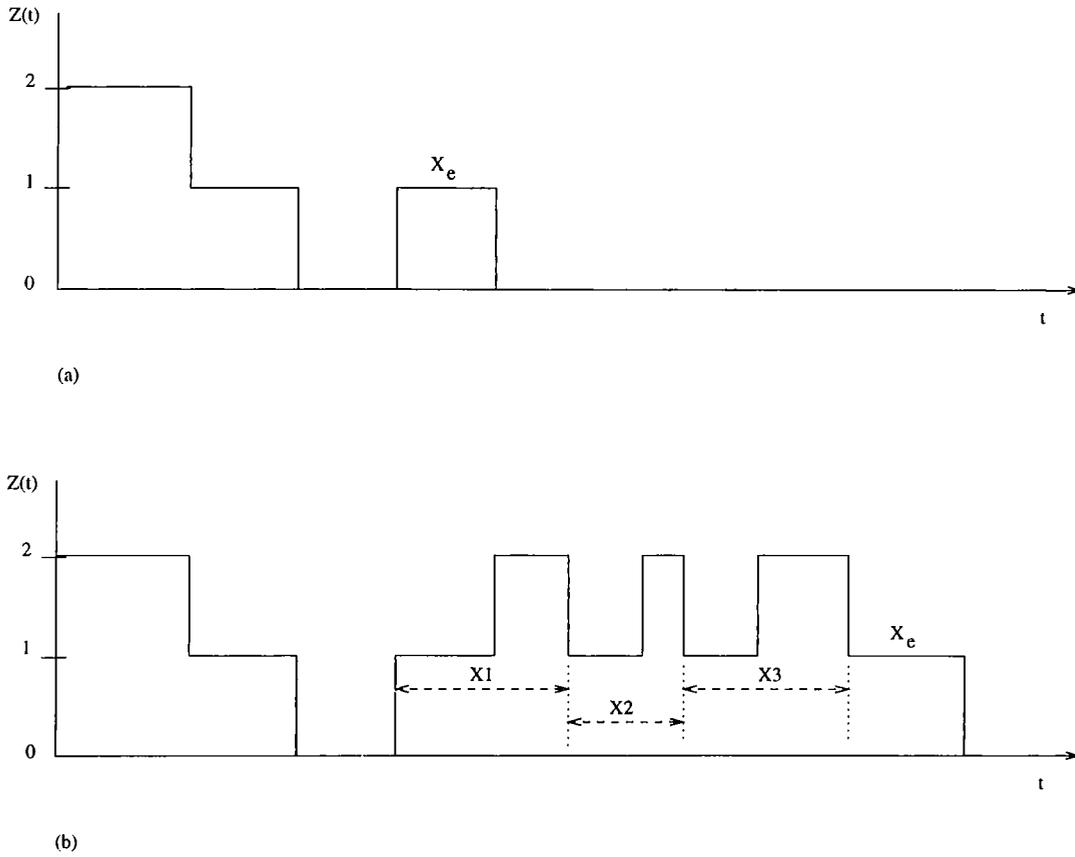


Figure 4.2: Cycle length: (a) The process  $Z(t)$  enters state 0 directly from state 1. (b) The process  $Z(t)$  visits state 2 before state 0.

$$T_1 = \begin{cases} X & \text{with probability } \rho \\ X + T_2 & \text{with probability } (1 - \rho) \end{cases}$$

where  $T_i, i = 1, 2, \dots$  are random variables.

$$CL = \begin{cases} Y + X_e & \text{with probability } \rho \\ Y + X_1 + X_e & \text{with probability } \rho(1 - \rho) \\ Y + X_1 + X_2 + X_e & \text{with probability } \rho(1 - \rho)^2 \\ \vdots & \\ Y + \sum_{i=0}^n X_i + X_e & \text{with probability } \rho(1 - \rho)^n \end{cases}$$

We can also write the following

$$CL = Y + X_1 + X_2 + \dots + X_n + X_e$$

$$= Y + \sum_{i=0}^N X_i + X_e$$

where  $N$  is a random variable.

$$F_{CL}(t) = P\{CL \leq t\} \quad (1)$$

$$\begin{aligned} &= P\{Y + X_e + \sum_{i=1}^N X_i \leq t\} \\ &= \sum_{n=0}^{\infty} P\{Y + X_e + \sum_{i=0}^n X_i \leq t \mid N = n\} P\{N = n\} \\ &= \sum_{n=0}^{\infty} P\{Y + X_e + \sum_{i=0}^n X_i \leq t\} \rho(1 - \rho)^n \\ &= \sum_{n=0}^{\infty} \rho(1 - \rho)^n f_Y * f_{X_e} * f_X^{(n)} \end{aligned} \quad (2)$$

where  $*$  denotes the convolution operator, and  $f_X^{(n)}$  is the  $n$ -fold convolution of lifetimes.

Let  $F_{X_e}(\cdot)$  be the cdf of the random variable  $X_e$ .

$$\begin{aligned} F_{X_e}(t) &= P\{X_e \leq t\} \\ &= P\{X \leq t \mid X \leq Y\} \\ &= \frac{P\{X \leq t, X \leq Y\}}{P\{X \leq Y\}} \\ &= \frac{\int_0^t e^{-\mu x} dF(x)}{P\{X \leq Y\}} \\ &= \rho^{-1} \int_0^t e^{-\mu x} dF(x) \end{aligned} \quad (3)$$

Using the Fundamental Theorem of Calculus, the pdf of  $X_e$  is

$$f_{X_e}(t) = \rho^{-1} e^{-\mu t} f(t) \quad (4)$$

### 4.1.2 Cycle Length - With Inspection

Recall that  $X$  and  $Y$  stand for the lifetime and repair time, respectively, and that repair time is exponentially distributed with parameter  $\mu$ . Assume that

inspection is carried out at intervals of finite length  $T$ . If an inspected unit is in the “critical” state it is sent to repair.

Let  $p$  denote the probability that the inspected unit is in the “Good” state, and  $(1 - p)$  be the probability that it is in the “critical” state.

There are many cases:

- $X < Y < T$
- $X < T < Y$
- $T < X < Y$
- $T < Y < X$
- $Y < T < X$
- $Y < X < T$

See Figure 4.3 for illustrations.

Let us now calculate the corresponding probabilities to events of concern.

We note that  $\{X < \min\{T, Y\}\} \equiv \{X < T < Y\}$  or  $\{X < Y < T\}$ . Therefore,

$$P\{X < \min\{T, Y\}\} = P\{X < T < Y\} + P\{X < Y < T\}$$

$$\begin{aligned} P\{X < T < Y\} &= \int_{x=0}^T \int_{y=T}^{\infty} \mu e^{-\mu y} f(x) dy dx \\ &= \int_{x=0}^T e^{-\mu T} f(x) dx \\ &= e^{-\mu T} \int_{x=0}^T f(x) dx \\ &= e^{-\mu T} F(T) \end{aligned}$$

$$P\{Y < T < X\} = \int_{x=T}^{\infty} \int_{y=0}^T \mu e^{-\mu y} f(x) dy dx$$

$$\begin{aligned}
&= \int_{x=T}^{\infty} (1 - e^{-\mu T}) f(x) dx \\
&= (1 - e^{-\mu T}) \int_{x=T}^{\infty} f(x) dx \\
&= (1 - e^{-\mu T}) \bar{F}(T)
\end{aligned}$$

$$\begin{aligned}
P\{X < Y < T\} &= \int_{x=0}^T \int_{y=x}^T \mu e^{-\mu y} f(x) dy dx \\
&= \int_{x=0}^T (e^{-\mu x} - e^{-\mu T}) f(x) dx \\
&= \int_{x=0}^T (e^{-\mu x} - e^{-\mu T}) dF(x)
\end{aligned}$$

$$\begin{aligned}
P\{T < Y < X\} &= \int_{x=T}^{\infty} \int_{y=T}^{y=x} \mu e^{-\mu y} f(x) dy dx \\
&= \int_T^{\infty} (e^{-\mu T} - e^{-\mu x}) f(x) dx \\
&= \int_T^{\infty} (e^{-\mu T} - e^{-\mu x}) dF(x)
\end{aligned}$$

$$\begin{aligned}
P\{T < X < Y\} &= \int_{x=T}^{\infty} \int_{y=x}^{\infty} \mu e^{-\mu y} f(x) dy dx \\
&= \int_T^{\infty} e^{-\mu x} f(x) dx \\
&= \int_T^{\infty} e^{-\mu x} dF(x)
\end{aligned}$$

$$\begin{aligned}
P\{Y < X < T\} &= \int_{x=0}^T \int_{y=0}^x \mu e^{-\mu y} f(x) dy dx \\
&= \int_0^T (1 - e^{-\mu x}) f(x) dx \\
&= \int_0^T (1 - e^{-\mu x}) dF(x)
\end{aligned}$$

Let  $N_X(T)$  and  $N_Y(T)$  be the random variables defined by

$$N_X(T) = \{n : nT \leq X < (n+1)T\}$$

$$N_Y(T) = \{n : nT \leq Y < (n+1)T\}$$

$$\begin{aligned}
P\{N_X(T) = 0\} &= F(T) \\
P\{N_X(T) = 1\} &= F(2T) - F(T) \\
P\{N_X(T) = 2\} &= F(3T) - F(2T) \\
&\vdots \\
P\{N_X(T) = k\} &= F[(k+1)T] - F(kT), k = 0, 1, 2, \dots
\end{aligned}$$

$$\begin{aligned}
F_{N_X(T)}(k) &= \sum_{i=0}^k \{F[(i+1)T] - F(iT)\} = F[(k+1)T] \\
E[N_X(T)] &= \sum_{k=0}^{\infty} \{1 - F[(k+1)T]\}.
\end{aligned}$$

Similarly,

$$E[N_Y(T)] = \sum_{k=0}^{\infty} \{e^{-(k+1)T}\}.$$

For the first two cases, the cycle length is  $CL = Y + X$ . That is we have  $CL = Y + T_1$ ,  $T_1 = X$  if  $X < \min\{T, Y\}$ .

