J. Phys. A: Math. Gen. **31** (1998) 6975–6994. Printed in the UK PII: S0305-4470(98)86614-6

Finite-dimensional Schwinger basis, deformed symmetries, Wigner function, and an algebraic approach to quantum phase

T Hakioglŭ
Physics Department, Bilkent University, 06533 Ankara, Turkey

Received 27 February 1998

Abstract. Schwinger's finite (D) dimensional periodic Hilbert space representations are studied on the toroidal lattice $Z_D \times Z_D$ with specific emphasis on the deformed oscillator subalgebras and the generalized representations of the Wigner function. These subalgebras are shown to be admissible endowed with the non-negative norm of Hilbert space vectors. Hence, they provide the desired canonical basis for the algebraic formulation of the quantum phase problem. Certain equivalence classes in the space of labels are identified within each subalgebra, and connections with area-preserving canonical transformations are examined.

The generalized representations of the Wigner function are examined in the finitedimensional cyclic Schwinger basis. These representations are shown to conform to all fundamental conditions of the generalized phase space Wigner distribution.

As a specific application of the Schwinger basis, the number-phase unitary operator pair in $Z_D \times Z_D$ is studied and, based on the admissibility of the underlying q-oscillator subalgebra, an *algebraic* approach to the unitary quantum phase operator is established. This being the focus of this work, connections with the Susskind–Glogower–Carruthers–Nieto phase operator formalism as well as standard action-angle Wigner function formalisms are examined in the infinite-period limit. The concept of continuously shifted Fock basis is introduced to facilitate the Fock space representations of the Wigner function.

1. Introduction

Recently, finite-dimensional quantum group symmetries find increasing physical applications in condensed matter systems. The Landau problem is known to have the ω_∞ symmetry [1] in the algebra satisfied by magnetic translation operators [2]. An $sl_q(2)$ realization of the same problem has also been recently studied [3]. The finite-dimensional representations of these algebras are parametrized by a discrete set of labels on a two-dimensional toroidal lattice $Z_D \times Z_D$. The action of the group elements on the Hilbert space vectors is cyclic, the periodicity of which is determined by the dimension of the corresponding algebra. In the Landau problem the periodicity is directly connected to the degeneracy of the Landau levels in the ground state [3]. In a more general framework, similar algebraic structures were examined long ago by Schwinger in the unitary cyclic representations of the Weyl–Heisenberg (WH) algebra [4]. Recently Floratos [5] examined the WH algebra parametrized by labelling the vectors on the toroidal lattice in terms of $D^2 - 1$ unitary traceless generators as a convenient representation of su(D). More generally, the elements of the discrete and finite-dimensional WH algebra are generators of the area-preserving diffeomorphisms on $Z_D \times Z_D$ which are known to respect the Fairlie–Fletcher–Zachos

0305-4470/98/336975+20$19.50

sine algebra [6]. The infinite-dimensional extension of this is the group of infinitesimal area-preserving diffeomorphisms which has been examined by Arnold in the theory of phase space formulation of classical Hamiltonian flow [7]. With the connection to areapreserving diffeomorphisms, the finite-dimensional WH algebra defines, in the quantum domain, namely, the set of linear canonical transformations on the discrete canonical phase space pair, the generalized coordinate and the momentum. This has been observed as an emerging *presymplectic* structure preserving the discrete phase space area of which the connection to the classical symplectic structure is established in the continuous limit as the dimension of the algebra is extended to infinity [8].

A more general frame for unitary cyclic representations of finite-dimensional algebras, of which a special case is the WH algebra, is Schwinger's finite, special-unitary-canonical basis [4]. Schwinger's approach proves to be a generalized realization for the group of discrete area-preserving transformations on the 2-torus. This basis has been used indirectly in various applications to physics, particularly to condensed matter [9–11] and field theory related problems [12] such as the discretized versions of the Chern–Simons theory [9, 10], the dynamics of Bloch electrons in two dimensions interacting with a constant uniform magnetic field [10], the quantum Hall effect [11] etc. Most of these applications refer to the discrete WH algebra although the results can be equally valid using the more general Schwinger basis which will be discussed first in section 2.

In this work we will follow a different route than the standard applications above and demonstrate that the Schwinger basis also provides an algebraic approach to the canonical phase space formulation of the well known quantum phase problem. As the first step in this route, the subalgebraic realizations of Schwinger's unitary operator basis will be constructed in sections 3.1 and 3.2 with a particular emphasis on the realizations in terms of the q-oscillator. It will be shown that in finite dimensions, these deformed oscillator realizations naturally lead to an admissible (i.e. non-negative) cyclic spectrum by the natural emergence of a positive Casimir operator. The net effect of the positive Casimir operator is to shift the spectrum to the admissible ranges, namely, a strong condition on the nonnegative norm of the vectors in the Hilbert space. The crucial role played by the admissible cyclic representations in the canonical formulation of the quantum phase problem will be examined. In order to complete the picture, we also briefly discuss the well known $u_q(sl(2))$ subalgebraic realizations of the Schwinger operator basis.

The equivalence classes and their connection to canonical transformations on the discrete lattice will be discussed in section 3.3. Section 4 is devoted to the application of the Schwinger basis in the Wigner–Kirkwood construction of the Wigner function. It will be shown that this generalized construction complies with all fundamental properties of the Wigner function. In section 5 we explore the applications of Schwinger's formalism in the unitary finite-dimensional number-phase operator basis. In this context, we elaborate more on the q-oscillator subalgebraic realizations of section 3.2. In sections 4.1–4.3 we examine the infinite-dimensional limit of the number-phase basis, the q-oscillator subalgebra and the Wigner function respectively. There, it will be shown that as the dimension of the unitary number-phase operator algebra is extended to infinity, the conventional phase operator formalism of Sussking–Glogower–Carruthers–Nieto is recovered. We consider this as the first step to establish the desired unification of the quantum phase problem with the canonical action-angle quantum phase space formalism. The admissible q-oscillator subalgebra is also investigated in the $D \to \infty$ limit and shown to have a linear equidistant spectrum accompanied by a typical spectral singularity at $D = \infty$. This singular behaviour is examined using Fujikawa's index theorem.

The representations of the Wigner function in the phase and number eigenbases are investigated in the finite and infinite Hilbert space dimensions. Within this unification scheme the concept of a continuously shifted finite-dimensional Fock basis is introduced in section 4.4. It is demonstrated that this concept facilitates the formulation of the Wigner function in the Fock eigenbasis. In the following, we start our discussion with a short study of Schwinger's cyclic unitary operator basis.

2. Finite-dimensional Schwinger operator basis

In this formulation [4, 8, 13], one considers a unitary cyclic operator $\hat{\mathcal{U}}$ acting on a finite-dimensional Hilbert space $\mathcal{H}_D$ spanned by a set of orthonormal basis vectors $\{|u\rangle_k\}_{k=0,\ldots,(D-1)}$ with the cyclic property $\hat{\mathcal{U}}^D = \hat{\mathcal{I}}$ as

$$\hat{\mathcal{U}}|u\rangle_k = |u\rangle_{k+1} \qquad |u\rangle_{k+D} = |u\rangle_k \qquad {}_k\langle u|u\rangle_{k'} = \delta_{k,k'}. \tag{1}$$

In the $\{|u\rangle_k\}$ basis, $\hat{\mathcal{U}}$ is represented by

$$\hat{\mathcal{U}} = \sum_{k=0}^{D-1} |u\rangle_{k+1}\,{}_k\langle u|. \tag{2}$$

The action of $\hat{\mathcal{U}}$ corresponds to a *rotation* in $\mathcal{H}_D$. The axis of rotation is along the direction in $\mathcal{H}_D$ given by the vector $|v\rangle_\ell$ of which the direction remains invariant under the action of $\hat{\mathcal{U}}$ as

$$\hat{\mathcal{U}}|v\rangle_\ell = e^{i\gamma_0\ell}|v\rangle_\ell \qquad |v\rangle_\ell = \frac{1}{\sqrt{D}}\sum_{k=0}^{D-1} v_k^{(\ell)}|u\rangle_k \qquad 0 \leqslant \ell \leqslant D-1 \tag{3}$$

where $v_k^{(\ell)} = e^{-i\gamma_0 k\ell}$ and $\gamma_0 = 2\pi/D$. On the other hand it was shown by Schwinger that the new set $\{|v\rangle_\ell\}_{\ell=0,\ldots,(D-1)}$ also forms an orthonormal set of vectors i.e. ${}_\ell\langle v|v\rangle_{\ell'} = \delta_{\ell,\ell'}$, for which one can define a second unitary operator $\hat{\mathcal{V}}$ such that $\hat{\mathcal{V}}^D = \hat{\mathcal{I}}$ and

$$\begin{aligned} \hat{\mathcal{V}}|v\rangle_\ell &= |v\rangle_{\ell+1} \qquad |v\rangle_{\ell+D} = |v\rangle_\ell \\ \hat{\mathcal{V}}|u\rangle_k &= e^{-i\gamma_0 k}|u\rangle_k \end{aligned} \tag{4}$$

where $k \in \mathbb{Z}$ and $0 \leqslant k \leqslant (D-1)$. The basis vectors $\{|u\rangle_k\}_{k=0,\ldots,(D-1)}$ and $\{|v\rangle_\ell\}_{\ell=0,\ldots,(D-1)}$ define two equivalent and *conjugate* representations in the sense that the representation in the $\{|u\rangle_k\}_{k=0,\ldots,(D-1)}$ basis in equation (3) is complemented by

$$|u\rangle_k = \frac{1}{\sqrt{D}}\sum_{\ell=0}^{D-1} u_\ell^{(k)}|v\rangle_\ell \qquad u_\ell^{(k)} = e^{i\gamma_0\ell k} = v_k^{(\ell)*}. \tag{5}$$

The corresponding operators $\hat{\mathcal{U}}$ and $\hat{\mathcal{V}}$ satisfy

$$\begin{aligned} &\hat{\mathcal{U}}^{m_1}\hat{\mathcal{V}}^{m_2} = e^{i\gamma_0 m_1 m_2}\hat{\mathcal{V}}^{m_2}\hat{\mathcal{U}}^{m_1} \\ &\hat{\mathcal{U}}^{m_1+D} = \hat{\mathcal{U}}^{m_1} \qquad \text{and} \qquad \hat{\mathcal{V}}^{m_2+D} = \hat{\mathcal{V}}^{m_2}. \end{aligned} \tag{6}$$

An operator Ψ, of which the projection in the $|u\rangle_k$ representation is $\Psi(u_k)$, is given in the $|v\rangle_\ell$ representation as $\tilde\Psi(v_\ell)$. These two conjugate representations are then connected by

$$\Psi(u_k) = \frac{1}{\sqrt{D}} \sum_{\ell=0}^{D-1} {}_k\langle u|v\rangle_\ell \, \tilde\Psi(v_\ell) \qquad \text{where} \qquad {}_k\langle u|v\rangle_\ell = e^{-i\gamma_0 \ell k}. \tag{7}$$

