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φ (Ric)-vector fields on warped product manifolds and applications

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Abstract
Sufficient and necessary conditions are provided on warped product manifolds and their base and fiber manifolds for a vector field φ_j to be a φ (Ric)-vector field , that is, $\nabla_i \varphi_j = \mu R_{ij}$ where R_{ij} is the Ricci tensor of M and μ is a scalar. Two warped product space-times admitting φ (Ric)-vector fields are considered. Lorentzian quasi-Einstein manifolds admitting a time-like φ (Ric)-vector field are shown to be either Ricci simple or a perfect fluid GRW space-time. The generators of a Lorentzian generalized quasi-Einstein manifold admitting a time-like φ (Ric)-vector field are eigenvectors of the Ricci tensor with zero eigenvalue.

Keywords φ (Ric)-vector field · Quasi-Einstein · Perfect fluid · Generalized Robertson–Walker space-time

Mathematics Subject Classification Primary 53C21 · 53C25; Secondary 53C50 · 53C80

1 Introduction

The concept of φ (Ric)-vector fields was first introduced in [11] as a generalization of concircular vector fields. It is proved that a Riemannian manifold has a constant scalar curvature whenever it admits a φ (Ric)-vector field having a constant length. Moreover, sufficient condition for the the existence of a φ (Ric)-vector field on Riemannian manifolds admitting a parallel Riemann curvature tensor were derived. The study is extended to conformally flat and subprojective spaces in [12]. Four years later, the same conditions were discussed on nearly

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quasi-Einstein manifolds [22, Theorem 4]. Kırık and Zengin studied φ (Ric)-vector field on quasi-Einstein manifolds in [13], on generalized quasi-Einstein(in brief GQE) manifolds in [14] and on GQE manifolds whose Ricci tensor is pseudo-Ricci symmetric in [15].

In the present work, Lorentzian manifolds admitting a φ (Ric)-vector field are investigated. A Lorentzian quasi-Einstein manifold having a time-like φ (Ric)-vector field is shown to be either Ricci simple or a perfect fluid GRW space-time. Then, we extend this study to generalized quasi-Einstein Lorentzian manifolds. Finally, a classification of φ (Ric)-vector fields on singly warped product manifolds and consequently on generalized Robertson–Walker (or simply GRW) space-times and standard static space-times are considered.

2 Preliminaries

In a non-Ricci flat Einstein manifold M, a φ (Ric)-vector field is, in some sense, a generalization of a concircular vector field, that is, the equations

$$\nabla_i \varphi_j = \alpha g_{ij}, \quad R_{ij} = \beta g_{ij}$$

where α, β are scalars show that

$$\nabla_i \varphi_j = \mu R_{ij} \tag{2.1}$$

where μ is a scalar. A vector field φ having property (2.1) is called a φ (Ric)-vector field. Conversely, a φ (Ric)-vector field is concircular if M is Einstein. A φ (Ric)-vector field is called proper if $\mu \neq 0$. Finally, φ is covariantly constant whenever $\mu = 0$. The symmetry of the Ricci tensor infers $\nabla_i \varphi_j = \nabla_j \varphi_i$, i.e. the corresponding 1-form of a φ (Ric)-vector field is closed (see [11]). This implies that φ is locally gradient, i.e. $\varphi_i = \nabla_i \omega$ for some function ω. This assertion will lead us to obtain (see [11] for further details)

$$\varphi_i R^i_k = \frac{\mu}{2} \nabla_k R, \tag{2.2}$$

$$\varphi_i R^i_{jkl} = \mu \nabla_m R^m_{jkl}. \tag{2.3}$$

The first equation with the defining property of φ (Ric)-vector fields show that a manifold M has a constant scalar curvature R if it admits a φ (Ric)-vector field of constant length. In [13, Theorem 2.3], the converse of this result is proved, that is, φ (Ric)-vector fields in Riemannian manifolds of constant scalar curvature has a constant length.