$$T_1 = \left\{ \begin{array}{ll} X & \text{if } \{X < \min\{T, Y\}\} \text{ or } \{T < X < Y; \text{ all good}\} \\ T & \text{if } \{T < X < Y; \text{ critical}\} \text{ or } \{T < Y < X; \text{ critical}\} \\ 2T & \text{if } \{2T < X < Y; \text{ good, critical}\} \\ & \text{or } \{2T < Y < X; \text{ good, critical}\} \\ 3T & \text{if } \{3T < X < Y; \text{ good, good, critical}\} \\ & \text{or } \{3T < Y < X; \text{ good, good, critical}\} \\ \vdots & \\ kT & \text{if } \{kT < X < Y; \text{ good, } \dots, \text{ good, critical}\} \\ & \text{or } \{kT < Y < X; \text{ good, } \dots, \text{ good, critical}\} \\ X + T_2 & \text{if } \{T < Y < X; \text{ all good}\} \\ & \text{or } \{Y < X < T\} \text{ or } \{Y < T < X; \text{ good}\} \\ T + T_2 & \text{if } \{Y < T < X; \text{ critical}\} \\ 2T + T_2 & \text{if } \{Y < 2T < X; \text{ good, critical}\} \\ 3T + T_2 & \text{if } \{Y < 3T < X; \text{ good, good, critical}\} \\ \vdots & \\ kT + T_2 & \text{if } \{Y < kT < X; \text{ good, } \dots, \text{ good, critical}\} \end{array} \right.$$

$$T_1 = \left\{ \begin{array}{ll} X & \text{if } \{X < \min\{T, Y\}\} \text{ or } \{T < X < Y, \text{all good}\} \\ T & \text{if } \{T < X < Y; \text{critical}\} \text{ or } \{T < Y < X; \text{critical}\} \\ 2T & \text{if } \{2T < X < Y; \text{good, critical}\} \\ & \text{or } \{2T < Y < X; \text{good, critical}\} \\ 3T & \text{if } \{3T < X < Y; \text{good, good, critical}\} \\ & \text{or } \{3T < Y < X; \text{good, good, critical}\} \\ \vdots & \\ kT & \text{if } \{kT < X < Y; \text{good, } \dots, \text{good, critical}\} \\ & \text{or } \{kT < Y < X; \text{good, } \dots, \text{good, critical}\} \\ X + T_2 & \text{if } \{kT < X < (k+1)T, T < Y < X; k \text{ good}\} \text{ or } \{Y < X < T\} \\ & \text{or } \{kT < X < (k+1)T, Y < T; k \text{ good}\} \\ T + T_2 & \text{if } \{Y < T < X; \text{critical}\} \\ 2T + T_2 & \text{if } \{Y < 2T < X; \text{good, critical}\} \\ 3T + T_2 & \text{if } \{Y < 3T < X; \text{good, good, critical}\} \\ \vdots & \\ kT + T_2 & \text{if } \{Y < kT < X; \text{good, } \dots, \text{good, critical}\} \end{array} \right.$$

where  $T_i$ ,  $i = 1, 2$  are iid random variables. Let  $I(A)$  be the indicator function of the event  $A$ ,

$$\begin{aligned} T_1 &= XI(X < T < Y) + XI(X < Y < T) + XI(T < X < Y; \text{allgood}) \\ &+ TI(T < X < Y; \text{critical}) + TI(T < Y < X; \text{critical}) \\ &+ 2TI(2T < X < Y; \text{good, critical}) \\ &+ 2TI(2T < Y < X; \text{good, critical}) \\ &+ 3TI(3T < X < Y; \text{good, good, critical}) \\ &+ 3TI(3T < Y < X; \text{good, good, critical}) \\ &\vdots \\ &+ kTI(kT < X < Y; \text{good, } \dots, \text{good, critical}) \\ &+ kTI(kT < Y < X; \text{good, } \dots, \text{good, critical}) \\ &+ XI(T < Y < X; \text{allgood}) + T_2I(T < Y < X; \text{allgood}) \\ &+ XI(Y < X < T) + T_2I(Y < X < T) \end{aligned}$$

$$\begin{aligned}
& + XI(Y < T < X; \text{good}) + T_2I(Y < T < X; \text{good}) \\
& + (T + T_2)I(Y < T < X; \text{critical}) \\
& + (2T + T_2)I(Y < 2T < X; \text{good, critical}) \\
& + (3T + T_2)I(Y < 3T < X; \text{good, good, critical}) \\
& \vdots \\
& + (kT + T_2)I(Y < kT < X; \text{good}, \dots, \text{good, critical}) \\
\\
T_1 = & XI(X < T < Y) + XI(X < Y < T) + XI(T < X < Y; \text{allgood}) \\
& + TI(T < X < 2T, X < Y; \text{critical}) + TI(T < Y < 2T, Y < X; \text{critical}) \\
& + 2TI(2T < X < Y; \text{good, critical}) \\
& + 2TI(2T < Y < X; \text{good, critical}) \\
& + 3TI(3T < X < Y; \text{good, good, critical}) \\
& + 3TI(3T < Y < X; \text{good, good, critical}) \\
& \vdots \\
& + kTI(kT < X < Y; \text{good}, \dots, \text{good, critical}) \\
& + kTI(kT < Y < X; \text{good}, \dots, \text{good, critical}) \\
& + XI(kT < X < (k+1)T, T < Y < X; \text{kgood}) \\
& + XI(Y < X < T) \\
& + XI(kT < X < (k+1)T, Y < T; \text{kgood}) \\
& + T_2I(kT < X < (k+1)T, T < Y < X; \text{kgood}) \\
& + T_2I(Y < X < T) \\
& + T_2I(kT < X < (k+1)T, Y < T; \text{kgood}) \\
& + (T + T_2)I(Y < T < X; \text{critical}) \\
& + (2T + T_2)I(Y < 2T < X; \text{good, critical}) \\
& + (3T + T_2)I(Y < 3T < X; \text{good, good, critical}) \\
& \vdots \\
& + (kT + T_2)I(Y < kT < X; \text{good}, \dots, \text{good, critical})
\end{aligned}$$

Upn taking the expectation and noting that  $E[T_1] = E[T_i]$ ,  $i = 1, 2, \dots$ , we have the following

$$\begin{aligned}
E[T_1] &= e^{-\mu T} \int_0^T x dF(x) + \int_0^T x [e^{-\mu x} - e^{-\mu T}] dF(x) \\
&+ \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} x e^{-\mu x} dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} x [e^{-\mu T} - e^{-\mu x}] dF(x) \\
&+ \int_0^T x [1 - e^{-\mu x}] dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} x [1 - e^{-\mu T}] dF(x) \\
&+ T(1-p) \sum_{k=2}^{\infty} k p^{k-1} \left( \int_{kT}^{\infty} e^{-\mu x} dF(x) + \int_{kT}^{\infty} [e^{-\mu kT} - e^{-\mu x}] dF(x) \right) \\
&+ T(1-p) e^{-\mu T} \bar{F}(T) \\
&+ T(1-p)(1 - e^{-\mu T}) \bar{F}(T) \\
&+ E[T_1] \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} [e^{-\mu T} - e^{-\mu x}] dF(x) \\
&+ E[T_1] \int_0^T (1 - e^{-\mu x}) dF(x) \\
&+ E[T_1] \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu T}) dF(x) \\
&+ E[T_1](1-p)(1 - e^{-\mu T}) \bar{F}(T) \\
&+ T(1-p) \sum_{k=2}^{\infty} k p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \\
&+ E[T_1](1-p) \sum_{k=2}^{\infty} p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT)
\end{aligned}$$

$$\begin{aligned}
E[T_1] &= \int_0^T e^{-\mu x} dF(x) + \int_0^T x(1 - e^{-\mu x}) dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} x dF(x) \\
&+ T(1-p) e^{-\mu T} \bar{F}(T) \\
&+ T(1-p) \sum_{k=2}^{\infty} k p^{k-1} e^{-\mu kT} \bar{F}(kT) \\
&+ T(1-p)(1 - e^{-\mu T}) \bar{F}(T) \\
&+ T(1-p) \sum_{k=2}^{\infty} k p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \\
&+ E[T_1] \int_0^T (1 - e^{-\mu x}) dF(x) \\
&+ E[T_1] \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \\
&+ E[T_1](1-p)(1 - e^{-\mu T}) \bar{F}(T)
\end{aligned}$$

$$\begin{aligned}
& + E[T_1](1-p) \sum_{k=2}^{\infty} p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \\
E[T_1] & = \int_0^T x dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} x dF(x) \\
& + T(1-p) \bar{F}(T) + T(1-p) \sum_{k=2}^{\infty} kp^{k-1} \bar{F}(kT) \\
& + E[T_1] \left( \int_0^T (1 - e^{-\mu x}) dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \right. \\
& \left. + (1-p) \sum_{k=2}^{\infty} p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) + (1-p)(1 - e^{-\mu T}) \bar{F}(T) \right)
\end{aligned}$$

The expected value of the random variable  $T_1$  is

$$E[T_1] = \frac{A}{B} \quad (5)$$

$$\begin{aligned}
A & = \int_0^T x dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} x dF(x) + T(1-p) \bar{F}(T) \\
& + T(1-p) \sum_{k=2}^{\infty} kp^{k-1} \bar{F}(kT) \quad (6)
\end{aligned}$$

$$\begin{aligned}
B & = 1 - \int_0^T (1 - e^{-\mu x}) dF(x) - \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \\
& - (1-p)(1 - e^{-\mu T}) \bar{F}(T) - (1-p) \sum_{k=2}^{\infty} p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \quad (7)
\end{aligned}$$

Finally, the expected value of the cycle length is

$$E[CL] = E[Y] + E[T_1] \quad (8)$$

Let us now, in a similar way, determine  $E(CC)$ , the expected cost incurred during a cycle. Let  $C_i$  denote the cost incurred during  $T_i$ ,  $i = 1, 2, \dots$ . Assume the following cost structure:

- $c_f$ : failure cost

- $c_p$ : inspection cost
- $c_d$ : down cost
- $c_r$ : repair cost.

$$C_1 = \left\{ \begin{array}{ll} c_f & \text{if } \{X < \min\{T, Y\}\} \\ c_f + c_r + C_2 & \text{if } \{Y < X < T\} \\ c_f + c_p N_X(T) & \text{if } \{T < X < Y, \text{ all good}\} \\ c_f + c_r + c_p N_X(T) + C_2 & \text{if } \{T < Y < X, \text{ all good}\} \text{ or } \{Y < T < X, \text{ all good}\} \\ c_p & \text{if } \{T < X < Y, \text{ critical}\} \text{ or } \{T < Y < X, \text{ critical}\} \\ 2c_p & \text{if } \{2T < X < Y, \text{ good, critical}\} \\ & \text{or } \{2T < Y < X, \text{ good, critical}\} \\ \vdots & \\ kc_p & \text{if } \{kT < X < Y, \text{ good, } \dots, \text{ good, critical}\} \\ & \text{or } \{kT < Y < X, \text{ good, } \dots, \text{ good, critical}\} \\ c_r + c_p + C_2 & \text{if } \{Y < T < X, \text{ critical}\} \\ c_r + 2c_p + C_2 & \text{if } \{Y < 2T < X, \text{ good, critical}\} \\ \vdots & \\ c_r + kc_p + C_2 & \text{if } \{Y < kT < X, \text{ good, } \dots, \text{ good, critical}\} \end{array} \right.$$

$$C_1 = \left\{ \begin{array}{ll} c_f & \text{if } \{X < \min\{T, Y\}\} \\ c_f + c_r + C_2 & \text{if } \{Y < X < T\} \\ c_f + c_p N_X(T) & \text{if } \{kT < X < (k+1)T, T < X < Y, \text{ all good}\} \\ c_f + c_r + c_p N_X(T) + C_2 & \text{if } \{kT < X < (k+1)T, T < Y < X, \text{ all good}\} \\ & \text{or } \{kT < X < (k+1)T, Y < T < X, \text{ all good}\} \\ c_p & \text{if } \{T < X < Y, \text{ critical}\} \text{ or } \{T < Y < X, \text{ critical}\} \\ 2c_p & \text{if } \{2T < X < Y, \text{ good, critical}\} \\ & \text{or } \{2T < Y < X, \text{ good, critical}\} \\ \vdots & \\ kc_p & \text{if } \{kT < X < Y, \text{ good}, \dots, \text{ good, critical}\} \\ & \text{or } \{kT < Y < X, \text{ good}, \dots, \text{ good, critical}\} \\ c_r + c_p + C_2 & \text{if } \{Y < T < X, \text{ critical}\} \\ c_r + 2c_p + C_2 & \text{if } \{Y < 2T < X, \text{ good, critical}\} \\ \vdots & \\ c_r + kc_p + C_2 & \text{if } \{Y < kT < X, \text{ good}, \dots, \text{ good, critical}\} \end{array} \right.$$