In analogy with the elements of the discrete Wigner–Kirkwood basis [14, 15], we now define the operator [4, 13]

$$\hat S_m \equiv e^{-i\gamma_0 m_1 m_2/2}\hat U^{m_1}\hat V^{m_2} = e^{i\gamma_0 m_1 m_2/2}\hat V^{m_2}\hat U^{m_1} \tag{8}$$

where $m = (m_1, m_2)$. We now represent the transformation in equations (3) and (5) between $\{|v\rangle_\ell\}_{0\leqslant\ell\leqslant(D-1)}$ and $\{|u\rangle_k\}_{0\leqslant k\leqslant(D-1)}$ bases using the unitary Fourier operator $\hat F$ defined as [16], $\{|v\rangle_k\} \equiv \hat F\{|u\rangle_k\}$, and $\{|u\rangle_k\} \equiv \hat F^{-1}\{|v\rangle_k\}$, where $\hat F^\dagger = \hat F^{-1}$. Then,

$$|u\rangle_k \xrightarrow{\hat F} |v\rangle_k \xrightarrow{\hat F} |u\rangle_{-k} \xrightarrow{\hat F} |v\rangle_{-k} \xrightarrow{\hat F} |u\rangle_k \tag{9}$$

$$|u\rangle_k \xrightarrow{\hat F^{-1}} |v\rangle_{-k} \xrightarrow{\hat F^{-1}} |u\rangle_{-k} \xrightarrow{\hat F^{-1}} |v\rangle_k \xrightarrow{\hat F^{-1}} |u\rangle_k.$$

The equations (9) produce a Fourier automorphism at the operator level as

$$\hat U \xrightarrow{\hat F} \hat V \xrightarrow{\hat F} \hat U^{-1} \xrightarrow{\hat F} \hat V^{-1} \xrightarrow{\hat F} \hat U \tag{10}$$

$$\hat U \xrightarrow{\hat F^{-1}} \hat V^{-1} \xrightarrow{\hat F^{-1}} \hat U^{-1} \xrightarrow{\hat F^{-1}} \hat V \xrightarrow{\hat F^{-1}} \hat U.$$

Next, we define a transformation $R_{\pi/2}$ in the space of the lattice vector m such that $R_{\pi/2} : (m_1, m_2) \to (-m_2, m_1)$. It is possible to show that

$$\hat F \hat S_m \hat F^{-1} = \hat S_{R_{\pi/2}:m} \qquad \hat F^4 = I \qquad \text{and} \qquad R_{\pi/2}^4 = I. \tag{11}$$

Equations (11) imply that equation (8) is invariant under simultaneous operations of $\hat F$ and $R^{-1}_{\pi/2}$. $\hat S_m$ has the properties

$$\begin{aligned}
&\hat S_m^\dagger = \hat S_{-m} \\
&\mathrm{Tr}\{\hat S_m\} = D\delta_{m,\mathbf{0}} \\
&\hat S_m \hat S_{m'} = e^{i\gamma_0 m\times m'/2}\hat S_{m+m'} \\
&(\hat S_m \hat S_{m'})\hat S_{m''} = \hat S_m(\hat S_{m'}\hat S_{m''}) \qquad \text{(associativity)} \\
&\hat S_{\mathbf{0}} = I \ \text{(unit element)} \qquad \hat S_m \hat S_{-m} = I \qquad \text{(inverse)}
\end{aligned} \tag{12}$$

where $m \times m' \equiv (m_1 m_2' - m_2 m_1')$. Using equations (8) and (12) it is possible to see that

$$(S_m)_D = S_{Dm} = S_{-Dm} = (-1)^{Dm_1m_2} I \tag{13}$$

where $\hat{S}_{Dm}$ commutes with all elements $\hat{S}_{m^0}$ for all m and m^0. With the associativity condition in equations (12) satisfied, the unitary Schwinger operator basis $\hat{S}_m$ defines a discrete projective representation of the Heisenberg algebra parametrized by the *discrete phase space* vector m in$^{m_1}Z_D^{m_2} \times Z_D$. Excluding m = **0** and if^2 D is a prime number, the elements of the basis $\hat{U}\,\hat{V}$ form a complete set of D − 1 unitary traceless matrices providing an irreducible representation for su(D). If D is not a prime, then the prime decomposition of D as $D = D_1D_2 \ldots D_i \ldots$, as shown in [4,13], permits the study of a physical system with a number of quantum degrees of freedom with each degree of freedom expressed in terms of an independent Schwinger basis with the cyclic property determined by the particular prime factor D_j. In what follows, we will assume that D is a prime number representing a single degree of freedom. Exceptional cases will be independently mentioned as needed.

The eigenspace of $\hat{S}_m$ is spanned by the eigenvectors $|\boldsymbol{m}, r\rangle_{\{0\leqslant r\leqslant(D-1)\}}$ $|\boldsymbol{m}$with eigenvalues

λ_r(6m)(.D −Using equations (1) and (4) we expand the eigenvectors1) in, for instance, the $|ui_k$ basis with coefficients $e_k^{(r)}$(m,r) i≡where 0_khu|m,r6i. r

From this definition and equation (1) it is clear that the coefficients are periodic, i.e. $e_k^{(r)}(\boldsymbol{m}) = e_{k+D}^{(r)}(\boldsymbol{m})$. The coefficients and eigenvalues are then determined by the recursion

$$\lambda_r(\boldsymbol{m})e_k^{(r)}(\boldsymbol{m}) = \mathrm{e}^{-\mathrm{i}\beta_k(m_1,m_2)}e_{k-m_1}^{(r)}(\boldsymbol{m}) \quad \text{where} \quad \beta_k(m_1,m_2) = \gamma_0 m_2(2k - m_1)/2 \tag{14}$$

which yields

$$e_k^{(r)}(\boldsymbol{m}) = \left\{\prod_{n=0}^{M-1}\lambda_r(\boldsymbol{m})\mathrm{e}^{-\mathrm{i}\beta_{k-nm_1}(m_1,m_2)}\right\}e_{k+1}^{(r)}(\boldsymbol{m})$$

$$\lambda_r(m) = e^{i\pi m_1 m_2} e^{-i\frac{2\pi r}{D}} \tag{15}$$

where M = [(modD) + 1]/m_1 and M ∈ Z. In deriving equation (15) from (14) we used the periodicity property $e_k^{(r)}(\boldsymbol{m}) = e_{k+D}^{(r)}(\boldsymbol{m})$. It should be noted that the diagonal representations |m,ri of $\hat{S}_m$ in the $|ui_k$ and $|vi_k$ bases are equivalent and consistent with equations (9) and (10) only for the case in which D is a prime number. We will come back to equation (15) when we examine the q-oscillator subalgebraic realizations of the Schwinger basis in section 3.2. We now turn to the subalgebraic structure of the Schwinger basis.

3. The deformed subalgebraic structure

It is well known that the $\hat{S}_m$ basis has an explicit deformed algebraic structure. Defining the operators $\hat{D}_m = D/2\pi\hat{S}_m$, the commutator

$$[\hat{D}_m, \hat{D}_n] = \mathrm{i}\frac{2}{\gamma_0}\sin\left(\frac{\gamma_0}{2}\boldsymbol{m}\times\boldsymbol{n}\right)\hat{D}_{m+n} \tag{16}$$

describes the Fairlie–Fletcher–Zachos sine algebra [6]. The generators of the algebra $\hat{J}_m$ can be represented by the Weyl matrices [17]

$$g = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \omega & 0 & \ldots & 0 \\ 0 & 0 & \omega^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \omega^{D-1} \end{pmatrix} \qquad h = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{17}$$

with $\hat{J}_m = \omega^{m_1 m_2/2} g^{m_1} h^{m_2}$ satisfying $hg = \omega gh$, $g^D = h^D = I$, with $\omega^D = 1$ and $\omega = e^{i\gamma_0}$. With these at hand, it is possible to verify that $[\hat{J}_m, \hat{J}_n] = i2/\gamma_0 \sin(\gamma_0/2\, m \times n)\hat{J}_{m+n}$.

The deformed $u_q(sl(2))$ subalgebraic realizations of the sine algebra have been under extensive investigation recently, based on the magnetic translation operator basis [9–12]. In the following we will present a brief account of this symmetry in the more general Schwinger basis.

3.1. The $u_q(sl(2))$ *subalgebraic realization*

We define the operators $\hat{A}$ and $\hat{A}^\dagger$ as

$$\hat{A} \equiv d\hat{S}_m + d'\hat{S}_{m'} \qquad \hat{A}^\dagger \equiv d^*\hat{S}_{-m} + d'^*\hat{S}_{-m'} \tag{18}$$

where d and d' satisfy

$$dd'^* = d^* d' = -(p^{1/2} - p^{-1/2})^{-2} \qquad p = e^{-i\gamma_0 m \times m'}. \tag{19}$$

We find that

$$\begin{gathered} \hat{A}\hat{S}_{m-m'} = p\hat{S}_{m-m'}\hat{A} \qquad \hat{A}\hat{S}^\dagger_{m-m'} = p^{-1}\hat{S}^\dagger_{m-m'}\hat{A} \\ \hat{S}_{m-m'} = s_p p^{\hat{J}_3} \qquad \text{where} \quad \hat{A}\hat{J}_3 \equiv (\hat{J}_3 + 1)\hat{A} \end{gathered} \tag{20}$$

and $s_p = e^{-i\pi m\times m'} = p^{D/2}$, such that equation (13) holds. It is also possible to realize in equations (20) that $\hat{S}_m\hat{S}_{-m'} = \tilde{s}_p p^{\hat{J}_3}$ such that $\tilde{s}_p = e^{-i\pi\gamma_0(D-1)m\times m'} = p^{(D-1)/2}$.

For both cases, a direct calculation yields

$$[\hat{A}, \hat{A}^\dagger] = -\frac{p^{\hat{J}_3 + D/2} - p^{-\hat{J}_3 - D/2}}{p^{1/2} - p^{-1/2}} \equiv -\left[\hat{J}_3 + \frac{D}{2}\right] \tag{21}$$

which, together with equations (20), implies a $u_{p^{1/2}}(sl(2))$ symmetry defined by the elements $\hat{A}, \hat{A}^\dagger, \hat{J}_3$. The Casimir operator for this subalgebra is given by

$$C_p = \hat{A}^\dagger\hat{A} + \left[\frac{1}{2}\left(\hat{J}_3 + \frac{D}{2} - \frac{1}{2}\right)\right]^2 = \hat{A}\hat{A}^\dagger + \left[\frac{1}{2}\left(\hat{J}_3 + \frac{D}{2} + \frac{1}{2}\right)\right]^2 \tag{22}$$

where $[x]$ is formally given in equation (21). The Hilbert space is spanned by the vectors $|j,j_3\rangle$ where $\hat{J}_3|j,j_3\rangle = j_3|j,j_3\rangle$ with $-j \leqslant j_3 \leqslant j$. If the lowest weight representations exist such that $\hat{A}|j,-j\rangle \equiv 0$, then j is determined by the value of the Casimir operator as $\hat{C}_p|j,-j\rangle = [\frac{1}{2}(D/2-\frac{1}{2}-j)]^2|j,-j\rangle$. The lowest weight representations are obtained by successive operations of $\hat{A}^\dagger$ on the state $|j,-j\rangle$. These representations are D-dimensional for the particular case $j = (D-1)/2$ such that $\hat{A}^\dagger|j,-j+(D-1)\rangle = \hat{A}^\dagger|j,j\rangle = 0$, where the highest and lowest weight representations coincide. In this case the representations are cyclic with period D. For this case, the Casimir operator vanishes. We close this section by referring to the extensive applications of the $u_q(sl(2))$ symmetry, for instance in [9–12] and move on to another subalgebraic realization of the Schwinger basis.