3 On singly warped product manifolds

The warped product manifold $M_1 \times_f M_2$ of two pseudo-Riemannian manifolds (M_i, g_i, ∇_i), $i = 1, 2$ with pseudo-Riemannian metric tensors g_i and Levi–Civita connections ∇_i is the product manifold $M_1 \times M_2$ with the metric tensor $g = g_1 \oplus f^2 g_2$ defined by

$$g = \pi_1^* (g_1) \oplus (f \circ \pi_1)^2 \pi_2^* (g_2),$$

where $\pi_i : M_1 \times M_2 \to M_i$ are two natural projections of $M = M_1 \times M_2$ onto M_i, * denotes the pull-back operator and $f : M_1 \to (0, \infty)$ is a smooth function (see [5,17,18,21]). f is called the warping function. It is noted that M_1 is isometric to the submanifold $M_1 \times \{y\}$ whereas M_2 is homothetic to $\{x\} \times M_2$ for every $x \in M_1$, $y \in M_2$. A tensor on M_i and its lift to M have the same notation. The reader is referred to [5,18] for the formulas of the Levi–Civita connection ∇, curvature tensor R and Ricci curvature Ric of g.

Let $\varphi = \varphi_1 + \varphi_2$ be a vector field on a singly warped product manifold (M, g, ∇) where $M = M_1 \times_f M_2$ furnished by $g = g_1 \oplus f^2 g_2$. Then

$$\begin{aligned} g\left(\nabla_X \varphi, Y\right) &= g\left(\nabla_{X_1}\varphi_1 + \nabla_{X_1}\varphi_2 + \nabla_{X_2}\varphi_1 + \nabla_{X_2}\varphi_2, Y_1 + Y_2\right) \\ &= g_1\left(\nabla^1_{X_1}\varphi_1, Y_1\right) + fX_1(f)\, g_2(\varphi_2, Y_2) + f\varphi_1(f)\, g_2(X_2, Y_2) \\ &\quad + f^2 g_2\left(\nabla^2_{X_2}\varphi_2, Y_2\right) - fY_1(f)\, g_2(X_2, Y_2). \end{aligned}$$

The Ricci tensor of the warped product manifold is given as follows:

Let $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ be two vector fields on a singly warped product manifold (M, g, ∇) where $M = M_1 \times_f M_2$ and $g = g_1 \oplus f^2 g_2$. Then

(1) $\mathrm{Ric}(X_1, Y_1) = \mathrm{Ric}^1(X_1, Y_1) - \frac{n_2}{f} H^f(X_1, Y_1)$, where n_2 is the dimension of M_2 and H^f is the Hessian of f.
(2) $\mathrm{Ric}(X_1, Y_2) = 0$,
(3) $\mathrm{Ric}(X_2, Y_2) = \mathrm{Ric}^2(X_2, Y_2) - f^\sharp g_2(X_2, Y_2)$, where

$$f^\sharp = f\Delta f + (n_2 - 1)\, g_1(\mathrm{grad} f, \mathrm{grad} f).$$

Now we will introduce the following tensor:

$$D(X, Y) = g(\nabla_X \varphi, Y) - \mu \mathrm{Ric}(X, Y).$$

It is clear that φ is a φ (Ric)-vector field with scalar μ if and only if $D = 0$.

Theorem 1 *Let $\varphi = \varphi_1 + \varphi_2$ be a φ (Ric)-vector field on $M_1 \times_f M_2$ where $\varphi_i \in \mathfrak{X}(M_i)$, for any $i = 1, 2$. Then, one of the following conditions holds.*

(1) f is constant and consequently φ_i is a φ_i (Ric)-vector field on M_i, $i = 1, 2$, or
(2) $\varphi_2 = 0$ and hence M_2 is Einstein. Moreover, φ_1 is a φ_1 (Ric)-vector field on M_1 if $H^f = 0$.