$$C_1 = \begin{cases} c_f & \text{wp } \int_0^T e^{-\mu x} dF(x) \\ c_f + c_r + C_2 & \text{wp } \int_0^T (1 - e^{-\mu x}) dF(x) \\ c_f + kc_p & \text{wp } p^k \int_{kT}^{(k+1)T} e^{-\mu x} f(x) dx \\ c_f + c_r + kc_p + C_2 & \text{wp } p^k \int_{kT}^{(k+1)T} [e^{-\mu T} - e^{-\mu x}] f(x) dx \\ & + p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu T}) f(x) dx \\ c_p & \text{wp } (1-p) \int_T^\infty e^{-\mu x} dF(x) \\ & + (1-p) \int_T^\infty [e^{-\mu T} - e^{-\mu x}] dF(x) \\ 2c_p & \text{wp } (1-p)p \int_{2T}^\infty e^{-\mu x} dF(x) \\ & + (1-p)p \int_{2T}^\infty [e^{-\mu 2T} - e^{-\mu x}] dF(x) \\ \vdots & \\ kc_p & \text{wp } (1-p)p^{k-1} \int_{kT}^\infty e^{-\mu x} dF(x) \\ & + (1-p)p^{k-1} \int_{kT}^\infty [e^{-\mu kT} - e^{-\mu x}] dF(x) \\ c_r + c_p + C_2 & \text{wp } (1-p)(1 - e^{-\mu T}) \bar{F}(T) \\ c_r + 2c_p + C_2 & \text{wp } (1-p)p(1 - e^{-\mu 2T}) \bar{F}(2T) \\ \vdots & \\ c_r + kc_p + C_2 & \text{wp } (1-p)p^{k-1}(1 - e^{-\mu kT}) \bar{F}(kT) \end{cases}$$

$$C_1 = \begin{cases} c_f & \text{wp } \int_0^T e^{-\mu x} dF(x) \\ c_f + c_r + C_2 & \text{wp } \int_0^T (1 - e^{-\mu x}) dF(x) \\ c_f + kc_p & \text{wp } p^k \int_{kT}^{(k+1)T} e^{-\mu x} dF(x), k = 1, 2, \dots \\ c_f + c_r + kc_p + C_2 & \text{wp } p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) f(x) dx, k = 1, 2, \dots \\ c_p & \text{wp } (1-p)e^{-\mu T} \bar{F}(T) \\ kc_p & \text{wp } (1-p)p^{k-1} e^{-\mu kT} \bar{F}(kT), k = 2, 3, \dots \\ c_r + c_p + C_2 & \text{wp } (1-p)(1 - e^{-\mu T}) \bar{F}(T) \\ c_r + kc_p + C_2 & \text{wp } (1-p)p^{k-1}(1 - e^{-\mu kT}) \bar{F}(kT), k = 2, 3, \dots \end{cases}$$

Upon taking the expectation we have

$$E[C_1] = cf \left( \int_0^T e^{-\mu x} dF(x) + \int_0^T (1 - e^{-\mu x}) dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} e^{-\mu x} dF(x) \right)$$

$$\begin{aligned}
& + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \\
& + c_r \left( \int_0^T (1 - e^{-\mu x}) dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \right. \\
& + (1-p)(1 - e^{-\mu T}) \bar{F}(T) + \sum_{k=2}^{\infty} (1-p)p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \left. \right) \\
& + c_p \left( \sum_{k=1}^{\infty} kp^k \int_{kT}^{(k+1)T} e^{-\mu x} dF(x) + \sum_{k=1}^{\infty} kp^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \right. \\
& + (1-p)e^{-\mu T} \bar{F}(T) + (1-p)(1 - e^{-\mu T}) \bar{F}(T) \\
& + \sum_{k=2}^{\infty} k(1-p)p^{k-1} e^{-\mu kT} \bar{F}(kT) + \sum_{k=2}^{\infty} k(1-p)p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \left. \right) \\
& + E[C_1] \left( \int_0^T (1 - e^{-\mu x}) dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \right. \\
& + (1-p)(1 - e^{-\mu T}) \bar{F}(T) + \sum_{k=2}^{\infty} (1-p)p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \left. \right)
\end{aligned}$$

$$\begin{aligned}
E[C_1] & = cf \left( F(T) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} dF(x) \right) \\
& + c_r \left( \int_0^T (1 - e^{-\mu x}) dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \right. \\
& + (1-p)(1 - e^{-\mu T}) \bar{F}(T) + \sum_{k=2}^{\infty} (1-p)p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \left. \right) \\
& + c_p \left( \sum_{k=1}^{\infty} kp^k \int_{kT}^{(k+1)T} dF(x) + (1-p) \bar{F}(T) + \sum_{k=2}^{\infty} k(1-p)p^{k-1} \bar{F}(kT) \right) \\
& + E[C_1] \left( \int_0^T (1 - e^{-\mu x}) dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \right. \\
& + (1-p)(1 - e^{-\mu T}) \bar{F}(T) + \sum_{k=2}^{\infty} (1-p)p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \left. \right)
\end{aligned}$$

The expected value of the random variable  $C_1$  is

$$E[C_1] = \frac{U}{V} \quad (9)$$

$$U = cf \left( F(T) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} dF(x) \right)$$

$$\begin{aligned}
& + c_r \left( \int_0^T (1 - e^{-\mu x}) dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \right. \\
& + (1 - p)(1 - e^{-\mu T}) \bar{F}(T) + \sum_{k=2}^{\infty} (1 - p)p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \left. \right) \\
& + c_p \left( \sum_{k=1}^{\infty} k p^k \int_{kT}^{(k+1)T} dF(x) + (1 - p) \bar{F}(T) + \sum_{k=2}^{\infty} k(1 - p)p^{k-1} \bar{F}(kT) \right) \\
V = & 1 - \left( \int_0^T (1 - e^{-\mu x}) dF(x) + \sum_{k=1}^{\infty} p^k \int_{kT}^{(k+1)T} (1 - e^{-\mu x}) dF(x) \right. \\
& + (1 - p)(1 - e^{-\mu T}) \bar{F}(T) + \sum_{k=2}^{\infty} (1 - p)p^{k-1} (1 - e^{-\mu kT}) \bar{F}(kT) \left. \right)
\end{aligned} \tag{10}$$

Finally, the expected value of the cycle length is

$$E[CC] = c_d + c_r + E[C_1]. \tag{11}$$

Our objective function is the expected total cost per unit time, in the long run, which is, using the Renewal-Reward Theorem,

$$C(T) = \frac{E(CC)}{E(CL)} \tag{12}$$

## 4.2 LIMITS

### 4.2.1 As $p \rightarrow 0$

As  $p \rightarrow 0$ , looking at the cycle length formula we can write

$$\begin{aligned}
A & = \int_0^T x dF(x) + T \bar{F}(T) \\
B & = 1 - \int_0^T (1 - e^{-\mu x}) dF(x) - (1 - e^{-\mu T}) \bar{F}(T) \\
& = e^{-\mu T} \bar{F}(T) + \int_0^T e^{-\mu x} dF(x)
\end{aligned}$$

$$E[T_1] \rightarrow \frac{\int_0^T x dF(x) + T\bar{F}(T)}{e^{-\mu T}\bar{F}(T) + \int_0^T e^{-\mu x} dF(x)}$$

Looking at the cycle cost formula we can write

$$U = c_f F(T) + c_r \left( \int_0^T (1 - e^{-\mu x}) dF(x) + (1 - e^{-\mu T})\bar{F}(T) \right) + c_p \bar{F}(T)$$

$$\begin{aligned} V &= 1 - \left( \int_0^T (1 - e^{-\mu x}) dF(x) + (1 - e^{-\mu T})\bar{F}(T) \right) \\ &= e^{-\mu T}\bar{F}(T) + \int_0^T e^{-\mu x} dF(x) \end{aligned}$$

$$E[C_1] \rightarrow \frac{c_f F(T) + c_r \left( \int_0^T (1 - e^{-\mu x}) dF(x) + (1 - e^{-\mu T})\bar{F}(T) \right) + c_p \bar{F}(T)}{e^{-\mu T}\bar{F}(T) + \int_0^T e^{-\mu x} dF(x)}$$

Notice that these results coincide with the results derived in Section 3.3.1 and Section 3.3.2 for Policy1.

#### 4.2.2 As $p \rightarrow 1$

As  $p \rightarrow 1$ , looking at the cycle length formula we can write

$$\begin{aligned} A &= \int_0^T x dF(x) + \int_T^\infty x dF(x) \\ &= \int_0^\infty x dF(x) \\ B &= 1 - \int_0^T (1 - e^{-\mu x}) dF(x) - \int_T^\infty (1 - e^{-\mu x}) dF(x) \\ &= \int_0^\infty e^{-\mu x} dF(x) \end{aligned}$$

$$E[T_1] \rightarrow \frac{\int_0^\infty x dF(x)}{\int_0^\infty e^{-\mu x} dF(x)}$$

Looking at the cycle cost formula we can write

$$\begin{aligned}
U &= c_f \left( F(T) + \int_T^\infty dF(x) \right) + c_r \left( \int_0^T (1 - e^{-\mu x}) dF(x) + \int_T^\infty (1 - e^{-\mu x}) dF(x) \right) \\
&+ c_p \sum_{k=1}^{\infty} k \int_{kT}^{(k+1)T} dF(x) \\
&= c_f + c_r \left( 1 - \int_0^\infty e^{-\mu x} dF(x) \right) + c_p \sum_{k=1}^{\infty} k \int_{kT}^{(k+1)T} dF(x) \\
V &= 1 - \int_0^\infty (1 - e^{-\mu x}) dF(x) \\
&= \int_0^\infty e^{-\mu x} dF(x) \\
E[C_1] &\rightarrow \frac{c_f + c_r \left( 1 - \int_0^\infty e^{-\mu x} dF(x) \right) + c_p \sum_{k=1}^{\infty} k \int_{kT}^{(k+1)T} dF(x)}{\int_0^\infty e^{-\mu x} dF(x)}
\end{aligned}$$

## 4.3 THE MARGINAL COSTS

### 4.3.1 Generalized Policy 1

Let us now derive the marginal cost function of this generalized policy.

**Lemma 5** *The cost incurred if an inspection is carried out at age  $x$ , is given by*

$$V_{1\Theta}(x) = C_p(x) + \Theta(x)\bar{H}(x)C_d. \quad (13)$$

*Proof:* The cost incurred includes inspection cost,  $C_p(x)$ . In addition, if the other unit is still in repair, and the inspected unit is in critical state, downtime cost  $C_d$  is incurred. The probability of the latter event is  $\Theta(x)\bar{H}(x)$ . No additional cost is incurred if the inspected unit is in good state, or if the standby unit is available.

**Lemma 6** *If the inspection is deferred to age  $x + \Delta$ , then the expected cost incurred is given by*

$$\begin{aligned} V_{2\Theta}(x, \Delta) &= \bar{H}(x)r(x)\Delta \left( \bar{H}(\Delta)C_d + H(\Delta)C_r \right) + r(x)\Delta C_f(x) \\ &+ [1 - r(x)\Delta] \left( C_p(x + \Delta) + \Theta(x + \Delta)\bar{H}(x + \Delta)C_d \right) \end{aligned}$$

*Proof:* Below, we provide a sketch of the proof.

- a) In the next  $\Delta$  time units (for infinitesimal  $\Delta$ ), the probability that a failure occurs, and the standby unit is not available is equal to  $\bar{H}(x)[r(x)\Delta]$ .
  - The probability that there will be no repair completion in  $(x, x + \Delta]$  is equal to  $\bar{H}(\Delta)$ . A downtime cost is incurred,  $C_d$ .
  - The probability that there will be repair completion in  $(x, x + \Delta]$  is equal to  $H(\Delta)$ . Repair cost,  $C_r$ , is incurred.
- b) If a failure occurs in  $(x, x + \Delta]$ , a failure cost  $C_f(x)$  is incurred. This event has a probability equal to  $r(x)\Delta$ .
- c) The probability that the (operating) unit will not fail in  $(x, x + \Delta]$  is equal to  $[1 - r(x)\Delta]$ .
  - At the end of the interval  $(x, x + \Delta]$  inspection cost is equal to  $C_p(x + \Delta)$ .
  - In addition, the probability that the unit inspected at age  $x + \Delta$  is in critical state and the other unit is still in repair is equal to  $\Theta(x + \Delta)\bar{H}(x + \Delta)$ . A downtime cost,  $C_d$ , is incurred.