3.2. *The spectrum shifted admissible* q*-oscillator realization*

Let us now consider the $\hat{A}$ and $\hat{A}^\dagger$ operators in equation (18) where d and d' are constant to be redetermined for the q-oscillator realization. Using equations (12) we construct the q-commutator

$$\hat{A}\hat{A}^\dagger - q\hat{A}^\dagger\hat{A} = (|d|^2+|d'|^2)(1-q) + dd'^*(\mathrm{e}^{-\mathrm{i}\gamma_0 \boldsymbol{m}\times\boldsymbol{m}'}-q)\hat{S}_{-\boldsymbol{m}'}\hat{S}_{\boldsymbol{m}} + d'd^*(\mathrm{e}^{\mathrm{i}\gamma_0 \boldsymbol{m}\times\boldsymbol{m}'}-q)\hat{S}_{-\boldsymbol{m}}\hat{S}_{\boldsymbol{m}'} \tag{23}$$

where $\boldsymbol{m}\times\boldsymbol{m}' \neq$ (mod$D$). Here, $|q| = 1$ and is otherwise arbitrary at this level. Equation (23) can be written as

$$\hat{A}\hat{A}^\dagger - q\hat{A}^\dagger\hat{A} = (|d|^2+|d'|^2)(1-q) + \hat{Q} \qquad \text{for } q = \mathrm{e}^{\pm\mathrm{i}\gamma_0 \boldsymbol{m}\times\boldsymbol{m}'} \tag{24}$$

where

$$\hat{Q} = \begin{cases} d\,d'^*(q^{-1}-q)\hat{S}_{-\boldsymbol{m}'}\hat{S}_{\boldsymbol{m}} & \text{if } q = \mathrm{e}^{\mathrm{i}\gamma_0 \boldsymbol{m}\times\boldsymbol{m}'} \\ d'd^*(q^{-1}-q)\hat{S}_{-\boldsymbol{m}}\hat{S}_{\boldsymbol{m}'} & \text{if } q = \mathrm{e}^{-\mathrm{i}\gamma_0 \boldsymbol{m}\times\boldsymbol{m}'}. \end{cases} \tag{25}$$

It can be shown that

$$\hat{A}\hat{Q} = q^{-1}\hat{Q}\hat{A} \qquad q = \mathrm{e}^{\pm\mathrm{i}\gamma_0 \boldsymbol{m}\times\boldsymbol{m}'} \tag{26}$$

which implies that a generalized number operator $\hat{N}$ can be defined in such a way that $\hat{A}\hat{N} \equiv (\hat{N}+1)\hat{A}$ and $\hat{Q} = c_q q^{-\hat{N}}$, where c_q is a proportionality constant whose value depends on the choice of d and d'. Equation (26) implies that $\hat{A}$, $\hat{A}^\dagger$ commute with all elements of the algebra. Since the cases q and q^{-1} give rise to identical results as far as the algebra is concerned, we only examine the case $q = \mathrm{e}^{-\mathrm{i}\gamma_0 \boldsymbol{m}\times\boldsymbol{m}'}$. In order to determine c_q we first make the choice

$$dd'^{*} = \frac{1}{q^{-1}-q} = \frac{1}{2\mathrm{i}\sin(\gamma_0 \boldsymbol{m}\times\boldsymbol{m}')} \qquad \text{hence} \qquad |d||d'| = \frac{1}{2|\sin(\gamma_0 \boldsymbol{m}\times\boldsymbol{m}')|}. \tag{27}$$

The constants d,d^0 are also undetermined up to a constant overall phase factor. Choosing their magnitudes symmetrically we can determine the real positive shift constant C as

$$C = |d|^2 + |d'|^2 = \frac{1}{|\sin(\gamma_0 \boldsymbol{m}\times\boldsymbol{m}')|}. \tag{28}$$

The first one in equations (25) leads to the same result in (28). From equations (25) we have $\hat{Q}^D = c_q^D q^{-D\hat{N}}$ iγ0(D 1)m . Then, making use of m /2 (D 1)/2 $q^D \equiv 1$ and equations (12) and (13), we find that c_q = e − × 0 = q− − . It can be seen that the net effect of the pure phase N (D 1)/2 c_q is to shift the spectrum of N^ by an overall constant (D−1)/2. Hence, Q^ = q− ^ − − .

With the generalized number operator as defined below equation (26), we have

$$\hat{A}\hat{A}^{\dagger} - q\hat{A}^{\dagger}\hat{A} = C(1-q) + q^{-\hat{N}-(D-1)/2}$$
$$\hat{A}\hat{N} = (\hat{N}+1)\hat{A} \qquad \hat{A}^{\dagger}\hat{N} = (\hat{N}-1)\hat{A}^{\dagger}. \tag{29}$$

Equations (29) describe the q-oscillator algebra with its spectrum shifted by the positive constant C as

$$\hat{A}^{\dagger}\hat{A} = C + [\hat{N}] \qquad \text{where} \qquad [\hat{N}] = \frac{q^{\hat{N}+(D-1)/2} - q^{-\hat{N}-(D-1)/2}}{q-q^{-1}} \tag{30}$$

where 0 6 kA^†A^k as required, and C is identified with the central invariant, which plays a crucial role in the existence of the admissible cyclic representations of the q-oscillator algebra endowed with a positive spectrum [18, 19]. In equations (29), the existence of the lowest (highest) weight vectors such that A^|n$_0$i = A^†|n$_0$ +D−1i = 0 crucially depends on the specific values of D and m×m^0. The condition for the existence of such |n$_0$i is given by C = −[n$_0$]. For C as given by (28), it can be checked in equation (30) that this condition is violated for D being an odd number. If D is an even number, such representations are permitted for m × m^0 = D/2(modD), however, in that case they are not irreducible. For D being a prime other than two, the situation is the same as when D is odd. We now examine how the q-oscillator algebra generators A,^ A^† and N^ act in the eigenspace of $\hat{S}_m$ operators. We first observe that if |m − m^0,ri is an eigenstate of $\hat{S}_{m-m_0}$ with eigenvalue λ_r(m − m^0) for 0 6 r 6 D − 1,

$$\hat{S}_{m-m_0}|m-m^0,r\rangle \equiv \lambda_r(m,m^0)|m-m^0,r\rangle$$
$$\hat{S}_{m}|m-m^0,r\rangle = g_r(m,m^0)|m-m^0,r-m\times m^0\rangle \tag{31}$$
$$\hat{S}_{m_0}|m-m^0,r\rangle = f_r(m,m^0)|m-m^0,r-m\times m^0\rangle$$

where the second and third equations can be deduced from equations (12). In the second and third equations, $g_r(\mathsf{m},\mathsf{m}^0)$ and $f_r(\mathsf{m},\mathsf{m}^0)$ are pure phase factors to be determined. Using equations (18) we compare the action of Aˆ in the q-oscillator eigenbasis |ni and in the eigenbasis $|\mathsf{m}$,ri as

$$\hat{A}|\boldsymbol{m}-\boldsymbol{m}',r\rangle = (dg+d'f)|\boldsymbol{m}-\boldsymbol{m}',r-\boldsymbol{m}\times\boldsymbol{m}'\rangle$$
$$\hat{A}|n\rangle = \sqrt{C+[n]}|n-1\rangle \tag{32}$$

where it is directly implied that a unit shift in n corresponds to a shift of r in units of $\mathsf{m}\times\mathsf{m}^0$. Since $\mathsf{m}\times\mathsf{m}^0$ 6= (modD) by construction, the set of integers $n\mathsf{m}\times\mathsf{m}^0$ for 0 6 n 6 (D−1) is the same as n itself in the same range. Then, all eigenvectors in the q-oscillator and the Schwinger bases are connected on a one-to-one basis with successive operations of Aˆ and Aˆ†. Since the eigenbasis $\{|\mathsf{m},\text{ri}\}_{06r6(D-1)}$ is normalized, equations (32) imply that

$$|(dg + d^0 f)|^2 = C + [n]. \tag{33}$$

We then apply equations (27) and (28) to obtain

$$\frac{|g|^2+|f|^2-\mathrm{i}(gf^*-g^*f)}{2|\sin(\gamma_0\boldsymbol{m}\times\boldsymbol{m}')|} = \frac{|1+\sin[\gamma_0(n+(D-1)/2)\boldsymbol{m}\times\boldsymbol{m}']|}{\sin[\gamma_0\boldsymbol{m}\times\boldsymbol{m}')} . \tag{34}$$

| |

Since |g| = |f| = 1, equation (34) yields

$$g_r(\boldsymbol{m},\boldsymbol{m}') = f_r^*(\boldsymbol{m},\boldsymbol{m}') = \mathrm{e}^{\mathrm{i}\frac{\gamma_0}{2}(n+(D-1)/2)\boldsymbol{m}\times_{\mathsf{m}_0}} \tag{35}$$

where it can be considered that $r = n\mathsf{m}\times\mathsf{m}^0$. Comparing$_0$ $\lambda_r(\mathsf{m})$ in equations (15) with the first equation in (31) we find that $\lambda_r(\mathsf{m},\mathsf{m}^0) = e^{i\gamma\,(n-D/2)\mathsf{m}\times\mathsf{m}_0}$. Equations (31)–(35) indicate that the admissible cyclic representations of the q-oscillator realization for a fixed value of the deformation parameter q 6= 1 have one-to-one correspondence with the diagonal representations of the Schwinger basis for fixed but arbitrary non-collinear vectors m,m^0.

To the author's knowledge, the admissible q-oscillator subalgebraic realizations of the Schwinger basis (or the magnetic translation basis) has not been studied before. The $su_q(2)$ realizations of two shifted and mutually commuting q-oscillators in the Schwinger boson representation has recently been studied by Fujikawa [20]. It will be demonstrated in section 5 that this particular realization plays a crucial role in the canonical formulation of the quantum phase problem.

3.3. Equivalence classes and canonical transformations on the lattice

In both the q-oscillator and the $u_{p^{1/2}}(sl(2))$ realizations examined here, there are sets of equivalence classes $E_{\mathsf{m}\times\mathsf{m}_0}$ incorporating those sets of subalgebras parametrized by different lattice vector pairs m and m^0 such that the deformation parameter remains invariant under unitary transformations within each such class.

