ROBUSTLY AND STRONGLY STABILIZING LOW ORDER CONTROLLER DESIGN FOR INFINITE DIMENSIONAL SYSTEMS

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By

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

ROBUSTLY AND STRONGLY STABILIZING LOW ORDER CONTROLLER DESIGN FOR INFINITE DIMENSIONAL SYSTEMS

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July 2018

This thesis deals with the robust stabilization of infinite dimensional systems by stable and low order controllers. The close relation between the Nevanlinna-Pick interpolation problem and the robust stabilization is well known in the literature. In order to utilize this relation, we propose a new optimal solution strategy for the Nevanlinna-Pick interpolation problem. Differently from the known suboptimal solutions, our method includes no mappings or transformations, it directly solves the problem in the right half plane. We additionally propose a method via suboptimal solutions of an associated Nevanlinna-Pick interpolation problem to robustly and strongly stabilize a set of plants which include the linearized models of two well known under actuated robots around their upright equilibrium points. In the literature, it is shown that the robust stabilization of an infinite dimensional system by stable controllers can be reduced to a bounded unit interpolation problem. In order to use this approach to design a finite dimensional controller, we propose a predetermined structure for the solution of the bounded unit interpolation problem. Aforementioned structure reduces the problem to a classical Nevanlinna-Pick interpolation problem which can be solved by the optimal solution strategy of this thesis. Finally, by combining the finite dimensional solutions of the bounded unit interpolation problem with the finite dimensional approximation techniques, we propose a method to design finite dimensional and stable controllers to robustly stabilize a given plant. Since time delay systems are one of the best examples of infinite dimensional systems, we provide numerical examples of various time delay systems for each proposed method.

Keywords: Robust stabilization, Strong stabilization, Stable controller, Finite dimensional controller, Infinite dimensional systems, Analytic interpolation, Nevanlinna-Pick interpolation, Modified Nevanlinna-Pick interpolation, Bounded unit interpolation.

ÖZET

SONSUZ BOYUTLU SİSTEMLER İÇİN DÜŞÜK DERECELİ GÜRBÜZ VE GÜÇLÜ DENETLEYİCİ TASARIMI

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Bu tez, sonsuz boyutlu sistemlerin düşük dereceli ve kararlı denetleyiciler ile gürbüz bir şekilde kararlılığının sağlanmasını konu almaktadır. Nevanlinna-Pick aradeğerlemesi ile gürbüz kontrol arasndaki yakın ilişki önceki çalışmalardan bilinmektedir. Bu ilişkiyi kullanmak için biz Nevanlinna-Pick aradeğerlemesi için yeni bir ideal çözüm stratejisi öneriyoruz. Bilinen diğer idealin altındaki çözümlerden farklı olarak bizim yöntemimiz hiçbir dönüşüm içermemektedir, problemi doğrudan sağ yarım düzlemde çözmektedir. Buna ek olarak, literatürde bilinen iki tip eksik tahrikli robotun dik denge noktaları etrafındaki doğrusallaştırılmış modelini de kapsayan bir sistem kümesinin ilgili Nevanlinna-Pick aradeğerleme probleminin idealin altındaki çözümleri ile gürbüz ve güçlü kararlılığının sağlanması için bir yöntem öneriyoruz. Kararlı denetleyiciler ile sonsuz boyutlu bir sistemin gürbüz kararlılığının sağlanmasının sınırlı birim aradeğerleme problemine daraltılabileceği literatürde gösterilmiştir. Bu yaklaşımı kullanarak sonlu boyutlu denetleyiciler tasarlamak adına sınırlı birim aradeğerleme problemi için önceden belirlenmiş bir yapı öneriyoruz. Bahsedilen önceden belirlenen yapı, problemi bu tezde anlatılan ideal çözüm stratejisi ile çözülebilecek bir Nevanlinna-Pick aradeğerleme problemine çevirmektedir. Son olarak, sınırlı birim aradeğerleme probleminin sonlu boyutlu çözümleri ile yaklaım teknikleri birleştirilerek verilen bir sistemin sonlu boyutlu ve kararlı denetleyiciler ile gürbüz bir şekilde kararlı hale getirilmesi için bir yöntem öneriyoruz. Zaman gecikmeli sistemler, sonsuz boyutlu sistemlerin en iyi örneklerinden olduğu için zaman gecikmeli sistemler içeren sayısal örnekler sağlıyoruz.

Anahtar sözcükler: Gürbüz kararlılık, Güçlü kararlılık, Kararlı denetleyici, Sonlu boyutlu denetleyici, Sonsuz boyutlu sistem, Analitik aradeğerleme, Nevanlinna-Pick aradeğerlemesi, Değiştirilmiş Nevanlinna-Pick aradeğerlemesi, Sınırlı birim aradeğerlemesi.

Dedicated to
my wife Seçil...

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Chapter 1

Introduction

Robust control theory is a branch of classical control theory which deals with the stabilization of uncertain plants and their closed loop performances [16]. There are no exact mathematical models to describe any physical system, each model comes with its own approximations and errors. Basically, it is possible to analyse such uncertainties in two groups [16]; structured and unstructured uncertainties. Unstructured uncertainties include additive and multiplicative uncertainties, whereas structured ones are generally in the form of parameter uncertainty. Robust stabilization aims to stabilize a feedback loop for a set of plants which can be grouped around a nominal plant with additional structured or unstructured uncertainties. Robust performance, on the other hand, aims to satisfy a predetermined performance level in addition to robust stability. Since all practical systems can be modelled up to a certain precision, robust stabilization and performance optimization is crucial in all real world applications.

Internal stabilization of a plant by a stable controller is called the strong stabilization. A stable controller brings a different kind of robustness to the feedback loop: robustness to sensor and closed loop failures. Since the controller is bounded input bounded output stable in strong stabilization operation, unbounded control responses are automatically avoided provided that the controller input is bounded, see [54] and [72] for details. Another basic advantage of a stable controller arises when the plant is unstable (e.g., aerospace applications) or it is expensive/dangerous to test a fresh controller directly on the plant [41]. For such cases, it is possible to test (or verify) the design of a controller in the open loop by frequency domain techniques since it is a stable transfer function, see[27], [43], [29] and [25] for further details on strong stability. A well known necessary condition under which it is possible to design a stable controller stabilizing the feedback loop is that the plant must satisfy the parity interlacing property. This property will be explained and analysed in the next section.

Finite dimensionality of the controller is vital in practical applications because it is difficult or impossible to realize an infinite dimensional controller. There are some approximation techniques to implement such functions, however, stability bounds might degrade due to finite dimensional approximations. Because of this fact, finite dimensional and proper controllers are desired for real world applications.

Robust stabilization of finite dimensional systems has been studied for a long time. It has been shown that the well known Nevanlinna-Pick interpolation problem is closely related to robust stabilization problem. All the internal stabilization conditions are equivalent to interpolation conditions on the complementary sensitivity function and robust stabilization can be reduced to an infinity norm bound on the same function. Although the Nevanlinna-Pick approach is just a sufficient condition for the robust stabilization, (i.e. a broader definition exists via Nyquist stability arguments) it has proven useful in the literature due to its ease of interpretation. We refer to [16] for details about robust stabilization.

Sensitivity shaping for both finite and infinite dimensional systems have been attractive and studied for over some decades. This problem has also been analysed together with strong stabilization condition. To the best of our knowledge, robust stabilization of infinite dimensional systems, with optimal sensitivity bounds, by stable controllers is still an open research problem. A recent contribution, [59], introduced a good insight to this topic. It has been shown that the robust stabilization of infinite dimensional systems by stable controllers can be reduced to a bounded unit interpolation problem. A brief summary of the contribution and the relevant mathematical preliminary will be given in the following section.

The main objective of this thesis is to design finite dimensional, proper and stable controllers to robustly stabilize infinite dimensional systems. In order to achieve this, analytic interpolation techniques together with finite dimensional approximation methods are utilized and sufficient conditions and performance bounds are derived for different types of systems. We propose a simple novel method for the computation of the optimal solution of the Nevanlinna-Pick interpolation problem [66], [68], which has a close relationship with the robust control problems. We also propose a sufficient condition under which it is possible to solve the modified Nevanlinna-Pick interpolation problem (in other words the bounded unit interpolation problem in $\mathcal{H}_\infty$) by a finite dimensional function, [70]. In addition to this sufficient condition, we also propose an algorithm to find the finite dimensional interpolating function when the problem is feasible. The rest of the thesis is organized as follows:

Mathematical basics about robust stabilization and stable controllers are given in Chapter 2 with relevant literature survey. Definitions of Nevanlinna-Pick interpolation problem, unit interpolation problem and modified version of the Nevanlinna-Pick interpolation problem are also defined in Chapter 2 together with the known solution methods of each problem from the literature.

In Chapter 3, we show that the central controller, which is designed via the parameterization of all suboptimal solutions of the associated Nevanlinna-Pick interpolation problem given in [8], is stable for a class of plants, [67]. This class of plants includes the linearized models of some underactuated robots which are widely used in the literature, i.e. Acrobot and Pendubot. With the proposed approach, we design stable and low-order controllers for these robots around their upright equilibrium point and compare the frequency response of the overall feedback loop with the ones from literature.

The new optimal solution strategy for the Nevanlinna-Pick interpolation problem in the open right half plane is described in detail in Chapter 4. There are some methods which solve the problem sub-optimally and to the best of our knowledge, previous optimal solutions for the right half plane interpolation data required a conformal map, and hence an introduction of an extra parameter. For this reason our direct solution is much simpler.

In Chapter 5, an iterative algorithm [65] is proposed to solve bounded unit interpolation problem in $\mathcal{H}_\infty$ which is based on the constructive method described in [16]. The main disadvantage of this algorithm is that, it is only applicable to real interpolation data. In addition to this, its computational complexity increases rapidly as the number of interpolation points increase. To overcome all these disadvantages, an algorithm which is based on the optimal solution of the Nevanlinna-Pick interpolation problem is proposed to solve real, rational and bounded unit interpolation problem with finite dimensional interpolants. Since this problem is shown to be equivalent to robust stabilization of infinite dimensional systems by stable controllers, our contribution is vital to design such controllers with finite dimensionality.

In Chapter 6, we propose a method to design proper, finite dimensional and stable controllers to robustly stabilize infinite dimensional systems. The proposed method uses finite dimensional approximation of some parts of the plant and defines a bounded unit interpolation problem including the approximation errors to design the desired controller. This chapter uses the finite dimensional solution algorithm of Chapter 5 to solve the associated bounded unit interpolation problem.

Chapter 7 concludes the study with a brief summary and some discussions on the proposed methods. Possible future extensions are also outlined in this chapter.

Chapter 2

Basic Concepts

2.1 Norms for Signals

We consider continuous functions which are defined from $[0, \infty)$ to $\mathbb{R}$ and assume that u and v are such functions. An operation which satisfies the following four properties on functions u and v and a real scaler a is called a norm, [16]:

- $\|u\| \geq 0$
- $\|u\| = 0 \iff u(t) = 0, \forall t$
- $\|au\| = |a|\|u\|, \forall a \in \mathbb{R}$
- $\|u + v\| \leq \|u\| + \|v\|$.

The 2-norm of a signal $u(t)$ in continuous time domain is defined as

$$\|u\|_2 = \left(\int_0^\infty u(t)^2 dt \right)^{1/2}. \tag{2.1}$$

In addition to this, we denote the Laplace transform of $u(t)$ by $U(s)$, and define the 2-norm of the Laplace transformed signal $U(s)$ as

$$\|U\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^\infty |U(j\omega)|^2 d\omega \right)^{1/2}. \tag{2.2}$$

The infinity norm (i.e. $\infty-$norm) of a signal in continuous time domain is defined as

$$\|u\|_\infty := \sup_t |u(t)|. \tag{2.3}$$

2.2 Norms for Systems

In this thesis, we consider causal, linear, time-invariant systems which have the following convolution type input-output relation between its input $r(t)$ and output $y(t)$

$$y(t) = g(t) * r(t) \tag{2.4}$$

which is

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau) r(\tau) d\tau. \tag{2.5}$$

The function $g(t)$ is the impulse response of the system and we denote its Laplace transform by $G(s)$. Causality means that $g(t) = 0$ for $t < 0$. $G(s)$ is proper if $|G(j\omega)|$ is bounded, strictly proper if $G(j\infty) = 0$ and bi-proper if both G and G^{-1} are proper. G is stable if it is bounded and analytic in the closed right half plane $\mathbb{C}_+$ i.e. $Re\{s\} \geq 0$ and the stability of G is denoted by $G \in \mathcal{H}_\infty$.

For transfer functions in the form $G = G_n/G_d$ where G_n and G_d are polynomials or quasi-polynomials, stability is equivalent to having all roots of the denominator G_d in $\mathbb{C}_-$. In particular, when G_n and G_d are polynomials, G is proper means $deg(G_n) \leq deg(G_d)$, strictly proper means $deg(G_n) < deg(G_d)$ and bi-proper means $deg(G_n) = deg(G_d)$, where $deg(.)$ denotes the degree of a polynomial. Degree of a polynomial is defined to be the degree of the highest order unknown within the polynomial. The $\infty-$norm of a system is defined as

$$\|G\|_\infty := \sup_{\omega \in \mathbb{R}} |G(j\omega)|. \tag{2.6}$$

Note that there exist an additional definition for the ∞-norm of a system as

$$\|G\|_\infty = \sup_{r \neq 0} \frac{\|y\|_2}{\|r\|_2} \tag{2.7}$$

since it is induced by the 2-norm of the input-output signals.

2.3 Stability of a Feedback Loop

Throughout this thesis, we consider the basic unity feedback loop shown in Figure 2.1, unless stated otherwise.

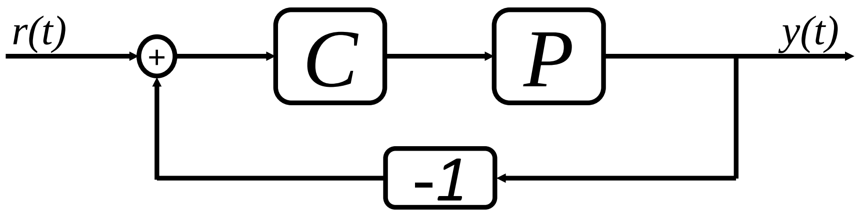


Figure 2.1: Basic unity feedback loop

We say that the controller C **internally stabilizes** the plant P if and only if the following conditions are satisfied:

$$\begin{aligned} S &\in \mathcal{H}_\infty \\ PS &\in \mathcal{H}_\infty \\ CS &\in \mathcal{H}_\infty \end{aligned} \tag{2.8}$$

where $S = (1+PC)^{-1}$ is the sensitivity function of the closed loop system. In the special case where P has finitely many distinct poles and zeros in the extended right half plane, $\mathbb{C}_{+e} = \mathbb{C}_+ \cup +\infty$ these conditions are equivalent to having

$$\begin{aligned} &T \in \mathcal{H}_\infty \\ &T(z_i) = 0, \forall i \\ &T(p_j) = 1, \forall j \end{aligned} \tag{2.9}$$

where z_i and p_j denote the zeros and poles of the plant P in $\mathbb{C}_{+e}$, respectively and $T = 1 - S$ is the complementary sensitivity function of the closed loop system.

2.4 Uncertainty and Robust Stability

Typically, it is not possible to fully characterize a physical system by a precise mathematical model. Because of this, in all practical applications we have to handle the uncertainties in order not to lose the stability of the feedback loop. It is convenient to model the uncertain plant as a set of plants around a nominal plant for unstructured uncertainties. For the scope of this thesis, we briefly go over the multiplicative uncertainty model and robust stabilization of such plant sets.

Consider a known nominal plant P and perturbed model $\tilde{P} = (1 + \Delta W)P$. This model constitutes a set of plants such that

$$\mathfrak{P}(P) = \{\tilde{P} = (1 + \Delta W)P : \Delta \in \mathcal{H}_\infty, \|\Delta\|_\infty < 1\} \tag{2.10}$$

where $W \in \mathcal{H}_\infty$ is a fixed known transfer function, namely the uncertainty weight. This kind of uncertainty is called the multiplicative uncertainty, see [16] for details and other types of uncertainties.

It is well known in the literature that a controller C **robustly stabilizes** the set of uncertain plants $\mathfrak{P}(P)$ if it can internally stabilize P (i.e. satisfies (2.9)) and satisfies the following norm condition

$$\|WT\|_\infty \leq 1. \tag{2.11}$$

2.5 Strong Stability

Strong stability requires a stable controller to be designed. A stable controller has two main advantages: it is robust to sensor failures as described by [16], [54] and it is testable stand-alone as mentioned by [41]. It is possible to test a stable controller by its input-output relationship practically by applying some test signals as an open-loop configuration before using it with the original plant to prevent catastrophic events that may occur due to controller implementation errors.

A controller C **strongly stabilizes** a given plant P if it can internally stabilize the plant P (i.e. satisfies (2.9)) in addition to being stable itself (i.e. $C \in \mathcal{H}_\infty$). A well known sufficient condition for the existence of strongly stabilizing controllers is the parity interlacing property (PIP) of the plant. The PIP is the property of having even number of poles between each pair of its zeros on the extended right half side of real line, see [16] for details. It is also notable that simultaneous stabilization of two plants is equivalent to strong stabilization of an auxiliary plant which is derived from the aforementioned plants of interest.

There is extensive literature on strong stabilization of finite dimensional plants, see e.g. [11], [13], [27], [43], [29] and also see [25] for sensitivity shaping of infinite dimensional systems by fixed order stable controllers.

2.6 Robust and Strong Stability

Let us assume that an uncertain plant set as in (2.10) is given. Assume that the set of controllers which stabilizes the given nominal plant P (i.e. which satisfies the conditions in (2.9)) is denoted by $\mathfrak{C}(P)$.

Problem 1. *Find a controller* $C \in \mathfrak{C}(P) \cap \mathcal{H}_\infty$ *satisfying* (2.11).

Problem 1 is called **robust and strong stabilization** problem. This problem has its roots in [30] and has been studied for different families of plants since then.

Problem 1 for infinite dimensional plant families has gained attraction by a recent contribution from [59]. In this study, it was shown that the robust and strong stabilization of a set of uncertain infinite dimensional plants having finitely many simple right half plane zeros is equivalent to bounded unit interpolation problem in $\mathcal{H}_\infty$. Prior to this study, equivalence of robust stabilization and Nevanlinna-Pick interpolation problem and equivalence of strong stabilization and unit interpolation in $\mathcal{H}_\infty$ was known. The importance of [59] is to combine aforementioned interpolation problems to propose a sufficient condition for Problem 1 when the nominal plant of interest is an infinite dimensional one. The disadvantage of the proposed method of [59] is that it proposes an infinite dimensional controller for an infinite dimensional plant.

2.7 Analytic Interpolation

Analytic interpolation refers to finding a transfer function in Laplace domain, which exactly satisfies a number of interpolation conditions in complex domain together with some higher level requirements such as being stable, being norm bounded, being minimum phase, etc. In general, stability of the interpolating function and interpolation points satisfy the internal stability of the closed loop whereas the other requirements stand for some further design requirements like the robustness of the feedback loop or the stability of the controller. There are a number of predefined analytic interpolation problems and we deal with Nevanlinna-Pick interpolation problem, unit interpolation problem and modified Nevanlinna-Pick (bounded unit) interpolation problem for the scope of this thesis.

2.7.1 Nevanlinna-Pick Interpolation Problem

The Nevanlinna-Pick interpolation problem is defined as follows:

Problem 2. *Given $\alpha_i \in \mathbb{C}_+$ and $\beta_i \in \mathbb{C}$ for $i \in \{1, \ldots, n\}$ find $F \in \mathcal{H}_\infty$ such that $F(\alpha_i) = \beta_i$ for all i and $\|F\|_\infty \leq \gamma$, for the smallest possible $\gamma > 0$.*

The smallest achievable norm is denoted by γ_{opt}. Generically, there is an admissible interpolant having $\|F\|_\infty = \gamma_{opt}$, which is an inner function of degree $n-1$, see e.g. [34] and [72]. Earlier studies as in [22], tried to solve Nevanlinna-Pick interpolation problem with a degree constraint where degree of the interpolant F satisfies $\deg(F) < n$. Other studies, like [19] and [33], considered some variations of the Nevanlinna-Pick problem with a degree constraint.

Model matching problem as described in [16], is a good example of Nevanlinna-Pick interpolation problem in control theory. Let W and M be stable and proper transfer functions, i.e. $W,\ M \in \mathcal{H}_\infty$. The model matching problem is to find a stable function Q such that $\|W - MQ\|_\infty$ is minimized, where typically M is an inner function of the form

$$M(s) = \prod_{i=1}^{n} \frac{z_i \ - \ s}{\bar{z}_i \ + \ s}$$

with $\Re(z_i) > 0,\ \forall i$. For the sake of simplicity, we assume that the set $\mathcal{Z} = \{z_1,\ \ldots,\ z_n\}$ consists of distinct elements. This way, higher order interpolation conditions do not appear in the problem. Moreover, to obtain solutions with real coefficients it is assumed that if $z_i \in \mathcal{Z}$ then $\bar{z}_i \in \mathcal{Z}$.

In this context $\|W - MQ\|_\infty = \gamma$ is the model matching error. When $n = 1$, the optimal solution is $Q(s) = (W(s) - W(z_1))/M(s)$, with $\gamma_{opt} = |W(z_1)|$. In the case where M has multiple zeros in $\mathbb{C}_+$, it is not trivial to find the optimal $Q \in \mathcal{H}_\infty$ and γ_{opt}.

Let us define $F = (W - MQ)$. The model matching problem has a solution with $\|W - MQ\|_\infty \leq \gamma$ if there exists F such that

$$F \in \mathcal{H}_\infty, \quad \|F\|_\infty \leq \gamma, \text{ and}$$

$$F(z_i) = W(z_i) \text{ for all } i \in \{1,\ \ldots,\ n\}\ .$$

Under these conditions, $Q = (W - F)M^{-1} \in \mathcal{H}_\infty$ is the solution of the model matching problem. This is a Nevanlinna-Pick problem with interpolation data $\alpha_i = z_i$, $\beta_i = W(z_i)$ for all $i \in \{1,\ \ldots,\ n\}$. This problem is solvable for some γ if and only if the associated Pick matrix

$$\mathsf{P}_\gamma = \mathsf{A} - \gamma^{-2}\mathsf{B}$$

is positive semi-definite where

$$[\mathsf{A}]_{ij} = \frac{1}{\alpha_i \ + \ \bar{\alpha}_j}, \qquad [\mathsf{B}]_{ij} = \frac{\beta_i \ \bar{\beta}_j}{\alpha_i \ + \ \bar{\alpha}_j} \tag{2.12}$$

for all $i, j \in \{1,\ \ldots,\ n\}$ and $\gamma_{opt} = \sqrt{\lambda_{max}}$, where λ_{max} is the largest eigenvalue of the matrix $\mathsf{A}^{-1}\mathsf{B}$, see [16]. The next step is to calculate the corresponding $F_{opt} \in \mathcal{H}_\infty$.