Proof The vector field φ is a φ (Ric)-vector field with scalar μ if and only if $D = 0$. Thus,

$$\begin{aligned} 0 &= D(X_1, Y_1) \\ &= g_1\left(\nabla^1_{X_1}\varphi_1, Y_1\right) - \mu \mathrm{Ric}^1(X_1, Y_1) + \frac{\mu n_2}{f} H^f(X_1, Y_1), \\ 0 &= D(X_1, Y_2) = fX_1(f)\, g_2(\varphi_2, Y_2), \end{aligned}$$

and

$$\begin{aligned} 0 &= D(X_2, Y_2) \\ &= \left[f\varphi_1(f) + \mu f^\sharp\right] g_2(X_2, Y_2) + f^2 g_2\left(\nabla^2_{X_2}\varphi_2, Y_2\right) - \mu \mathrm{Ric}^2(X_2, Y_2) \end{aligned}$$

where $\varphi = \varphi_1 + \varphi_2$ and $\varphi_i \in \mathfrak{X}(M_i)$, for any $i = 1, 2$. This infers $fX_1(f)\, g_2(\varphi_2, Y_2) = 0$, that is, f is constant. The rest equations become

$$\begin{aligned} 0 &= g_1\left(\nabla^1_{X_1}\varphi_1, Y_1\right) - \mu \mathrm{Ric}^1(X_1, Y_1) \\ 0 &= f^2 g_2\left(\nabla^2_{X_2}\varphi_2, Y_2\right) - \mu \mathrm{Ric}^2(X_2, Y_2). \end{aligned}$$

This completes the proof. □

The following result discusses the converse of the previous one.

Theorem 2 *Let $\varphi = \varphi_1 + \varphi_2$ be a vector field on $M_1 \times_f M_2$ where $\varphi_i \in \mathfrak{X}(M_i)$, for $i = 1, 2$. Then, φ is φ (Ric)-vector field with scalar μ if*

(1) *f is constant and φ_i is a φ_i (Ric)-vector field on M_i, for any $i = 1, 2$ with scalars μ and $\frac{\mu}{f^2}$ (respectively), or*
(2) *$\varphi_2 = 0$ and*

$$\mu \mathrm{Ric}^1 (X_1, Y_1) = g_1 \left(\nabla^1_{X_1}\varphi_1, Y_1\right) + \frac{\mu n_2}{f} H^f (X_1, Y_1),$$
$$\mu \mathrm{Ric}^2 (X_2, Y_2) = \left[f\varphi_1 (f) + \mu f^\sharp\right] g_2 (X_2, Y_2).$$

Proof The tensor D vanishes if

$$D (X_1, Y_1) = g_1 \left(\nabla^1_{X_1}\varphi_1, Y_1\right) - \mu \mathrm{Ric}^1 (X_1, Y_1) + \frac{\mu n_2}{f} H^f (X_1, Y_1),$$
$$D (X_1, Y_2) = f X_1 (f) g_2 (\varphi_2, Y_2),$$
$$D (X_2, Y_2) = \left[f\varphi_1 (f) + \mu f^\sharp\right] g_2 (X_2, Y_2) + f^2 g_2 \left(\nabla^2_{X_2}\varphi_2, Y_2\right) - \mu \mathrm{Ric}^2 (X_2, Y_2)$$

vanish since it is linear in each slot. For a constant function f, it is

$$D (X_1, Y_1) = g_1 \left(\nabla^1_{X_1}\varphi_1, Y_1\right) - \mu \mathrm{Ric}^1 (X_1, Y_1),$$
$$D (X_1, Y_2) = 0,$$
$$D (X_2, Y_2) = f^2 g_2 \left(\nabla^2_{X_2}\varphi_2, Y_2\right) - \mu \mathrm{Ric}^2 (X_2, Y_2)$$

and so all components of D will be zero if φ_i is a φ_i (Ric)-vector field on M_i, for any $i = 1, 2$ with scalars μ and $\frac{\mu}{f^2}$ respectively.