Let us assume a transformation $\mathbf{m}, \mathbf{m}' \in Z_D \times Z_D$ into a new one $\mathbf{m}^*, \mathbf{m}^{*\prime} \in Z_D \times Z_D$ of which the effective action is to map the pair $\mathbf{m}, \mathbf{m}'$ as

$$R_{\mathbf{m},\mathbf{m}';\mathbf{m}^*,\mathbf{m}^{*\prime}}\, f(\mathbf{m},\mathbf{m}') = f(\mathbf{m}^*,\mathbf{m}^{*\prime}) \tag{36}$$

such that $\mathbf{m} \times \mathbf{m}' = \mathbf{m}^* \times \mathbf{m}^{*\prime}$, hence $\mathbf{m},\mathbf{m}';\mathbf{m}^*,\mathbf{m}^{*\prime} \in E_{\mathbf{m}\times\mathbf{m}'}$. Here f represents an arbitrary function. If $R^p_{\mathbf{m},\mathbf{m}';\mathbf{m}^*,\mathbf{m}^{*\prime}}$ is represented in equation (36) by the 2×2 integer matrix R, then the R matrix satisfies

$$R^t P R = P \qquad \text{where} \qquad P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{37}$$

with $\det R = +1$. Here R^t corresponds to the ordinary transpose of R. Equation (37) implies that both $\mathbf{m}$ and $\mathbf{m}'$ are to be transformed by the same transformation

$$R f(\mathbf{m},\mathbf{m}') = f(R\mathbf{m}, R\mathbf{m}') = f(\mathbf{m}^*,\mathbf{m}^{*\prime}) \tag{38}$$

and besides the unimodularity of R there is no further restriction. Hence, R is an element of $sl(2,Z_D)$. The product $\mathbf{m} \times \mathbf{m}'$ corresponds to the exact cocycle [8] in the Schwinger operator basis which is proportional to the discrete phase space area spanned by the vectors $\mathbf{m},\mathbf{m}'$. Hence, R plays the role of a class of area-preserving canonical transformations. As a result of the projective character of the Schwinger basis, any unitary transformation acting on the basis elements preserves the phase space area hence the symplectic structure as described by the matrix P in equation (37). The action of R on the lattice is then equivalent to the reflection of such unitary transformations in the operator space.

At this point, we find it necessary to mention briefly that there are implications of the equivalence classes in the construction of the generalized coproduct $\Delta(.^{\otimes})$ for two deformed subalgebras parametrized by different lattice vectors. Let us denote by $\hat{X}_1, \hat{X}_1^\dagger, \hat{H}_1$ and $\hat{X}_2, \hat{X}_2^\dagger, \hat{H}_2$ as the generators in two such algebras from the same equivalence class. It is possible to write for their tensor product algebra a generalized coproduct as

$$\begin{aligned}
\Delta(\hat{X}^{\otimes}) &= \hat{X}_1 \otimes p^{\hat{H}_2/2} + p^{-\hat{H}_1/2} \otimes \hat{X}_2 \\
\Delta(\hat{X}^{\dagger\otimes}) &= \hat{X}_1^\dagger \otimes p^{\hat{H}_2/2} + p^{-\hat{H}_1/2} \otimes \hat{X}_2^\dagger \\
\Delta(\hat{H}^{\otimes}) &= \hat{H}_1 \otimes I + I \otimes \hat{H}_2
\end{aligned} \tag{39}$$

where $\Delta(\hat{X}^{\otimes}), \Delta(\hat{X}^{\dagger\otimes}), \Delta(\hat{H}^{\otimes})$ respect the same deformed algebra.

Keeping their labels on the lattice explicit, we now consider all operators $\hat{A}_{\mathbf{n},\mathbf{n}'}$, $\hat{A}^\dagger_{\mathbf{n},\mathbf{n}'}$ and $\hat{S}_{\mathbf{n}-\mathbf{n}'}$ on the translated lattice by $\mathbf{r}$ such that $\mathbf{n} = \mathbf{m}+\mathbf{r}$ and $\mathbf{n}' = \mathbf{m}'+\mathbf{r}$. The algebra on this translated lattice space is given by

$$\begin{gathered}
\hat{A}_{\mathbf{m}+\mathbf{r},\mathbf{m}'+\mathbf{r}}\hat{S}_{\mathbf{m}-\mathbf{m}'} = p'\hat{S}_{\mathbf{m}-\mathbf{m}'}\hat{A}_{\mathbf{m}+\mathbf{r},\mathbf{m}'+\mathbf{r}} \qquad \hat{A}^\dagger_{\mathbf{m}+\mathbf{r},\mathbf{m}'+\mathbf{r}}\hat{S}_{\mathbf{m}-\mathbf{m}'} = p'^{-1}\hat{S}_{\mathbf{m}-\mathbf{m}'}\hat{A}^\dagger_{\mathbf{m}+\mathbf{r},\mathbf{m}'+\mathbf{r}} \\
\hat{S}_{\mathbf{m}-\mathbf{m}'} = s_{p'}\, p'^{J_3} \qquad p' = p\, e^{i\gamma_0\delta\alpha} \qquad \delta\alpha = \mathbf{r} \times (\mathbf{m}-\mathbf{m}').
\end{gathered} \tag{40}$$

Here, p is given in equation (19). The equations (40) define the elements of $u_{p'^{1/2}}(sl(2))$ with a different deformation parameter p'. It is clear that translations on the lattice are not in the class of area-preserving transformations defined above, and they cannot be realized by any unitary transformation on the Schwinger basis. Such transformations act as a bridge between the two projective representations characterized by two different cocycles. In our formalism here, this effectively corresponds to transforming the elements of those subalgebras belonging to one equivalence class $E_{m\times m_0}$ into those of the other one $E_{n\times n_0}$. In the example of equations (40), these two subalgebras are $u_{p^{1/2}}(sl(2))$ and $u_{p'^{1/2}}(sl(2))$ with deformations p and $p' = pe^{i\gamma_0\delta\alpha}$ respectively.

We now shift our attention to a more general structure of linear canonical transformations implicitly generated by R on the lattice. The similarity tranformation induced by the Fourier operator $\hat F$ in equations (9)–(11) has been shown in section 2 to effectively generate the simplest example of canonical transformations, i.e. a $\pi/2$ rotation on $Z_D \times Z_D$. Let us now seek general canonical transformations on the lattice generated by an operator $\hat G$ such that

$$\hat{\mathcal{G}}\hat{S}_m\hat{\mathcal{G}}^{-1} = \hat{S}_{R:m} = \hat{S}_{m'} \qquad \text{where} \qquad R = \begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix} \in sl(2, \mathbb{Z}_D) \tag{41}$$

with $s \times t = \det R = 1$, where $s = (s_1,s_2)$ and $t = (t_1,t_2)$ are two vectors on $Z_D \times Z_D$.

Such a transformation $\hat G$ can be given more explicitly in the $\hat U,\hat V$ basis by $\hat G\hat U\hat G^{-1} = \hat S_s \qquad \hat G\hat V\hat G^{-1} = \hat S_t$

$$\hat U \xrightarrow{\hat G} \hat S_s \xrightarrow{\hat G} \hat S_{s_1 s + s_2 t} \xrightarrow{\hat G} \cdots \tag{42}$$

$$\hat V \xrightarrow{\hat G} \hat S_t \xrightarrow{\hat G} \hat S_{t_1 s + t_2 t} \xrightarrow{\hat G} \cdots .$$

Using equations (42) and the results in section 2, the action of the $\hat G$ operator on the basis vectors $\{|u_k\rangle\}_{0\leqslant k\leqslant(D-1)}$ and $\{|v_k\rangle\}_{0\leqslant k\leqslant(D-1)}$ can be found to be

$$\hat G|v_k\rangle = |s,k\rangle \qquad \hat G|u_k\rangle = |t,-k\rangle \tag{43}$$

where, similarly to the first one of equations (31), $|s,k\rangle$ and $|t,-k\rangle$ are the eigenvectors of $\hat S_s$ and $\hat S_t$ with eigenvalue indices k and $-k$ respectively. Hence $\hat G$ converts the vectors in the eigenbasis of $\hat U$ and $\hat V$ into those in $\hat S_s$ and $\hat S_t$ respectively. The similarity transformation in equations (9)–(11) is a special case of the transformation in (41) and (42) for $s = (0,1)$ and $t = (-1,0)$.

4. Applications to the Wigner–Kirkwood basis and the generalized Wigner function

We now consider the discrete Wigner–Kirkwood operator basis [13–15] $\hat{\Delta}(\boldsymbol{V})$ acting on the quantum phase space spanned by the vectors $\boldsymbol{V} = (V_1, V_2)$. Phase space representations in the Wigner–Kirkwood and the Schwinger bases are connected by the dual form

$$\hat{\Delta}(\boldsymbol{V}) = \frac{1}{D^2}\sum_{\boldsymbol{m}} \mathrm{e}^{-\mathrm{i}\gamma_0(\boldsymbol{m}\times\boldsymbol{V})}\hat{S}_{\boldsymbol{m}} \qquad \hat{S}_{\boldsymbol{m}} = \int \mathrm{d}\boldsymbol{V}\,\mathrm{e}^{\mathrm{i}\gamma_0\boldsymbol{m}\times\boldsymbol{V}}\,\hat{\Delta}(\boldsymbol{V}) \tag{44}$$

where $\boldsymbol{m}\times\boldsymbol{V} = m_1V_2 - m_2V_1$ and the range of the integral over $\boldsymbol{V}$ is the entire 2-torus. Similar constructions in the discrete formalism have also been made, for instance in [8,13]. The Wigner function $W_\psi(\boldsymbol{V})$ is defined as the projection of $\hat{\Delta}(\boldsymbol{V})$ in a physical state $|\psi\rangle$ as

$$W_\psi(\boldsymbol{V}) = \langle\psi|\hat{\Delta}(\boldsymbol{V})|\psi\rangle. \tag{45}$$

Any operator $\hat{F}(\hat{U},\hat{V})$ with $\|\hat{F}\| < \infty$ can then be associated with a classical function $f(\boldsymbol{V})$ as

$$\frac{1}{D}\langle\psi|\hat{F}|\psi\rangle = \int \mathrm{d}\boldsymbol{V}\, f(\boldsymbol{V})W_\psi(\boldsymbol{V}) \qquad f(\boldsymbol{V}) = \mathrm{Tr}\{\hat{F}\hat{\Delta}^\dagger(\boldsymbol{V})\} \qquad \mathrm{d}\boldsymbol{V} = \mathrm{d}V_1\,\mathrm{d}V_2. \tag{46}$$

Hereupon, the particular normalization we will use is based on $\int_0^{2\pi} \mathrm{d}x\,\mathrm{e}^{\mathrm{i}xn} = 2\pi\delta_{n,0}$ and in the continuous limit $\lim_{D\to\infty}\sum_{n=0}^{D-1}\mathrm{e}^{\mathrm{i}xn} = 2\pi\delta(x)$.

Let us now consider the action of $\hat{\mathcal{F}}$ in section 2. The action of the Fourier operator $\hat{\mathcal{F}}$ on the Wigner–Kirkwood basis can be found using equation (44) to be $\hat{F}\hat{\Delta}(\boldsymbol{V})\hat{F}^{-1} = \hat{\Delta}(R_{\pi/2}^{-1} : \boldsymbol{V})$, where $R_{\pi/2}^{-1} : \boldsymbol{V} = (V_2, -V_1)$ with $R_{\pi/2}$ as given in equation (11). This is one of the simplest non-trivial canonical transformations corresponding to the rotation of the vector $\boldsymbol{V}$ by $\pi/2$ on the quantum phase space. As an extension of the finite transformations generated by the operator $\hat{F}$, one can find in equation (41) explicit unitary transformations generated by $\hat{G}$ of which the reflections on the quantum phase space are linear canonical ones on the quantum phase space observables.