Model matching problem appears in $\mathcal{H}_\infty$ control as well as in identification problems for which we refer the readers to [15], [14], [23], [39], [45], [64]. In particular it arises in $\mathcal{H}_\infty$ control after the parametrization of all stabilizing controllers, [3], [18], [24].

The original Nevanlinna-Pick formulation (interpolation on the unit disc) is directly applicable to discrete time systems. However, for continuous time systems, a mapping between the unit disc and right half plane is needed. There are three well known solutions to the Nevanlinna-Pick interpolation problem in the literature. The solution outlined in Section 2.7.1.1 is the original method dealing with interpolation on unit disc. The solution method given in Section 2.7.1.2 deals with continuous time system formulations (right half plane interpolation data) and parameterizes all suboptimal solutions to the problem. The method summarized in Section 2.7.1.3 uses a conformal map and solves the problem over the unit disc. This method also includes an inverse conformal mapping for the interpolant to be converted to the right half plane (continuous time). These mappings can be costly (in terms of computation time) and numerically problematic (precision of the interpolation). To overcome these problems, the conformal mapping must be chosen wisely.

2.7.1.1 Nevanlinna's Algorithm

In [16] a suboptimal solution is described through a series of Möbius transforms. The core idea of the method is to transform the interpolation problem of n data points to a problem involving $n-1$ data points through a Möbius transform which is derived from the n^{th} interpolation pair. Applying this idea iteratively results in an interpolation problem of a single data point and it is easily solved as mentioned above. Then, this solution is back transformed through n inverse Möbius transforms in order to find the original interpolating function. If it can be identified by Pick matrix that the original problem with n data is solvable then this method gives the solution after n forward plus n inverse Möbius transforms. More explanation and some informative examples can be found in [16] and [36].

2.7.1.2 Parametrization of All Suboptimal Solutions

In [8] all suboptimal solutions are characterized as follows. For a given $\gamma > \gamma_{opt}$, define $\mathsf{D}_\alpha(s)$ as the diagonal matrix whose non-zero entries are

$$[\mathsf{D}_\alpha(s)]_{ii} = (s - \alpha_i)^{-1},$$

and compute transfer functions $\Theta_{ij}(s)$ from

$$\begin{bmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathsf{Q}_1 \mathsf{D}_\alpha(s)\ \mathsf{P}_\gamma^{-1} \mathsf{Q}_2 \tag{2.13}$$

where $\mathsf{Q}_1 = \begin{bmatrix} \beta_1/\gamma & \cdots & \beta_n/\gamma \\ 1 & \cdots & 1 \end{bmatrix}$ and $\mathsf{Q}_2 = \begin{bmatrix} -\overline{\beta}_1/\gamma & 1 \\ \vdots & \vdots \\ -\overline{\beta}_n/\gamma & 1 \end{bmatrix}$. Then, all feasible solutions are in the form

$$F(s) = \gamma \, \frac{\Theta_{11}(s)\ G(s) + \Theta_{12}(s)}{\Theta_{21}(s)\ G(s) + \Theta_{22}(s)} \tag{2.14}$$

where $G \in \mathcal{H}_\infty$ is a free parameter with $\|G\|_\infty \leq 1$, see [8] for details.

Note that as γ decreases to γ_{opt} the matrix P_γ becomes singular, and hence the right hand side of (2.13) becomes ill-conditioned in the (near)-optimal case. This ill-conditioning problem is analyzed in [38] where some suggestions are given to fix this problem. Moreover, in the parameterization (2.14), which $G(s)$ gives the optimal solution $F_{opt}(s)$ is not apparent. However, there exist some study on this issue in the literature. Particularly, [5] uses a state space approach to get formulas for solution of the minimal possible norm.

2.7.1.3 Solution Through Conformal Map

For the case where the interpolation data is defined on the unit disc, $\mathbb{D}$, and one tries to find an analytic function on the unit disc, with $|F(z)| \leq \gamma$ for all $z \in \mathbb{D}$, the optimal and suboptimal solutions are given and discussed in detail in [21], see also [72]. However, for an interpolation problem defined on $\mathbb{C}_+$ this method requires a conformal map between $\mathbb{C}_+$ and $\mathbb{D}$. The numerical properties of this approach for the given case depend heavily on the choice of the conformal map. The simplest possible conformal map is $\varphi(\alpha) = \frac{\alpha - r}{\alpha + r} : \mathbb{C}_+ \to \mathbb{D}$, where $r > 0$ is a free parameter to be chosen. It is important to choose r judiciously to avoid numerical problems. In practice, r should not lie within a small neighborhood of any α_i. For small scale problems, i.e. $2 < n < 10$, it may be relatively easy to find a "good" value for r; but as the dimensionality of the problem increases, it becomes harder to find such an r. Due to this difficulty, the problem becomes prone to numerical errors as the dimensionality increases. To our knowledge, there does not exist an automated method to choose r for a given set of $(\alpha_1, \ \ldots, \ \alpha_n)$.

2.7.2 Unit Interpolation Problem

The unit interpolation problem is defined as follows:

Problem 3. *Given $\alpha_i \in \mathbb{C}_+$ and $\beta_i \in \mathbb{C}$ for $i \in \{1, \ldots, n\}$ find $U, U^{-1} \in \mathcal{H}_\infty$ such that $U(\alpha_i) = \beta_i$ for all i.*

In [16], it is shown through parameterization of all stabilizing controllers that the strong stabilization of a given plant is equivalent to unit interpolation problem and a constructive method is outlined to generate a unit interpolating function, under some constraints related to parity interlacing property.

In order to briefly summarize the method in [16], assume that for some $1 < k < n$ we have U_k such that

$$U_k(\alpha_i) = \beta_i \tag{2.15}$$

for all $i = 1, ..., k$ and $U_k, U_k^{-1} \in \mathcal{H}_\infty$.

Then it is possible to write U_{k+1} as

$$U_{k+1} = (1 + c_{k+1} H_{k+1})^{l_{k+1}} U_k \tag{2.16}$$

where $H_{k+1}(\alpha_i) = 0$ for $i = 1, ..., k$. This choice of H_{k+1} makes $U_{k+1}(\alpha_i) = U_k(\alpha_i)$ for $i = 1, ..., k$ independent of c_{k+1} and l_{k+1}. As a result, U_{k+1} is guaranteed to satisfy the interpolation data $U_{k+1}(\alpha_i) = \beta_i$ for $i \in \{1, \ldots, k\}$. If it is possible to choose c_{k+1} and l_{k+1} such that $U_{k+1}(\alpha_{k+1}) = \beta_{k+1}$ and $|c_{k+1}| < 1/\|H_{k+1}\|_\infty$ then U_{k+1} satisfies the interpolation data $U_{k+1}(\alpha_i) = \beta_i$ for $i \in \{1, \ldots, k+1\}$ and it is outer. At the end of the algorithm, $U = U_n$ satisfies all the interpolation data and is outer if at each stage the condition $|c_i| < 1/\|H_i\|_\infty$ was satisfied for $i = 1, ..., n$.

A Simple Example

Let us assume that the interpolation data is given as $(\alpha_i, \beta_i) = \{(1, 4), (0.5, 3)\}$. Find $U(s)$ which satisfies the given interpolation data and which is outer.

Solution:

- Let $U_1(s) = 4$. It naturally satisfies the first interpolation condition; $U_1(1) = 4$.
- Take $H_2(s) = (s - 1)/(s + 1)$ which satisfies $H_2(1) = 0$.
- Write $U = U_2 = (1 + c_2 H_2)^{l_2} U_1$ with the given H_2 and $l_2 = 1$.
- Need to satisfy the second interpolation condition as $U_2(0.5) = (1+c_2(0.5-1)/(0.5+1))4 = 3$.
- This yields $(1 - (1/3)c_2) = 3/4$ and finally $c_2 = 0.75 < 1/\|H_2\|_\infty$ where $\|H_2\|_\infty = 1$.
- As a result, $U(s) = (7s + 1)/(s + 1)$ satisfies the interpolation data as $U(1) = 4$ and $U(0.5) = 3$, in addition to this we have both $U, U^{-1} \in \mathcal{H}^\infty$.

As the example illustrates, by this method, it is possible to generate outer interpolating functions. It is also possible to tune the degree of the function by changing the value of the parameter l when c does not satisfy the norm condition. One additional important point is the choice of H_i functions. Due to the imposed requirements, zero locations of the function are fixed (i.e. zeros of H_k have to be at α_i for $i = 1, ..., k-1$), however, pole locations are not constrained by the requirements. This means, it might also be a parameter to shape the frequency response of resulting interpolation function.

In the literature, there exist a relaxed version of unit interpolation problem, namely, positive real interpolation. In [8], the parameterization of all solutions to the positive real interpolation problem is defined in terms of 4 transfer functions which are defined by the interpolation data. Note that, any positive real rational function F is also a unit in $\mathcal{H}_\infty$ (i.e. $F, F^{-1} \in \mathcal{H}_\infty$). In addition to this, [7] and [22] formulated the problem of positive real interpolation as a maximization problem with a generalized entropy criterion. The dual of this problem is a convex optimization problem in a finite dimensional space.

2.7.3 Modified Nevanlinna-Pick Interpolation Problem

Problem 4. *Consider Problem 2 with the following additional constraint*

$$F^{-1} \in \mathcal{H}_\infty$$

and determine whether such F exists.

Problem 4 is called the modified Nevanlinna-Pick interpolation problem (mN-PIP) and is shown to be solvable for $\gamma = 1$ if and only if the associated Pick matrix

$$[\mathsf{P_M}([l_1, \ldots, l_n])]_{ij} = \frac{-\ln \beta_i - \ln \bar{\beta}_j + j2\pi(l_j - l_i)}{1 - \alpha_i \bar{\alpha}_j}$$

is positive semi-definite for some integer set $[l_1, \ldots, l_n]$ and for all $i, j \in \{1, \ldots, n\}$, see [9] and [52] for details.

This problem is also called as the bounded unit interpolation problem in $\mathcal{H}_\infty$ because it is also possible to define this problem as follows:

Problem 5. *Consider Problem 3 with the following additional constraint*

$$\|U\|_\infty < \rho \tag{2.17}$$

for the smallest possible ρ and determine whether such U exists.

The necessary and sufficient conditions for an infinite dimensional bounded unit interpolating function is given in [9], [52] through a modified Pick matrix. In [27], [40], a solution method for the infinite dimensional case is discussed.

In [1], sufficient conditions to find a solution for Problem 5 are derived. The conservatism of these conditions are represented by a two point interpolation problem in [1] and by a three point interpolation problem in [4].

There have been some efforts in the literature which try to solve the bounded unit interpolation problem through positive real functions: [7] and [22] formulated the problem of positive real interpolation as a maximization problem with a generalized entropy criterion. The dual of this problem is a convex optimization problem in a finite dimensional space. Bound on the infinity norm of the interpolating function is modelled as a constraint to the minimization problem; [19] utilizes these ideas to find a passive finite dimensional approximate for originally passive systems by analytic interpolation. The method of [19] produces positive real interpolating functions with finite dimension which closely approximates the frequency response of the original system. Furthermore, [33] also uses the same approach about analytic interpolation and solves the finite dimensional bounded interpolation problem with a possibly non-minimum phase but stable interpolating function. Although all of these studies are related to analytic interpolation problem, none of them directly addresses Problem 5.

Chapter 3

Central Nevanlinna-Pick Solution Approach for a Class of Plants

Publication Notice: The materials of this section are at least partially covered in the publication [67] which was published by the author and his advisor during the study time of this thesis dissertation.

Underactuated robotics, inspired by the fast and unconscious movements of human body, studies the possibilities of doing things more efficiently than it may be done under full control. It aims to control a mechanism having more degrees of freedom (i.e. joints) than the number of actuators (i.e. motors). There are two famous examples of underactuated robots in the literature. First of these examples is the Acrobot [47]. The Acrobot is a basic and simple model of a human body on a high bar [62]. The underactuated joint is a model of the hand on the bar. The second example is the Pendubot [50], in which the second joint is an unactuated pendulum.

Swinging up the robots to the upright equilibrium point have been studied in the literature and there are many results both for Acrobot [47, 61] and Pendubot [20]. Another important aim is to design a stabilizing controller to balance the robot at the equilibrium point. In the literature, nonlinear control theory is generally utilized to achieve upright control [61] and the linearized model of the robots around their equilibrium point is used to design a stabilizing controller for the equilibrium point.

The papers by Xin et.al. [62] and [60] are the first effort in the literature to design a low order and stable controllers for Acrobot and Pendubot's upright equilibrium point. By making use of the linearized models of both robots, they first proved that the linearized model is stabilizable by a stable controller and then proposed a method to design such controllers.

In this chapter, inspired by the studies of [62], the problem of robust stabilization of finite dimensional SISO plants by a stable controller is revisited. A method to design reduced order controllers is proposed via the well known NPIP and the proposed method is tested on the examples of Acrobot and Pendubot. Third order stable controllers are designed for both examples. The stability range of the parameter uncertainty is compared for both examples and it is shown that the proposed method outperforms the prior robustness performance of the controllers found in [62] with an order increase of one without violating strong stability. In addition to these, fourth order stable controllers are designed for both robots to track step-like inputs by simply shifting the interpolation problem to conform the conditions of the proposed solution.

The chapter is organized as follows: Section 3.1 represents two famous examples of underactuated robots, namely Acrobot and Pendubot. Section 3.2 defines the problem of robust stabilization of SISO plants via NPIP. Section 3.3 discusses the stability criteria of the controller which is defined in Section 3.2. Section 3.4 is about the order of the proposed controller and discusses the constraints which yield to a low order controller. Section 3.5 is about integral action of the controller and a method to design a fourth order controller with integrator action is proposed. Section 3.6 concludes the chapter with some discussion.

3.1 Motivating Examples

Two examples of underactuated robots, Acrobot and Pendubot, will be revisited in this chapter. As explained in the introduction, the stabilization of these robots on their upright equilibrium points is generally studied over the linearized model of the robots at the point of interest. To the best of our knowledge, for the first time in the literature Xin et.al. have proposed a second order stable controller design for Acrobot and Pendubot in [62]. The generic plant structure of the linearized models can be expressed as

$$P(s) = \frac{\rho(s+z_1)(s-z_1)}{(s+p_1)(s+p_2)(s-p_1)(s-p_2)} \tag{3.1}$$

and the numerical values of the parameters are calculated by the physical properties of the robots.

3.1.1 Acrobot

With the parameters of the Acrobot in [47] and linearized model in [62],

$$P_a(s) = \frac{-1.3545(s-1.281)(s+1.281)}{(s-6.101)(s-2.24)(s+6.101)(s+2.24)} \tag{3.2}$$

Since there are two poles on the positive real line, ($p_1 = 2.24,\ p_2 = 6.101$) between positive zeros ($z_1 = 1.281,\ \infty$), the plant satisfies the PIP hence it is strongly stabilizable.

The second order controller designed in [62] is

$$C_{a2}(s) = \frac{-131.4411(s+2.24)(s+6.101)}{(s+1.281)(s+19.01)} \tag{3.3}$$

and for the resulting complementary sensitivity
$T_{a2} = C_{a2}P_a/(1+C_{a2}P_a)$, we have $\|T_{a2}\|_\infty = 7.5468$. Note that poles of C_{a2} are in open left half plane, hence it is a strongly stabilizing controller of order two.

3.1.2 Pendubot

With the parameters of Pendubot in [74, 44] and linearized model in [62], plant for the given robot becomes

$$P_p(s) = \frac{245.9467(s - 3.261)(s + 3.261)}{(s - 11.48)(s - 6.374)(s + 11.48)(s + 6.374)} \tag{3.4}$$

Again we have two real poles ($p_1 = 6.374,\ p_2 = 11.48$) between two consecutive positive zeros ($z_1 = 3.261,\ \infty$); hence, the plant satisfies the PIP.

The second order controller designed for (3.4) in [62] is

$$C_{p2}(s) = \frac{3.257(s + 6.374)(s + 11.48)}{(s + 40.29)(s + 3.261)} \tag{3.5}$$

and the resulting complementary sensitivity
$T_{p2} = C_{p2}P_p/(1 + C_{p2}P_p)$, leads to $\|T_{p2}\|_\infty = 7.8685$. Note that poles of C_{p2} are in open left half plane, hence it is a strongly stabilizing controller of order two.

These examples will be revisited at the end of each section.

3.2 Problem Definition

Internal stability of a feedback system can be achieved by a controller

$$C(s) = \frac{1}{P(s)}\left(\frac{T(s)}{1 - T(s)}\right)$$

provided that we find a transfer function $T = PC/(1+PC)$ such that the following conditions hold for all right half plane zeros z_i and poles p_j of $P(s)$

$$T \in \mathcal{H}_\infty \tag{3.6}$$

$$T(z_i) = 0 \tag{3.7}$$

$$T(p_j) = 1 \tag{3.8}$$

It is well known in the literature that the robust stabilization of a class of plants as described in (2.10) can be achieved if $\|WT\|_\infty \leq 1$. In this chapter, we assume $W(s) = K$ where K is a constant and robust stabilization condition is $\|T\|_\infty \leq 1/K$. The technique proposed here applies to more general weight $W(s)$ by proper change of interpolation conditions. In order to optimize the robustness level we try to find such T for the largest possible $K > 0$.

The robust stabilization problem can be reformulated as a NPIP. Assuming $\gamma > \gamma_{opt}$ and $\gamma_{opt} = 1/K_{max}$, all T satisfying (3.6), (3.7), (3.8) and $\|T\|_\infty \leq \gamma$ for $z_i, p_j < \infty$ can be parameterized as follows:

$$T(s) = \gamma \, \frac{\Theta_{11}(s)\ G(s) + \Theta_{12}(s)}{\Theta_{21}(s)\ G(s) + \Theta_{22}(s)} \tag{3.9}$$

where $G \in \mathcal{H}_\infty$ is a free parameter with $\|G\|_\infty \leq 1$, see [8] for details.

In general, for the plant (3.1) we have

$$\begin{aligned}
\Theta_{11}(s) &= \frac{(s-n_1)(s-n_2)(s-z_1)}{(s+z_1)(s+p_1)(s+p_2)}, \\
\Theta_{12}(s) &= \frac{-\sigma(s-n_3)(s-z_1)}{(s+z_1)(s+p_1)(s+p_2)}, \\
\Theta_{21}(s) &= \frac{\sigma(s+n_3)}{(s+p_1)(s+p_2)}, \\
\Theta_{22}(s) &= \frac{(s+n_1)(s+n_2)}{(s+p_1)(s+p_2)}.
\end{aligned} \tag{3.10}$$

Note that n_1, n_2, n_3, σ are some real positive numbers and z_1 is the only finite zero of the plant in right half plane and p_1, p_2 are the poles of the plant in the right half plane.

It is important to note that the interpolation conditions should also include the zeros at infinity to obtain a proper controller. However, including these zeros at infinity leads to a boundary interpolation problem and should be tackled differently. For the purposes of this chapter, this relative degree problem will be solved by adjusting the free parameter G, as discussed in the following sections.

Parametrization of each example is presented in the following subsections:

3.2.1 Acrobot

Using the parametrization of all solutions to NPIP in [8], with the interpolation data $\{z_1, p_1, p_2\} = \{1.281, 6.101, 2.24\}$ the following transfer functions can be computed for $\gamma = 1.01\gamma_{opt}$ where $\gamma_{opt} = 5.6231$ (this is the optimal performance level over all stabilizing controllers; once the stability and order of the controller are taken into account there is naturally a performance degradation from this optimal level),

$$\begin{aligned}
\Theta_{11}(s) &= \frac{(s-86.69)(s-11.1)(s-1.281)}{(s+1.281)(s+2.24)(s+6.101)}, \\
\Theta_{12}(s) &= \frac{-87.156(s-11.04)(s-1.281)}{(s+1.281)(s+2.24)(s+6.101)}, \\
\Theta_{21}(s) &= \frac{87.156(s+11.04)}{(s+2.24)(s+6.101)}, \\
\Theta_{22}(s) &= \frac{(s+86.69)(s+11.1)}{(s+2.24)(s+6.101)}.
\end{aligned} \tag{3.11}$$

3.2.2 Pendubot

Similarly, for the Pendubot, using the parametrization of all solutions to NPIP in [8], with the interpolation data $\{z_1, p_1, p_2\} = \{3.261, 6.374, 11.48\}$ the following transfer functions can be computed for $\gamma = 1.01\gamma_{opt}$ where $\gamma_{opt} = 5.5511$,

$$\begin{aligned}
\Theta_{11}(s) &= \frac{(s-221.1)(s-23.19)(s-3.261)}{(s+11.48)(s+6.374)(s+3.261)}, \\
\Theta_{12}(s) &= \frac{-221.93(s-23.1)(s-3.261)}{(s+11.48)(s+6.374)(s+3.261)}, \\
\Theta_{21}(s) &= \frac{221.93(s+23.1)}{(s+11.48)(s+6.374)}, \\
\Theta_{22}(s) &= \frac{(s+221.1)(s+23.19)}{(s+11.48)(s+6.374)}.
\end{aligned} \tag{3.12}$$

3.3 Stability of the Controller

Recall that it is also possible to write the parametrization of all controllers satisfying robust stability of the feedback system via $\|T\|_\infty \leq 1/K = \gamma$ as

$$C = \frac{T}{PS}, \quad \text{where} \quad S = 1 - T. \tag{3.13}$$

Note that by using (3.9) we obtain

$$S = 1 - T = \frac{G(\Theta_{21} - \gamma\Theta_{11}) + (\Theta_{22} - \gamma\Theta_{12})}{G\Theta_{21} + \Theta_{22}}. \tag{3.14}$$

Hence, if

$$R = \frac{\Theta_{21} - \gamma\Theta_{11}}{\Theta_{22} - \gamma\Theta_{12}} \in \mathcal{H}_\infty \tag{3.15}$$

and $G \in \mathcal{H}_\infty$, with $\|GR\|_\infty \leq 1$ then the controller is stable. Recall that the conditions $G \in \mathcal{H}_\infty$ and $\|G\|_\infty < 1$ imply stability of the feedback system.