The MCF here is dependent on  $\Theta$ , the deterioration state of the unit. Now recalling Equation 1, we write

$$\eta_{\Theta}(x) = \lim_{\Delta \rightarrow 0} \left( \frac{V_{2\Theta}(x, \Delta) - V_{1\Theta}(x)}{\Delta} \right) \quad (14)$$

Using the results of Lemma 5 and Lemma 6, we derive the following,

**Theorem 4** *The MCF for the generalized Policy 1 is given by,*

$$\begin{aligned} \eta_{\Theta}(x) = & r(x) \left( \bar{H}(x)(1 - \Theta(x))C_d + C_f(x) - C_p(x) \right) + C'_p(x) + \Theta'(x)C_d \\ & - C_d \left( \Theta(x)h(x) + H(x)\Theta'(x) \right) \end{aligned}$$

where the operator ' stands for the derivative, and  $r(\cdot)$  denotes the failure rate function of the lifetime of a unit.

Notice that if 0 is substituted for  $\Theta(x)$  in the above MCF, we obtain the MCF of the simple Policy 1, derived in Section 3.6.1, Theorem 2.

### 4.3.2 Generalized Policy 2

Let us assume that when an inspected unit (operating) is found in critical state and the other unit is still in repair, the inspected unit keeps operating, to avoid system failure. Recall that in the original policy (Policy 1), such a unit is sent to repair, and a down cost,  $C_d$ , is incurred. Apart from this assumption, the policy is similar to the previous. See Figure 4.4 for a typical realization of the system, under this policy.

**Lemma 7** *The cost,  $V_{1\Theta}(x)$ , incurred if an inspection is carried out at age  $x$ , is given by*

$$V_{1\Theta}(x) = C_p(x) \tag{15}$$

*Proof:* The cost incurred includes inspection cost,  $C_p(x)$ . In addition, if the other unit is still in repair, and the inspected unit is in critical state, no cost is incurred. The probability of the latter event is  $\bar{\Theta}(x)\bar{H}(x)$ . No additional cost is incurred if the inspected unit is in good state, or if the standby unit is available.

**Lemma 8** *If the inspection is deferred to age  $x + \Delta$ , then the expected cost,  $V_{2\Theta}(x, \Delta)$ , incurred is given by*

$$\begin{aligned} V_{2\Theta}(x, \Delta) &= \bar{H}(x) r(x) \Delta \left( \bar{H}(\Delta) C_d + H(\Delta) C_r \right) + r(x) \Delta C_f(x) \\ &+ [1 - r(x) \Delta] C_p(x + \Delta) \end{aligned}$$

*Proof:*

Below, we provide a sketch of the proof.

- a) In the next  $\Delta$  time units (for infinitesimal  $\Delta$ ), the probability that a failure occurs, and the standby unit is not available is equal to  $\bar{H}(x)[r(x)\Delta]$ .
  - The probability that there will be no repair completion in  $(x, x + \Delta]$  is equal to  $\bar{H}(\Delta)$ . A downtime cost is incurred,  $C_d$ .
  - The probability that there will be repair completion in  $(x, x + \Delta]$  is equal to  $H(\Delta)$ . Repair cost is incurred,  $C_r$ .
- b) If a failure occurs in  $(x, x + \Delta]$ , failure cost  $C_f(x)$  is incurred. This will occur with probability  $r(x)\Delta$ .
- c) The probability that the (operating) unit will not fail in  $(x, x + \Delta]$  is equal to  $[1 - r(x)\Delta]$ .
  - At the end of the interval  $(x, x + \Delta]$  inspection cost is equal to  $C_p(x + \Delta)$ .
  - In addition, the probability that the unit inspected at age  $x + \Delta$  is in critical state and the other unit is still in repair is equal to  $\bar{\Theta}(x + \Delta)\bar{H}(x + \Delta)$ . The unit will continue operation, so no cost (downtime cost) is incurred.

Using the results of Lemma 7 and Lemma 8, we derive the following,

**Theorem 5** *The MCF for the altered policy is given by,*

$$\eta_{\Theta}(x) = r(x) \left( \bar{H}(x)C_d + C_f(x) - C_p(x) \right) + C'_p(x) \quad (16)$$

where the operator ' stands for the derivative, and  $r(\cdot)$  denotes the failure rate function of the lifetime of a unit.

Notice that this result coincides with the MCF of the simple Policy 2, derived in Section 3.6.2, Theorem 3. This result suggests that for Policy 2, as far as the MCF is concerned, the probability  $p$  that an inspected unit is in the good state is not important. Therefore, the MCF does not reflect one important aspect of the system.

We provide the proof of Theorem 4.

*Proof:*

By definition of the MCF,

$$\begin{aligned} \eta_{\Theta}(x) &= \lim_{\Delta \rightarrow 0} \left( \frac{V_{2\Theta}(x, \Delta) - V_{1\Theta}(x)}{\Delta} \right) \\ \eta_{\Theta}(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[C_d - H(\Delta)C_d + H(\Delta)C_r] + r(x)\Delta C_f(x) \right. \\ &\quad + [1 - r(x)\Delta][C_p(x + \Delta) + \Theta(x + \Delta)\bar{H}(x + \Delta)C_d] - C_p(x) \\ &\quad \left. + \Theta(x)H(x)C_d(x) - \Theta(x)C_d \right) \\ \eta_{\Theta}(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[C_d - H(\Delta)C_d + H(\Delta)C_r] + r(x)\Delta C_f(x) \right. \\ &\quad + C_p(x + \Delta) + \Theta(x + \Delta)\bar{H}(x + \Delta)C_d - r(x)\Delta C_p(x + \Delta) \\ &\quad \left. - r(x)\Delta\Theta(x + \Delta)\bar{H}(x + \Delta)C_d - C_p(x) - \Theta(x)C_d + \Theta(x)H(x)C_d \right) \\ \eta_{\Theta}(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[C_d - H(\Delta)C_d + H(\Delta)C_r] + r(x)\Delta C_f(x) \right. \\ &\quad + C_p(x + \Delta) - C_p(x) + \Theta(x + \Delta)C_d - \Theta(x + \Delta)H(x + \Delta)C_d \\ &\quad - r(x)\Delta C_p(x + \Delta) - r(x)\Delta\Theta(x + \Delta)C_d + r(x)\Delta\Theta(x + \Delta)H(x + \Delta)C_d \\ &\quad \left. - \Theta(x)C_d + \Theta(x)H(x)C_d \right) \end{aligned}$$

$$\begin{aligned}
\eta_{\Theta}(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[C_d - H(\Delta)C_d + H(\Delta)C_r] + r(x)\Delta C_f(x) \right. \\
&\quad + C_p(x + \Delta) - C_p(x) + \Theta(x + \Delta)C_d - \Theta(x)C_d - r(x)\Delta C_p(x + \Delta) \\
&\quad - r(x)\Delta\Theta(x + \Delta)C_d + r(x)\Delta\Theta(x + \Delta)H(x + \Delta)C_d \\
&\quad \left. - \Theta(x + \Delta)H(x + \Delta)C_d + \Theta(x)H(x)C_d \right) \\
\eta_{\Theta}(x) &= \lim_{\Delta \rightarrow 0} \left( \bar{H}(x)r(x)[C_d - H(\Delta)C_d + H(\Delta)C_r] + r(x)C_f(x) + C_p'(x) \right. \\
&\quad \left. + \Theta'(x)C_d - r(x)C_p(x + \Delta) - r(x)\Theta(x + \Delta)C_d + r(x)\Theta(x + \Delta)H(x + \Delta)C_d - A \right)
\end{aligned}$$

$$\begin{aligned}
A &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \Theta(x + \Delta)H(x + \Delta)C_d - \Theta(x)H(x)C_d \right) \\
A &= C_d \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \Theta(x + \Delta)H(x + \Delta) - \Theta(x)H(x) \right) \\
A &= C_d \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \Theta(x + \Delta)H(x + \Delta) - \Theta(x)H(x) + \Theta(x + \Delta)H(x) - \Theta(x + \Delta)H(x) \right) \\
A &= C_d \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \Theta(x + \Delta)H(x + \Delta) - \Theta(x + \Delta)H(x) + [\Theta(x + \Delta)H(x) - \Theta(x)H(x)] \right) \\
A &= C_d \left( \Theta(x)h(x) + \Theta'(x)H(x) \right)
\end{aligned}$$

$$\begin{aligned}
\eta_{\Theta}(x) &= r(x) \left( \bar{H}(x)(1 - \Theta(x))C_d + C_f(x) - C_p(x) \right) + C_p'(x) + \Theta'(x)C_d \\
&\quad - C_d[\Theta(x)h(x) + H(x)\Theta'(x)]
\end{aligned}$$

Next, we provide the proof of Theorem 5.

*Proof:*

By definition of the MCF,

$$\begin{aligned}
\eta_{\Theta}(x) &= \lim_{\Delta \rightarrow 0} \left( \frac{V_{2\Theta}(x, \Delta) - V_{1\Theta}(x)}{\Delta} \right) \\
\eta_{\Theta}(x) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta[\bar{H}(\Delta)C_d + H(\Delta)C_r] + r(x)\Delta C_f(x) \right)
\end{aligned}$$

$$\begin{aligned}
& + [1 - r(x) \Delta]C_p(x + \Delta) - C_p(x) \Big) \\
\eta_{\Theta}(x) & = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left( \bar{H}(x)r(x)\Delta\bar{H}(\Delta)C_d + \bar{H}(x)r(x)\Delta H(\Delta)C_r + r(x)\Delta C_f(x) \right. \\
& \quad \left. + C_p(x + \Delta) - C_p(x) - r(x)\Delta C_p(x + \Delta) \right) \\
\eta_{\Theta}(x) & = \bar{H}(x)r(x)C_d + C'_p(x) + r(x)C_f(x) - r(x)C_p(x) \\
\eta_{\Theta}(x) & = r(x) \left( \bar{H}(x)C_d + C_f(x) - C_p(x) \right) + C'_p(x)
\end{aligned}$$

As a consequence of Theorem 4, we have the following.

**Corollary 1** *The case with instantaneous repair reduces to the age dependent model, whose MCF is given by*

$$\eta(x) = r(x)[C_f(x) - C_p(x)] + C'_p(x) + \Theta'(x)C_d \quad (17)$$

*Proof:*

The proof is straightforward. The service being infinitely fast means that  $\bar{H}(x) = 0 \forall x \geq 0$ . This says that there is always a unit in standby. Therefore, the MCF reduces to  $\eta(x) = r(x)[C_f(x) - C_p(x)] + C'_p(x) + \Theta'(x)C_d$ .

The following is a consequence of Theorem 5.

**Corollary 2** *The case with instantaneous repair reduces to the age dependent model, whose MCF is given by (Berg [11])*

$$\eta(x) = r(x)[C_f(x) - C_p(x)] + C'_p(x) \quad (18)$$

*Proof:*

The case of infinitely fast repair is characterized by  $\bar{H}(x) = 0 \forall x \geq 0$ , i.e.,  $H(x) = 1 \forall x \geq 0$ . Hence, the MCF becomes  $\eta(x) = r(x)[C_f(x) - C_p(x)] + C'_p(x)$ .