The properties of a generalized phase space Wigner function have been enlisted by Hillery *et al* [21] under several fundamental conditions. Most of these conditions can be checked by employing the appropriate canonical transformations $\hat{G}$ and the corresponding R. In the following we will check these conditions for equation (44) using the properties of the Schwinger basis.

(i) The Wigner function is real: $W_\psi(\boldsymbol{V}) = W_\psi^*(\boldsymbol{V})$.

Using the first equation in (12) it can easily be proven that $\hat{\Delta}(\boldsymbol{V})$ is a self-adjoint operator. Hence, $W_\psi(\boldsymbol{V})$ is real.

(ii) Integration over one phase space variable V_i yields the marginal probability

distribution of the physical state in the eigenbasis of the other variable|h | i| | i = | i $_j$ = = | i

= | i V=$_j$: R dV=$_i$ $W_\psi(V)$ =

V_j ψ ^{2} where V_j v v for (i 1,j 2) and V_j u v_j for (i 2,j 1).

To prove this property using equation (44), perform the integral over V_i to obtain $D\delta_{m_j,0}$. Then express $\hat{S}_m \cdot \delta_{m_j,0}$ in the $\hat{U},\hat{V}$ basis where only the m_jth power of $\hat{U}$ or $\hat{V}$ appears.

Write $\hat{U}$ or $\hat{V}$ raised to the power m_j in terms of its eigenbasis using equations (1)–(3) or (4), (5). Following the $m_i(i \neq j)$ summation, perform the summation over the eigenvector index k to obtain the proof. Note that this condition is true for any canonically transformed $V = (V_1,V_2)$ such that $V_i \to (R : V)_i, V_j \to (R : V)_j$, which can be easily done using $\hat{G}$ and R in equation (41).

(iii) $W_\psi(V)$ should be covariant under Galilean translations on the phase space.

Since the phase space spanned by the vectors m is discrete, the translations are generated by the integer powers of $\hat{U}$ and $\hat{V}$ operators as $\hat{U}^{n_1}|u\rangle_k = |u\rangle_{k+n_1}$ and $\hat{V}^{n_2}|v\rangle_k = |u\rangle_{k+n_2}$. In the Galilean translated physical state $|\psi'\rangle = \binom{\hat{\mathcal{U}}^{n_1}}{\hat{\mathcal{V}}^{n_2}}|\psi\rangle$, the Wigner function is given by

$$W'_\psi(\boldsymbol{V}) = \langle\psi|\begin{pmatrix}\hat{\mathcal{U}}^{-n_1}\\ \hat{\mathcal{V}}^{-n_2}\end{pmatrix}\hat{\Delta}(\boldsymbol{V})\begin{pmatrix}\hat{\mathcal{U}}^{n_1}\\ \hat{\mathcal{V}}^{n_2}\end{pmatrix}|\psi\rangle \tag{47}$$

where the upper and lower cases correspond to the translations performed independently in either the $|u\rangle_k$ or the $|v\rangle_k$ basis. Using the properties of the $\hat{U}$ and $\hat{V}$ operators as well as equations (12) it can be shown that

$$W'_\psi(\boldsymbol{V}) = W_\psi(\boldsymbol{V}') \qquad \text{where } V' = \begin{pmatrix}V_1+n_1,\ V_2\\ V_1,\ V_2+n_2\end{pmatrix}. \tag{48}$$

Hence, equation (44) is covariant under Galilean translations on the lattice. (iv)

$W_\psi(V)$ should be covariant under space and/or time inversions.

To prove this, we assume that the time inversion is defined by $(m_1, m_2) \xrightarrow{T^*} (m_1, -m_2)$ and the space inversion is given by $(m_1,m_2) \xrightarrow{P} (-m_1,-m_2)$. The time inversion is a $\det T^* = -1$ type improper canonical transformation. Following a similar derivation in the time inverted, i.e. $|\psi'\rangle = \hat{T}^*|\psi\rangle$, or space inverted, i.e. $|\psi'\rangle = \hat{P}|\psi\rangle$, physical state $|\psi'\rangle$, it is possible to see that

$$W_{\hat{T}^*:\psi}(V) = W_\psi(V') \qquad V' = (V_1,-V_2) \tag{49}$$

$$W_{\hat{P}:\psi}(V) = W_\psi(V') \qquad V' = (-V_1,-V_2).$$

In particular we notice that the transformation corresponding to space inversion is identical to the successive operations of the Fourier operator in equation (11) twice, namely, $_{\psi}$ V) and W_{ψ^0} ($\hat{P} = \hat{F}^2$.

(v) If W (V) are two Wigner functions corresponding to the physical states |ψi and |ψ^{0}i respectively, then

$$\int \mathrm{d}\boldsymbol{V}\, W_{\psi}(\boldsymbol{V}) W_{\psi'}(\boldsymbol{V}) = \frac{1}{D} |\langle\psi|\psi'\rangle|^2 . \tag{50}$$

We present the proof starting from

$$\int \mathrm{d}\boldsymbol{V}\, W_{\psi}(\boldsymbol{V}) W_{\psi'}(\boldsymbol{V}) = \frac{1}{D^4} \sum_{\boldsymbol{m},\boldsymbol{m}'} \int \mathrm{d}\boldsymbol{V}\, \mathrm{e}^{-\mathrm{i}\gamma_0(\boldsymbol{m}+\boldsymbol{m}')\times\boldsymbol{V}} \langle\psi|\hat{S}_{\boldsymbol{m}}|\psi\rangle\langle\psi'|\hat{S}_{\boldsymbol{m}'}|\psi'\rangle . \tag{51}$$

We then express |ψi and |ψ^{0}i, for instance in the {|ui$_k$}$_{0\leqslant k\leqslant(D-1)}$ basis as

$$|\psi\rangle = \sum_k \psi_k |u\rangle_k \qquad |\psi0\mathrm{i} = \mathrm{X}\psi_{k0}|\mathrm{ui}_k. \tag{52}$$

k

The V integral yields $D^2\delta_{m,-m_0}$. Then, using $\hat{S}_{m}|u\rangle_k = \mathrm{e}^{-\mathrm{i}\gamma_0/2(2k+m_1)m_2}|u\rangle_{k+m_1}$ and performing the summations over m_1,m_2 we obtain the right-hand side of equation (50).

(vi) If $\hat{Y}$ and $\hat{Z}$ are two dynamical operators of $\hat{U}$ and $\hat{V}$, then

$$\frac{1}{D}\,\mathrm{Tr}\{\hat{Y}\hat{Z}\} = \int \mathrm{d}\boldsymbol{V}\, y(\boldsymbol{V}) z(\boldsymbol{V}) \tag{53}$$

where y(V) and z(V) are classical functions on the phase space corresponding to $\hat{Y}$ and $\hat{Z}$.

The proof of this condition can be done using equation (46) and $\mathrm{Tr}\{\hat{S}_{m}\} = D\delta_{m,0}$.

We thus suggest that the realizations of the generalized Wigner–Kirkwood basis in terms of the elements of the Schwinger basis as expressed in equation (44) satisfies all fundamental conditions to represent the Wigner function in a more generalized form.

The connection between the unitary transformations in the Schwinger basis and canonical area-preserving ones on the quantum phase space have been intensively studied recently. We refer to [8] for a detailed analysis of this connection. The Wigner function on $Z_D \times Z_D$ has been examined by Wooters [22] and applications to action-angle case and the problems therein have been recently studied in detail by Bizzaro [23] and Vaccaro [24].

The discrete Wigner function we examined in this section is based on the particular normalization adopted in equations (44), (i.e. the $1/D^2$ factor in the first equation). Using a different normalization, it is also possible to examine the case in which one of the two (or both) continuous phase space variables V = (V_1,V_2) is (are) replaced by the discrete ones. The former is more convenient in the case in which the canonical variables correspond to the action-angle pair, whereas the latter should be used when the discrete phase space variables are considered on equal footing (i.e. canonical linear discrete coordinate and momentum [13]). It should be noted that in sections 5 and 6 we will use the normalization adopted for the action-angle variables and replace the $1/D^2$ factor in equations (44) by $1/(2\pi D)$ in order to obtain the conventional action-angle Wigner function in the continuous limit.

5. Applications to the unitary number-phase basis and connection to the quantumphase problem

It is known that a finite-dimensional admissible cyclic algebra

$$
\begin{aligned}
&\hat{a}|n\rangle = f(n)^{1/2}|n-1\rangle \quad n \neq 0 \qquad \hat{a}^{\dagger}|n\rangle = f(n+1)^{1/2}|n+1\rangle \quad n \neq (D-1)\\
&\hat{a}|0\rangle = f(0)^{1/2}\beta|D-1\rangle \qquad |\beta| = 1; |D\rangle \equiv |0\rangle \\
&\hat{a}^{\dagger}|D-1\rangle = f(D)^{1/2}\beta^{*}|0\rangle \\
&0 \leqslant f(n) \qquad n \in \mathbb{Z}\ (\mathrm{mod}\ D)
\end{aligned}
\tag{54}
$$

provides a well-defined algebraic basis for the quantum phase operator [19, 20]. Here $\hat{a}$ and $\hat{a}^{\dagger}$ are spectrum lowering and raising operators and $f(n)$ is a generalized spectrum with the cyclic property that $f(n+D) = f(n)$. The admissibility condition is enforced by the last equation in (54).

The unitary phase operator $\hat{\mathcal{E}}_{\phi}$ is given in the generalized cyclic number basis by [19, 20]

$$
\hat{\mathcal{E}}_{\phi} = \sum_{n=0}^{D-1} |n-1\rangle\langle n| \qquad |n+D\rangle \equiv |n\rangle \quad \text{for all } n
\tag{55}
$$

where the discrete phase eigenvalues and eigenstates are with $0 \leqslant \ell \leqslant D-1$. The phase

$$
\hat{\mathcal{E}}_{\phi}|\phi\rangle_{\ell} = \mathrm{e}^{\mathrm{i}\gamma_0 \ell}|\phi\rangle_{\ell} \qquad |\phi\rangle_{\ell} = \frac{1}{\sqrt{D}}\sum_{n=0}^{D-1} \mathrm{e}^{\mathrm{i}\gamma_0 n\ell}|n\rangle \qquad |\phi\rangle_{\ell+D} \equiv |\phi\rangle_{\ell}
\tag{56}
$$

eigenbasis is orthonormal and resolves the identity as

$$
{}_{\ell_0}\langle\phi|\phi\rangle_{\ell} = \delta_{\ell_0,\ell} \qquad \text{and} \qquad I = \sum_{\ell=0}^{D-1} |\phi\rangle_{\ell\,\ell}\langle\phi|.
\tag{57}
$$