For the plants in the form (3.1) the controllers obtained from the parametrization (3.9) with the choice of $G = 0$ are stable, but they are improper. One way to handle this problem is to multiply the resulting controller by a term $1/(1 + \varepsilon s)^\ell$ for small enough $\varepsilon > 0$ and sufficiently large integer ℓ, see [62]. In this chapter, we use the free parameter G in the form

$$G = \frac{g_1}{g_2 s + g_3} \tag{3.16}$$

and adjust g_1, g_2, g_3 to obtain a stable and proper controller.

3.4 Low Order and Proper Controller Design

In order to have a proper and low order controller, extra conditions are required on T for especially strictly proper plants. Irrespective of the relative degree of the plant, Θ_{12} and Θ_{21} are strictly proper and have relative degree one, moreover Θ_{11} and Θ_{22} are bi-proper.

From (3.13), if plant is proper with relative degree two and T is proper with relative degree n, then C has a relative degree of $n - 2$. Note that if $n < 2$, the controller is improper. As in [62], the aim of this chapter is to find a proper controller, hence we need to satisfy $n = 2$.

We know that Θ_{12} has a relative degree one, and Θ_{11} is bi-proper. By a choice of G of the form (3.16), $\Theta_{11}G$ is of relative degree one. It is easy to see that the condition $g_1 = \sigma g_2$ ensures that the relative degree of $\Theta_{11}G + \Theta_{12}$ is two. This and the fact that Θ_{22} is bi-proper lead to a T whose relative degree is $n = 2$.

In addition to the condition $g_1 = \sigma g_2$, G has to satisfy $G \in \mathcal{H}_\infty$ and $\|G\|_\infty \leq 1$. That leads to the constraints $-g_3/g_2 < 0$ and $\sigma g_2/g_3 < 1$. The designs for each plant will be done by taking these constraints into consideration.

3.4.1 Acrobot

For the design of $G = g_1/(g_2 s + g_3)$, we know that $g_1 = \sigma g_2$ is required for properness of the controller. Let us take $g_2, g_3 > 0$ to ensure $-g_2/g_3 < 0$. And finally take $g_3 = \sigma g_2 + d$ where d is some positive constant ($d = 1$ for this chapter).

For $g_2 = 1$, $G = 87.16/(s + 88.16)$ and the corresponding controller is

$$C_{a3}(s) = \frac{63917(s - 11.07)(s + 6.101)(s + 2.24)}{(s + 1.281)(s^2 + 195.6s + 1.056 \times 10^5)} \tag{3.17}$$

and $T_{a3} = C_{a3}P_a/(1 + C_{a3}P_a)$ yields $\|T_{a3}\|_\infty = 5.6793$. It is important to note that the newly proposed controller is stable, proper and third order. Compared to the controller obtained in [62], (3.3), for which we had $\|T_{a2}\|_\infty = 7.5468$, we now have about 25% improvement in the robust stability level. Figure 3.1 shows the magnitude Bode plots of the transfer functions T_{a2} and T_{a3}.

Both C_{a2} and C_{a3} stabilize the plant P_a for nominal values of $z_1 = 1.281$, $p_1 = 6.101$ and $p_2 = 2.24$. The values of the parameters for which the systems with C_{a2} and C_{a3} remain stable are given in Table 3.1. Note that C_{a3} also provides extra robustness to parameter variations, i.e. the allowable range of z_1 (when p_1 and p_2 are at their nominal values) is larger for C_{a3}, $(0\ ,\ 1.3894)$, compared to that of C_{2a}, which is $(0.6735\ ,\ 1.3672)$. Similar conclusions are deduced for variations in p_1 and p_2. In order to calculate these intervals, we fix all the parameters apart from the tested one and by turn increase and decrease the tested parameter to find upper and lower bounds beyond which the stability is lost.

Table 3.1: Robustness margins for the parameters of Acrobot.

	Min z_1	Max z_1	Min p_1	Max p_1	Min p_2	Max p_2
C_{a2}	0.6735	1.3672	5.7163	6.7055	2.0988	2.6492
C_{a3}	0	1.3894	5.6258	6.9286	2.0655	2.9864

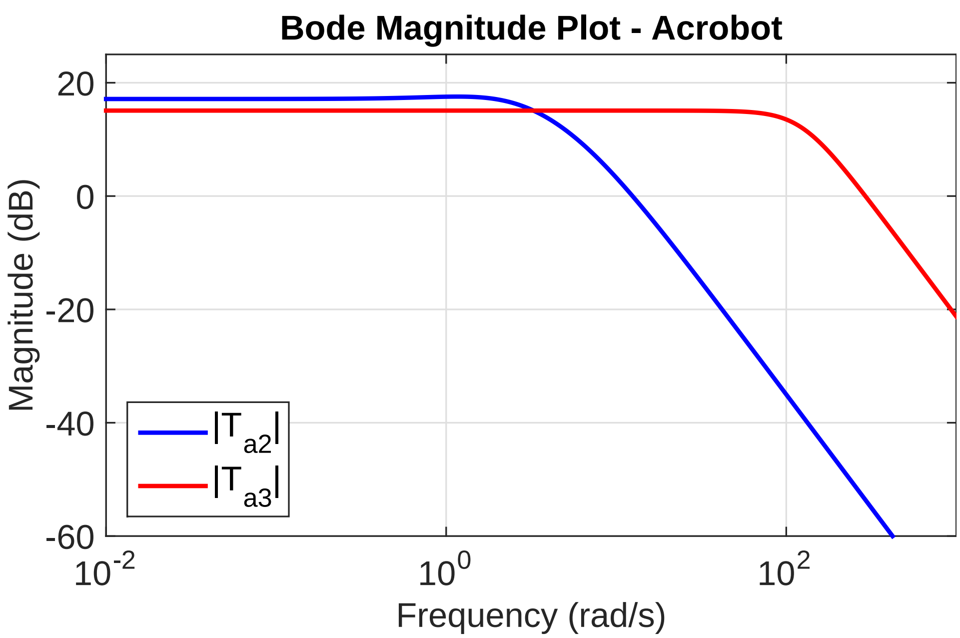


Figure 3.1: Bode magnitude plot of T_{a2} and T_{a3}

3.4.2 Pendubot

For the design of $G = g_1/(g_2 s + g_3)$, we know from Section 3.4 that $g_1 = \sigma g_2$ is required for properness of the controller. Using same arguments as Acrobot and $g_2 = 1$ we have $G = 221.9/(s + 222.9)$. Note that $G \in \mathcal{H}_\infty$, $\|G\|_\infty = 0.9955 < 1$

and $\|GR\|_\infty = 0.9956 < 1$. With this choice of G the corresponding controller is

$$C_{p3}(s) = \frac{-2246.8(s - 23.14)(s + 11.48)(s + 6.374)}{(s + 3.261)(s^2 + 488.3s + 6.716 \times 10^5)} \tag{3.18}$$

and $T_{p3} = C_{p3}P_p/(1 + C_{p3}P_p)$ gives $\|T_{p3}\|_\infty = 5.6066$. It is important that the same level of improvement has been obtained as in Acrobot, compared to the controller proposed in [62] i.e. (3.5). Figure 3.2 shows the Bode magnitude plots of the transfer functions T for the controllers (3.5) and (3.18).

Both C_{p2} and C_{p3} stabilize the plant P_p for nominal values of $z_1 = 3.261$, $p_1 = 11.48$ and $p_2 = 6.374$. The values of the parameters for which the systems with C_{p2} and C_{p3} remain stable (while all other parameters are kept unchanged) are given in Table 3.2. Note that C_{p3} provides extra robustness.

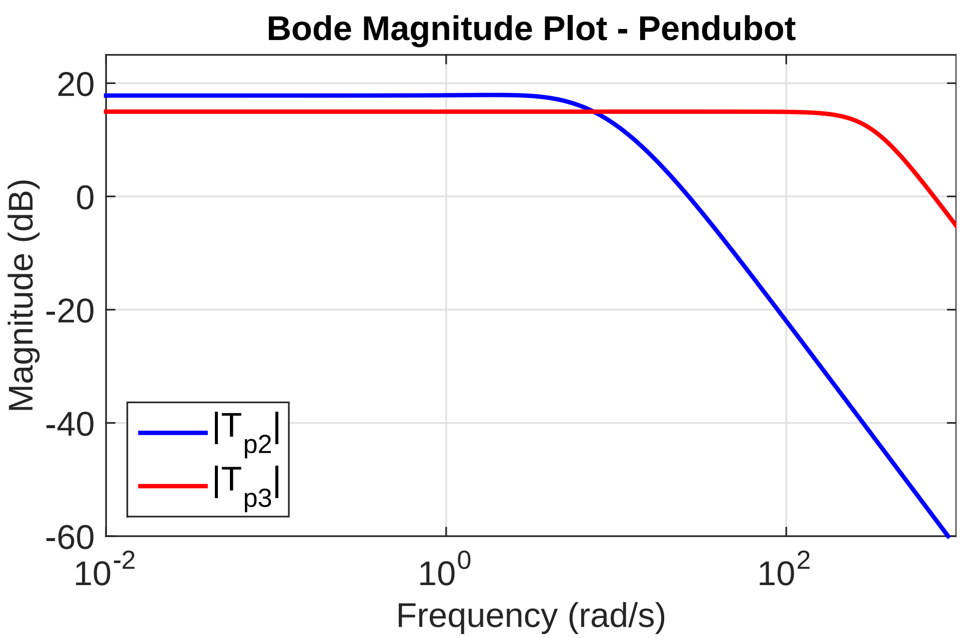


Figure 3.2: Bode magnitude plot of T_{p2} and T_{p3}

3.5 Including Integral Action in the Controllers

Tracking of step-like reference signals is a desired property of feedback loops. In order to track the step-like inputs, $T(0) = 1$ must also be satisfied. This

Table 3.2: Robustness margins for the parameters of Pendubot

	Min z_1	Max z_1	Min p_1	Max p_1	Min p_2	Max p_2
C_{p2}	2.0054	3.4644	10.8062	12.7041	5.999	7.3367
C_{p3}	0	3.5399	10.5755	13.2613	5.8718	8.0567

interpolation condition is a boundary condition in Nevanlinna-Pick type setting. There are several ways to incorporate boundary interpolation conditions into this extension of the NPIP, see [21]. A simple approach is to shift all interpolation conditions. Let us consider this situation directly over examples and design low order, proper and stable (except for the pole at $s = 0$) controllers for Acrobot and Pendubot.

3.5.1 Acrobot

We use the parametrization of all suboptimal solutions of the NPIP from [8]. In order to define the problem we choose a sufficiently small positive number ε (e.g. $\varepsilon = 10^{-3}$) and consider the interpolation data as

$$\{z_1 + \varepsilon, p_1 + \varepsilon, p_2 + \varepsilon, \varepsilon\} \longrightarrow \{0, 1, 1, 1\}.$$

Then, as in (3.9)–(3.10), parametric solutions can be computed and shifted by $-\varepsilon$ to get the transfer functions, $\Theta_{11}, \Theta_{12}, \Theta_{21}, \Theta_{22}$ with the suboptimal level chosen as $\gamma = 1.01\gamma_{opt}$, where $\gamma_{opt} = 5.6930$. Recall that in the absence of the additional interpolation condition $T(\varepsilon) = 1$ we had $\gamma_{opt} = 5.6231$ as the smallest achievable $\|T\|_\infty$ among all stabilizing controllers.

For $G = 87.31/(s + 88.31)$ the controller C_{a4} is given as

$$C_{a4}(s) = \frac{64300\ (s - 11.05)(s + 6.101)(s + 2.24)(s - 0.0005)}{s\ (s + 1.281)\ (s^2 + 200s + 1.06 \times 10^5)}.$$

This controller contains an integral action and stabilizes the feedback system. Other than the integrator the controller does not contain any unstable modes. It leads to a complementary sensitivity function whose H_∞ norm is 5.7056, which is within 1% of the smallest achievable norm 5.6930 as expected. The step response of the associated system is shown in Figure 3.3.

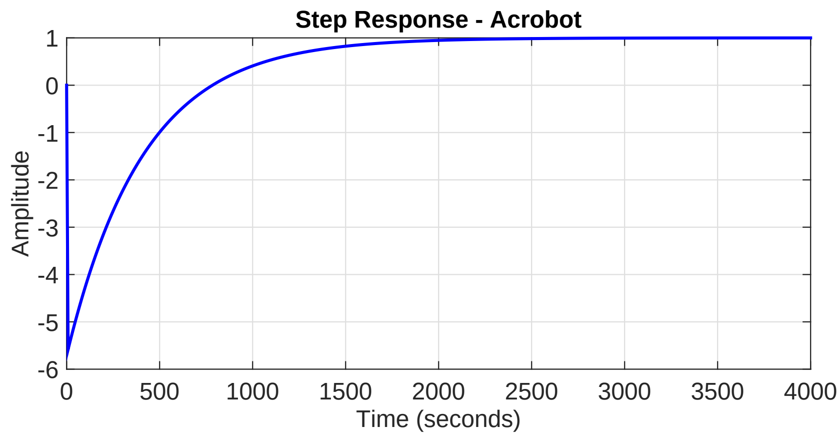


Figure 3.3: Step response plot of $T_{a4} = C_{a4}P_a/(1 + C_{a4}P_a)$

3.5.2 Pendubot

For the Pendubot, using similar arguments we obtain an integral action controller, C_{p4}, with the design parameters $\gamma_{opt} = 5.6120$, $G = 222.1/(s + 223.1)$

$$C_{p4}(s) = \frac{2250\ (23.13 - s)(s + 11.48)(s + 6.374)(s - 0.0005)}{s\ (s + 3.261)\ (s^2 + 500s + 6.73 \times 10^5)}.$$

This leads to a stable feedback system whose complementary sensitivity function H_∞ norm is 5.6264, which is within 0.26% of the smallest achievable norm 5.6120. The step response of the associated system is shown in Figure 3.4.

3.6 Discussions

For the well known underactuated robots Acrobot and Pendubot; third order, stable and proper controllers are designed to minimize the $\mathcal{H}_\infty$ norm of the complementary sensitivity function to maximize multiplicative uncertainty in the plant models. The results are compared with the stable controllers designed using other

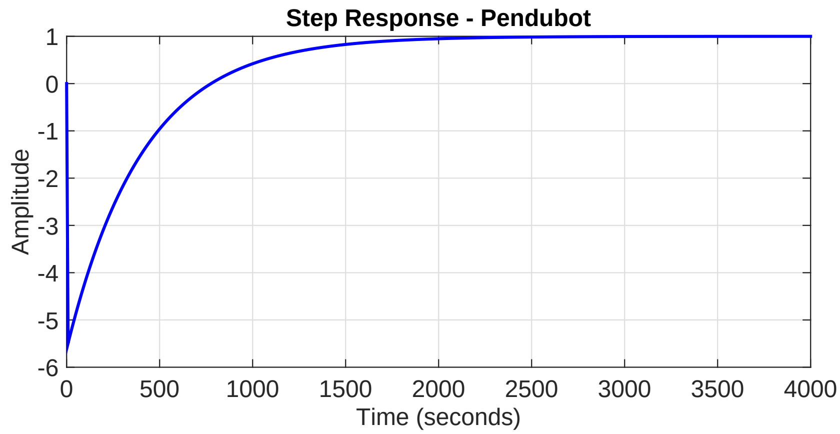


Figure 3.4: Step response plot of $T_{p4} = C_{p4}P_p/(1 + C_{p4}P_p)$

techniques, from the literature, using other design objectives. It is shown that for both systems approximately 25% improvements are obtained in terms of the closed loop system $\mathcal{H}_\infty$ norms. Moreover, the controllers designed here provide larger stability robustness to individual parameter perturbations in the pole and zero locations.

Chapter 4

Optimal Solution of Nevanlinna-Pick Interpolation

Publication Notice: The materials of this section are at least partially covered in the publications [66] and [68] which was published by the author and his advisor during the study time of this thesis dissertation.

This chapter deals with the optimal NPIP, which appears in robust control. Early papers [35], [52] and [71] defined various robust stabilization problems as an analytic interpolation problem where interpolation constraints ensure the internal stability of the nominal feedback system and a norm bound handles the robustness of the feedback loop.

The proposed method of this chapter solves the optimal NPIP, as it is given in Problem 2, directly with the right half plane interpolation data and obtains $F_{opt} \in \mathcal{H}_\infty$ with no approximations nor intermediate transformations. Let us recall Problem 2:

Problem 2:

Given $\alpha_i \in \mathbb{C}_+$ *and* $\beta_i \in \mathbb{C}$ *for* $i \in \{1, \ldots, n\}$ *find* $F \in \mathcal{H}_\infty$ *such that* $F(\alpha_i) = \beta_i$ *for all* i *and* $\|F\|_\infty \leq \gamma$, *for the smallest possible* $\gamma > 0$.

Some robust control examples are solved using the proposed method in the following subsections.

4.1 The Optimal Nevanlinna-Pick Interpolant

It is well known that for a NPIP involving n interpolation conditions, the optimal interpolant is a rational inner function of order $n-1$, see e.g. [21], [34], [72] and their references. Therefore, we must have

$$F_{opt}(s) = g\,\frac{[s^{n-1} \quad \ldots \quad s \quad 1]\;\mathsf{J}\;\Phi}{[s^{n-1} \quad \ldots \quad s \quad 1]\;\Phi} \tag{4.1}$$

where

$$g = \pm\gamma_{opt}, \tag{4.2}$$

$$\Phi = [\phi_{n-1} \quad \ldots \quad \phi_0]^{\mathsf{T}} \in \mathbb{R}^n \text{ and} \tag{4.3}$$

$$\mathsf{J} = \begin{bmatrix} (-1)^0 & & 0 \\ & \ddots & \\ 0 & & (-1)^{n-1} \end{bmatrix}. \tag{4.4}$$

Moreover, all roots of the polynomial

$$D_F(s) := [s^{n-1} \quad \ldots \quad s \quad 1]\;\Phi \tag{4.5}$$

have to be in $\mathbb{C}_-$. Thus, to compute F_{opt}, we must compute Φ and determine the sign of g. This is done by constructing an eigenvalue-eigenvector problem using the interpolation conditions, as follows.

The pairs $(\alpha_i,\ \beta_i)$ satisfy the interpolation conditions

$$g\;[\alpha_i^{n-1} \quad \ldots \quad \alpha_i \quad 1]\;\mathsf{J}\;\Phi = \beta_i\;[\alpha_i^{n-1} \quad \ldots \quad \alpha_i \quad 1]\;\Phi$$

for $i \in \{1,\ \ldots,\ n\}$. These equations can be written in a compact form as

$$(g\mathsf{V}_\alpha\mathsf{J} - \mathsf{D}_\beta\mathsf{V}_\alpha)\,\Phi = 0 \tag{4.6}$$

where

$$\mathsf{V}_\alpha := \begin{bmatrix} \alpha_1^{n-1} & \dots & \alpha_1 & 1 \\ \vdots & \dots & \vdots & \vdots \\ \alpha_n^{n-1} & \dots & \alpha_n & 1 \end{bmatrix}, \quad \mathsf{D}_\beta := \begin{bmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_n \end{bmatrix}.$$

Since α_i's are distinct, the Vandermonde matrix V_α is invertible. So, the set of equations (4.6) can be re-written as

$$(g\mathsf{I} - \mathsf{J}\ \mathsf{V}_\alpha^{-1}\mathsf{D}_\beta\mathsf{V}_\alpha)\ \Phi = 0. \tag{4.7}$$

Let us define

$$\mathsf{L} := \mathsf{J}\ \mathsf{V}_\alpha^{-1}\mathsf{D}_\beta\mathsf{V}_\alpha\ . \tag{4.8}$$

Thus, in order to have a nontrivial solution (i.e. $\Phi \neq 0$) for (4.7), the constant g must be an eigenvalue of L and Φ should be the corresponding eigenvector. The above discussion can be summarized with the following.

Proposition 1. The optimal interpolant $F_{opt}(s)$ is given by (4.1) where $g = \pm\gamma_{opt}$, with γ_{opt} being the square root of the largest eigenvalue of the matrix $\mathsf{A}^{-1}\mathsf{B}$, where A and B are defined in (2.12); the sign of g and Φ are determined as the feasible eigenvalue and eigenvector pair of the matrix L, (4.8), such that $D_F(s)$ defined by (4.5) is a Hurwitz polynomial.

4.1.1 Connections with Sarason's and Nehari's theorems

According to Sarason's theorem, $\gamma_{opt} = \|W(\mathbf{T})\|$ and

$$F_{opt} = \frac{W(\mathbf{T})G_o}{G_o}$$

where G_o is the singular vector corresponding to the largest singular value of $W(\mathbf{T})$, where $\mathbf{T}$ denotes the compressed shift operator defined on the subspace $\mathcal{H}(M) := \mathcal{H}_2 \ominus M\mathcal{H}_2$. If M is finite dimensional then so is $\mathbf{T}$, and a basis for $\mathcal{H}(M)$ can be determined from functions $(s + \bar{\alpha}_i)^{-1}$. Moreover, in this case G_o, as well as $W(\mathbf{T})G_o$, can be expressed in terms of the basis functions. These computations are rather complicated and they do not illustrate the resulting F_{opt}

in the compact form of Proposition 1. On the other hand, the power of Sarason's approach becomes clear when one deals with an infinite dimensional M and a finite dimensional W. See e.g. [21] and its references.