The above result's importance is conspicuous. Note the case where the function  $\Theta(x)$  is independent of  $x$ , the MCF reduces to the well known result derived by Berg [11]. It demonstrates that our maintenance policy applied to our model reduces to the age replacement policy (ARP), extensively studied by researchers in the field of reliability and maintenance. Therefore, ours is an extension of the ARP.

At this point, to gain insight, let us turn our attention to the ARP. Berg [11] applied the MCA to the ARP with minimal repair. Let us remain faithful to his notations, and denote by  $c_1$  and  $c_2$  the failure cost and the preventive replacement cost, respectively. To justify preventive maintenance, failure cost should be larger than preventive maintenance cost, ie  $c_1 > c_2$ . Using the Renewal-Reward Theorem (see e.g. [47]), the long-run expected cost per unit of time, if the preventive maintenance is performed at age  $T$ , is given by

$$C(T) = \frac{c_1 F(T) + c_2 \bar{F}(T)}{\int_0^T \bar{F}(u) du} \quad (19)$$

The numerator equals the expected costs incurred during a cycle, and the denominator equals the expected cycle length. This result is very well known in the literature. It was derived firstly by Barlow & Proschan [6].

The problem is solved by determining the value of  $T$ ,  $T^*$ , that minimizes  $C(T)$ , that is solving  $\min_T C(T)$ . The optimal solution satisfies  $C(T^*) \leq C(T) \forall T \geq 0$ .

**Theorem 6** *The optimal replacement age,  $T^*$ , satisfies the equation*

$$r(T) \int_0^T \bar{F}(u) du + \bar{F}(T) = \frac{c_1}{c_1 - c_2} \quad (20)$$

*Proof:*

Differentiate  $C(T)$  with respect to  $T$ , obtain  $C'(T)$  and set it equal to zero. For convenience, rewrite  $C(T)$  as

$$C(T) = \frac{c_1 + (c_2 - c_1)\bar{F}(T)}{\int_0^T \bar{F}(u) du}.$$

$$C'(T) = \frac{\partial C(T)}{\partial T} = 0$$

implies

$$-(c_2 - c_1)f(T) \int_0^T \bar{F}(u) du = [c_1 + (c_2 - c_1)\bar{F}(T)]\bar{F}(T)$$

$$\frac{-(c_2 - c_1)}{\bar{F}(T)}f(T) \int_0^T \bar{F}(u) du = c_1 + (c_2 - c_1)\bar{F}(T)$$

$$(c_1 - c_2)r(T) \int_0^T \bar{F}(u) du = c_1 + (c_2 - c_1)\bar{F}(T)$$

$$r(T) \int_0^T \bar{F}(u) du = \frac{c_1}{c_1 - c_2} - \bar{F}(T)$$

$$r(T) \int_0^T \bar{F}(u) du + \bar{F}(T) = \frac{c_1}{c_1 - c_2}$$

The optimality equation given by Theorem 6 is derived by Berg [11] using the MCA. Let us elaborate on its derivation.

The cost,  $V_1(x)$ , of performing a preventive maintenance action at age  $x$ , and the expected cost,  $V_2(x, \Delta)$ , incurred in  $(x, x + \Delta]$  if the preventive replacement is deferred to age  $x + \Delta$ , are

$$V_1(x) = c_2, \tag{21}$$

$$V_2(x, \Delta) = r(x)\Delta c_1 + [1 - r(x)\Delta]c_2 \tag{22}$$

The MCF of the ARP is now obtained,

$$\eta(x) = (c_1 - c_2)r(x) \tag{23}$$

Consider the renewal process whose renewal interval, having cdf  $G(\cdot)$ , is the replacement epochs. We have

$$\bar{G}(x) = \begin{cases} \bar{F}(x) & \text{if } x < T \\ 0 & \text{if } x \geq 0 \end{cases}$$

The expected cost during the service life of an item is

$$\begin{aligned}
 D(T) &= \int_0^T \eta(x) \bar{G}(x) dx + c_2 \\
 &= \int_0^T (c_1 - c_2) r(x) \bar{F}(x) dx + c_2 \\
 &= (c_1 - c_2) \int_0^T f(x) dx + c_2 \\
 &= (c_1 - c_2) F(T) + c_2 \\
 &= c_1 F(T) + (1 - F(T)) c_2 \\
 &= c_1 F(T) + c_2 \bar{F}(T)
 \end{aligned} \tag{24}$$

The expected service life is

$$U(T) = \int_0^\infty \bar{G}(x) dx = \int_0^T \bar{F}(x) dx \tag{25}$$

Recall that the long-run expected cost per unit of time is

$$C(T) = \frac{D(T)}{U(T)} \tag{26}$$

Using the above equations, we obtain

$$C(T) = \frac{c_1 F(T) + c_2 \bar{F}(T)}{\int_0^T \bar{F}(x) dx} \tag{27}$$

Observe that this equation coincides with Equation 19.

According to the MCA, the optimal replacement age  $T^*$  satisfies

$$C(T) = \eta(T)$$

Substituting for  $C(T)$  and  $\eta(T)$ , we get the optimality equation

$$C(T) = \frac{c_1 F(T) + c_2 \bar{F}(T)}{\int_0^T \bar{F}(x) dx} \tag{28}$$

which is equivalent to

$$r(T) \int_0^T \bar{F}(u) du + \bar{F}(T) = \frac{c_1}{c_1 - c_2} \tag{29}$$

Note this equation coincides (as it must) with Equation 20, the optimality equation derived using differential (calculus) optimization.

## 4.4 ASYMPTOTIC BEHAVIOR

We study the behavior of the objective cost function  $C(T)$  and that of the marginal cost function  $\eta(T)$  as the inspection interval  $T$  gets large. We begin with the behavior of the expected cycle length.

$$\begin{aligned} E[T_1] &\rightarrow \frac{\int_0^\infty x dF(x)}{\int_0^\infty e^{-\mu x} dF(x)} \\ &\rightarrow \frac{E[X]}{E[e^{-\mu X}]} \end{aligned} \quad (30)$$

Therefore,

$$ECL(T) \rightarrow E[Y] + \frac{E[X]}{E[e^{-\mu X}]} \quad (31)$$

Now, we proceed with the behavior of the expected cycle cost.

$$\begin{aligned} E[C_1] &\rightarrow \frac{c_f + c_r(1 - \int_0^\infty e^{-\mu x} dF(x))}{\int_0^\infty e^{-\mu x} dF(x)} \\ &\rightarrow \frac{c_f + c_r(1 - E[e^{-\mu X}])}{E[e^{-\mu X}]} \end{aligned} \quad (32)$$

Therefore,

$$ECC(T) \rightarrow c_d + c_r + \frac{c_f + c_r(1 - E[e^{-\mu X}])}{E[e^{-\mu X}]} \quad (33)$$

Finally, we have

$$\begin{aligned} C_T &\rightarrow \frac{c_d + c_r + \frac{c_f + c_r(1 - E[e^{-\mu X}])}{E[e^{-\mu X}]}}{E[Y] + \frac{E[X]}{E[e^{-\mu X}]}} \\ &\rightarrow \frac{c_f + c_r + c_d E[e^{-\mu X}]}{E[X] + E[Y] E[e^{-\mu X}]} \end{aligned} \quad (34)$$

We turn our attention to the marginal cost function of our model.

$$\eta(T) = r(T) \left( p c_d e^{-\mu T} + c_f - c_p \right) - (1 - p) \mu e^{-\mu T} c_d \quad (35)$$

As  $T \rightarrow \infty$ ,  $\eta(T) \rightarrow r(T)(c_f - c_p) \rightarrow \infty$ , assuming an increasing failure rate  $r(T)$ .

Transitions Among States			
$Z(t)$	Event (Change) in $(t, t + \Delta)$	$Z(t + \Delta)$	Probability
2	nothing occurs	2	$1 - \lambda\Delta$
1	repair completed	2	$\mu\Delta$
2	failure occurs	1	$\lambda\Delta$
1	nothing occurs	1	$1 - (\lambda + \mu)\Delta$
0	repair completed	1	$\mu\Delta$
1	failure occurs	0	$\lambda\Delta$
0	nothing occurs	0	$1 - \mu\Delta$

Table 4.1: Analytical Results for Exponential Lifetime and repair Distributions

## 4.5 SPECIAL CASE: EXPONENTIAL LIFE-TIMES

This case is quite well known in the literature. The interested reader is referred to Ravichandran [46]. Here, lifetime and repair time are independent random variables. They are exponentially distributed with parameters  $\lambda$  and  $\mu$ , respectively. Recall that  $\{Z(t), t \geq 0\}$  and  $\{R(t), t \geq 0\}$  denote, respectively, the number of units available and the number of units in the repair facility, at time  $t$ . These are stochastic processes, with state spaces  $S_Z = \{0, 1, 2\}$  and  $S_R = \{0, 1, 2\}$ . Due to the memoryless property, we observe that

$$P\{Z(t+s) = j \mid Z(u), 0 \leq u \leq t\} = P\{Z(t+s) = j \mid Z(t)\}, j \in S_Z = \{0, 1, 2\}. \quad (36)$$

Hence, the Markovian property holds, so  $\{Z(t), t \geq 0\}$  is a Markov process. Let us study its behavior in an infinitesimal time  $\Delta$ .

Denote by  $P_j(t)$  the probability that the system is in state  $j$  at time  $t$ , given that all units were operational at the time origin.

$$P_j(t) = P\{Z(t) = j \mid Z(0) = 2\}$$

From the Table 4.1 above, we can write

$$\begin{aligned}
P_0(t + \Delta) &= P_0(t)(1 - \mu\Delta) + P_1(t)\lambda\Delta \\
P_0(t + \Delta) - P_0(t) &= -P_0(t)\mu\Delta + P_1(t)\lambda\Delta \\
\lim_{\Delta \rightarrow 0^+} \left[ \frac{1}{\Delta} (P_0(t + \Delta) - P_0(t)) \right] &= P_0(t)\mu + P_1(t)\lambda \\
P_0'(t) &= P_1(t)\lambda - P_0(t)\mu
\end{aligned} \tag{37}$$

Similarly, we derive

$$P_1'(t) = P_2(t)\lambda + P_1(t)\mu - P_0(t)(\lambda + \mu) \tag{38}$$

$$P_2'(t) = P_1(t)\mu - P_2(t)\lambda \tag{39}$$

The steady state distribution of the process  $\{Z(t), t \geq 0\}$  is obtained by setting the above time derivatives in the above Equations 37, 38 and 39 equal to zero.

$$\begin{aligned}
\lambda\pi_1 - \mu\pi_0 &= 0 \\
\lambda\pi_2 + \mu\pi_1 - (\lambda + \mu)\pi_0 &= 0 \\
\mu\pi_1 - \lambda\pi_2 &= 0 \\
\pi_0 + \pi_1 + \pi_2 &= 0
\end{aligned}$$

These have the solution

$$\pi_0 = \frac{\lambda^2}{\lambda^2 + \lambda\mu + \mu^2} \tag{40}$$

$$\pi_1 = \frac{\lambda\mu}{\lambda^2 + \lambda\mu + \mu^2} \tag{41}$$

$$\pi_2 = \frac{\mu^2}{\lambda^2 + \lambda\mu + \mu^2} \tag{42}$$

The stationary availability of the system corresponds the long run fraction of time that at least one unit is operational,

$$\alpha = \pi_1 + \pi_2 = \frac{\mu^2 + \lambda\mu}{\lambda^2 + \lambda\mu + \mu^2} \quad (43)$$