We now define the unitary operator $\hat{\mathcal{E}}_N = \mathrm{e}^{-\mathrm{i}\gamma_0 \hat{N}}$ with $\hat{N}$ describing the number operator such that $\hat{N}|n\rangle = n|n\rangle$. Then, $\hat{E}_N = \mathrm{e}^{-\mathrm{i}\gamma_0 \hat{N}}$ has

$$
\hat{E}_N|\phi\rangle_{\ell} = |\phi\rangle_{\ell-1} \qquad \hat{E}_N|n\rangle = \mathrm{e}^{-\mathrm{i}\gamma_0 n}|n\rangle \quad \text{where} \quad |n\rangle = \frac{1}{\sqrt{D}}\sum_{\ell=0}^{D-1} \mathrm{e}^{-\mathrm{i}\gamma_0 n\ell}|\phi\rangle_{\ell}.
\tag{58}
$$

The properties of the unitary phase and number operators $\hat{E}_{\phi}$ and $\hat{E}_N$ have been recently studied from this algebraic point of view [19]. Here, in addition to these properties, they also establish a particular application of Schwinger's operator basis. Among the four equivalent choices in equations (10), we examine the particular case

$$
\begin{pmatrix} \hat{\mathcal{U}} \\ \hat{\mathcal{V}} \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{\mathcal{E}}_N \\ \hat{\mathcal{E}}_{\phi} \end{pmatrix}.
\tag{59}
$$

Using this, and following (8), we construct the operators $\hat{S}_m$ in the number-phase basis as

$$\hat{S}_{\boldsymbol{m}} \equiv \mathrm{e}^{-\mathrm{i}\gamma_0 m_1 m_2/2} \hat{\mathcal{E}}_N^{m_1} \hat{\mathcal{E}}_\phi^{m_2} \qquad \text{where} \quad \hat{\mathcal{E}}_N^{m_1} \hat{\mathcal{E}}_\phi^{m_2} = \mathrm{e}^{\mathrm{i}\gamma_0 m_1 m_2} \hat{\mathcal{E}}_\phi^{m_2} \hat{\mathcal{E}}_N^{m_1}. \quad (60)$$

All properties of the cyclic Schwinger unitary operator basis studied in sections 2 and 3 are satisfied in the unitary number-phase basis. In addition to these properties, a strong limitation exists on the admissibility of the representations in H_D to make the mapping in (59) an acceptable one.

The q-oscillator algebra in section 3.1 defined by the elements $\hat{A}$, $\hat{A}^\dagger$, and $\hat{N}$ for a fixed m and m' with $q = e^{\pm i\gamma_0 m\times m'}$ and $\gamma_0 = 2\pi/D$ is an admissible cyclic algebra which provides a natural realization of equations (54) with $\hat{a} \to \hat{A}$, $\hat{a}^\dagger \to \hat{A}^\dagger$, and $\hat{N}$. In this case, the admissible algebra in equations (54) is given by the shifted q-oscillator algebra in equations (29) where

$$f(n) \to [n] + C = \frac{q^{n+(D-1)/2} - q^{-n-(D-1)/2}}{q - q^{-1}} + C \qquad C = \frac{1}{|\sin(\gamma_0 \boldsymbol{m} \times \boldsymbol{m}')|} \neq 0. \quad (61)$$

Now, let us consider a real cyclic operator $F(\hat{N})$ with $0 \leqslant \|F(\hat{N})\|$ such that $F(\hat{N}) = F(\hat{N}+D)$ of which the eigenvalues in the number basis $\{|n\rangle\}_{0\leqslant n\leqslant(D-1)}$ are given by $f(n)$. We consider the expansion of $F(\hat{N})$ as

$$F(\hat{N}) = \frac{1}{D} \sum_{k=0}^{D-1} \tilde{f}_k q^{-\hat{N}k} \qquad q = e^{-i\gamma_0 m\times m'}. \quad (62)$$

The sets of integers $\{k\, m \times m'; m \times m' \neq 0 \pmod D\}_{0\leqslant k\leqslant(D-1)}$ and $\{k\}_{0\leqslant k\leqslant(D-1)}$ are equivalent for any m, m'. Thus, equations (62) is nothing but the operator Fourier expansion of $F(\hat{N})$. Using equation (25), and the fact that m and m' are not to be collinear, the operator $q^{-\hat{N}}$ can be realized as the third element $c_q^{-1}\hat{S}_{-m'}\hat{S}_m$ of the qoscillator subalgebra. Hence, equation (62) can be equivalently written as

$$F(\hat{N}) = \frac{1}{D} \sum_{k=0}^{D-1} \tilde{f}_k \hat{S}_{(k\boldsymbol{m}-\boldsymbol{m}')} \qquad \tilde{f}_k = \mathrm{Tr}\{\hat{S}^\dagger_{(k\boldsymbol{m}-\boldsymbol{m}')} F(\hat{N})\} = \sum_{n=0}^{D-1} \mathrm{e}^{\mathrm{i}\gamma_0 k n} f(n) \quad (63)$$

where we redefined $\tilde{f}_k$ as $\tilde{f}_k \to \tilde{f}_k c_q^{-1} q^{1/2} = \tilde{f}_k q^{D/2}$. Since the vectors m, m' are fixed but undetermined, equation (63) is the expansion of $F(\hat{N})$ in an arbitrary but fixed qoscillator subalgebra based on a fixed m and m' of the Schwinger basis with the deformation parameter $q = e^{-i\gamma_0 m\times m'}$.

As a specific application of section 4, and making use of the correspondence in (59), we construct the Schwinger realization of the discrete Wigner–Kirkwood operator basis in the number-phase space as

$$\hat{} = \frac{1}{2\pi D} 1(J,\theta) \sum_{m} \mathrm{e}^{\mathrm{i}(\gamma_0 m_1 J - m_2\theta)} \mathrm{e}^{-\mathrm{i}\frac{1}{2}\gamma_0 m_1 m_2} \hat{\mathcal{E}}_N^{m_1} \hat{\mathcal{E}}_\phi^{m_2} \quad (64)$$

where we used the particular $1/(2\pi D)$ normalization to examine the action-angle Wigner function and J,θ are introduced as the generalized *action-angle variables* as a physical realization of the phase space vector $V \to (\theta/\gamma_0, I)$ in equations (44). The change in the normalization factor from equation (44) to equation (64) is then simply the Jacobian of the transformation $dV \to dI\, d\theta$. The Wigner–Kirkwood basis $\hat{\Delta}(J,\theta)$ has the cyclic property that $\hat{\Delta}(J,\theta) = \hat{\Delta}(J(\mathrm{mod}\, D), \theta(\mathrm{mod}\, 2\pi))$. Let us now insert the identity operator in (57) on both sides of the basis operators in (64). Using equations (56) and (58) repeatedly m_2 and m_1 times, equation (64) becomes

$$\hat{\Delta}(J,\theta) = \frac{1}{2\pi D}\sum_{m}\sum_{\ell=0}^{D-1} e^{i(\gamma_0 m_1 J - m_2\theta)} e^{i\gamma_0 \ell m_2} e^{i\gamma_0 m_1 m_2/2} |\varphi_\ell\rangle\langle\varphi_{\ell+m_1}|. \tag{65}$$

The action-angle Wigner function in any particular finite-dimensional Hilbert space state $|\psi\rangle$ is then given as in (45) by

$$W_\psi(J,\theta) = \langle\psi|\hat{\Delta}(J,\theta)|\psi\rangle \tag{66}$$

with all required conditions for the generalized Wigner function satisfied. In section 6.3 we will examine the continuous limit of equation (66) as $D \to \infty$.

6. The limit to continuum

The large D limit of the sine algebra has been extensively studied initially, for instance in [5,6], and later by many other workers. We will not present these results here. We will consider the $D \to \infty$ limit with the condition that D remains a prime number.

6.1. The number-phase basis

In the limit $D \to \infty$ the spectra of $\hat{U}$ and $\hat{V}$ become arbitrarily dense and approach a continuously uniform distribution on the unit circle. Hence, for both unitary operators, the strong convergence is clearly guaranteed from those with discrete spectra to those with continuous spectra [25, 26]. In particular, the continuous limits of $\hat{\mathcal{E}}_\phi$ and $\hat{\mathcal{E}}_N$ will be identified as

$$\begin{aligned} &\lim_{D\to\infty} \hat{\mathcal{E}}_N^{m_1} \to \hat{\tilde{\mathcal{E}}}{}_N^{\gamma} \equiv \mathrm{e}^{-\mathrm{i}\gamma \hat{N}} &&\text{where} \quad \gamma \equiv \lim_{D\to\infty} \frac{2\pi m_1}{D} \in \mathbb{R} \\ &\lim_{D\to\infty} \hat{\mathcal{E}}_\phi^{m_2} \equiv \hat{\tilde{\mathcal{E}}}{}_\phi^{m_2} && 0 \leqslant m_2 < \infty,\ m_2 \in \mathbb{Z} \end{aligned} \tag{67}$$

where $\hat{\tilde{\mathcal{E}}}_N$ and $\hat{\tilde{\mathcal{E}}}_\phi$ are now corresponding unitary operators with continuous spectra. On the other hand, in the limit to continuity we must restrict the physical states that $\hat{\mathcal{E}}_\phi$ and $\hat{\mathcal{E}}_N$ act upon to those everywhere differentiable and continuous functions in the infinite-dimensional Hilbert space. For all such acceptable states $|\psi\rangle$, the condition for weak convergence

$$\lim_{D\to\infty} |(\hat{\mathcal{E}}_N^{m_1} - \hat{\tilde{\mathcal{E}}}{}_N^{\gamma})|\psi\rangle\|^2 < \epsilon \qquad \text{where } \epsilon < 0^+ \text{ (arbitrarily small)} \tag{68}$$

and similarly for $\hat{\tilde{\hat{\mathcal{E}}}}_\phi$, must be respected. In particular, it was shown in [25] that the eigenstates of $\tilde{E}_N$ and $\tilde{E}_\varphi$ are good examples of such $|\psi i$ and the convergence in (68) in the limit $D \to \infty$ is known to exist. Considering the $D \to \infty$ limit of equations (56) and (58), the eigenstates of $\hat{\tilde{\hat{\mathcal{E}}}}_N$ and $\hat{\tilde{\hat{\mathcal{E}}}}_\phi$ are

$$|n\rangle = \int \frac{\mathrm{d}\phi}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}\phi n}|\phi\rangle \qquad |\phi\rangle = \frac{1}{\sqrt{2\pi}} \lim_{D\to\infty} \sum_{n=0}^{D-1} \mathrm{e}^{\mathrm{i}\phi n}|n\rangle \tag{69}$$

where we have defined

$$\begin{aligned} &\lim_{D\to\infty} |n\rangle = |n\rangle \qquad 0 \leqslant n < \infty \\ &\lim_{D\to\infty} \frac{1}{\sqrt{\gamma_0}}|\phi\rangle_\ell \equiv |\phi\rangle \qquad \phi = \lim_{D\to\infty} \frac{2\pi\ell}{D} \in \mathbb{R} \quad \text{and } 0 \leqslant \varphi < 2\pi \end{aligned} \tag{70}$$

with the proper normalizations $h\varphi^0|\varphi i = \delta(\varphi - \varphi^0)$ and $hn^0|ni = \delta_{n^0,n}$. Remember that the periodic boundary conditions are still valid in the limit (i.e. $|\varphi i \equiv |\varphi + 2\pi i$ and $|ni = \lim_{D\to\infty} |n + Di$). For a generally acceptable state $|\psi\rangle = \sum_{\ell=0}^{D-1} \psi_\ell |\phi\rangle_\ell$ with $k|\psi ik = 1$, a similar weak convergence condition as in (68) stated for the phase operator

requires

$$\lim_{D\to\infty} |(\hat{\mathcal{E}}_\phi - \hat{\tilde{\hat{\mathcal{E}}}}_\phi)\psi\rangle\|^2 = \lim_{D\to\infty} \sum_{\ell=0}^{D-1} |\psi_\ell(\mathrm{e}^{\mathrm{i}\gamma_0\ell} - \mathrm{e}^{\mathrm{i}\phi})|^2 < \epsilon. \tag{71}$$

Since $|\psi_\ell| \leqslant 1$, and the convergence

$$\lim_{D\to\infty} |(\hat{\mathcal{E}}_\phi - \hat{\tilde{\hat{\mathcal{E}}}}_\phi)|\psi\rangle\|^2 = \lim_{D\to\infty} \sup\{|(\mathrm{e}^{\mathrm{i}\gamma_0\ell} - \mathrm{e}^{\mathrm{i}\phi})|^2 : 0 \leqslant \ell < (D-1)\} < \epsilon \tag{72}$$

is guaranteed because of equations (69) and (70), the only condition for the existence for such acceptable states is that in the limit $D \to \infty$, the wavefunction ψ_ℓ is sufficiently well behaved and everywhere differentiable. Once the weak convergence condition in equation (68) is satisfied for an acceptable state $|\psi i$ expressed in one basis (i.e. in $|ni$ or $|\varphi i$), the weak convergence in the other basis is guaranteed by equations (69).