Nehari's theorem computes γ_{opt} as the norm of the Hankel operator whose symbol is $M(-s)W(s)$. It also gives the resulting F_{opt} as follows (this material is taken from [16], the reader is referred to the relevant references in there). Let $M(-s)W(s) =: R_u(s) + R_s(s)$ be a decomposition, such that R_u is unstable, and R_s is stable. Consider a minimal realization $R_u(s) = C(sI - A)^{-1}B$, and let $\mathsf{W_c}$ and $\mathsf{W_o}$ be the associated controllability and observability grammians, i.e. solutions of

$$A\mathsf{W_c} + \mathsf{W_c}A^* = BB^*, \qquad A^*\mathsf{W_o} + \mathsf{W_o}A = C^*C.$$

Then, γ_{opt}^2 is the largest eigenvalue of $\mathsf{W_c}\mathsf{W_o}$. Let $x_{\max}$ be the corresponding eigenvector. It gives

$$F_{opt}(s) = \gamma_{opt} M(s) \frac{C(sI - A)^{-1} x_{\max}}{B^*(sI + A^*)^{-1}\gamma_{opt}^{-1} W_o x_{\max}}. \tag{4.9}$$

Assuming that all the α_i's are distinct, we can choose a particular realization

$$R_u(s) = \sum_{i=1}^{n} \frac{r_i}{(s - \alpha_i)} = C(sI - \Lambda)^{-1}B$$

$\Lambda = \mathrm{diag}\{\alpha_1, \cdots, \alpha_n\}$, $C = [1 \ \cdots \ 1]$ and $B = [\bar{r}_1 \ \cdots \ \bar{r}_n]^*$. Then, computation of r_i's yield

$$[\mathsf{W_o}]_{i,j} = \frac{1}{\bar{\alpha}_i + \alpha_j}$$

$$[\mathsf{W_c}]_{i,j} = \frac{\beta_i \bar{\beta}_j}{\alpha_i + \bar{\alpha}_j} \, \frac{\prod_{k=1}^{n}(\alpha_i + \bar{\alpha}_k)}{\prod_{k \neq i}(\alpha_k - \alpha_i)} \, \frac{\prod_{k=1}^{n}(\bar{\alpha}_j + \alpha_k)}{\prod_{k \neq j}(\bar{\alpha}_k - \bar{\alpha}_j)}$$

The resulting F_{opt} can thus be determined from (4.9) using the appropriate eigenvector of $\mathsf{W_c}\mathsf{W_o}$. The computation of L in (4.11) is simpler than this product. Also, the fact that eigenvector Φ of L directly determines F_{opt} via (4.1) makes the new formula obtained in this chapter more attractive than the above mentioned alternatives from the literature.

4.1.2 Remarks on numerical issues

As seen from (4.7), computation of the matrix L requires inversion of a Vandermonde matrix V_α. But such matrices are known to be ill-conditioned, when n is large, or two data points α_i and α_j are close to each other. As the number of interpolation data points increase, it is well known that the Vandermonde matrix becomes more and more ill-conditioned, i.e. hard to invert precisely. We know that the condition number of a matrix is a good indication of the accuracy of the results from matrix inversion. A condition number near 1 indicates that the associated matrix is numerically suitable to be inverted. As the condition number increases the matrix becomes ill-conditioned. For the efficiency of the calculations, we use the built-in MATLAB function `rcond` which stands for the reciprocal condition number, i.e. the matrix is more ill-conditioned as the reciprocal condition number decreases. This MATLAB function returns an estimate of the reciprocal condition number rc which is the reciprocal of the 1-norm condition number c. Reciprocal condition number of a matrix V_α is defined as

$$rc = \frac{1}{c} = \frac{1}{\|\mathsf{V}_\alpha\|_1 \|\mathsf{V}_\alpha^{-1}\|_1}. \tag{4.10}$$

Reciprocal condition numbers of the Vandermonde matrices are shown in the Figure 4.1. As it is seen in the figure, as the number of data points increases reciprocal condition number of the Vandermonde matrix decreases dramatically. In Figure 4.1, data points are generated uniformly and randomly in the region [0,1] with a multiplier R given in the legend. For each iteration 500 different data point sets are generated and the average reciprocal condition number is plotted. If a matrix is well conditioned then the reciprocal condition number associated to that matrix is near 1, it gets smaller and smaller as the matrix becomes ill-conditioned.

Due to this conditioning property of Vandermonde matrices, in the literature, there has been significant effort to find analytical, or numerically reliable methods to compute inverses of Vandermonde matrices using their special structures.

In the scope of this thesis, we are going to compare three different methods to

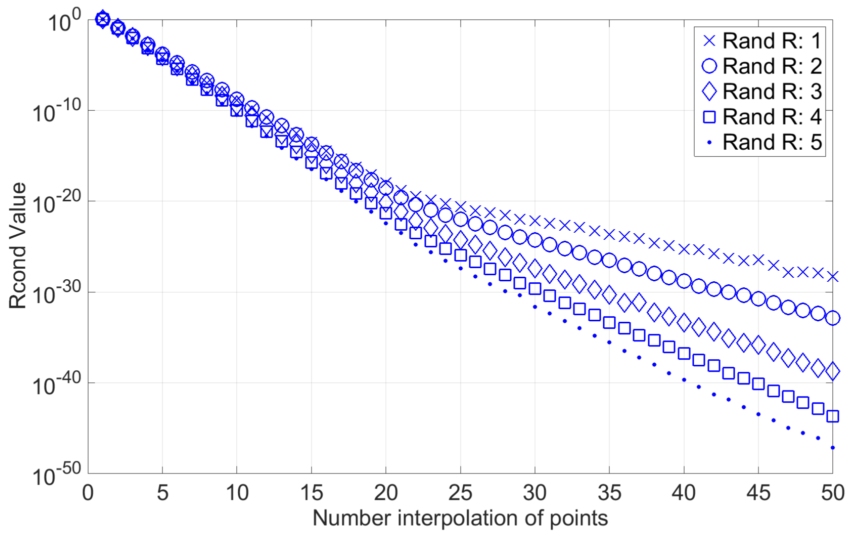


Figure 4.1: Reciprocal condition number of the Vandermonde matrix generated by different number of data points.

invert a Vandermonde matrix:

- Classical `MATLAB` inverse
- Special inverse from [32]
- Scaled inverse

Besides these inversion methods, another point which can be exploited is the observation that in many control problems β_i is defined from a given function $W(s)$ as $W(\alpha_i) =: \beta_i$. When this is the case, it is possible to avoid inversion of the Vandermonde matrix. This approach is explained as a fourth method to obtain optimal interpolant and its accuracy will be compared with other inversion methods.

Classical Matrix Inverse

This inversion method is a built-in function of `MATLAB` and is used as a benchmark inversion method for the purpose of this thesis.

Special and Accurate Vandermonde Matrix Inverse

This inversion method is a special method which is designed to form the inverse of a Vandermonde matrix directly from its entries. Details of the method is explained in [32] and example `MATLAB` implementations can be downloaded [1].

Classical Weighted Matrix Inverse

Weighting a matrix prior to inversion may lead to a less ill-conditioned situation in some cases. This strategy was proposed to generate V_x from V_α with $\alpha_{gm} = \left(\prod_{i=1}^{n} \alpha_i\right)^{1/n}$ and

$$V_x = V_\alpha D_s^{-1}$$

where $D_s = diag(\{\alpha_{gm}^0, \alpha_{gm}^1, \dots \alpha_{gm}^{n-1}\})$.

Using V_x instead of V_α in (4.8) and pre-multiplying eigenvectors of L in (4.8) by D_s^{-1} gives the resulting inversion method.

Avoiding Inverse of Vandermonde Matrix

If β_i is defined from a given function $W(s)$ as $\beta_i = W(\alpha_i)$ for all i then in order to avoid inversion of V_α define the coefficients $a_1, \dots, a_n \in \mathbb{R}$ from the polynomial

$$\prod_{i=1}^{n}(s - \alpha_i) =: s^n + a_1 s^{n-1} + \cdots + a_n \ .$$

Let I_k denote the identity matrix of dimension k and define

$$\mathsf{A}_d := \begin{bmatrix} -a_1 & & & \\ \vdots & & \mathsf{I}_{n-1} & \\ -a_n & 0 & \cdots & 0 \end{bmatrix} .$$

Then, it is a simple exercise to verify that

$$\mathsf{A}_d = \mathsf{V}_\alpha^{-1}\ \mathsf{D}_\alpha\ \mathsf{V}_\alpha$$

[1] `https://people.sc.fsu.edu/~jburkardt/m_src/vandermonde/vandermonde.html`

and since $\beta_i = W(\alpha_i)$ we have that

$$\mathsf{V}_\alpha^{-1}\ \mathsf{D}_\beta\ \mathsf{V}_\alpha = W(\mathsf{A}_d) \quad \text{and} \quad \mathsf{L} = \mathsf{J}\ W(\mathsf{A}_d). \tag{4.11}$$

In many cases it is easier to evaluate $W(\mathsf{A}_d)$ than trying to find the right hand side of (4.8) using the inverse of V_α.

4.1.3 Comparison of the Numerical Methods

In order to be able to compare the performance of the four numerical methods described above, Algorithm 1 is used (see below).

Algorithm 1 Numerical Performance Comparison Method

```
W_set = {W_1, ..., W_K}
for k ∈ {1, ..., K} do {select interpolation data generator function}
  W = W_k
  for t ∈ tRange do {select region of the random generator}
    for n ∈ {2, ..., N} do {number of interpolation points}
      for m ∈ {1, ..., M} do {number of simulation repetitions}
        α = tR(n)
        β = W(α)
        for l ∈ {1, 2, 3, 4} do {repeat for all numerical methods}
          Find solution F_l(s) to the problem with proposed method
          Record the maximum error e_l(m) = max_i(|F_l(α_i) − β_i|)
        end for
      end for
      Find average maximum error for each solver as A_l = (1/M) ∑ e_l
      Plot (N, A_l) on the corresponding subplot
    end for
    Put the title of the sub-plot
  end for
  Save and close figure
end for
```

In this algorithm, $R(n)$ is a randomly generated real vector of size n which is uniformly distributed between $(0, 1)$. For the rest of the simulations, t value, which is used to determine the random number generating interval, is printed in titles of relevant sub-plots. Also the interpolation data generating functions,

Table 4.1: Figure legend and subsection correspondence

Legend Name	**Marker**
Classical	$\times$
Special	$\circ$
Weighted	$\Diamond$
Avoidance	$\Box$

$W(s)$, are printed in the captions of the sub-plots. Additionally, a legend is provided for all figures to point the marker-solver relations. Corresponding sections to each marker is given in Table 4.1.

Interpolation data generating functions $W(s)$ are selected as the benchmark transfer functions which are frequently encountered in control theory problems:

- $W(s) = \frac{s+1}{s+r}$ for $r > 1$ is a transfer function with high pass characteristic which is typically a bounding function in robust stability problems
- $W(s) = \frac{s+r}{s+1}$ for $r > 1$ is a transfer function with low pass characteristic which is typically a bounding function in sensitivity minimization problems
- $W(s) = \frac{1}{1-q(s)\exp^{-hs}}$ for $h > 0$ and $q(s)$ is low pass with $q(0) = 1 - \epsilon$ for sufficiently small ϵ, is a transfer function which is typically a bounding function in repetitive control problems

Figures 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7 present the numerical performances of the aforementioned techniques when they are used to solve the NPIP via the proposed optimal solution method. In each figure, error is the total interpolation error, i.e.

$$e = \sum_{i=1}^{n} |\beta_i - F(\alpha_i)|$$

where F is the solution of the interpolation problem. As it is clear in all figures, performance of classical and weighted inverse are equivalent to each other, special inverse is better than both of them and in many cases avoiding the inversion is the top performer among all when total interpolation error is considered. There

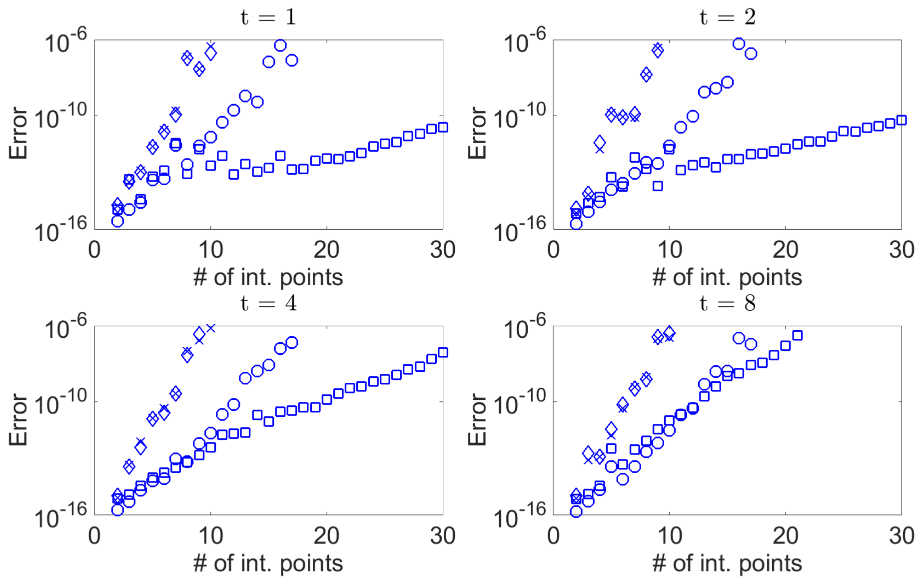


Figure 4.2: Interpolation data generator function $W(s) = \frac{1}{1-\frac{s+1-\epsilon}{10s+1}\exp^{-s}}$

are some cases (i.e. Figure 4.4 and 4.5 when $t = \{20, 50\}$) where special inverse performs better than the avoidance method. Apart from these cases, we can conclude that it is good to avoid the inversion of V_α and use $\mathsf{L} = \mathsf{J}W(\mathsf{A_d})$ when it is possible.

4.2 Examples

4.2.1 Robust Repetitive Control Problem

Repetitive control problem deals with the tracking of periodic signals (or the rejection of periodic disturbances). In this section, a periodic reference tracking problem with the block diagram depicted in Figure 4.8 will be discussed.

The repetitive control problem is to find a controller $C = C_o C_u$ such that the feedback system (C, P) is stable (i.e. $S = (1 + PC)^{-1},\ PS,\ CS$ are in $\mathcal{H}_\infty$) and the system can track periodic reference input $r(t)$ with a determined period of L, i.e. the steady state error $e(t) = r(t) - y(t)$ should be as small as possible (ideally

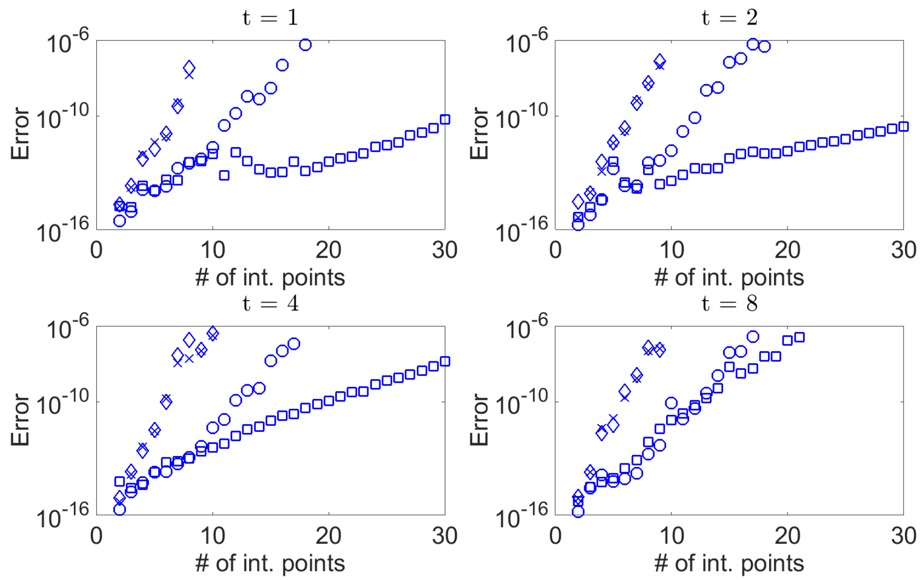


Figure 4.3: Interpolation data generator function $W(s) = \frac{1}{1-\frac{s+1-\epsilon}{10s+1}\exp^{-2s}}$

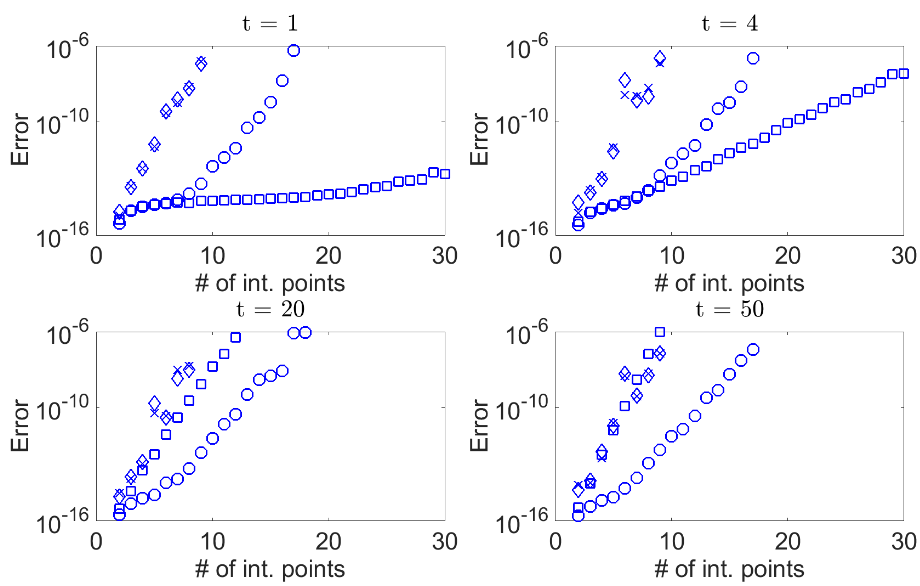


Figure 4.4: Interpolation data generator function $W(s) = \frac{s+4}{s+1}$

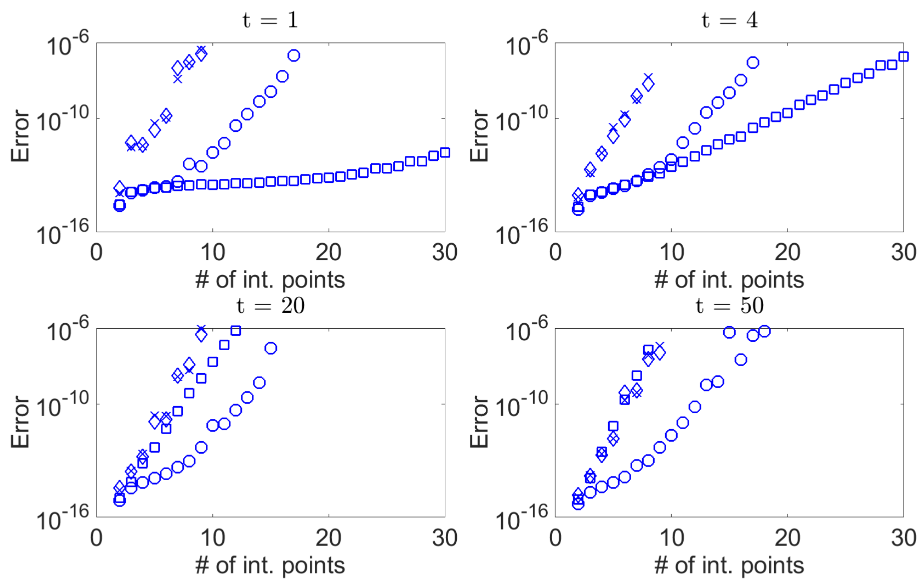


Figure 4.5: Interpolation data generator function $W(s) = \frac{s+20}{s+1}$

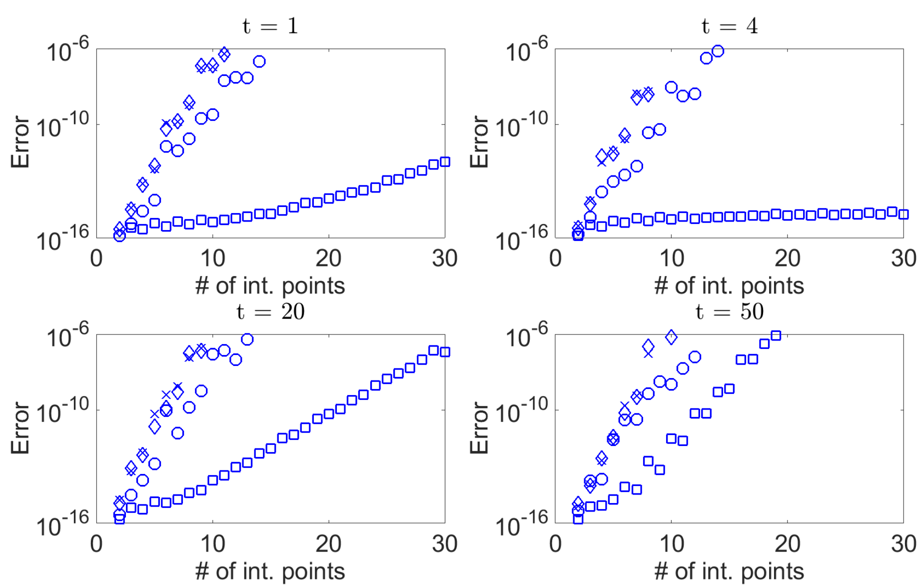


Figure 4.6: Interpolation data generator function $W(s) = \frac{s+1}{s+4}$

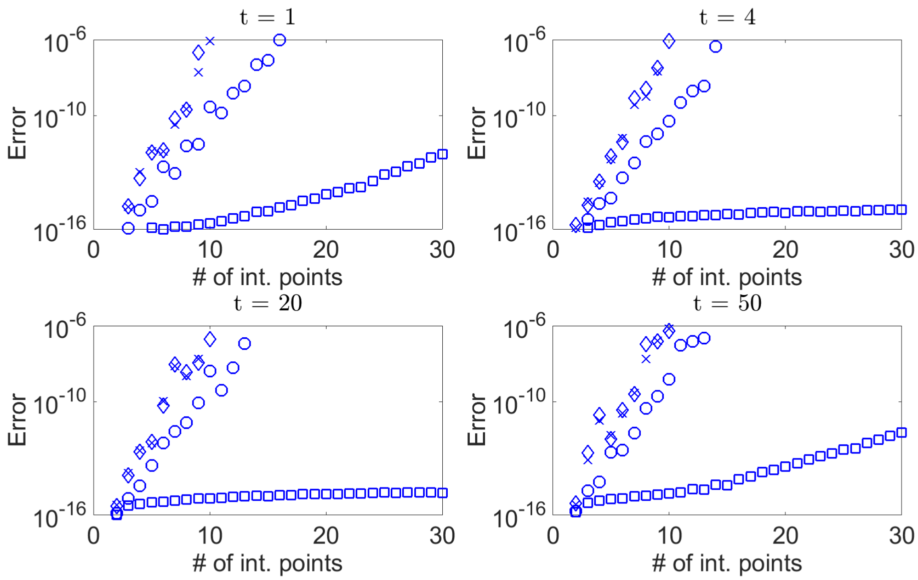


Figure 4.7: Interpolation data generator function $W(s) = \frac{s+1}{s+20}$

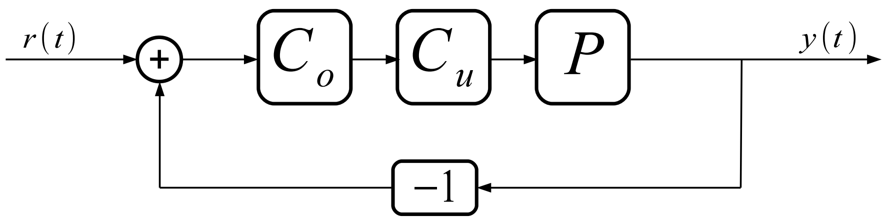


Figure 4.8: Block diagram for the repetitive control problem

zero). It is well known, [31] and [48], that an ideal choice for C_u in repetitive control problem is

$$C_u(s) = \frac{1}{1 - e^{-Ls}}.$$

Note that C_u has infinitely many poles that are located at

$$d_k = \pm jk2\pi/L, \qquad k = 0, 1, \ldots.$$

In this configuration C_o should be a stabilizing controller for the auxiliary plant C_uP. Such systems do not admit a stabilizing C_o if P is strictly proper, see e.g. [28] and [63].