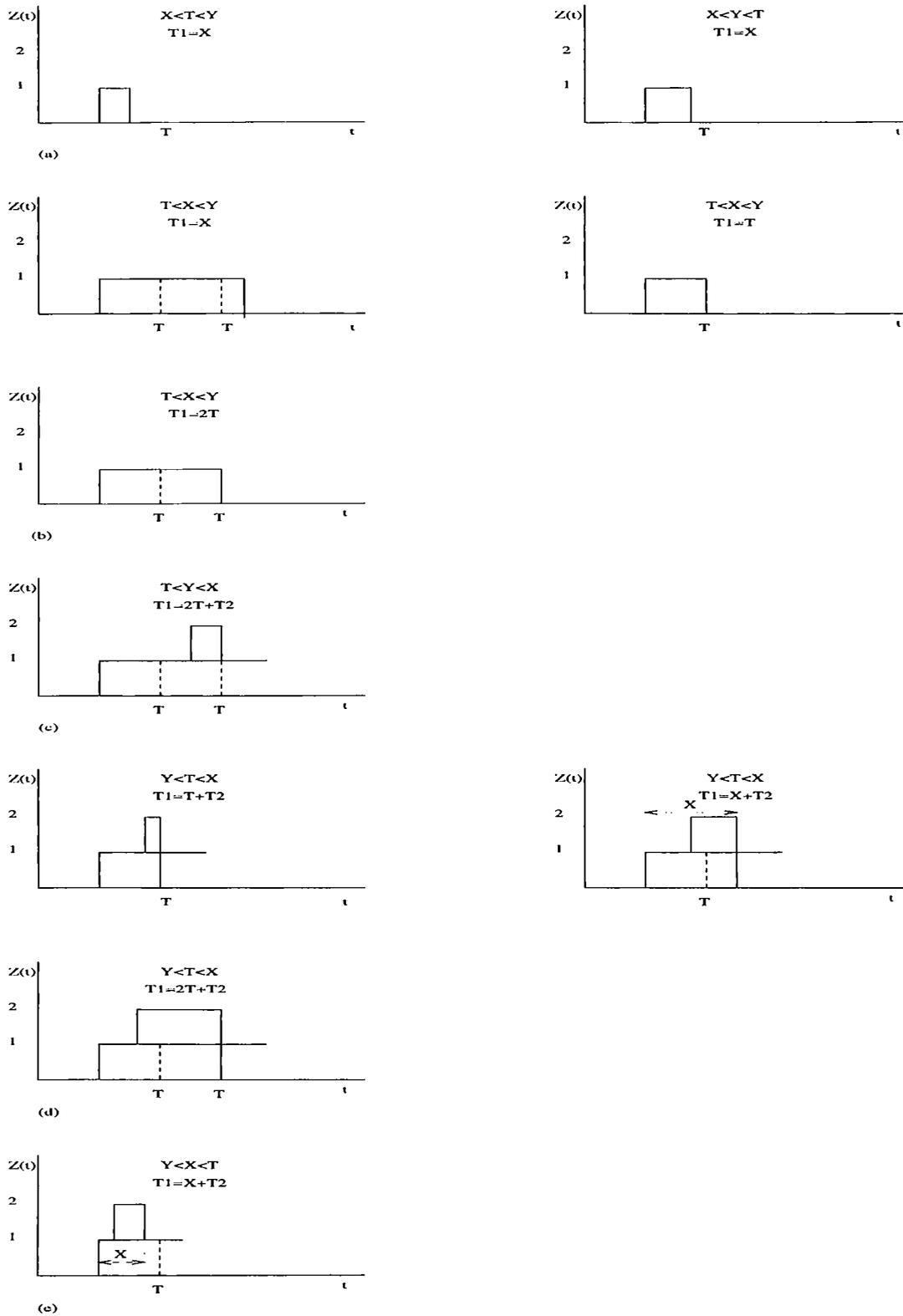


Figure 4.3: Cycle length, generalized policy 1, with inspection carried out at intervals of length  $T$ : (a)  $X < \min\{T, Y\}$ ; (b)  $T < X < Y$ ; (c)  $T < Y < X$ ; (d)  $Y < T < X$ ; (e)  $Y < X < T$ .

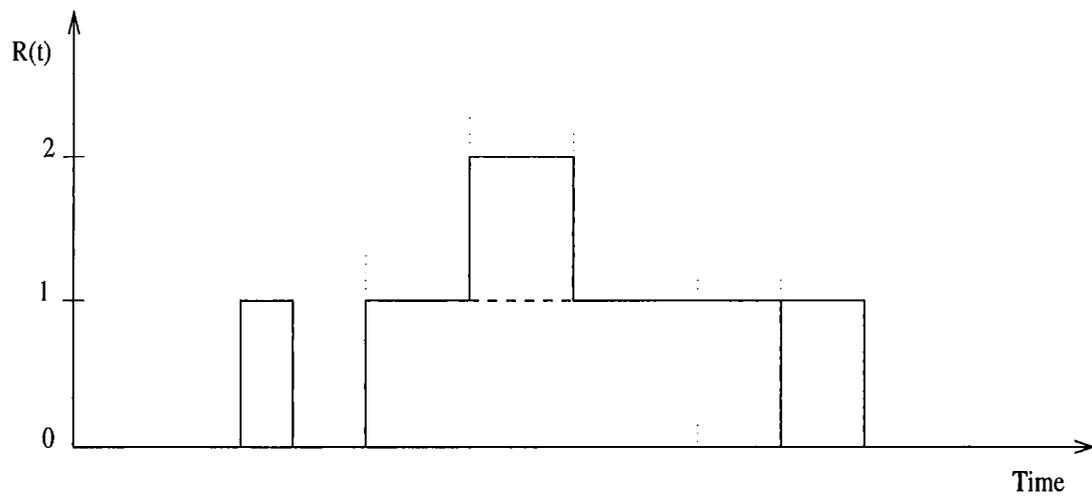
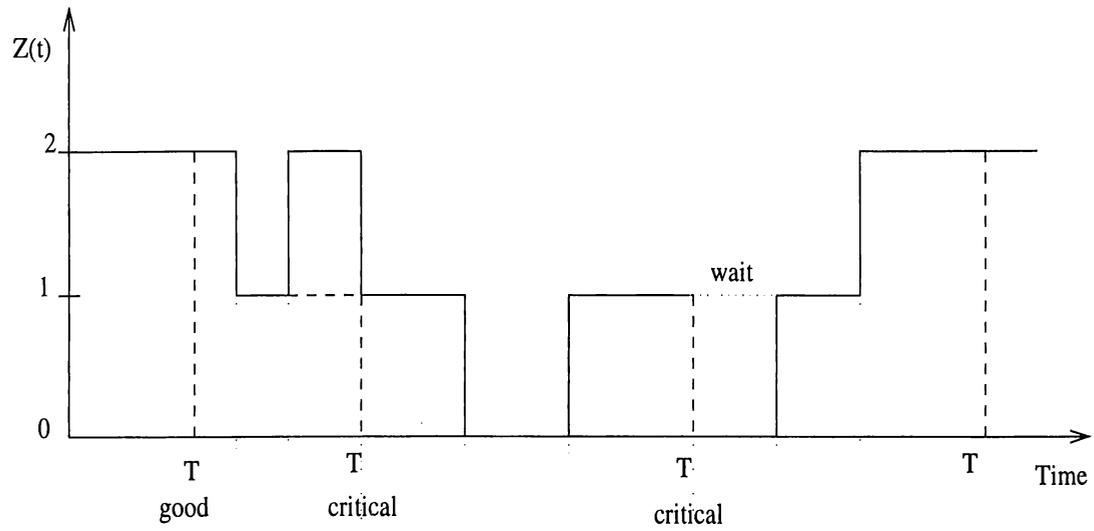


Figure 4.4: A typical realization of the system, for generalized Policy 2

# Chapter 5

## NUMERICAL RESULTS

### 5.1 MODEL PARAMETERS

After determining the expected cycle costs and expected cycle length, let us experiment with the model, with the generalized Policy 1. Consider the following costs:

- $c_p=0.1$ ;
- $c_f=3$ ;
- $c_r=0.5$ ;
- $c_d=5$ ;

Recall that an inspected unit is in the critical state with probability  $1 - p$ . To justify inspections, it is plausible to assume that the inspection cost,  $c_p$ , is much lower than the failure cost,  $c_f$ .

The lifetime distribution is Weibull, whose parameters we shall denote by  $\alpha$  and  $\beta$ . The hazard rate function is

$$r(x) = \frac{\beta}{\alpha} \left(\frac{x}{\beta}\right)^{\beta-1}.$$

This distribution is widely used in reliability analysis, as it covers IFR ( $\beta > 1$ ) and DFR ( $\beta < 1$ ) cases, and the exponential distribution as a particular case ( $\beta = 1$ ). The Raleigh distribution is also a particular case of the Weibull distribution ( $\beta = 2$ ), and its hazard rate function is linear. The mean value of the Weibull distribution is

$$E(X) = \alpha\Gamma(1 + \beta^{-1})$$

where  $\Gamma(\cdot)$  is the Gamma function. We let the parameters of the lifetime distribution,  $X$ , be  $(\alpha, \beta) = (5, 2)$ . Recall that our work assumes that the repair time distribution is exponential with parameter  $\mu$ . We take  $\mu = 0.25$ . Hence, the expected values are  $E(X) = 5\sqrt{\pi}/2$ , and  $E(Y) = 4$ . This prevents congestion of the repair facility, since  $E[Y] < E[X]$ .

## 5.2 $p$ CONSTANT

Expected cycle length, expected cycle cost and expected cost per unit time, in the long run are provided in Table 5.2, for  $p = 0.5$ . The objective function  $C(T)$ , the expected cost per unit time, in the long run, and the MCF  $\eta(T)$  are provided in Figure 5.2 for  $p = 0.5$ . We note that the optimal inspection time,  $T^*$ , almost occurs at the intersection of the two functions. This result parallels the similar results obtained by Berg [10] [11] for the Age Replacement maintenance policy. Table 5.4 gives the marginal cost values. Note that these marginal cost values are increasing with  $T$ , since the lifetime is an IFR distribution. For the special case  $p = 1$ , it is clear that inspection is not economical, since whenever inspected, the unit is found in the good state. This is a degenerate case, and inspection should not be done. Table 5.1 and Figure 5.1 show this. Finally, Table 5.3 and Figure 5.3 show results for the special case  $p = 0$ , which reduces to the simple Policy 1, studied in Chapter 3.