Now, assume that $\varphi_2 = 0$, one gets

$$D (X_1, Y_1) = g_1 \left(\nabla^1_{X_1}\varphi_1, Y_1\right) - \mu \mathrm{Ric}^1 (X_1, Y_1) + \frac{\mu n_2}{f} H^f (X_1, Y_1),$$
$$D (X_1, Y_2) = 0,$$
$$D (X_2, Y_2) = \left[f\varphi_1 (f) + \mu f^\sharp\right] g_2 (X_2, Y_2) - \mu \mathrm{Ric}^2 (X_2, Y_2).$$

Thus, the conditions

$$\mu \mathrm{Ric}^1 (X_1, Y_1) = g_1 \left(\nabla^1_{X_1}\varphi_1, Y_1\right) + \frac{\mu n_2}{f} H^f (X_1, Y_1),$$
$$\mu \mathrm{Ric}^2 (X_2, Y_2) = \left[f\varphi_1 (f) + \mu f^\sharp\right] g_2 (X_2, Y_2).$$

guarantee $D = 0$. □

We now define generalized Robertson–Walker space-times (GRW) and standard static space-times (SSS-T) to characterize their φ (Ric)-vector fields. We begin by fixing some notation for the rest of the paper.

Assume that (M, g) is an $n-$dimensional Riemannian manifold and I is an open connected interval of $\mathbb{R}$. Moreover, b and f are assumed to be smooth functions on I and M, respectively where $b > 0$ on I and $f > 0$ on M and also $\mathrm{d}t^2$ denotes the usual Euclidean metric tensor on I.

Then $(n + 1)$-dimensional warped product manifold $I \times M$ equipped with the metric tensor

$$\bar{g} = -\mathrm{d}t^2 \oplus b^2 g$$

is said to be a generalized Robertson–Walker space-time and is denoted by $\bar{M} = I \times_b M$. This class of space-times can be considered as a generalization of the well-known Robertson–Walker space-times (see [10,19,20]).

Likewise, we define standard static space-times. The $(n+1)$-dimensional product manifold $I \times M$ furnished with the metric tensor

$$\bar{g} = -f^2 \mathrm{d}t^2 \oplus g$$

is called a standard static space-time and is denoted by $\bar{M} = I_f \times M$. Roughly speaking a standard static space-time can be regarded as an extension of the Einstein static universe (see [1–4]).

From now on, $\frac{\partial}{\partial t} \in \mathfrak{X}(I)$ is denoted by ∂_t to state related formulas and expressions in simpler forms.

Remark 1 A vector field of the form $h\partial_t$ on $(I, -\mathrm{d}t^2)$ is a φ (Ric)-vector field where $h \in \mathcal{C}^\infty(I)$ if and only if $\dot{h} = 0$ on I.

The tensor $\bar{D}$ is defined on $\bar{M}$ as

$$\bar{D}\left(\bar{X}, \bar{Y}\right) = \bar{g}\left(\bar{\nabla}_{\bar{X}}\bar{\varphi}, \bar{Y}\right) - \mu \bar{R}\mathrm{ic}\left(\bar{X}, \bar{Y}\right).$$

where $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$.

Proposition 1 *Let $\bar{M} = I \times_b M$ be an $(n+1)$-dimensional generalized Robertson–Walker space-time equipped with the metric tensor $\bar{g} = -\mathrm{d}t^2 \oplus b^2 g$. If X and Y are vector fields on M, then a vector field of the form $\bar{\varphi} = h\partial_t + \varphi$ on $(\bar{M}, \bar{g})$ where $h \in \mathcal{C}^\infty(I)$ and $\varphi \in \mathfrak{X}(M)$ satisfies*

(1) $\bar{D}(\partial_t, \partial_t) = -\dot{h} + \frac{\mu n \ddot{b}}{b}$,
(2) $\bar{D}(X, \partial_t) = b\dot{b} g(X, \varphi)$,
(3) $\bar{D}(X, Y) = \left[hb\dot{b} - \mu b\ddot{b} - \mu(n-1)\dot{b}^2\right] g(X, Y) + b^2 g(\nabla_X \varphi, Y) - \mu \mathrm{Ric}(X, Y)$

Theorem 3 *Let $\bar{\varphi} = h\partial_t + \varphi$ be a $\bar{\varphi}$ (Ric)-vector field on a GRW space-time of the form $I \times_b M$ with scalar μ. Then, one of the followings holds*

(1) $h = a$ for some $a \in \mathbb{R}$ where b is constant and φ is a φ (Ric)-vector field on M with factor μ/b^2,
(2) $\varphi = 0$ where

$$\dot{\phi}\backslash\dot{h} = \frac{\mu n \ddot{b}}{b}$$
$$\mu \mathrm{Ric}(X, Y) = \left[hb\dot{b} - \mu b\ddot{b} - \mu(n-1)\dot{b}^2\right] g(X, Y),$$

for any vector field X and Y on M.