Following the ideas of [16], it is possible to convert repetitive control problem to an analytic interpolation problem by making some approximations as follows:

- *Internal Stability:* Assume that the plant P has simple poles in $\mathbb{C}_+$, p_i, $i \in \{1, 2, \ldots, n\}$, and simple zeros in $\mathbb{C}_+$, z_j, $j \in \{1, 2, \ldots, m\}$. Then, the feedback system is internally stable if and only if $T(p_i) = 1$ and $T(z_j) = 0$ for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ and $T \in \mathcal{H}_\infty$.
- *Repetitive Performance:* The feedback system tracks periodic reference input $r(t)$ of period L if $T(d_k) = 1$ for all $k \in \{0, 1, 2, \ldots\}$ where each d_k is a pole of C_u in $\mathbb{C}_+$.

Note that this is an infinite dimensional interpolation problem. Any subset of d_k for $k \in \{0, 1, 2, \ldots, n_f\}$ reduces the problem to a finite dimensional interpolation problem which approximates the theoretical repetitive performance. Approximate performance of the controller to periodic references for different values of n_f will be discussed and simulation results will be illustrated.

Optimal Robust Repetitive Control Problem (ORRCP): Let $P = P_o \ \Delta_m$ where P_o is the nominal plant model and Δ_m is the multiplicative uncertainty on the nominal plant. Assume that there exist W_m such that $|W_m(j\omega)| > |\Delta_m(j\omega)|$ for all ω and W_m has no poles or zeros in the right half plane; if P_o is strictly proper, W_m has to be improper with a relative degree opposite of that of P_o, so that the resulting controller is proper. The task is to find a controller $C = C_o C_u$ such that (C, P_o) solves the repetitive control problem and $\|T_o \ W_m\|_\infty \leq 1/K$ for $K > 0$ where $T_o = 1 - S_o$ and $S_o = (1 + P_o C)^{-1}$. The largest K achieving this inequality is the largest achievable uncertainty level, while satisfying closed loop stability and repetitive control tracking objectives.

Solution of the (ORRCP): Let $n_f \in \mathbb{N}$ be fixed such that

$$\frac{2\pi}{L} n_f < W_{BW} < \frac{2\pi}{L}(n_f + 1)$$

where W_{BW} is the bandwidth of the function $W_m^{-1}(j\omega)$. The robust repetitive control problem has a solution if there exists a transfer function F such that the conditions

(1) F is real, rational, stable

(2) $F(p_i) = W_m(p_i)$ for all i

(3) $F(z_j) = 0$ for all j

(4) $F(d_k) = W_m(d_k)$ for all $k \in \{0, 1, \ldots, n_f\}$

(5) $\|F\|_\infty \leq 1/K$

are satisfied for the largest possible K, (we call this value of K maximal allowable uncertainty). Once such F is constructed, then $T_o = F/W_m$. Solution of the robust repetitive control problem is very similar to NPIP with the interpolation data $\alpha = \{p_i,\ z_j,\ d_k\}$, $\beta = \{W_m(p_i),\ 0,\ W_m(d_k)\}$ and $\gamma = 1/K$. However, this data violates the assumption of Nevanlinna-Pick formulation by $d_k \notin \mathbb{C}_+$ (i.e we would like to avoid boundary interpolation conditions). To overcome this, a slight variation is considered: choose a small number $\varepsilon > 0$ and modify the problem as

(1') F_s is real, rational, stable

(2') $F_s(\varepsilon + p_i) = W_m(p_i)$ for all i

(3') $F_s(\varepsilon + z_j) = 0$ for all j

(4') $F_s(\varepsilon + d_k) = W_m(d_k)$ for all $k \in \{0, 1, \ldots, n_f\}$

(5') $\|F_s\|_\infty \leq 1/K$

and redefine $F(s) = F_s(s + \varepsilon)$. If F_s is a solution of the modified problem then F is the solution of the original robust repetitive control problem. The boundary condition avoidance method was also used in [49]. For other boundary condition tackling strategies readers are directed to [21].

We now consider two numerical examples and solve them with the optimal solution strategy proposed in this chapter. Let the nominal plant be

$$P_o = \frac{e^{-hs}}{s^2 - bs + c}$$

where $h > 0$, $b > 0$, $c > 0$. Note that nominal plant has no finite zeros, and has two poles (i.e. p_1, p_2) in $\mathbb{C}_+$. We will handle the effect of time delay separately without imposing interpolation conditions (see below). Throughout the simulations, we are going to use two different nominal plants, one (P_{Ro}) having two distinct real poles (with $b = 10, c = 24 \Longrightarrow p_1 = 4, p_2 = 6$) and the other ($P_{Io}$) having complex conjugate poles (with $b = 10, c = 50 \Longrightarrow p_{1,2} = 5 \pm 5j$). Let us further assume that the closed loop system is required to track L periodic references with $L = 1$ sec and hence $C_u = 1/(1 - e^{-s})$, and $d_k = \pm j2\pi k$, $k = 0, 1, 2, \cdots$.

Let the multiplicative uncertainty be defined as

$$W_m(s) = \frac{s+\delta}{s+1}\,(s^2 + bs + c)$$

where $\delta = 0.1$ for all computations.

Since the plant has the form $P_o = e^{-hs}\bar{P}_o$ then for the closed loop system to be causal we must have $T_o(s) = e^{-hs}\bar{T}_o$ for some $\bar{T}_o \in \mathcal{H}_\infty$. It follows that $\|T_o W_m\|_\infty = \|e^{-hs}\bar{T}_o W_m\|_\infty = \|\bar{T}_o W_m\|_\infty$. Hence, it is possible to write the solution procedure for the optimal robust repetitive control problem for time delay systems as follows. Find F_{sd} satisfying

(1") F_{sd} is real, rational, stable

(2") $F_{sd}(\varepsilon + p_i) = e^{hp_i} W_m(p_i)$ for all i

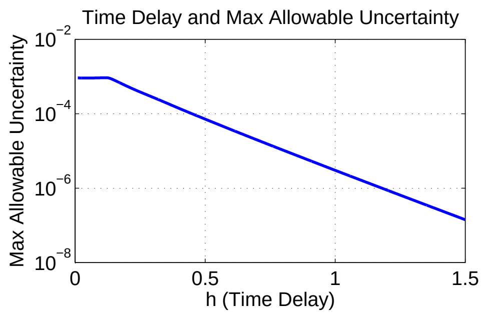
(3") $F_{sd}(\varepsilon + z_j) = 0$ for all j

(4") $F_{sd}(\varepsilon + d_k) = e^{hd_k} W_m(d_k)$ for all $k \in \{0, 1, \ldots, n_f\}$

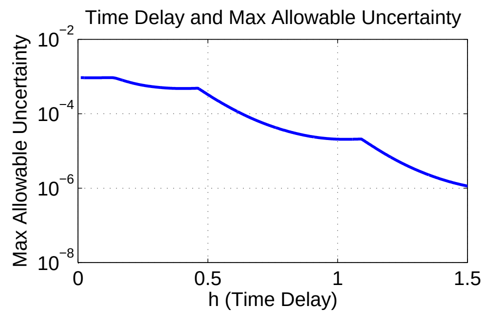
(5") $\|F_{sd}\|_\infty \leq 1/K$

Once such F_{sd} is constructed then $F_d(s) = F_{sd}(s + \varepsilon)$ and $\bar{T}_o = F_d/W_m$ and finally $T_o(s) = e^{-hs}\bar{T}_o(s)$. The above selection of W_m leads to a T_o which is strictly proper with relative degree 2, that leads to a proper controller.

Note that for the given W_m, maximum allowable uncertainty K can be computed as a function of the time delay h for each of the plants defined above. Figure 4.9 shows this relation for P_{Ro} and P_{Io}.



(a) P_{Ro}



(b) P_{Io}

Figure 4.9: Maximum allowable uncertainty, K, versus time delay, h; (a) for P_{Ro} and (b) for P_{Io}. Each calculation is conducted with $n_f = 10$.

For time domain response simulations of both plants (P_{Ro} and P_{Io}), related NPIP are solved by the method proposed in this chapter. Input signal is fixed throughout all simulations and one period of this input signal is shown in Figure 4.10.

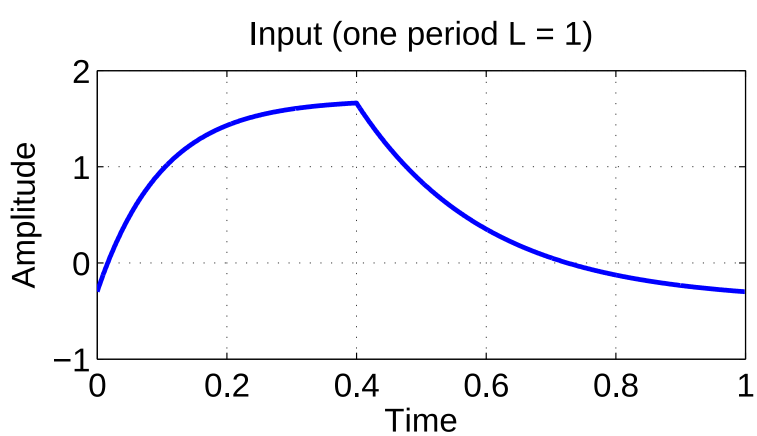


Figure 4.10: One period of input signal for simulations

Figures 4.11 and 4.12 show that the time domain responses of nominal feedback systems for both plants converge to the periodic input signal shown in Figure 4.10. In these simulations the parameters are chosen as $h = 0.1$, $\varepsilon = 0.5$ and $n_f = 5$.

It is clear from Figures 4.11 and 4.12 that closed loop systems track L-periodic

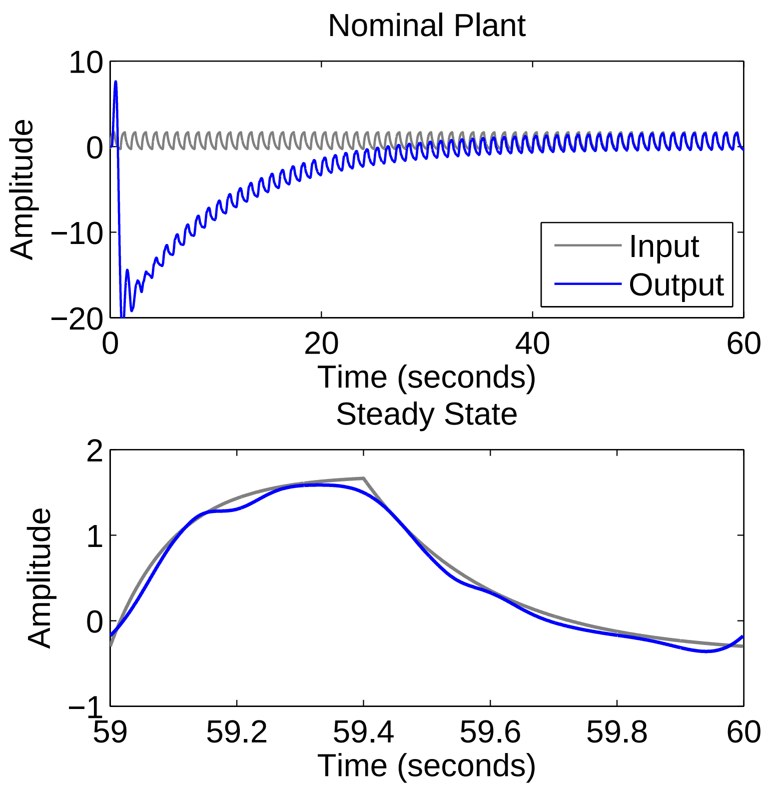


Figure 4.11: Time domain responses of nominal plant P_{Ro}

reference input with a small steady state error. This error is due to finite number (n_f) of modes taken in the interpolation. As we shall see below, the steady state error decreases with increasing n_f.

Next step is to check the robustness of the solution for both cases. In order to test robustness, let us introduce 6 different plants as

- $P_{R1} = P_{Ro} + \Delta_1$ where $\Delta_1 = K\frac{s+0.1}{2(s+1)}$
- $P_{R2} = P_{Ro}\Delta_2$ where $\Delta_2 = \frac{144.5s^2+2988s+743.8}{s^2+373s+580800}$
- $P_{R3} = \frac{e^{-h_1 s}}{s^2-b_1 s+c_1}$ where $h_1 = 1.02h$, $b_1 = 1.03b$, $c_1 = 1.001c$, $h = 0.1$, $b = 10$ and $c = 24$
- $P_{I1} = P_{Io} + \Delta_1$
- $P_{I2} = P_{Io}\Delta_2$

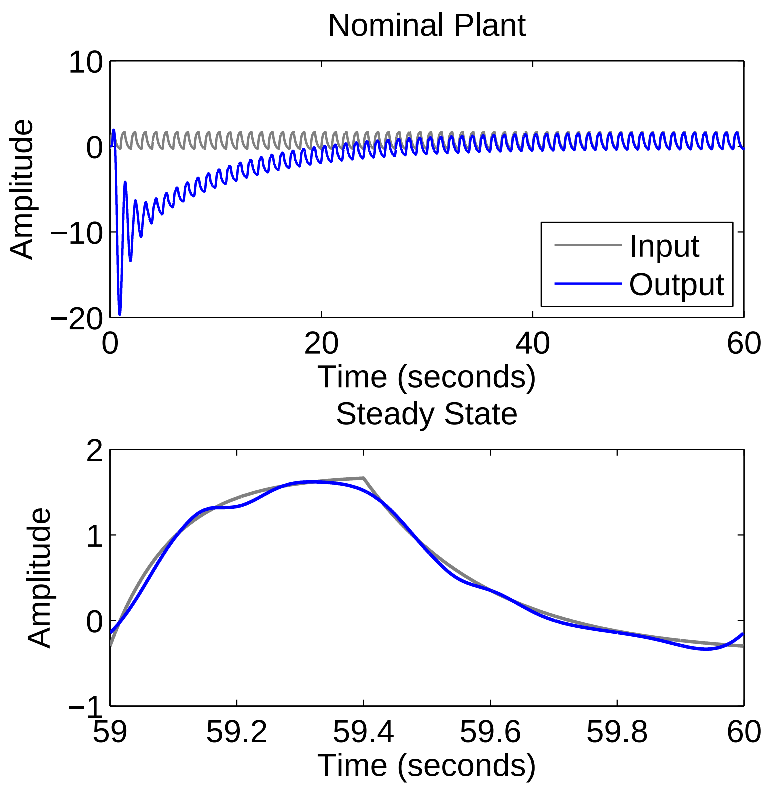


Figure 4.12: Time domain responses of nominal plant P_{Io}.

- $P_{I3} = \frac{e^{-h_1 s}}{s^2 - b_1 s + c_1}$ where $h_1 = 1.02h$, $b_1 = 1.03b$, $c_1 = 1.001c$, $h = 0.1$, $b = 10$ and $c = 50$

Figures 4.13 and 4.14 show time domain responses of the closed loop systems with those 6 perturbed plants. It is clear from the figures that the designed system is robust to uncertainty in the plant and achieves tracking of periodic reference with a small steady state error.

There are three basic parameters in the solution of robust repetitive control problem that directly influence the analytic interpolation formulation. These parameters are time delay of the nominal plant, h, amount of shift ε, applied to poles of C_u in order to comply to the assumptions of NPIP, and the number of modes of C_u included in the interpolation, n_f. In the next subsections, the effect of each parameter to the overall performance will be discussed. For the following simulations, P_{Io} is used as the nominal plant and $h = 0.1$, $\varepsilon = 0.5$ and $n_f = 5$

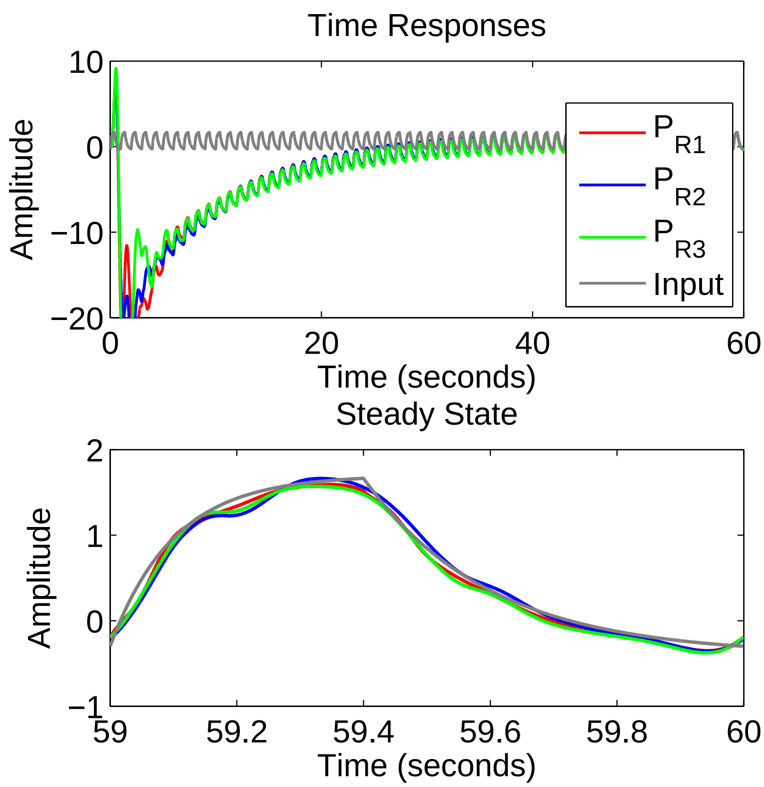


Figure 4.13: Time domain response of P_{R1}, P_{R2}, P_{R3}

unless mentioned otherwise.

4.2.1.1 Effect of h

Time delay of the nominal plant (h) effects the interpolation conditions because it appears as an exponential multiplier in β. As seen in the Figures 4.15 and 4.16, peak amplitude of the time domain response and the settling time increase as h increases. In addition to this, it is notable that as h increases, steady state tracking performance of the closed loop system becomes poorer. Also recall from Figure 4.9 that as h increases maximum allowable uncertainty level K decreases.

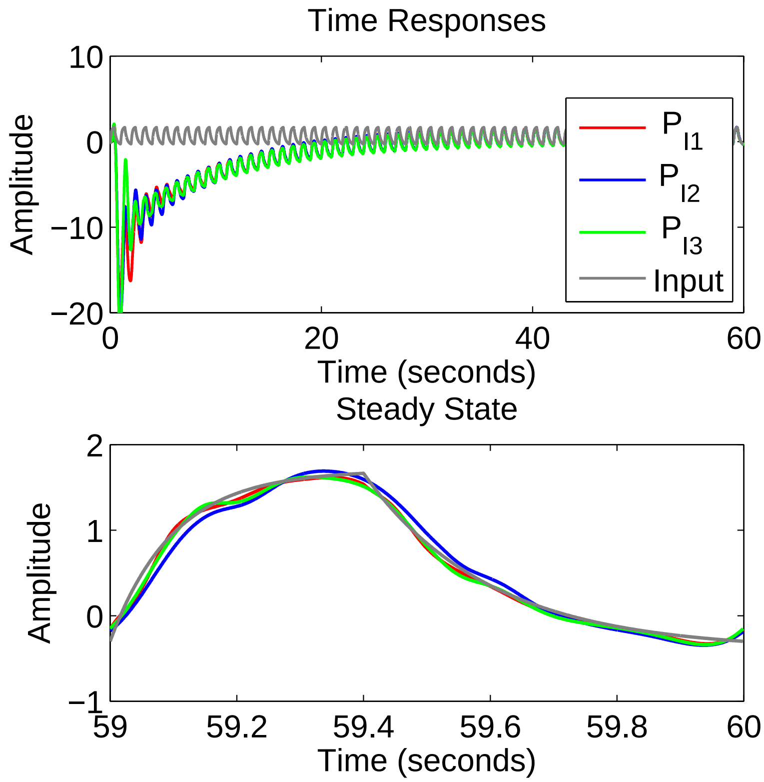


Figure 4.14: Time domain response of P_{I1}, P_{I2}, P_{I3}

4.2.1.2 Effect of ε

Amount of the shift in the interpolation data (ε) effects the interpolation conditions (i.e. α) because it directly shifts the points to $\mathbb{C}_+$. As seen in the Figures 4.17 and 4.18, peak amplitude of the time domain response and the settling time increase as ε decreases. This is a natural consequence because the design methodology places the closed loop poles to the left of ε in the real axis. As ε increases, dominant poles of the closed loop system move away from imaginary axis.