$\alpha = 5, \beta = 2, \mu = 0.25, p = 1$			
T	ECL(T)	ECC(T)	C(T) ( $10^{-3}$ )
1	15.564	15.160	974
2	15.564	14.582	937
3	15.564	14.389	924
4	15.564	14.293	918
5	15.564	14.235	914
6	15.564	14.197	912
7	15.564	14.171	910
8	15.564	14.154	909
9	15.564	14.144	908
10	15.564	14.139	908
11	15.564	14.136	908
12	15.564	14.135	908
13	15.564	14.135	908
14	15.564	14.134	908
15	15.564	14.134	908
16	15.564	14.134	908
17	15.564	14.134	908
18	15.564	14.134	908
19	15.564	14.134	908
20	15.564	14.134	908

Table 5.1: Numerical results, Generalized Policy 1, for the degenerate case  $p = 1$

$\alpha = 5, \beta = 2, \mu = 0.25, p = 0.50$			
T	ECL(T)	ECC(T)	C(T) ( $10^{-3}$ )
1	6.637	6.766	1019
2	9.388	8.512	907
3.0	11.570	10.145	877
3.1	13.086	11.559	876
3.2	13.223	11.667	875
3.3	13.355	11.771	874
3.4	13.480	11.872	873
3.5	13.599	11.970	873
3.6	13.713	12.065	872
3.7	13.821	12.157	872
3.8	13.823	12.245	872
3.9	14.021	12.331	872
4.0	13.132	11.448	872
4.1	13.086	11.559	872
4.2	13.223	11.667	872
4.3	13.355	11.771	872
4.4	13.480	11.872	873
4.5	13.599	11.970	873
4.6	13.713	12.065	874
4.7	13.821	12.157	874
4.8	13.823	12.245	875
4.9	14.021	12.331	875
5	14.171	12.414	876
6	14.816	13.092	884
7	15.188	13.542	892
8	15.387	13.821	898
9	15.486	13.980	903
10	15.532	14.063	905
11	15.552	14.104	907
12	15.560	14.122	908
13	15.563	14.130	908
14	15.564	14.133	908
15	15.564	14.134	908
16	15.564	14.134	908
17	15.564	14.134	908
18	15.564	14.134	908

Table 5.2: Numerical results, generalized Policy1, with  $p = 0.50$

$\alpha = 5, \beta = 2, \mu = 0.25, p = 0$			
T	ECL(T)	ECC(T)	C(T) ( $10^{-3}$ )
1	5.263	5.913	1124
2	7.041	6.648	944
3	9.152	7.843	857
3.5	10.219	8.568	838
3.6	10.426	8.719	836
3.7	10.632	8.871	834
3.8	10.834	9.024	833
3.9	11.032	9.178	832
4.0	11.227	9.331	831
4.1	11.418	9.485	831
4.2	11.604	9.638	831*
4.3	11.787	9.791	831
4.4	11.964	9.942	831
4.5	12.137	10.092	832
4.6	12.305	10.241	832
4.7	12.467	10.388	833
4.8	12.624	10.532	834
4.9	12.777	10.675	835
5.0	12.923	10.814	837
5.1	13.065	10.952	838
5.2	13.201	11.086	840
5.3	13.332	11.218	841
5.4	13.457	11.346	843
5.5	13.577	11.471	845
6	14.102	12.044	854
7	14.819	12.926	872
8	15.212	13.489	887
9	15.409	13.815	896
10	15.500	13.988	902
11	15.540	14.072	906
12	15.555	14.110	907
13	15.563	14.125	908
14	15.563	14.131	908
15	15.564	14.133	908
16	15.564	14.134	908
17	15.564	14.134	908
18	15.564	14.134	908

Table 5.3: Numerical results, generalized Policy 1, for the special case  $p = 0$

$\mu = 0.25$								
$\eta(T)$								
Main Policy								Alt. Policy
T	$p = 0$	$p = 1$	$p = 0.95$	$p = 0.90$	$p = 0.50$	$p = 0.10$	$p = 0.01$	<i>all p</i>
1	-0.742	0.544	0.479	0.415	-0.099	-0.613	-0.729	0.544
2	-0.294	0.949	0.887	0.825	0.328	-0.170	-0.282	0.949
3	0.106	1.263	1.205	1.147	0.684	0.221	0.117	1.263
4	0.468	1.517	1.464	1.412	0.992	0.573	0.479	1.517
5	0.802	1.733	1.686	1.640	1.267	0.895	0.811	1.733
6	1.113	1.928	1.887	1.846	1.520	1.194	1.121	1.928
7	1.407	2.110	2.075	2.040	1.759	1.477	1.414	2.110
8	1.687	2.289	2.259	2.229	1.988	1.747	1.693	2.289
9	1.956	2.467	2.442	2.416	2.212	2.007	1.961	2.467
10	2.217	2.648	2.627	2.605	2.433	2.260	2.222	2.648
11	2.472	2.833	2.815	2.797	2.653	2.508	2.476	2.833
12	2.722	3.023	3.008	2.993	2.872	2.752	2.725	3.023
13	2.967	3.218	3.205	3.193	3.092	2.992	2.970	3.218
14	3.210	3.417	3.407	3.396	3.314	3.231	3.212	3.417
15	3.451	3.621	3.612	3.604	3.536	3.468	3.452	3.621
16	3.689	3.829	3.822	3.815	3.759	3.703	3.690	3.829
17	3.926	4.041	4.035	4.030	3.984	3.938	3.927	4.041
18	4.162	4.256	4.251	4.246	4.209	4.172	4.163	4.256
19	4.397	4.474	4.470	4.466	4.435	4.405	4.398	4.474
20	4.632	4.694	4.690	4.688	4.663	4.638	4.632	4.694
21	4.865	4.916	4.914	4.911	4.891	4.870	4.866	4.916
22	5.099	5.140	5.138	5.136	5.119	5.103	5.099	5.140
23	5.332	5.365	5.364	5.362	5.349	5.335	5.332	5.365
24	5.565	5.592	5.590	5.589	5.578	5.568	5.565	5.592
25	5.798	5.819	5.818	5.817	5.808	5.800	5.798	5.819
26	6.030	6.048	6.047	6.046	6.039	6.032	6.030	6.048
27	6.262	6.277	6.276	6.275	6.270	6.264	6.263	6.277
28	6.495	6.506	6.506	6.505	6.500	6.496	6.495	6.506
29	6.727	6.736	6.736	6.735	6.732	6.728	6.727	6.736
30	6.959	6.967	6.966	6.966	6.963	6.960	6.959	6.967
31	7.191	7.197	7.197	7.197	7.194	7.192	7.192	7.197
32	7.424	7.428	7.428	7.428	7.426	7.424	7.424	7.428
33	7.656	7.659	7.659	7.659	7.658	7.656	7.656	7.659
34	7.888	7.891	7.891	7.890	7.889	7.888	7.888	7.891
35	8.120	8.122	8.122	8.122	8.121	8.120	8.120	8.122

Table 5.4: Marginal Cost Values for Various Values of  $p$

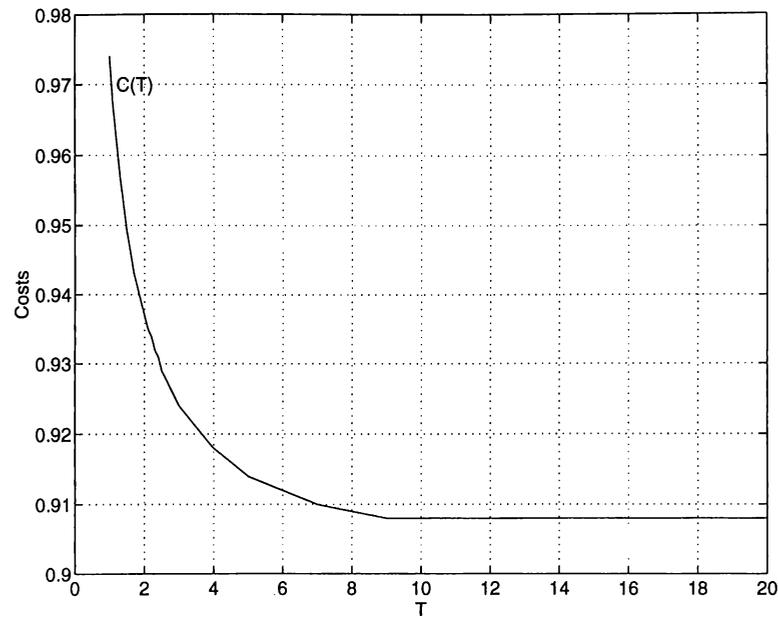


Figure 5.1: Cost curve,  $C(T)$ , versus inspection interval  $T$ , for the degenerate case  $p = 1$

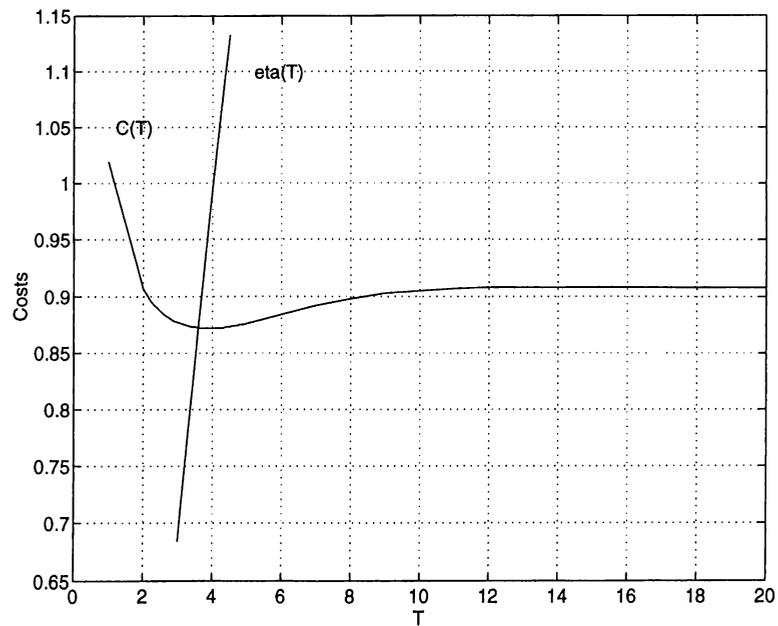


Figure 5.2: Cost curve,  $C(T)$ , and marginal cost function,  $\eta(T)$ , versus inspection interval  $T$ , with  $p = 0.50$

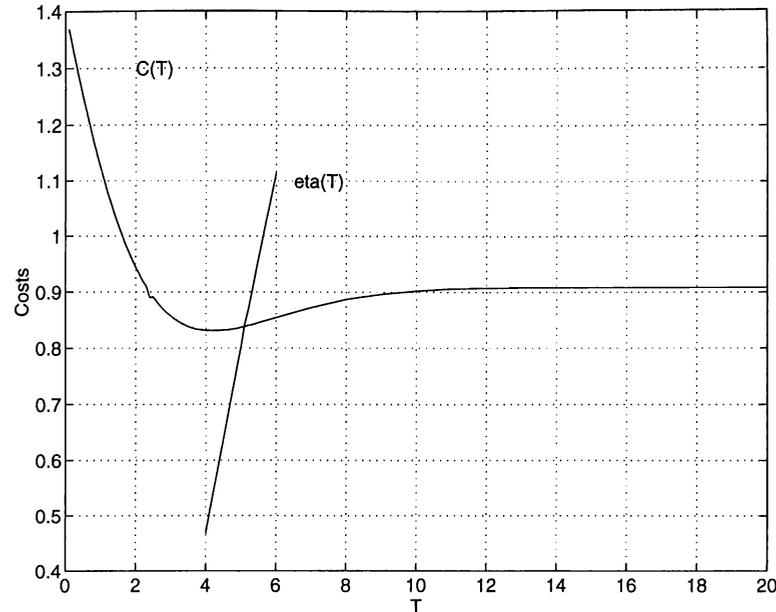


Figure 5.3: Cost curve,  $C(T)$ , and marginal cost function,  $\eta(T)$ , versus inspection interval  $T$ , with  $p = 0$

### 5.3 $p$ A FUNCTION OF $T$

We now assume  $p$  to be a decreasing function of inspection time  $T$ . We let  $p(T) = e^{-\delta T}$ , where  $\delta$  is a parameter. As a result, a unit that has survived the first inspection is less likely to survive the next inspection. Actually, the motivation behind assuming this functional form is that it is plausible to assume  $p(kT) = [P(T)]^k$ ,  $k=1,2,\dots$ . The exponential function is a continuous function that nicely suits the purpose. To handle this realistic extension we only substitute  $p(T)$  for  $p$  in our formulas. The results are similar, and the optimal inspection interval  $T^*$  occurs near at the intersection of the objective function and the marginal cost curves  $C(T)$  and  $\eta(T)$ . For the Age Replacement policy, Berg [10] [11] showed that the optimal replacement age occurs at the intersection of the two curves. The interpretation of this finding is that the marginal cost function is below the average cost function in the region where the wastage of the remaining lifetimes of the units is higher than the gain in averting failures; the marginal cost function is above the average cost function in the region where the gain in averting failures is higher than the wastage of

the remaining lifetimes of the units.