The actions of the operators in (67) on the infinite-dimensional Hilbert space spanned by the vectors in (69) are therefore

$$\hat{\tilde{E}}_{\gamma N}|\varphi i = |\varphi - \gamma i \qquad \hat{\tilde{E}}_{\gamma N}|ni = e^{-i\gamma n}|ni \qquad \hat{\tilde{E}}_{\ell\varphi}|ni = |n - \ell i \qquad \hat{\tilde{E}}_{\ell\varphi}|\varphi i = e^{i\ell\varphi}|\varphi i. \tag{73}$$

In this continuous limit, equation (60) implies that

$$\mathrm{e}^{-\mathrm{i}\gamma\hat{N}}\hat{\tilde{\hat{\mathcal{E}}}}{}_\phi^\ell = \mathrm{e}^{\mathrm{i}\ell\gamma}\hat{\tilde{\hat{\mathcal{E}}}}{}_\phi^\ell \mathrm{e}^{-\mathrm{i}\gamma\hat{N}}. \tag{74}$$

Differentiating (74) with respect to γ and considering the limit $\gamma \to 0$ we find that

$$[\hat{N}, \hat{\tilde{\hat{\mathcal{E}}}}{}_\phi^\ell] = -\ell\hat{\tilde{\hat{\mathcal{E}}}}{}_\phi^\ell \tag{75}$$

which is the Susskind–Glogower–Carruthers–Nieto phase-number commutation relation [27] with $\hat{\mathcal{E}}_\phi$ describing the unitary phase operator with a continuous spectrum as given in (73). The expansion of equation (74) for all orders in γ is consistent with the first-order term described in equation (75). The coefficient of the O(γ^r) term reproduces the rth-order commutation relations between $\hat{N}$ and $\hat{\mathcal{E}}_\phi^\ell$. In this respect, equation (74) or, more generally, its discrete version in equation (6) should be treated as generalized canonical commutation relations.

6.2. The spectrum shifted q*-oscillator*

To study the D → ∞ limit of the q-oscillator we first consider, in the numerator of [n] in (61), the equivalence of the sets of integers {n m×m0;m×m0(modD)}06n6(D−1) and {n}06n6(D−1) for any m,m0. If m×m0 6= 1, this equivalence amounts to folding the value of nm×m^0 into the first *Brillouin zone* n for 0 6 n 6 (D − 1). In the limit, the spectrum is given by

$$f(n) = \lim_{D\to\infty} \frac{1 \pm \sin(\gamma_0 n)}{|\sin(\gamma_0 \boldsymbol{m} \times \boldsymbol{m}')|}. \qquad (76)$$

Depending on m×m^0, the sine term in the numerator takes continuous values in the range [0,1) for 0 6 n 6 (D − 1). Two limiting cases can be identified depending on the basis vectors m,m^0 by

$$f(n) = \begin{cases} \lim_{D\to\infty} [1/\gamma_0 \pm n] & \text{if } m \times m^0 = 1 \\ \lim_{D\to\infty} [1 \pm \gamma_0 n] & \text{if } m \times m^0 = (D-1)/4 \end{cases} \qquad (77)$$

∈ Z.

The first case is identical to the continuous limit considered by Fujikawa [18]. The spectrum is linear and unbounded, and the admissibility condition implies an unbounded positive shift by $\lim_{D\to\infty} 1/\gamma_0$. This is somewhat an infinitely shifted harmonic oscillator spectrum. Whereas, in the second case in (77), one obtains a continuous, finite, and linear spectrum. The limit D → ∞ has other interesting features. Fujikawa has shown that the vanishing of the index [28]

$$I = \sum_{n=0}^{D-1} \{\mathrm{e}^{-f(n)} - \mathrm{e}^{-f(n+1)}\} \qquad (78)$$

is a stringent condition for the existence of the unitary phase operator. Using this index condition for the general admissible algebra in (54), it was previously shown [19, 28] that the limit D → ∞ has a singular behaviour in the spectrum at D = ∞. This typical transition to a singular behaviour is also visible here if we compare the two indexes in (78) once calculated using equation (76) and then (77). The former correctly yields I = 0, whereas for the latter I 6= 0. Hence, in transition from (76) to (77), the vanishing index condition is violated. This proves that the spectrum as expressed in (77) is not admissible at the limit D = ∞. The admissible form of (77) is given by

$$f(n) = \begin{cases} \lim_{D\to\infty} 1/\gamma_0[1 \pm \sin(\gamma_0 n) &] \quad \text{if } m \times m^0 = 1 \\ \lim_{D\to\infty} [1 \pm \sin(\gamma_0 n) &] \quad \text{if } m \times m^0 = (D-1)/4 \in Z \end{cases} \tag{79}$$

so that the vanishing index condition is respected. Thus, we learn that the vanishing index requires the information on the cyclic properties of the algebra to be maintained for all D including the transition to infinity. For a more general consideration of the index theorem, we refer to [28]. Before closing this section, we remark that the second limiting case in (79) is somewhat similar to tight binding energy spectra in certain condensed matter systems.

6.3. The Wigner function in the phase eigenbasis

Let us define in (64) the variables$^{1/2}$ $\varphi = \lim_{D\to\infty} \varphi_\ell$, $\varphi + \gamma = \lim_{D\to\infty} \varphi_{\ell+m_1}$ with $\varphi,\gamma \in R$ as well as $|\varphi\rangle = \lim_{D\to\infty} \gamma_0^{-}|\varphi\rangle_\ell$ in accordance with equations (67) and (70). Since φ,γ are continuous, we can replace the summation over m_1, in the limit, by an integral over γ such that lim $_{D\to\infty} 1/D\sum_m \to \lim_{D\to\infty}\sum_{m_2=0}^{-1}\int d\gamma/2\pi$ D . Combining everything, we find for the Wigner function in this limit

$$W_\psi(J,\theta) = \int \frac{d\gamma}{2\pi} e^{iJ\gamma}\langle\psi|\theta-\gamma/2\rangle\langle\theta+\gamma/2|\psi\rangle \tag{80}$$

which is the conventional action-angle Wigner function represented in the continuous phase basis. Recently, a similar construction of the continuous Wigner function based on the continuous WH basis was suggested in [13] as well as in [29] in very close correspondence with the results obtained here. Equation (80) can be realized as the action-angle analogue of [29]. If one starts in the generalized dual form represented by equation (44) with the symmetric normalization, the discrete WH representation of 1(V) is obtained which leads in the continuous limit to Wolf's Wigner function formulation in [29]. The continuous WH representation as the standard representation of the Wigner function has also been examined by Schwinger [4] as well as in [8].

6.4. The continuously shifted finite-dimensional Fock spaces and the Wigner function in thegeneralized Fock representation

Let us consider the cyclic algebra in (54) with the unitary phase and number operators as defined in equations (56) and (58). We consider the phase operator $\hat{\mathcal{E}}_\phi^{-\alpha}$ in $\mathcal{H}_D$ as

$$\hat{\mathcal{E}}_\phi^{-\alpha}|\phi\rangle_\ell = e^{-i\gamma_0\ell\alpha}|\phi\rangle_\ell \qquad \text{and} \qquad \hat{\mathcal{E}}_\phi^{-\alpha}|n\rangle \equiv |n+\alpha\rangle \tag{81}$$

where $\alpha \in R[0,1)$ and $|n+\alpha\rangle$ is defined by

$$|n+\alpha\rangle \equiv \frac{1}{\sqrt{D}}\sum_{\ell=0}^{D-1} e^{-i\gamma_0(n+\alpha)\ell}|\phi\rangle_\ell \, . \tag{82}$$

Since $\alpha \in R[0,1)$, the states $\{|n+\alpha\rangle\}_{0\leqslant n\leqslant(D-1)}$ do not belong to the set of vectors spanning

finite-dimensional Fock spacethe finite-dimensional conventional Fock space $\mathcal{F}_D^{(\alpha)}$ where $\{|n+\alpha$ F i0D6n. We now define a continuously shifted$_{6(D\ 1)}$;$\alpha \in R$;$|n+\alpha+D\rangle \equiv |n+\alpha\rangle\} \in$
(α) −

F_D . It can be readily verified that the following relations are satisfied by equation (82) for all continuous values of α:

$$\text{hn} + \alpha|\text{n}_0 + \alpha\text{i} = \delta_{\text{n,n}_0} \qquad \sum_{n=0}^{D-1} |\text{n} + \alpha\text{ihn} + \alpha| = \text{I}. \tag{83}$$

This implies that for a fixed α ∈ R, the shifted Fock space $\mathcal{F}_D^{(\alpha)}$ is also spanned by a complete orthonormal set of vectors $\{|\text{n} + \alpha\text{i}\}_{06\text{n}6(\text{D}-1)}$ and it can equivalently be used in$_{(\alpha)}$ the generalized Fock representation of a physical state. The overlap between F_D and F_D clearly respects the condition |hn|n + αi| 6 1 for all α ∈ R, and the extreme limits of α → 0 and D → ∞ are commutative and well behaved:

$$|\langle n|n+\alpha\rangle| = \begin{cases} |\sin\pi\alpha|/(\pi\alpha) & \text{if D} \to \infty,\ 0\ 6\ \alpha\ 6\ 1 \\ 1-(1-1/D)(\pi\alpha)^2/3! & \text{if D} < \infty,\ \alpha \to 0. \end{cases} \tag{84}$$