4.2.1.3 Effect of n_f

Number of modes included in the interpolation problem (n_f) determines the number of interpolation points, which is $n + m + (1 + 2n_f)$, where n (respectively m) is the number of poles (respectively zeros) of the nominal plant in $\mathbb{C}_+$. Table 4.2

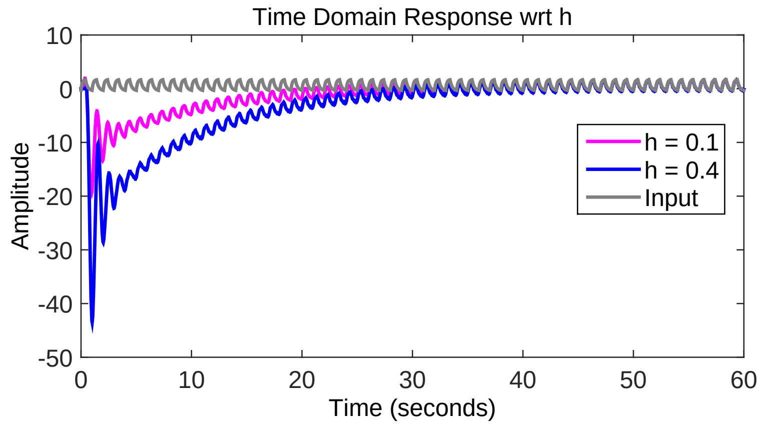


Figure 4.15: Effect of time delay of the nominal plant h on the time domain response of the overall system - All time view

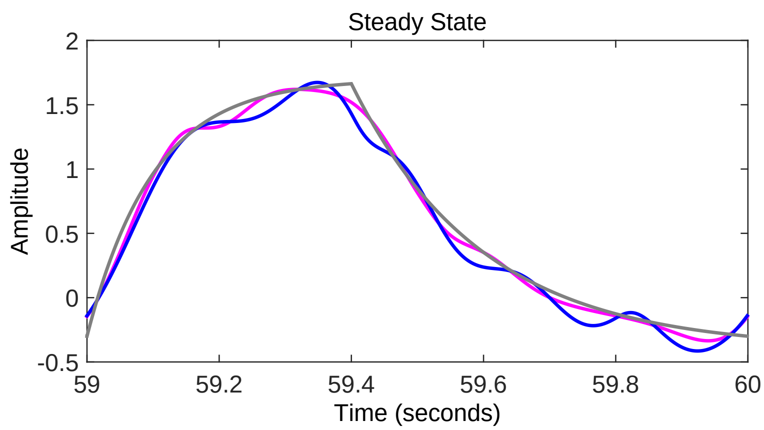


Figure 4.16: Effect of time delay of the nominal plant h on the time domain response of the overall system - Steady state view

shows that as n_f increases the steady state tracking performance of the closed loop system improves as expected: here e_{99-100} denotes the 2-norm of $e(t)$ over one period between time instants 99 sec and 100 sec, scaled by the 2-norm of $r(t)$ over one period. On the other hand, the maximum allowable uncertainty level, K, decreases as the number of interpolation data points increases, as quantified in Table 4.2. This is expected because as n_f increases, we require more bandwidth from the controller, yet the uncertainty becomes significant in high frequencies. Hence, we can quantify the well-known trade-off between the performance (steady state tracking error) and robustness (maximum allowable uncertainty level) in this example.

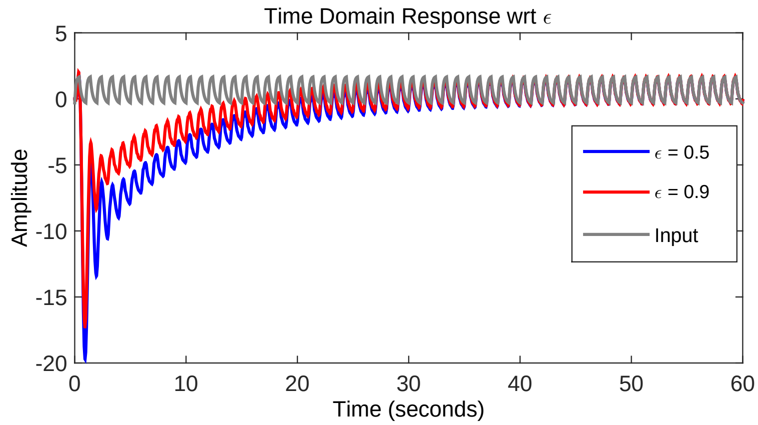


Figure 4.17: Effect of amount of the shift in interpolation data ε on the time domain response of the overall system - All time view

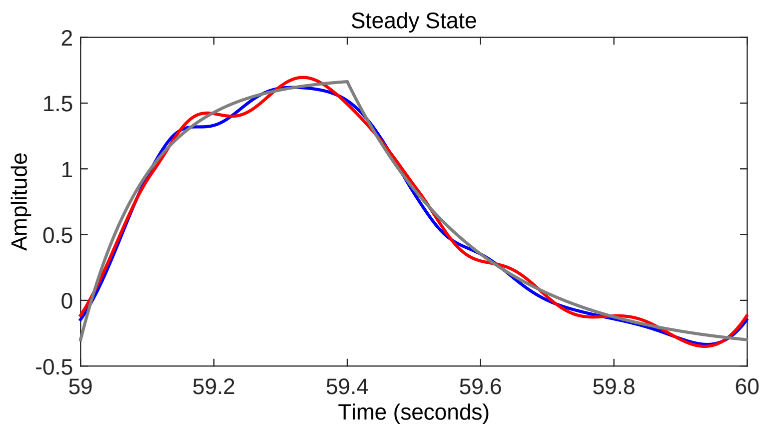


Figure 4.18: Effect of amount of the shift in interpolation data ε on the time domain response of the overall system - Steady state view

4.2.2 Delay Margin Optimization

One of the most interesting problems in robust control is the *delay margin optimization*, [12]. The goal is to find an optimal controller C_o for a given finite dimensional plant P_o, such that the feedback system (C_o, P) is stable for all $P(s) = P_o(s)e^{-\tau s}$ where τ is uncertain in $[0\ ,\ h)$, with the maximal delay $h > 0$. The objective of the delay margin optimization is to design C_o solving this problem for the maximal allowable h. When P_o has low number of poles and zeros in $\mathbb{C}_+$ an analytical expression can be found for an upper bound on the maximal allowable h, see [37]. A solution of this problem for a general P_o can be found as

Table 4.2: Steady state error and maximum allowable uncertainty as n_f increases.

n_f	3	5	8	10
e_{99-100}	1.40×10^{-2}	0.50×10^{-2}	0.24×10^{-2}	0.19×10^{-2}
K	1.14×10^{-3}	0.93×10^{-3}	0.38×10^{-3}	0.24×10^{-3}

follows:

(C, P) is stable if

- (C, P_o) is stable
- $\|W_h T_o\|_\infty \leq 1$

where $W_h(s)$ is an upper bound on the plant uncertainty

$$\left|\frac{P(j\omega)}{P_o(j\omega)} - 1\right| = |e^{-j\omega\tau} - 1| \leq W_h(j\omega)$$

The parametrization of all stabilizing controllers for a given nominal plan P_o is well known in the literature and given as

$$C = \frac{X + D_p Q}{Y - N_p Q}$$

where $Q \in \mathcal{H}_\infty$ is a free parameter and $P_o = N_p/D_p$ can be written where both $N_p, D_p \in \mathcal{H}_\infty$ and mutually coprime. $X \in \mathcal{H}_\infty$ and $Y \in \mathcal{H}_\infty$ are calculated from Bézout identity (i.e. $XN_p + YD_p = 1$).

If the stabilizing controller model is plugged in second item of the robust control problem, the following result can be obtained

$$\|W_h T_o\|_\infty = \|W_h N_p X + W_h N_p D_p Q\|_\infty \leq 1$$

where $T_o = P_o C S_o$ and $S_o = (1 + P_o C)^{-1}$. This is identical to model matching problem which was explained in Section 2.7.1 with the matching data $T_1 = W_h N_{po} X$ and $T_2 = W_h N_{po} D_p$ where $N_p = N_{pi} N_{po}$ is an inner-outer factorization of the transfer function N_p. Once an interpolating transfer function

$F = W_h N_{po} X + W_h N_{po} D_p Q$ is calculated by solving the NPIP, it is an easy exercise to calculate C_o.

Let p_i be the poles of P_o in $\mathbb{C}_+$, i.e. unstable zeros of T_2. Then, for a given h, a feasible C_o can be found if one can find a transfer function $F \in \mathcal{H}_\infty$, such that $\|F\|_\infty < 1$ and $F(p_i) = W_h(p_i)/N_{pi}(p_i)$ for all $\mathbb{C}_+$ poles P_o, where $W_h(s)$ is an outer function such that [12]

$$|W_h(j\omega)| = \begin{cases} 2\sin(\omega h/2) & \omega \in [0, \pi/h) \\ 2 & \omega \geq \pi/h \ . \end{cases}$$

Clearly, the above problem can be solved using the optimal NPIP. The difficulty in this formulation is the construction of $W_h(s)$ which corresponds to the tight bound given above. For this reason, approximations are used: $\widetilde{W}_h(s)$, such that $|\widetilde{W}_h(j\omega)| \geq |W_h(j\omega)|$, for all ω, [12]. More precisely, if $F_{opt}(s)$ is the optimal interpolant, then the optimal controller is

$$C_{opt}(s) = \frac{F_{opt}(s) N_{pi}(s)}{P_o(s)\ (\ \widetilde{W}_h(s) - F_{opt}(s) N_{pi}(s)\ )} \ .$$

Here we use Matlab's built-in outer function construction tool to derive a bounding weight for $h = 1$:

$$\widetilde{W}_h(s) = \frac{(2+\delta)(s+\epsilon)(s+2.5)}{s^2 + (2+\sqrt{3})s + 5 + 2\delta}. \tag{4.12}$$

with $\delta = 0.028$ and $\epsilon = 2 \times 10^{-6}$. This gives a maximum error bound of 4%, more precisely,

$$\begin{array}{ll} |W_h(j\omega)| \leq |\widetilde{W}_h(j\omega)| \leq 1.04\ |W_h(j\omega)| & \omega \geq 10^{-5}, \\ |W_h(j\omega)| \leq |\widetilde{W}_h(j\omega)| \leq 1.04 \times 10^{-5} & \omega < 10^{-5}. \end{array}$$

So, the conservatism introduced by this second order approximate bound is relatively low. Next, we illustrate this point with a numerical example. Let us now consider the following family of plants, with poles in $\mathbb{C}_+$ as follows:

$$P_N(s) = G_o(s) \prod_{i=1}^{N} \frac{s + p_i}{s - p_i}$$

with $N = 3, 5, 7$, and $G_o(s)$ is an arbitrary outer function with no poles or zeros on the Im-axis, and

$$p_1 = 0.3\ \kappa,$$

$$p_{2,3} = (0.15 \pm j\ 0.15)\ \kappa,$$

$$p_{4,5} = (0.25 \pm j\ 0.35)\ \kappa,$$

$$p_{6,7} = (0.05 \pm j\ 0.01)\ \kappa,$$

where $\kappa > 0$ is a scaling factor on the magnitude of the unstable poles.

We investigate the largest allowable $\kappa > 0$ such that the delay margin is greater or equal to $h = 1$. From the NPIP, it is computed that the largest allowable κ which makes $\gamma_{opt} < 1$ is equal to 1.7725, 0.9176, and 0.8473 for P_3, P_5 and P_7, respectively. The corresponding controllers lead to

$$\|W_h T_N\|_\infty \leq \gamma_{opt}/1.04 =: \widetilde{\gamma},$$

where $T_N = P_N C_{opt}(1 + P_N C_{opt})^{-1}$. Since $\gamma_{opt} \nearrow 1$ we have that $\widetilde{\gamma} = 0.96$ which means that there is still some room to increase κ, until the condition $\|W_h T_N\|_\infty < 1$ is violated.

In this problem, κ can be seen as a frequency scaling factor: in other words, for an arbitrary $h > 0$ and $h \neq 1$, we may define $\hat{s} = hs$, i.e. $s = \hat{s}/h$, then P_N has poles in the $\hat{s}$-plane at $p_i h$. So, with $\kappa = h$ we can solve the original delay margin optimization problem. Thus, the weight (4.12) can be used universally for any delay value h, by scaling the poles. In particular, when P_o contains only one pole in $\mathbb{C}_+$, a lower bound for the largest allowable (ph) can be calculated using the coefficients of $\widetilde{W}_h$ as the largest (ph) which satisfies

$$\frac{(2+\delta)((ph)+\epsilon)((ph)+2.5)}{(ph)^2 + (2+\sqrt{3})(ph) + 5 + 2\delta} < 1$$

where $\delta = 0.028$ and $\epsilon = 2 \times 10^{-6}$. The above inequality is satisfied for all $ph \in [0\ ,\ 1.66)$, i.e. a lower bound of the maximum allowable (ph) is found to be 1.66. This result is consistent with [37], where the achievable delay margin is found as $(ph) = 2$. With a higher order $\widetilde{W}_h$, it is possible to obtain (tighter) lower bounds closer to 2.0, [46].

4.2.3 Robust Stabilization of Time Delay Systems

Let us now consider a retarded time delay system

$$P(s) = \frac{1}{s + 1 + 4e^{-h_o s}} \qquad h_o = 2.7 \ sec.$$

This plant has four poles in $\mathbb{C}_+$ denoted as $p_1 = \bar{p}_2$, $p_3 = \bar{p}_4$, where

$$p_1 \approx 0.3327 + j0.9367 \qquad p_3 \approx 0.0801 + j3.0355 \ .$$

The locations of these poles are computed using `QPmR.m` (alternatively, one may compute these using `YALTA` or `DDE-BIFTOOL`), see [2, 17, 56]. We consider uncertainty in the delay by replacing h_o with $h \in [2.7 \ \ 2.74]$, and let $P_\Delta(s)$ denote the uncertain transfer function obtained by taking any value of h in the given interval. It can be verified, again using the above mentioned software tools, that $P_\Delta(s)$ has four poles in $\mathbb{C}_+$. The locations of the dominant poles for $h = 2.7$ and $h = 2.74$ are shown in Figure 4.19.

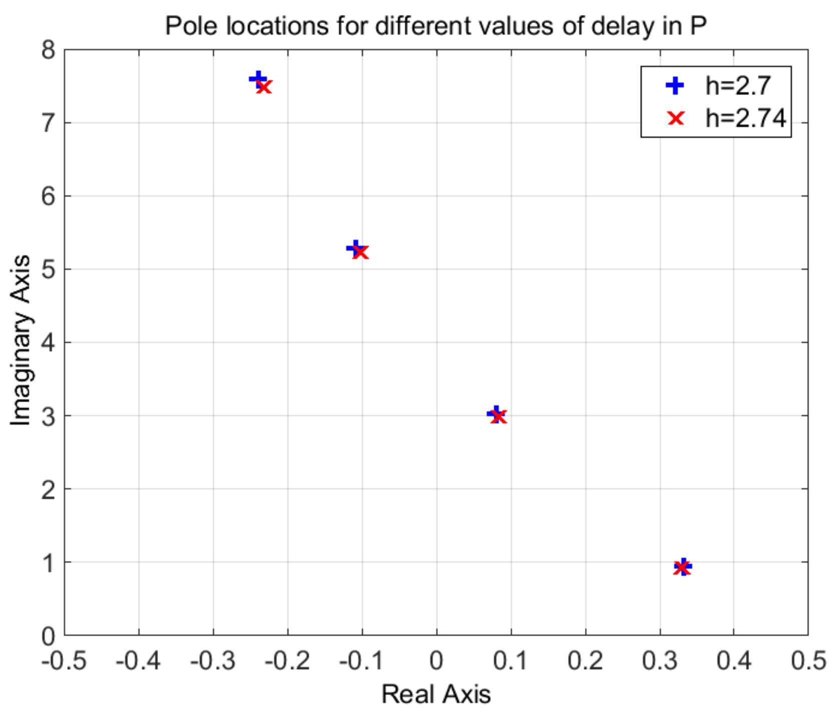


Figure 4.19: Location of the poles of P for different values of delay $h = \{2.7, 2.74\}$

An upper bound for $|P(j\omega) - P_\Delta(j\omega)|$ can be computed as $|W_a(j\omega)|$ where

$$W_a(s) = \frac{2.0152\ (s + 2007)(s + 5.162)(s + 2.646)(s + 0.009832)}{10^4\ (s^2 + 0.3569s + 7.29)(s^2 + 0.3391s + 28.16)}.$$

Now define a co-prime factorization $P(s) = N(s)/D(s)$ where

$$D(s) = \prod_{k=1}^{4} \frac{s - p_k}{s + p_k} \qquad \text{and} \qquad N(s) = P(s)D(s)\ .$$

A robustly stabilizing controller can be found for this system if a function $F \in \mathcal{H}_\infty$ can be found such that $\|F\|_\infty < 1$ and $F(p_i) = W_a(p_i)/N(p_i)$, see [16]. Applying the optimal interpolation method given above the following transfer function is obtained as the optimal solution:

$$F_{opt}(s) = 0.98528\ \frac{(s - 1.072)(s^2 - 0.4351s + 0.9697)}{(s + 1.072)(s^2 + 0.4351s + 0.9697)}.$$

Since $\|F_{opt}\|_\infty = 0.98528 < 1$, it can be deduced that a robustly stabilizing controller can be found.

The resulting optimal controller can be written as

$$C_{opt} = \frac{R}{1 - PR} \qquad \text{where} \qquad R = \frac{DF_{opt}}{W}.$$

The closed loop transfer functions are

$$S = (1 + PC_o)^{-1} = (1 - NF_{opt}/W) \qquad CS = DF_{opt}/W$$

$$PS = N\ \frac{1 - NF_{opt}/W}{D} \qquad T = 1 - S = NF_{opt}/W\ .$$

Note that $PS \in \mathcal{H}_\infty$ due to the interpolation conditions $F_{opt}(p_i) = W_a(p_i)/N(p_i)$.

In the above approach, when the maximal delay increases to 2.75 sec, with a similar 4th order weight we find $\gamma_{opt} > 1$, hence, a robustly stabilizing controller cannot be computed. On the other hand, with an infinite dimensional weight it may be possible to get $\gamma_{opt} < 1$ for maximal delay values as high as 3.0 sec.

4.3 Discussions

An alternative solution method is proposed for computing the optimal Nevanlinna-Pick interpolant when the problem data consists of distinct points in the right half plane. The optimal interpolant is given in a direct manner without resorting to transformations or approximations of suboptimal solutions.

Three examples from robust control problems are represented. In the first example, two different nominal plants are introduced and robust repetitive control problem is solved for each nominal plant. Time domain performances for these nominal plants (a clue for internal stability) and for 6 different perturbed plants (a clue for robust stability) are illustrated. Finally, effects of time delay h, amount of shift ε and number of modes n_f on the closed loop performance are discussed. In the second example, a delay margin optimization problem is considered and a solution method is proposed through the optimal solution of NPIP. In the last example, robust control of infinite dimensional systems is discussed.

Chapter 5

Bounded Unit Interpolation in $\mathcal{H}_\infty$

This section studies the solution of Problem 5 which is closely related to strong stabilization and simultaneous stabilization problems in feedback control theory, see [1], [4], [26], [59], [60] and [73]. In [30], the bounded unit interpolation problem is defined and additionally the close relation between this problem and sensitivity shaping or robust stabilization by a stable controller is discussed.

This section aims to find a sufficient condition for the finite dimensional case and to derive an algorithm for the desired interpolating function. The conservatism of the proposed method is also compared with the infinite dimensional case, as well.

The rest of the section is organized as follows: Section 5.1 proposes an interpolation method to solve Problem 5 by modifying the method defined in [16] to handle the norm bound. The main disadvantage of this method is being applicable only to real interpolation data. Section 5.2 proposes a different interpolation scheme to solve Problem 5 by making use of the optimal Nevanlinna-Pick problem solution method. Section 5.2.1 is the novel contribution of this chapter as it explains the method to generate bounded unit interpolating function having a predetermined form if the necessary conditions are satisfied. At the end of the section, an algorithm is provided as a brief summary of the proposed method for practical purposes. Two simple interpolation problems from literature are also revisited in this section to illustrate the performance comparison of the proposed method. Four different illustrative examples from strong stabilization literature are revisited and solved by the proposed method in Section 5.2.2. Finally, Section 5.3 concludes by some discussion and possible future studies.

5.1 Bounded Unit Interpolating Function for Real Data

Publication Notice: The materials of this section are at least partially covered in the publication [65] which was published by the author and his advisor during the study time of this thesis dissertation.

5.1.1 Solution Through Modified Unit Interpolation

In this section, we try to make use of the method proposed by [16] in order to find a norm constrained function to solve the mNPIP defined in Chapter 2. Consider Problem 5 with $\rho = 1$. Note that this problem is equivalent to solve Problem 3 and find a solution U satisfying $\|U\|_\infty < 1$. We briefly summarized the solution method of [16] to Problem 3 in Section 2.7.2. We noted that this constructive solution strategy has two parameters, l_i values and the pole locations of H_i functions, to shape the frequency response of the interpolation function. A solution for Problem 5 can be found by the method summarized in Algorithm 2.

Algorithm 2 Proposed Algorithm to solve Problem 5

Interpolation Data: (α_i, β_i), $i \in \{1, \ldots, n\}$
Define $\mathbf{r} = (r_1, ..., r_{n-1})$ to be the relative pole location for the corresponding α_i within each $H_i(s) = \prod_{j=1}^{i} \frac{s-\alpha_j}{s+r_j\bar{\alpha}_j}$ which will be a part of the interpolation operation. (i.e. $\mathbf{r} = (1, ..., 1)$ is the case called as fixed pole location)
Define l_{max} to be the maximum allowable relative degree for the interpolating functions U_i.
Let $U_1(s) = \beta_1$
for $k = 2 : n$ **do**
    $l_k = 1$
    while $(l_k < l_{max})$ **do**
        $U_k = (1 + c_k H_k)^{l_k} U_{k-1}$
        Find c_k from $U_k(\alpha_k) = \beta_k$
        if $((\|U_k\|_\infty < 1)$ and $(c_k < 1/\|H_k\|_\infty))$ **then**
            Break while
        else
            $l_k = l_k + 1$
        end if
    end while
end for
$U = U_n$

5.1.2 An Illustrative Example

In [59] the authors have studied on a numerical example and derived upper and lower bounds for the multiplicative uncertainty under which a stable controller can be generated. The example is formed by the plant

$$P(s) = \frac{(s-\alpha)(s-4e^{-s}+1)}{(s-10)(s-15)(2e^{-s}+1)} \tag{5.1}$$

where the factorization is in the form

$$\begin{aligned} M_n(s) &:= \frac{(s-\alpha)(s-p)}{(s+\alpha)(s+p)} \\ M_d(s) &:= \frac{(s-10)(s-15)(2e^{-s}+1)}{(s+10)(s+15)(e^{-s}+2)} \\ N_o(s) &:= \frac{(s+\alpha)(s+p)(s-4e^{-s}+1)}{(s-p)(s+10)(s+15)(e^{-s}+2)} \end{aligned} \tag{5.2}$$

and $\mathbb{C}_+$ roots of the quasi-polynomials in the numerator and denominator can be calculated by using qpmr or Yalta packages. ([2], and [57])

Remark: $\mathcal{H}_\infty$-stability and stability of such plants are discussed in detail in [6].