Example 1 Let $M = I \times_b \mathbb{R}$ be a warped product manifold endowed with the metric tensor $g = -\mathrm{d}t^2 + b^2 \mathrm{d}s^2$. Suppose that $\varphi = \varphi_1 \partial_t + \varphi_2 \partial_s$ is a φ (Ric)-vector field on (M, g). Then

$$\varphi = (\zeta_1 t + \eta_1)\partial_t + (\zeta_2 s + \eta_2)\partial_s$$

if $\dot{b} = 0$ or else

$$\varphi = \frac{k}{b}\partial_t.$$

Proposition 2 *Let $\bar{M} = I_f \times M$, be an $(n+1)$-dimensional standard static space-time equipped with the metric tensor $\bar{g} = -f^2 \mathrm{d}t^2 \oplus g$. If X and Y are vector fields on M, then a vector field of the form $\bar{\varphi} = h\partial_t + \varphi$ on $(\bar{M}, \bar{g})$ where $h \in \mathcal{C}^\infty(I)$ and $\varphi \in \mathfrak{X}(M)$ satisfies*

(1) $\bar{D}(\partial_t, \partial_t) = -f[\varphi(f) + \mu\Delta(f)] - f^2\dot{h}$
(2) $\bar{D}(X, \partial_t) = -\phi f X(f)$
(3) $\bar{D}(X, Y) = g(\nabla_X \varphi, Y) - \mu \mathrm{Ric}(X, Y) + \frac{\mu}{f} H^f(X, Y)$

Theorem 4 *Let $\varphi = h\partial_t + \varphi$ be a φ (Ric)-vector field on a standard static space-time of the form $I_f \times M$ with scalar μ. Then, one of the followings holds*

(1) *f is constant and φ is a φ (Ric)-vector field on M with scalar μ, and also $h = a$ for some $a \in \mathbb{R}$,*
(2) *$h = 0$ where*

$$\varphi(f) = -\mu\Delta(f)$$
$$\mu \mathrm{Ric}(X, Y) = g(\nabla_X \varphi, Y) + \frac{\mu}{f} H^f(X, Y)$$

for any vector field X and Y on M.

4 On Lorentzian manifolds

Let M be a Lorentzian manifold admitting a time-like proper φ (Ric)-vector field. Let M be a quasi-Einstein manifold, i.e. the Ricci tensor is non-flat and takes the form

$$R_{ij} = ag_{ij} + bu_i u_j, \tag{4.1}$$

where a, b are scalars, $b \neq 0$ and u is a unit covariant vector [6]. However throughout the paper we assume that a and b are constants. Then we have

$$\nabla_i \varphi_j = \mu a g_{ij} + \mu b u_i u_j. \tag{4.2}$$

A contraction with φ^j yields

$$\varphi^j \nabla_i \varphi_j = \mu \left[a\varphi_i + bu_i \left(\varphi^j u_j \right) \right].$$

The scalar curvature $R = na + \varepsilon b$, $\varepsilon = \pm 1$ is constant since we assume that the associated scalars a, b are constants. Now, M has constant scalar curvature R and consequently φ has a constant length, i.e.