Expected cycle length, expected cycle cost and expected cost per unit time, in the long run are provided in Table 5.5-5.5 for  $\delta = 0.95$  and for  $\delta = 0.50$ , respectively. The objective function  $C(T)$  and the MCF  $\eta(T)$  are provided in Figures 5.4-5.5 for  $\delta = 0.95$  and for  $\delta = 0.50$ , respectively. We observe that the optimal inspection time,  $T^*$ , is near the intersection of the two functions. Table 5.7 provides the marginal cost values.

$\alpha = 5, \beta = 2, \mu = 0.25, \delta = 0.95$			
T	ECL(T)	ECC(T)	C(T) ( $10^{-3}$ )
1	6.148	6.430	1046
2	7.554	7.015	929
3.0	9.378	8.045	858
3.1	9.572	9.559	853
3.2	9.768	9.704	850
3.3	9.964	9.850	846
3.4	10.159	9.995	843
3.5	10.354	10.140	840
3.6	10.548	10.283	838
3.7	10.740	10.425	836
3.8	10.930	10.565	835
3.9	11.118	10.704	834
4.0	11.303	9.413	833
4.1	11.485	9.559	832
4.2	11.664	9.704	832
4.3	11.839	9.850	832*
4.4	12.010	9.995	832
4.5	12.177	10.140	833
4.6	12.340	10.283	833
4.7	12.498	10.425	834
4.8	12.652	10.565	835
4.9	12.800	10.704	836
5.0	12.944	10.840	837
6	14.106	12.051	854
7	14.820	12.928	872
8	15.212	13.490	887
9	15.409	13.815	896
10	15.500	13.988	902
11	15.540	14.072	906
12	15.555	14.110	907
13	15.561	14.125	908
14	15.563	14.131	908
15	15.564	14.133	908
16	15.564	14.134	908
17	15.564	14.134	908
18	15.564	14.134	908

Table 5.5: Numerical results, generalized Policy 1, with  $\delta = 0.95$

$\alpha = 5, \beta = 2, \mu = 0.25, \delta = 0.50$			
T	ECL(T)	ECC(T)	C(T) ( $10^{-3}$ )
1	7.292	7.257	995
2	8.564	7.810	912
3	10.093	8.703	862
3.1	10.254	9.966	859
3.2	10.416	10.089	856
3.3	10.578	10.212	853
3.4	10.741	10.335	851
3.5	10.903	10.458	848
3.6	11.064	10.580	847
3.7	11.225	10.702	845
3.8	11.384	10.824	844
3.9	11.542	10.944	842
4.0	11.699	9.844	841
4.1	11.854	9.966	841
4.2	12.006	10.089	840
4.3	12.156	10.212	840
4.4	12.303	10.335	840*
4.5	12.447	10.458	840
4.6	12.588	10.580	840
4.7	12.726	10.702	841
4.8	12.861	10.824	842
4.9	12.992	10.944	842
5.0	13.119	11.063	843
6	14.172	12.146	857
7	14.841	12.963	873
8	15.218	13.501	887
9	15.410	13.819	896
10	15.501	13.989	902
11	15.540	14.072	906
12	15.556	14.110	907
13	15.561	14.125	908
14	15.563	14.131	908
15	15.564	14.133	908
16	15.564	14.134	908
17	15.564	14.134	908
18	15.564	14.134	908

Table 5.6: Numerical results, generalized Policy 1, with  $\delta = 0.50$

$\mu = 0.25$							
$\eta(T)$							
Gen Policy 1							Policy 2
T	$\delta = 0.95$	$\delta = 0.90$	$\delta = 0.80$	$\delta = 0.50$	$\delta = 0.10$	$\delta = 0.01$	<i>all p</i>
1	1.186	1.206	1.236	1.219	0.774	0.569	0.544
2	0.323	0.362	0.447	0.721	0.972	0.954	0.949
3	0.302	0.326	0.382	0.627	1.138	1.252	1.263
4	0.531	0.542	0.571	0.734	1.294	1.493	1.517
5	0.822	0.826	0.840	0.937	1.454	1.701	1.733
6	1.119	1.121	1.127	1.181	1.621	1.890	1.928
7	1.409	1.410	1.412	1.411	1.799	2.071	2.110
8	1.687	1.688	1.689	1.704	1.988	2.249	2.289
9	1.956	1.956	1.957	1.965	2.186	2.428	2.467
10	2.217	2.217	2.218	2.222	2.391	2.611	2.648
11	2.472	2.472	2.472	2.474	2.603	2.798	2.833
12	2.722	2.722	2.722	2.723	2.820	2.991	3.023
13	2.968	2.968	2.968	2.968	3.041	3.189	3.218
14	3.210	3.210	3.210	3.210	3.265	3.391	3.417
15	3.451	3.451	3.451	3.451	3.491	3.598	3.621
16	3.689	3.689	3.689	3.689	3.719	3.809	3.829
17	3.926	3.926	3.926	3.926	3.948	4.024	4.041
18	4.162	4.162	4.162	4.162	4.178	4.241	4.256
19	4.397	4.397	4.397	4.397	4.409	4.461	4.474
20	4.632	4.632	4.632	4.632	4.640	4.683	4.694
21	4.865	4.865	4.865	4.865	4.871	4.907	4.916
22	5.099	5.099	5.099	5.099	5.103	5.132	5.140
23	5.332	5.332	5.332	5.332	5.335	5.358	5.365
24	5.565	5.565	5.565	5.565	5.567	5.586	5.592
25	5.798	5.798	5.798	5.598	5.800	5.814	5.819
26	6.030	6.030	6.030	6.030	6.031	6.044	6.048
27	6.262	6.262	6.262	6.262	6.263	6.273	6.277
28	6.495	6.495	6.495	6.495	6.496	6.503	6.506
29	6.727	6.727	6.727	6.727	6.728	6.734	6.736
30	6.959	6.959	6.960	6.959	6.960	6.965	6.967
31	7.191	7.191	7.191	7.191	7.191	7.196	7.197
32	7.424	7.424	7.424	7.424	7.424	7.427	7.428
33	7.656	7.656	7.656	7.656	7.656	7.658	7.659
34	7.888	7.888	7.888	7.888	7.888	7.890	7.891
35	8.120	8.120	8.120	8.120	8.120	8.121	8.122

Table 5.7: Marginal Cost Values for Various Values of  $\delta$

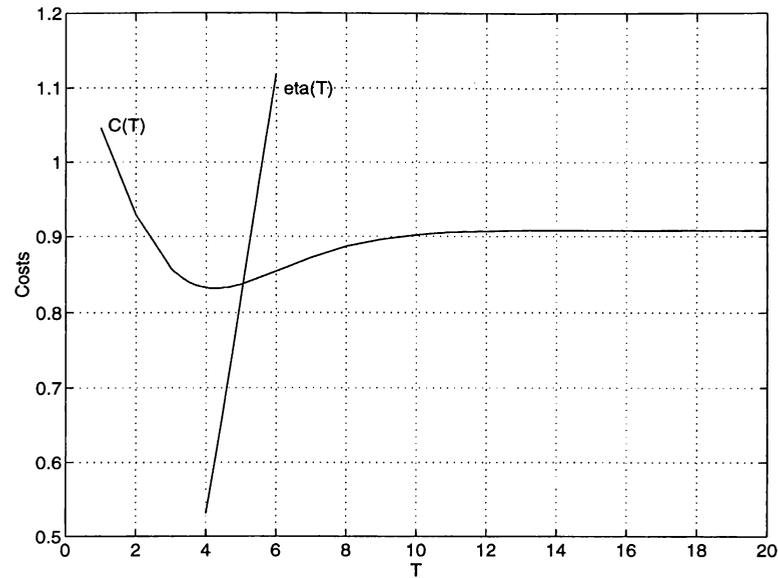


Figure 5.4: Cost curve,  $C(T)$ , and marginal cost function,  $\eta(T)$ , versus inspection interval  $T$ , with  $\delta = 0.95$

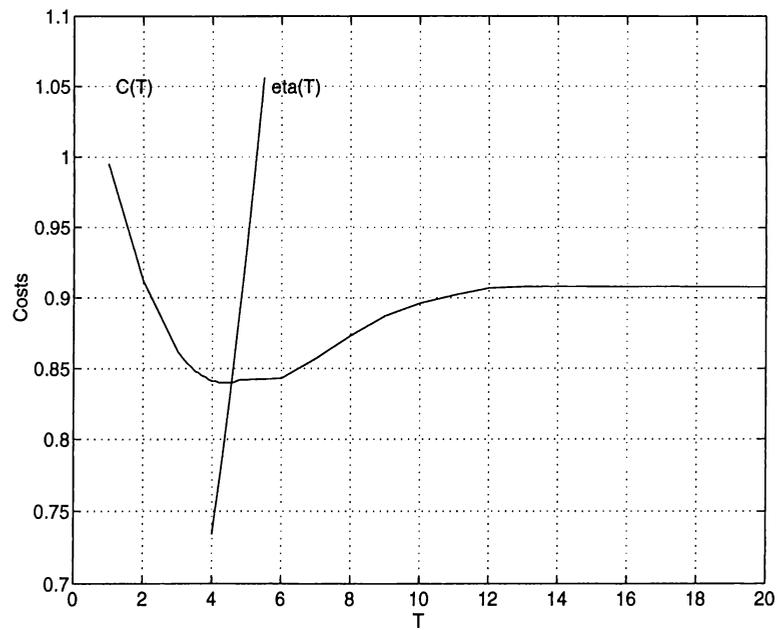


Figure 5.5: Cost curve,  $C(T)$ , and marginal cost function,  $\eta(T)$ , versus inspection interval  $T$ , with  $\delta = 0.50$

## Chapter 6

# CONCLUSION

In this thesis, we suggested and studied two simple maintenance policies of a two unit cold redundant standby system. We applied the Marginal Cost Approach to these policies, derived the marginal cost function  $\eta(T)$ . We compared these policies. Next, we generalized the policies, and derived their marginal cost functions. This study is the first that applies the Marginal Cost Analysis to a model other than the Age Replacement and the Block Replacement models. Our objective was to determine the inspection interval  $T^*$  that yields the minimal expected long run cost per unit time  $C(T)$ . The results we obtained for this relatively complex model are compared to those obtained by Berg [10, 11] for the Age Replacement and the Block Replacement preventive maintenance policies. The optimal inspection time (interval)  $T$  occurs near the intersection of the two curves  $\eta(T)$  and  $C(T)$ . We also witnessed how flat the objective function curve is about  $T^*$ . Consequently, using the marginal cost function  $\eta(T)$  to minimize  $C(T)$ , i.e. setting  $C(T) = \eta(T)$ , is more efficient than just looking at  $C(T)$ . As Berg himself admits, the better insight into the model is perhaps the most important benefit of the marginal cost analysis.

As avenues of future research, we suggest generalizing Policy 2, as we did for Policy 1. We also suggest applying the marginal cost analysis to yet more complex maintenance models, such as the warm standby system. In addition, it may be investigated whether the cost objective function  $C(T)$

could be expressed in terms of the marginal cost function  $\eta(T)$ , as it is the case for the age and the block replacement policies.

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