The problem is to find a stable controller to robustly stabilize the set of uncertain plants defined by the given uncertainty weight W

$$\begin{aligned} &W(s) = K\frac{s+1}{s+10}, \quad K > 0 \\ &2 \leq \alpha < 10, \qquad p = 0.7990 \end{aligned} \tag{5.3}$$

See Figure 2 of [59] for the lower and upper bounds of the largest allowable K for which the robust and strong stabilization problem is solvable with data given in (5.3) for the plant (5.1), with $\alpha \in (2\ ,10)$. In what follows we illustrate the application of the algorithm proposed in Algorithm 2. Our objective is to find a finite dimensional G as an alternative to the infinite dimensional one constructed in [59].

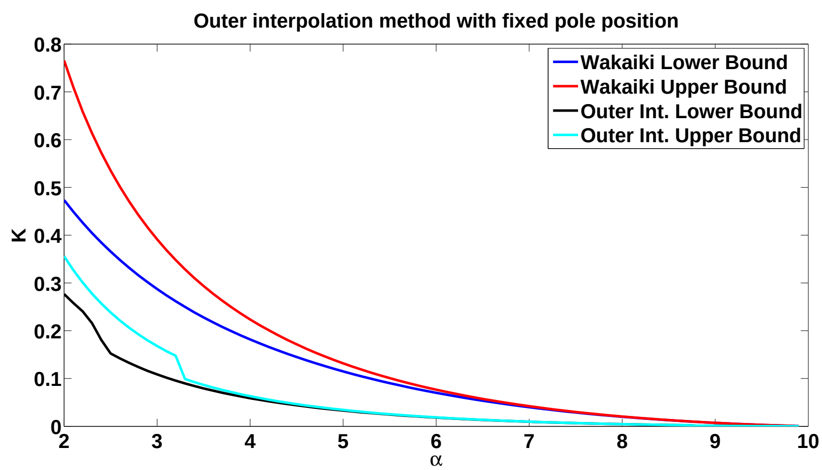


Figure 5.1: Upper and lower bounds for K using inner $H(s)$ having degree $l < 5$

5.1.2.1 Outer Interpolation by Inner $H(s)$

For the given numerical example, W_s and W_n in [59] are generated by `Matlab` built-in function *fitmagfrd* and the interpolation data calculated as $\beta_1 = W(\alpha)/(M_d(\alpha)W_s(\alpha))$ and $\beta_2 = W(p)/(M_d(p)W_s(p))$ where $p = 0.7990$ is the only unstable zero of the infinite dimensional part of the plant and α is the simple zero of the plant which is used as a parameter to calculate the maximum allowable multiplicative uncertainty K under which a stable controller can be generated. W_s is replaced by W_n for upper bound calculations. Since we have 2 interpolation conditions, and both are real, we just need to design one H function for each interpolation phase. The simplest and immediate choice for H is $(s - \alpha)/(s + \alpha)$ for which H becomes inner. With this choice of H the resulting upper and lower bound calculations are shown in Figure 5.1.

5.1.2.2 Outer Interpolation by Varying Pole Location

After having the unsatisfactory results which are explained in Section 5.1.2.1 and shown in Figure 5.1, we introduced a new parameter to the problem as the relative

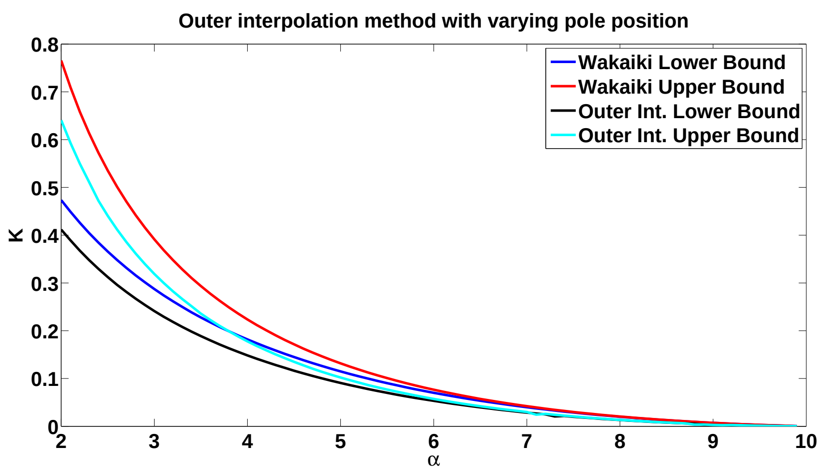


Figure 5.2: Upper and lower bounds for K using adjustable $H(s)$ having degree $l < 5$

pole location of the interstage function H. Instead of having

$$H(s) = \frac{s - \alpha}{s + \alpha}$$

we decided to replace its pole by a factor of r as

$$H(s) = \frac{s - \alpha}{s + r\alpha}$$

and searching for the optimum r which maximizes the upper and lower bounds for a fixed α. The results of this choice is shown in the Figure 5.2.

As it is clearly seen when Figure 5.2 and Figure 5.1 are compared, letting pole location to vary instead of fixing it to make H inner significantly improves both the upper and lower bound approximations. It is also important to understand what upper and lower bounds mean in terms of the solution of the mNPIP. In the formulation derived by [59], lower bound is the bound under which the defined problem is certainly solvable by the given solution method (i.e. by calculating an infinite dimensional G). Similarly, upper bound is the bound above which the defined problem is certainly not solvable by the given solution method. In other words, for a fixed α in Figure 2 of [59], the optimum multiplicative uncertainty K under which a stable but infinite dimensional controller can be generated is

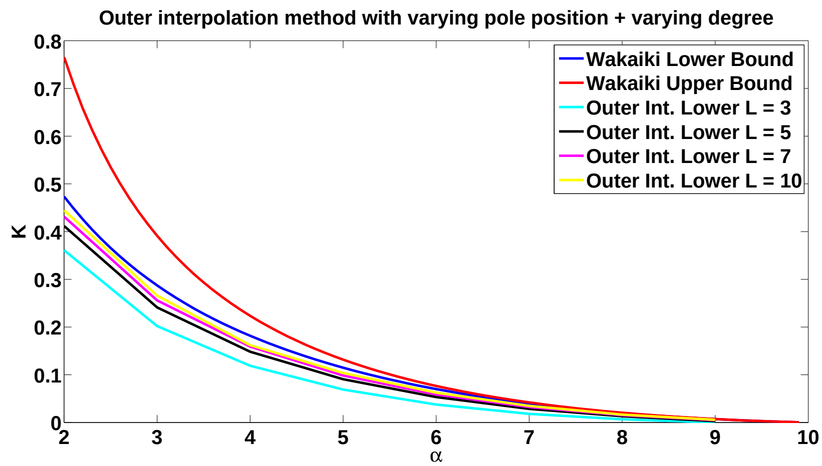


Figure 5.3: Lower bounds for K using adjustable $H(s)$ having degree $l \in [3, 5, 7, 10]$

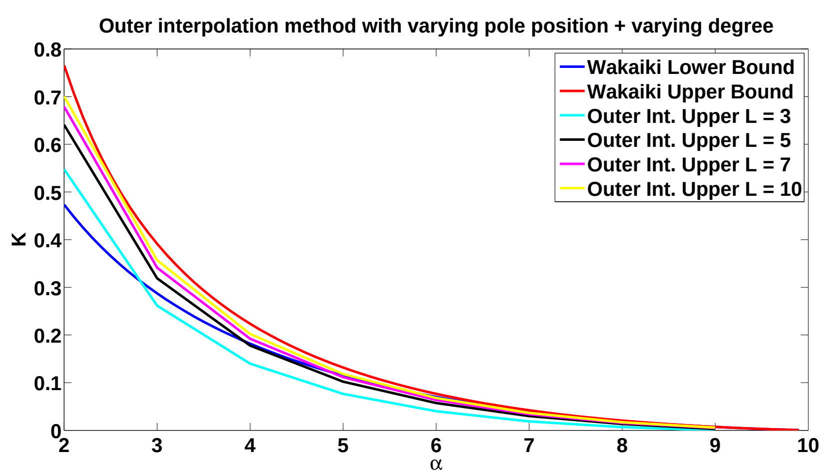


Figure 5.4: Upper bounds for K using adjustable $H(s)$ having degree $l \in [3, 5, 7, 10]$

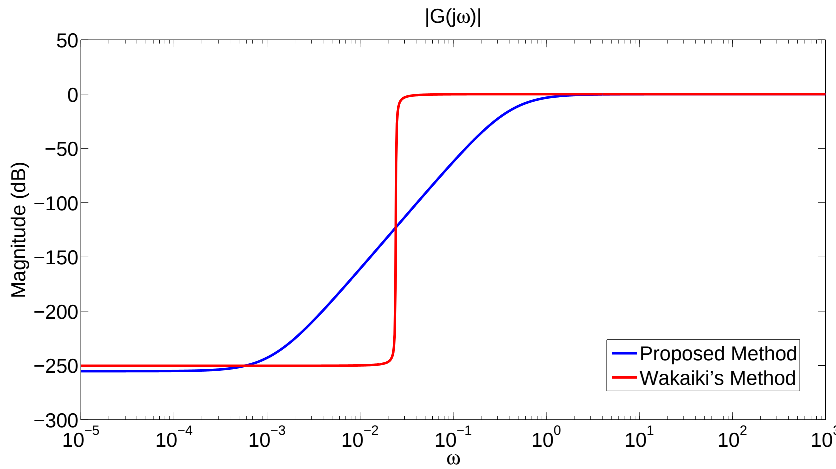


Figure 5.5: The magnitude plots of $G(j\omega)$ obtained by [59] and by the proposed algorithm, using $l = 5$, when $\alpha = 2$.

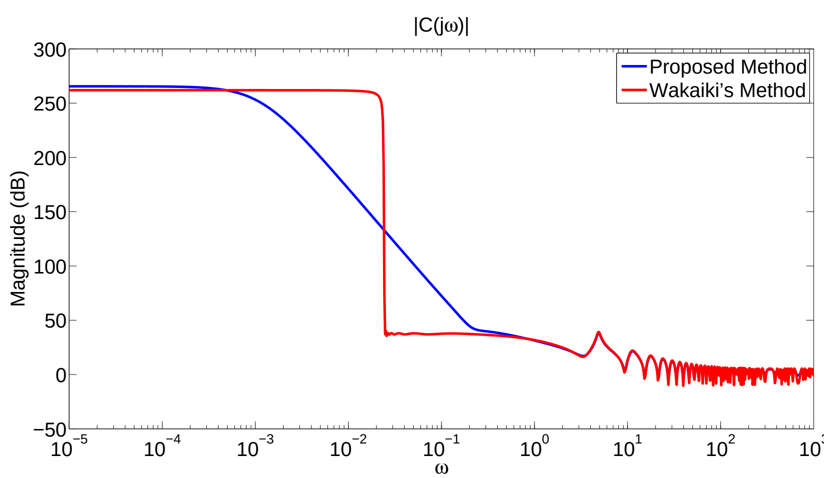


Figure 5.6: The magnitude plot of the calculated $C(s)$ functions by both methods with $\alpha = 2$, and $l = 5$ for the proposed method.

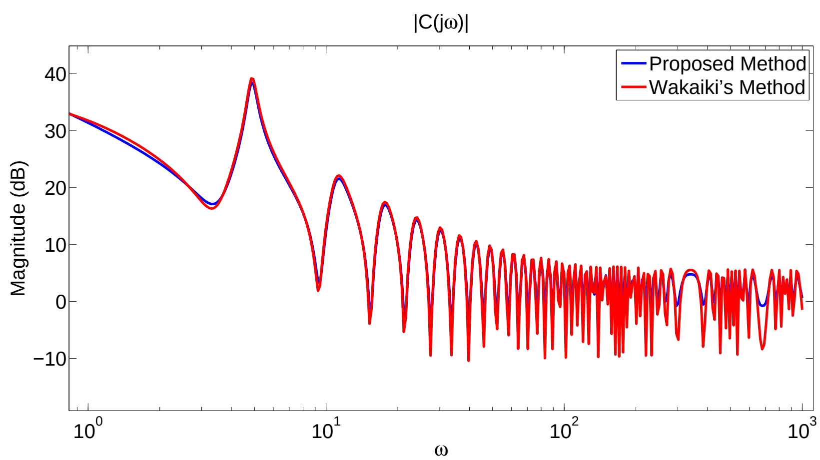


Figure 5.7: Just a zoomed version of the Figure 5.6

between the defined upper and lower bounds. When this explanation is considered, the expected behavior of the approximated upper and lower bounds is to approach from above and from below, respectively. Figure 5.2 suggests that approximate lower bound behaves as expected whereas the approximate upper bound also approaches from below. To be sure about the approaching direction of the bounds an extra simulation is computed. In this simulation, the varying pole location technique described in this section is used for some different values of $l \in [3, 5, 7, 10]$ to visualize the approaching direction of the approximate bounds calculated by the technique defined by [16]. Figure 5.3 clearly shows that the approximate lower bound approaches from below to the original lower bound calculated by [59] as expected. However, as Figure 5.4 depicts the approximate upper bound also approaches to the original upper bound calculated by [59] from below. This unexpected behavior seems to grant a better upper bound than the original bound as a first impression, however, since the problems which are solved to generate the bounds are a relaxed version of the original problem, an approximate solution for the upper bound calculation is not that much meaningful. On the contrary, any approximate solution for the lower bound which stays strictly below the original bound is a suboptimal solution to the original problem. We make use of this fact to generate a low order interpolation function G for the case when $\alpha = 2$, $l = 5$, $K = 0.4117$, $r = 0.2046$ as given in (5.4) and the controller is given by (5.5). The comparison of the controller and G function designed by [59] and designed by the newly proposed method are given in Figures 5.5, 5.6, and 5.7. It is clear from (5.4) that the resulting finite dimensional G is outer, i.e. both G, $G^{-1} \in \mathcal{H}^{\infty}$. The interpolation data that is required for the given value of $K = 0.4117$ is calculated by $\beta_{1,2} = W(z_{1,2})/M_d(z_{1,2})W_s(z_{1,2})$ for $z_1 = \alpha = 2$ and $z_2 = p = 0.7990$ that turn out to be $(\alpha_i, \beta_i) = \{(0.7990, 0.1275), (2, 0.3955)\}$. It is possible to show that $G(0.7990) = 0.1275$ and $G(2) = 0.3955$ by using (5.4). As a final remark, Figure 5.5 shows that $\|G\|_{\infty} \leq 1$. As a result, this G function is a solution to the mNPIP.

$$G(s) = \frac{(s + 0.001147)^5}{(s + 0.4091)^5} \tag{5.4}$$

$$C(s) = \frac{W(s) - M_d(s)W_s(s)G(s)}{M_n(s)N_0(s)W_s(s)G(s)} \tag{5.5}$$

5.2 Bounded Unit Interpolation

Publication Notice: The materials of this section are at least partially covered in the publication [70] which was published by the author and his advisor during the study time of this thesis dissertation.

5.2.1 Solution Through Optimal Nevanlinna-Pick Interpolant

In this section we consider a form of $F(s)$ given as

$$F(s) = [\hat{F}(s)]^n \tag{5.6}$$

where $\hat{F}(s) = \frac{G(s)+1}{G(s)+\rho}$ with $\rho \in \mathbb{R}$ and $G \in \mathcal{H}_\infty$.

Proposition 1. *$\hat{F}$ defined by (5.6) is a unit function in $\mathcal{H}_\infty$, i.e. $\hat{F} \in \mathcal{H}_\infty$ and $\hat{F}^{-1} \in \mathcal{H}_\infty$, if $\rho > 1$, $G \in \mathcal{H}_\infty$ with $\|G\|_\infty < 1$. Moreover, under these conditions, we have $\|\hat{F}\|_\infty < 1$.*

Proof. The fact that $\hat{F}$ is a unit in $\mathcal{H}_\infty$ comes from the small gain theorem. In order to prove that $\|\hat{F}\|_\infty < 1$, let us follow the definition of the norm

$$\|\hat{F}(s)\|_\infty = \sup_\omega |\hat{F}(j\omega)| = \sup_\omega \left|\frac{G(j\omega)+1}{G(j\omega)+\rho}\right|. \tag{5.7}$$

Using (5.7) we can rewrite the statement $\|\hat{F}\|_\infty < 1$ as

$$\sup_\omega \left|\frac{x(\omega)+jy(\omega)+1}{x(\omega)+jy(\omega)+\rho}\right| < 1 \longleftrightarrow \frac{(x(\omega)+1)^2+y^2(\omega)}{(x(\omega)+\rho)^2+y^2(\omega)} < 1, \quad \forall\omega \tag{5.8}$$

where $G(j\omega) = x(\omega) + jy(\omega)$, and $x(\omega),\ y(\omega) \in \mathbb{R}$ for all ω. By simple algebra and assuming $\rho = 1 + \varepsilon$ for some $\varepsilon > 0$, we need to prove

$$2\varepsilon(x(\omega)+1) + \varepsilon^2 > 0, \quad \forall\omega. \tag{5.9}$$

Note that the condition

$$\|G(s)\|_\infty = \sup_\omega |G(j\omega)| = \sup_\omega \sqrt{x^2(\omega)+y^2(\omega)} < 1 \tag{5.10}$$

implies that $|x(\omega)| < 1$ for all ω. Putting together $\varepsilon > 0$ and $(x(\omega) + 1) > 0$ for all ω, (5.9) is proven. □

Proposition 2. *If $\hat{F}$ satisfies conditions $F, F^{-1} \in \mathcal{H}_\infty$ and $\|F\|_\infty < 1$, then these conditions also hold for any positive integer power of $\hat{F}$, i.e. for $F = \hat{F}^n$ for some positive integer n.*

Proof. The case for conditions $F, F^{-1} \in \mathcal{H}_\infty$ is straight forward since F has same zeros and poles as $\hat{F}$ with multiplicity n. For $\|F\|_\infty < 1$, we can rewrite $\hat{F}(j\omega) = r(\omega)e^{j\theta(\omega)}$ and $F(j\omega) = r^n(\omega)e^{jn\theta(\omega)}$ where $r(\omega) \in \mathbb{R}$ and $\theta(\omega) \in \mathbb{R}$ for all ω. With this interpretation

$$\|\hat{F}\|_\infty = \sup_\omega |\hat{F}(j\omega)| = \sup_\omega |r(\omega)| < 1 \tag{5.11}$$

implies that $|r(\omega)| < 1$ for all ω. Hence, we can conclude that

$$\|F\|_\infty = \sup_\omega |F(j\omega)| = \sup_\omega |r^n(\omega)| = \sup_\omega |r(\omega)|^n < 1 \tag{5.12}$$

since $|r(\omega)| < 1$ for all ω and $n > 0$. □

Let us consider the arguments in [42] for $F, F^{-1} \in \mathcal{H}_\infty$ and $F(\alpha_i) = \beta_i$ for all i. Given interpolation data (α_i, β_i) for $i \in \{1, 2, \ldots, n\}$ as in Problem 3, a unit interpolating function of degree kn_0 exists for a positive integer n_0, if the following Nevanlinna-Pick matrix

$$\mathsf{P_{ij}} = \left[\frac{\beta_i^{1/n} + \bar{\beta}_j^{\,1/n}}{\alpha_i + \bar{\alpha}_j}\right]_{i,j\in\{0,1,\ldots,k\}} \tag{5.13}$$

is positive definite for $n = n_0$. As explained in [42], n-th root is calculated in such a way that if $\beta_i \in \mathcal{B}$ and $\beta_j \in \mathcal{B}$ are conjugate pairs, so are $\beta_i^{1/n}$ and $\beta_j^{1/n}$. All possible combinations of n-th roots of complex interpolation pairs should be checked to decide if P is positive definite. It is proven in [42] that every unit interpolation problem has a solution in the integer interval $n_0 \leq n < \infty$ if the problem is feasible for some integer $n_0 > 0$. Note that this condition is only for existence of an interpolating unit, however, it says nothing about the infinity norm ($\|F\|_\infty$) of the interpolating function. The following proposition defines a sufficient condition for the solution of Problem 5.

Proposition 3. *In order to solve Problem 5, let* R *be a Pick matrix defined as*

$$\mathsf{R_{ij}} = \left[\frac{1-\gamma_i\bar{\gamma}_j}{\alpha_i+\bar{\alpha}_j}\right]_{i,j\in\{0,1,\ldots,k\}} \tag{5.14}$$

where

$$\gamma_i = \frac{\rho\beta_i^{1/n}-1}{1-\beta_i^{1/n}} \tag{5.15}$$

for $i \in \{0,1,\ldots,k\}$*. If* R *is positive definite for some* $\rho > 1$ *and* $n = n_0$*, where* n_0 *is a positive integer, then a real rational bounded unit interpolating function* F *with degree* kn_0 *exists.*

Proof. To prove this proposition, let us first note that if R is positive definite for some integer $n = n_0 > 0$ then it is possible to find a rational function $G \in \mathcal{H}_\infty$ of order k which satisfies the interpolation conditions $G(\alpha_i) = \gamma_i$ for all $i \in \{0,1,\ldots,k\}$ and $\|G\|_\infty < 1$. For the calculation of optimal $G(s)$, readers are directed to [68]. By using this G, we can write $\hat{F} = (G+1)/(G+\rho)$ as in (5.6) and this $\hat{F}$ satisfies $\hat{F}(\alpha_i) = \beta_i^{1/n_0}$ for all $i \in \{0,1,\ldots,k\}$. Note that $\hat{F}$ has degree k and it satisfies $F, F^{-1} \in \mathcal{H}_\infty$ and $\|F\|_\infty < 1$ by Proposition 1. As a final step, if we write $F = \hat{F}^{n_0}$ then it satisfies $F(\alpha_i) = \beta_i$ for all $i \in \{0,1,\ldots,k\}$. F also satisfies Proposition 2, hence F is a solution of Problem 5 with degree kn_0. □

It is important to note that, having R positive definite is a sufficient condition to have a solution for real rational bounded unit interpolation problem provided that the necessary conditions (parity interlacing property and $|\beta_i| < 1$ for all $i \in \{0,1,\ldots,k\}$) are satisfied.