$$0 = a\varphi_i + bu_i \left(\varphi^j u_j \right).$$

This equation implies either u and φ are orthogonal and consequently $a = 0$ and M is Ricci simple, i.e. $R_{ij} = bu_i u_j$ or u is a time-like field parallel to φ. The later case implies

$$\nabla_i \varphi_j = \alpha g_{ij} + \beta \varphi_i \varphi_j. \tag{4.3}$$

Moreover, it is

$$\varphi^j \nabla_i \varphi_j = 0 = \mu \varphi^j R_{ij},$$

i.e. φ is an eigenvector of the Ricci tensor with zero eigenvalue. Applying Chen's simple characterization of GRW space-times (see [7,16]), M turns out to be a GRW space-time. We also have $u^i R_{ij} = 0$ and so $a = b$. The Ricci curvature becomes

$$R_{ij} = a\left(g_{ij} + u_i u_j \right).$$

A contraction with g^{ij} implies $a = \frac{R}{n-1}$, i.e.

$$R_{ij} = \frac{R}{n-1}\left(g_{ij} + u_i u_j\right).$$

This discussion leads to the following result.

Theorem 5 *A Lorentzian quasi-Einstein manifold admitting a time-like φ (Ric)-vector field is Ricci simple and the Ricci tensor takes the form $R_{ij} = bu_i u_j$ or it is a perfect fluid GRW space-time and the Ricci tensor takes the form*

$$R_{ij} = \frac{R}{n-1}\left(g_{ij} + u_i u_j\right)$$

where $u_i = \frac{\varphi_i}{\sqrt{-\varphi^m \varphi_m}}$.

Now, suppose that M is a generalized quasi-Einstein manifold, i.e. the Ricci curvature has the form

$$R_{ij} = ag_{ij} + bu_i u_j + cv_i v_j, \tag{4.4}$$

where a, b, c are scalars and u_i, v_i are two orthonormal 1-forms [8,9]. However throughout the paper we assume that a, b and c are constants. Following the same strategy for a proper time-like φ (Ric)-vector field, one can obtain

$$\nabla_i \varphi_j = \mu a g_{ij} + \mu b u_i u_j + \mu c v_i v_j \tag{4.5}$$

and so

$$\varphi^j \nabla_i \varphi_j = \mu\left[a\varphi_i + bu_i\left(\varphi^j u_j\right) + cv_i\left(\varphi^j v_j\right)\right].$$

Since we assume that the associated scalars are constants, M has constant scalar curvature $R = na + \varepsilon b + \varepsilon c$, $\varepsilon = \pm 1$ and consequently φ has a constant length i.e.

$$0 = a\varphi_i + bu_i\left(\varphi^j u_j\right) + cv_i\left(\varphi^j v_j\right). \tag{4.6}$$

Since φ is time-like, one of the generators, say u, must be time-like and consequently v is space-like. Transvestind this equation twice by u^i and v^i, one gets

$$0 = (a-b)\left(u^i \varphi_i\right) \tag{4.7}$$

$$0 = (a+c)\left(v^i \varphi_i\right) \tag{4.8}$$

The vector fields u and φ could not be orthogonal and so $a = b$. Thus,

$$0 = a\left(\varphi_i + u_i\left(\varphi^j u_j\right)\right) + cv_i\left(\varphi^j v_j\right)$$

$$R_{ij} = a\left(g_{ij} - u_i u_j\right) + cv_i v_j.$$

For a non-zero a, Eq. (4.6) shows that φ is a linear combination of both u and v.

From Eq. (4.8), either $a = -c$ or v and φ are orthogonal. The first case implies

$$R_{ij} = a\left(g_{ij} + u_i u_j - v_i v_j\right)$$

A contraction with g^{ij} implies $a = \frac{R}{n-2}$, i.e.

$$R_{ij} = \frac{R}{n-2}\left(g_{ij} + u_i u_j - v_i v_j\right).$$

In this case, it is clear that $u^i R_{ij} = v^i R_{ij} = 0$. The following result rises.

Theorem 6 *In a Lorentzian generalized quasi-Einstein manifold admitting a proper time-like* φ (Ric)*-vector field,* φ *has a constant length and the generators are eigenvectors of the Ricci tensor with zero eigenvalue and the Ricci tensor is given by*

$$R_{ij} = \frac{R}{n-2}\left(g_{ij} + u_i u_j - v_i v_j\right),$$

provided that a is non-zero and v *and* φ *are not orthogonal.*

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