Since $\mathcal{F}_D$ and $\mathcal{F}_D^{(\alpha)}$ are spanned by cyclic vectors, $\alpha = 1$ and $\hat{\alpha} = 0$ correspond to the identical Fock space representations. The action of the operatoris, therefore, equivalent to a continuous shift of the origin in $F_{D\alpha}$ Eby$_\varphi{}^\beta$ on the vectors inβ ∈ R such that $F_D{}^{(\alpha)}$

$$\hat{\mathcal{E}}_\phi^\beta : \mathcal{F}_D^{(\alpha)} \to \mathcal{F}_D^{(\alpha-\beta)}. \tag{85}$$

Hence, a continuous shift β induced by the operator $\hat{\mathcal{E}}_\phi^\beta$ is effectively equivalent to carrying vectors from the Fock space $F_D{}^{(\alpha)}$ into the other one $F_D{}^{(\alpha-\beta)}$, and the limit β → 0 is continuous and analytic. Therefore, equation (85) decribes an isomorphism between two inequivalent Fock spaces with equal dimensions. The physical implication of the state |αi is that it corresponds to the *vacuum state* in $F_{D(\alpha)}$ and, unless α = 0, it is not the conventional vacuum |0i. The Fock space of the q-oscillator in equation (29) is a typical example in which such a vacuum state is observed where we specifically have $\mathcal{F}_D^{(D-1)/2}$. For D being an odd integer, the conventional Fock representations in F_D are obtained. For D being an even integer, the Fock space of the q-oscillator is $F_D{}^{1/2}$ and the vacuum state is $|\frac{1}{2}\rangle$ correponding to $\alpha = \frac{1}{2}$. One crucial application of this is to examine the projection of the Wigner–Kirkwood basis onto the shifted Fock space $F_D{}^{(\alpha)}$. Let us now insert the identity operator in (83) on both sides of the unitary number-phase basis operators in (64) yielding

$$\hat{} \quad = \frac{1}{2\pi D} 1(J,\theta) X e^{i(\gamma_0 m_1 J - m_2\theta)} e^{-i\gamma_0 m_1 m_2/2}$$

m

$$\times \left\{ \sum_{n=0}^{D-1} |n+\alpha\rangle\langle n+\alpha| \right\} \hat{\mathcal{E}}_{\mathcal{N}}^{m_1} \hat{\mathcal{E}}_{\phi}^{m_2} \left\{ \sum_{n'=0}^{D-1} |n'+\alpha\rangle\langle n'+\alpha| \right\}. \tag{86}$$

So far, the continuous shift α was arbitrary. Now, we adopt a particular set of values of α for each J independently in such a way that $2(J-\alpha) \in \mathbb{Z}$. Since equations (83) are valid for all $\alpha \in \mathbb{R}$, this adaptive choice for α does not spoil the properties of the Wigner function studied in section 5. Now, considering the limit $D \to \infty$ and following a similar calculation leading to (80), we obtain the Wigner function

$$W_\psi(J,\theta) = \frac{1}{2\pi} \lim_{D\to\infty} \sum_{m_2=0}^{D-1} e^{-im_2\theta} \langle\psi|J - m_2/2\rangle\langle J + m_2/2|\psi\rangle \tag{87}$$

which is expressed in the shifted Fock bases in the limit $D \to \infty$. For the choice of α as $2(J-\alpha) \in \mathbb{Z}$, we have for the basis vectors $\{|J \pm m_2/2\rangle; m_2 = odd\} \in \mathcal{F}_D^{(\alpha\pm\frac{1}{2})}$ and $\{|J \pm m_2/2\rangle; m_2 = even\} \in \mathcal{F}_D^{(\alpha)}$. Note that because of the cyclic property of the vectors in H_D, the shifted Fock space $\mathcal{F}_D^{\alpha+1/2}$ shares the same vectors with $\mathcal{F}_D^{\alpha-1/2}$ for all $D < \infty$ and $\alpha \in \mathbb{R}[0,1)$. Thus, $\mathcal{F}_D^{\alpha+1/2}$ and $\mathcal{F}_D^{\alpha-1/2}$ are indeed the same shifted Fock space. This discussion implies that if in the summation in (87), the even and odd values of m_2 are separated, the Wigner function becomes a sum of two contributions $W^{(\mathrm{even})}$ and $W^{(\mathrm{odd})}$ projected onto $\mathcal{F}_D^{\alpha}$ and $\mathcal{F}_D^{\alpha+1/2}$ for even and odd m_2 respectively as

$$W_\psi(J,\theta) = W_\psi^{(\mathrm{even})}(J,\theta) + W_\psi^{(\mathrm{odd})}(J,\theta). \tag{88}$$

Since each contribution is based on a different D-dimensional shifted Fock basis, they are properly normalized. It is interesting to note that a similar decomposition of the Wigner function in the Fock representation has been recently proposed by Luks and Peřinová [30] as well as by Vaccaro [24] in order to avoid certain superficial anomalies of the Wigner function they use in mixed physical states. Using continuously shifted Fock spaces, the decomposition they propose follows naturally. To elaborate more on the resolution of the anomalous behaviour of the Wigner function using the shifted Fock spaces exceeds our purpose here. It can be shown that the concept of continuously shifted Fock basis can also be generalized to the continuously shifted discrete Schwinger basis vectors $\{|u\rangle_k\}$ and $\{|v\rangle_k\}$. This subtle point certainly deserves much more attention in the generalized formulation of the Wigner function and quantum canonical transformations, which we intend to present in a forthcoming work.

7. Conclusions

The central theme of this work was to demonstrate that conceptual foundation of the quantum phase lies in the algebraic properties of the canonical transformations on the generalized quantum phase space. In this context:

(1) It is shown that the Schwinger operator basis provides subalgebraic realizations of the admissible q-oscillators in addition to the known deformed su(2) symmetries labelled by

the lattice vectors in $Z_D \times Z_D$. The intensively studied magnetic translation operator algebra is a specific physical realization of Schwinger's operator algebra. In this context, some interesting physics might be found in the realization of the shifted q-oscillator subalgebra in terms of the magnetic translation operators as applied to the Bloch electron problem. To the author's knowledge, the nearest approach to this idea has recently been made by Fujikawa *et al* (see the second reference in (18)).

(2) Certain equivalence classes within each subalgebra, using different lattice labels, are identified in terms of area-preserving transformations. A general formulation of such discrete, linear canonical transformations is presented.

(3) The dual form between the Schwinger operator basis and the generalized discrete Wigner–Kirkwood basis is examined and the connection to the general area-preserving canonical transformations on $Z_D \times Z_D$ is briefly studied.

(4) The application of the Schwinger operator basis on the number-phase basis is discussed and shown that it provides an algebraic approach to the formulation of the quantum phase problem. The admissibly shifted q-oscillator realizations of the Schwinger basis are studied from this algebraic point of view. In this context, the algebraic canonical phase space formulation of quantum phase appears to be a unique example in which natural applications of a quantum algebra in the resolution of a physical problem is explicitly found. The generalized Wigner–Kirkwood basis is examined in the unitary number-phase basis and the limit to the conventional formulation of the action-angle Wigner function is investigated as the size of the lattice tends to infinity, or reciprocally, as the lattice spacing $2\pi/D$ tends to zero.

(5) Finally, much work has to be done on understanding the quantum phase problem within the canonical quantum phase space formalism. This problem is also evidently connected to the recent research areas such as classical and quantum integrability, the deformation quantization, theory of nonlinear quantum canonical transformations and the Lie algebraic representations of the Wigner function.

Acknowledgments

The author is grateful to Professor T Dereli (Middle East Technical University), Professor M Arik (Bogaziçi University), and Professor A Klyachko (Bilkent University) for useful discussions and critical comments.

References

[1] Shen X 1992 *Int. J. Mod. Phys.* A **7** 3717
[2] Zak J 1964 *Phys. Rev.* **134** 1602
Brown E 1964 *Phys. Rev.* **133** 1038
Dereli T and Verçin A 1992 *Phys. Lett.* B **288** 109
Dereli T and Verçin A 1993 *J. Phys. A: Math. Gen.* **26** 6961
[3] Wiegmann P B and Zabrodin A V 1994 *Phys. Rev. Lett.* **72** 1890
Guang-Hong Chen, Le-Man Kuang and Mo-Lin Ge 1996 *Phys. Rev.* B **53** 9540
Guang-Hong Chen, Le-Man Kuang and Mo-Lin Ge 1996 *Phys. Lett.* A **213** 231
[4] Schwinger J 1960 *Proc. Natl. Acad. Sci.* **46** 883
Schwinger J 1960 *Proc. Natl. Acad. Sci.* **46** 1401
[5] Floratos E G 1989 *Phys. Lett.* B **228** 335
[6] Fairlie D, Fletcher P and Zachos C 1989 *Phys. Lett.* B **218** 203 Falirlie D B and Zachos C 1989 *Phys. Lett.* B **224** 101
[7] Arnold V 1966 *Ann. Inst. Fourier* **XVI** 319
Arnold V 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)

[8] Aldrovandi R and Galetti D 1990 *J. Math. Phys.* **31** 2987
[9] Kogan I I 1994 *Int. J. Mod. Phys.* A **9** 3887
[10] Choon-Lin Ho 1996 *J. Phys. A: Math. Gen.* **29** L107
[11] Ito S, Karabali D and Sakita B 1993 *Phys. Lett.* B **296** 143 Sakita B 1993 *Phys. Lett.* B **315** 124
[12] Kogan I I 1992 *Mod. Phys. Lett.* A **7** 3717
[13] Galetti D and de Toledo Piza A F R 1988 Physica **149A** 267
[14] Wigner E P 1932 *Phys. Rev.* **40** 749
[15] Kirkwood J G 1933 *Phys. Rev.* **44** 31
[16] Balian R and Itzykson C 1986 *C. R. Acad. Sci., Paris* **303** 773
[17] Weyl H 1931 *The Theory of Groups in Quantum Mechanics* (New York: Dover)
[18] Fujikawa K, Kwek L C and Oh C H 1995 *Mod. Phys. Lett.* A **10** 2543 [19] Hakioglu T 1998 *J. Phys. A: Math. Gen.* **31** 707
Hong-Chen Fu and Ryu Sasaki 1996 *J. Phys. A: Math. Gen.* **29** 4049
[20] Fujikawa K and Harunobu Kubo 1997 *Mod. Phys. Lett.* A **12** 403
Fujikawa K and Harunobu Kubo 1998 *Phys. Lett.* A **239** 21
[21] Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
[22] Wooters W K 1987 *Ann. Phys.* **176** 1
[23] Bizzaro J P 1994 *Phys. Rev.* A **49** 3255
[24] Vaccaro J 1995 *Phys. Rev.* A **52** 3474
[25] Lerner E C, Huang H W and Walters G E 1970 *J. Math. Phys.* **11** 1679
[26] Riesz F and Nagy B Sz 1955 *Functional Analysis* (New York: Ungar)
[27] Susskind L and Glogower J 1964 *Physics* **1** 49
Carruthers P and Nieto M M 1965 *Phys. Rev. Lett.* **14** 387
Carruthers P and Nieto M M 1965 *Rev. Mod. Phys.* **40** 411
[28] Fujikawa K 1995 *Phys. Rev.* A **52** 3299
[29] Wolf K B 1996 *Opt. Commun.* **132** 343
[30] Luks A and Peřinová V 1993 *Phys. Scr.* T **48** 94