Proposition 3 has two parameters, ρ and n, in order to satisfy R being positive definite. In general, we need to conduct a search on ρ vs. n plane to find the region on which R is positive definite. However, in this study, we want to find the lowest possible degree interpolating function, i.e. minimum possible n. In order to achieve this, throughout this chapter, we will first find the smallest possible n for which R can be made positive definite. But first, let us figure out the effect of $\rho = 1+\varepsilon$ on $\|\hat{F}\|_\infty$.

Proposition 4. *If $\varepsilon = \varepsilon_0$ solves Problem 5 for some positive integer $n = n_0$ then there exists $\varepsilon_1 > \varepsilon_0$ for which the problem is feasible.*

Proof. Let us assume that $\mathsf{R}^{(0)}$ which is defined by (5.14) for $\rho = 1+\varepsilon_0$ is positive definite (i.e. $\mathsf{R}^{(0)} > \phi I$ where I is the identity matrix of proper size and $\phi > 0$).

Write $\mathsf{R}^{(1)}$ using (5.14) for $\rho = 1 + \varepsilon_1$ as

$$\begin{aligned}
\mathsf{R}^{(1)}_{\mathsf{ij}} &= \left[\frac{1-(\varepsilon_0 w_i - 1 + \delta w_i)(\varepsilon_0 \bar{w}_j - 1 + \delta \bar{w}_j)}{\alpha_i + \bar{\alpha}_j}\right] \\
&= \mathsf{R}^{(0)}_{\mathsf{ij}} + \delta\left[\frac{w_i + \bar{w}_j - 2\varepsilon_0 w_i \bar{w}_j}{\alpha_i + \bar{\alpha}_j}\right] - \delta^2\left[\frac{w_i \bar{w}_j}{\alpha_i + \bar{\alpha}_j}\right] \\
&= \mathsf{R}^{(0)}_{\mathsf{ij}} + \delta\Delta^{(1)}_{ij} + \delta^2\Delta^{(2)}_{ij} \\
\mathsf{R}^{(1)} &> I(\phi - \delta\Delta - \delta^2\Delta) \qquad (5.16)
\end{aligned}$$

where

$$\varepsilon_1 = \varepsilon_0 + \delta, \quad \delta > 0, \quad w_i = \frac{\beta_i^{1/n_0}}{1-\beta_i^{1/n_0}}$$
$$\Delta = \max(\|\Delta^{(1)}\|_\infty, \|\Delta^{(2)}\|_\infty).$$

For $\delta = 0$, we know that $\phi > 0$, hence $\mathsf{R}^{(1)} = \mathsf{R}^{(0)}$ and $\mathsf{R}^{(0)}$ is positive definite by assumption. As δ increases right hand side of (5.16) decreases, however, $\mathsf{R}^{(1)}$ is positive definite until it reaches zero. Assume that $\delta_f > 0$ is the point which makes right hand side of (5.16) zero. Hence, it is proven that the problem is feasible when $\delta \in [0, \delta_f) \to \varepsilon \in [\varepsilon_0, \varepsilon_1)$ where $\varepsilon_1 = \varepsilon_0 + \delta_f > \varepsilon_0$. □

Proposition 5. *If Problem 5 is feasible for some $n = n_0$ and $\varepsilon = \varepsilon_0$, then it is possible to decrease the norm of the interpolating function by some $\varepsilon_1 > \varepsilon_0$ if ε_1 also solves the interpolation problem.*

Proof. The result is obtained directly from the proofs of Propositions 1 and 4. □

Putting all these together, we can divide the problem into two parts:

- Fix ε as some sufficiently small number and search linearly over n and find smallest possible n_0 which makes R in (5.14) positive definite

- Using the idea in Proposition 5, fix $n = n_0$ this time and conduct a search on ε to find largest possible ε for which R in (5.14) stays positive definite

This interpretation of the solution leads us to find the solution to Problem 5 by the proposed method with the smallest possible degree. The proposed method is summarized in Algorithm 3 in detail.

Algorithm 3 Bounded Unit Interpolation

1: Interpolation Data: (α_i, β_i), $i \in \{0, 1, \ldots, k\}$
2: Maximum Degree Desired: n_{max}
3: Continue if P.I.P. is satisfied, jump to Step 19 if not.
4: Continue if all $|\beta_i| < 1$ for all $i \in \{0, 1, \ldots, k\}$, jump to Step 19 if not.
5: $\rho_1 = 1 + \varepsilon$ where $\varepsilon = 10^{-3}$.
6: $M = floor(n_{max}/k)$
7: $n = 0$
8: **while** $n < M$ **do**
9:     $n = n + 1$
10:     Calculate γ_i for all $i \in \{0, 1, \ldots, k\}$ using ρ_1 and n as in (5.15)
11:     **if** R in (5.14) is positive definite **then**
12:         Set $n_0 = n$
13:         Set ρ_2 as a big number. (in most practical cases $\rho_2 = 100$ is sufficiently large)
14:         Binary search on $\rho \in [\rho_1, \rho_2]$ by using n_0 to find the range over which R is positive definite $\rightarrow (\rho_{low}, \rho_{high})$
15:         Use [68] to calculate the optimal interpolating function for given ρ_{high} and n_0.
16:         **return**
17:     **end if**
18: **end while**
19: No feasible interpolating function exists, exit

In order to understand the conservatism introduced by this sufficient condition, we can compare it to some other sufficient conditions in the literature. In [1], a method to generate bounded unit interpolating functions is introduced. The interpolation problem of

$$(\alpha, \beta) = \{(1, 0.29984), (2, 0.130588)\}$$

is solved by a 5^{th} order unit interpolating function having an infinity norm of 0.8473. By the method proposed in this chapter, it is possible to solve the same problem with a 3^{rd} order bounded unit function having an infinity norm of 0.9745. It is important to note that, since the infinity norm of both solutions remain below 1, having a smaller degree is an advantage of the proposed method. A 28^{th} order unit is designed by the same method in [4] to solve the interpolation problem with the data

$$(\alpha, \beta) = \{(1, 0.1), (3, 0.2), (5, 0.15)\}.$$

It is indeed possible to solve this problem with an 18^{th} order unit by using the method of this chapter.

5.2.2 Examples

The test cases of the proposed algorithm and the conservatism caused by the proposed sufficient condition will be explained by four different examples.

5.2.2.1 Example 1

Let us revisit the example in [59] with a slight modification. The plant definition and co-prime factorization of the plant is given as

$$\begin{aligned}
P(s) &= \frac{(s-\alpha)(s+1)(s-4e^{-s}+1)}{(s-10)(s-15)(e^{-s}+0.2s+0.1)} \\
M_n(s) &= \frac{(s-\alpha)(s-p)}{(s+\alpha)(s+p)} \\
M_d(s) &= \frac{(s-10)(s-15)(s^2-1.4446s+4.9233)}{(s+10)(s+15)(s^2+1.4446s+4.9233)} \\
N_0(s) &= P(s)M_d(s)/M_n(s)
\end{aligned} \tag{5.17}$$

where $p = 0.7990$ is the only zero of the term $(s - 4e^{-s} + 1)$ in $\mathbb{C}_+$. Note that N_0 is outer (i.e. $N_0, 1/N_0 \in \mathcal{H}_\infty$). Let us further assume that we are given a robustness weight as

$$W(s) = K\frac{s+1}{s+10}$$

(i.e. $\|WT\|_\infty < 1$ for $T = PC/(1+PC)$ is required for robust stability) which satisfy $\|W\|_\infty < 1$ when $K < 1$ and there exists a finite dimensional outer approximation W_s such that $|W_s(j\omega)| < 1 - |W(j\omega)|$ for all ω. It was proven that for such a plant P, a robustly stabilizing stable controller can be designed if it is possible to find a bounded unit interpolating function U such that

$$U(z_i) = \frac{W(z_i)}{M_d(z_i)W_s(z_i)}$$

where $z_1 = \alpha$ and $z_2 = p$ are the only simple zeros of the plant in $\mathbb{C}_+$, see [59] for details. The maximum allowable uncertainty bound (i.e. K_{max}) calculated with the method defined in [59] for each value of α is given in Figure 5.8. Note that this bound shows the maximum value by an infinite dimensional bounded interpolating function U.

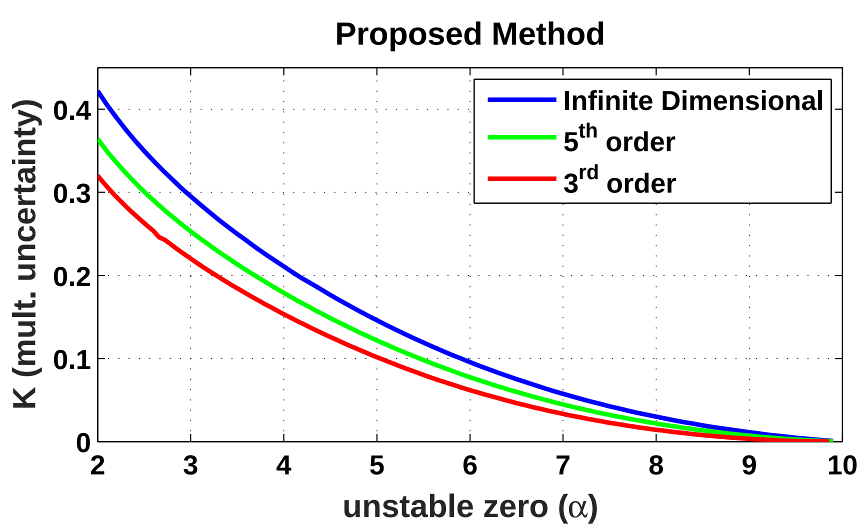


Figure 5.8: Maximum allowable multiplicative uncertainty with respect to real part of the unstable zeros, see [59] for details

We should also note that [65] attempts to find finite dimensional bounded interpolating functions for this problem. The disadvantage of the method of [65] is that it only applies to real interpolation data. Otherwise, it gives a good approximation of the maximum allowable uncertainty bound with a 5^{th} order U for each α. Figure 5.8 also shows the maximum allowable uncertainty bound calculated by a 3^{rd} and 5^{th} order U which is designed by the proposed method of this chapter. It is obvious that the results of this method are similar to the results of [65] and in addition the newly proposed method is also capable of handling complex interpolation data. This is a superior feature of the proposed method. It is also important to note that, the proposed method approximates the infinite dimensional behaviour better as the order of the interpolating function increases. This is a natural and expected feature of an interpolation method.

5.2.2.2 Example 2

Let us consider a different example as shown below.

$$\begin{aligned}
P(s) &= \frac{(s-100)(s-1-j\omega)(s-1+j\omega)}{(s-10)(s+1)(s+10)} \\
M_n(s) &= \frac{(s-100)(s-1-j\omega)(s-1+j\omega)}{(s+100)(s+1-j\omega)(s+1+j\omega)} \\
M_d(s) &= \frac{(s-10)}{(s+10)} \\
N_0(s) &= P(s)M_d(s)/M_n(s)
\end{aligned} \tag{5.18}$$

Note that P has two complex and one real zeros in $\mathbb{C}_+$. Because of complex zeros, the method of [65] is not applicable. Figure 5.9 shows the maximum allowable uncertainty bound for each value of ω using an infinite dimensional interpolator, a 4^{th} order interpolator and an 8^{th} order interpolator. Controller design method and robustness weight W are same as Example 1. As expected, maximum allowable uncertainty bound approaches the infinite dimensional interpolator case as the degree of interpolator increases.

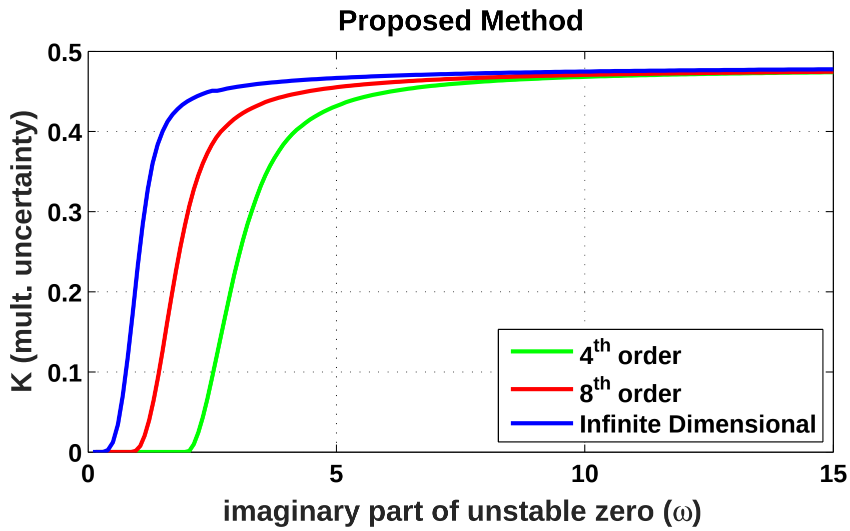


Figure 5.9: Maximum allowable multiplicative uncertainty with respect to real part of the unstable zeros

5.2.2.3 Example 3

This example is taken from [73], where a method to design finite dimensional stable controllers, which have same degree as the plant, is introduced. The example is given to illustrate the MIMO case of the proposed method. In [73], a MIMO plant P is defined as

$$P = \left[\frac{(s+1)(s-2-j\theta)(s-2+j\theta)}{(s+2+j)(s+2-j)(s-1)(s-5)}, \right.$$

$$\left. \frac{(s+5)(s-2-j\theta)(s-2+j\theta)}{(s+2+j)(s+2-j)(s-1)(s-5)} \right]. \tag{5.19}$$

It was shown that, as θ decreases, it becomes more difficult to find a stable controller by the method of [73] and indeed, for $\theta < 12$ the method becomes numerically fragile.

Let us assume that we rewrite the plant P as

$$P = [P_1,\ P_2]$$

and define

$$P_0 = \frac{(s+3)(s-2-j\theta)(s-2+j\theta)}{(s+2+j)(s+2-j)(s-1)(s-5)}. \tag{5.20}$$

where $P_1 = P_0(1 + W_1)$ and $P_2 = P_0(1 + W_2)$. Then we can also define

$$W = \frac{2(10^{-5}s + 1)}{s + 3}$$

where

$$|W(j\omega)| > |W_1(j\omega)| = |W_2(j\omega)|, \quad \text{for all } \omega$$

is satisfied as shown in Figure 5.10.

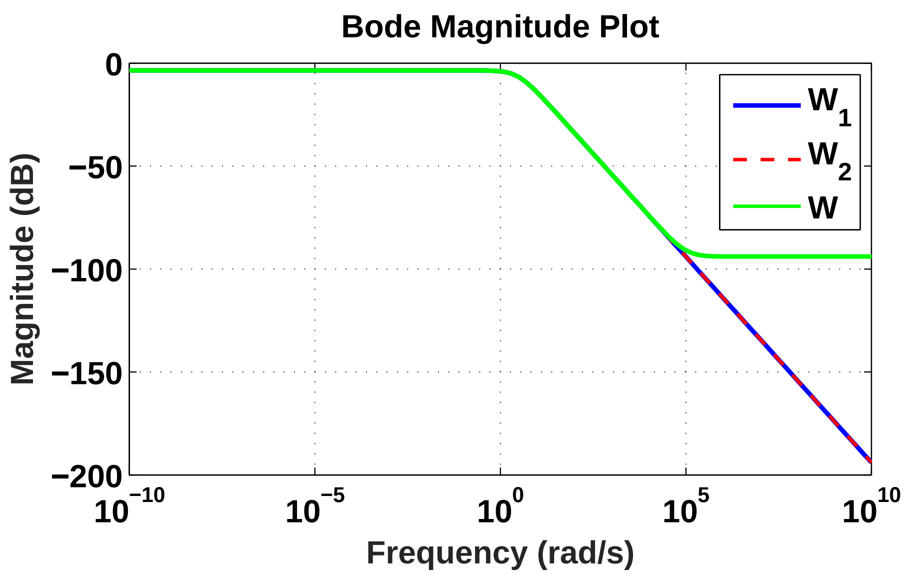


Figure 5.10: Bode magnitude plots of W_1, W_2, W

Note that, if it is possible to find a stable controller C, which internally stabilizes P_0 and satisfies $\|WT\|_\infty < 1$ for $T = P_0C/(1 + P_0C)$, then this C will strongly stabilize P. Since $\|W\|_\infty < 1$ as in Figure 5.10, then we can apply the ideas in [59] to find finite dimensional stable C. An important observation is that, since P_0 is strictly proper, the controller will be improper. However, it is always possible to adjust an improper controller to make it bi-proper without losing stability, see [60] for details.

Figure 5.11 shows the order of the bounded unit interpolating function which was designed by the proposed method of this chapter with respect to θ (i.e. imaginary part of the zeros of P_0 in $\mathbb{C}_+$). As the discussions in [73] and [51], the degree of the unit interpolating function increases as the P.I.P. comes closer to violation (i.e. as θ decreases).

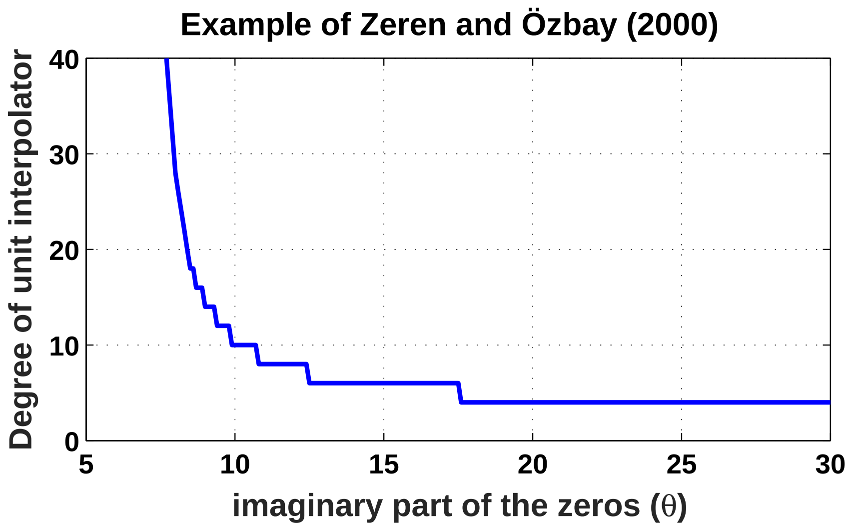


Figure 5.11: Degree of the interpolator with respect to imaginary part of the unstable zeros, see [73] for details

As seen from Figure 5.11, the proposed method of this chapter is capable of finding a stable controller for some relatively small values of θ (i.e. $\theta < 12$), where the original study was not able to give a numerically stable solution strategy. However, the degree of the controller becomes impractically high as $\theta \to 0$. The biggest disadvantage of the proposed method is that it can find a 4^{th} order interpolating function at its best, which yields a possibly 6^{th} order controller where the method of [73] can find a 4^{th} order controller. Some further study can be conducted to find conditions which will focus on the degree of the resulting controller.

5.2.2.4 Example 4

In [26], a method to design stable controllers for sensitivity minimization is proposed by bounded unit interpolation. Let us revisit an example from that paper. We need to find a real, rational transfer function F such that $F(s_i) = w_i/\gamma$ for $i = 1, 2$ where F is also a bounded unit function, $s_{1,2} = 0.3125 \pm 0.8548j$ and $w_{1,2} = 0.79 \mp 0.42j$. [26] have proposed a search algorithm to find F and they showed that for $\gamma > 1.08$ it is always possible to find a third order F satisfying all conditions.

By the proposed method of this chapter, as shown in Figure 5.12, it is possible to find some high degree F for $\gamma > 1.088$. Besides this disadvantage, for $\gamma > 1.124$ the proposed method of this chapter is capable of finding F of degree three or less. This might be an advantage to design low order controllers despite some performance degradation, i.e. for larger γ.

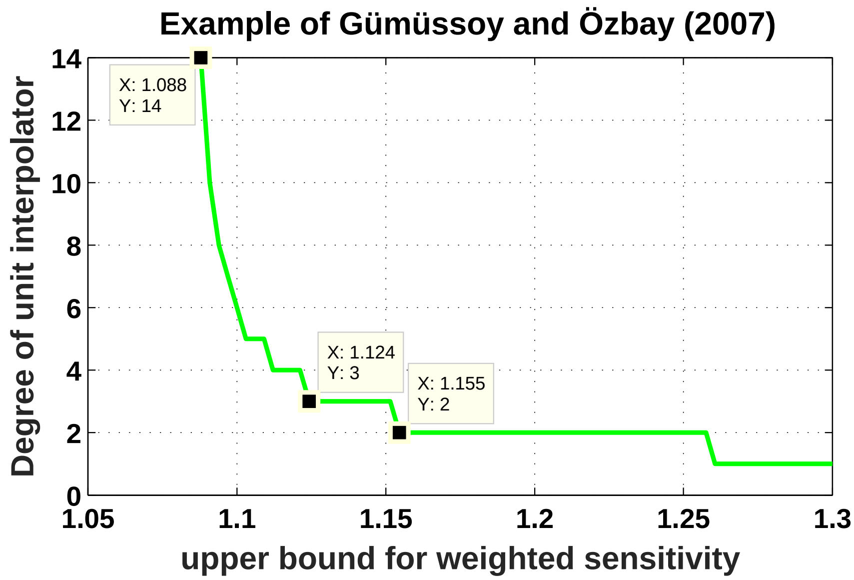


Figure 5.12: Degree of the interpolating function with respect to norm of the weighted sensitivity, see [26] for details

5.3 Discussions

An alternative approach to solve finite dimensional, real, rational, bounded unit interpolation problem is proposed. The proposed approach starts by a predetermined form for the interpolating function given by (5.6) and converts the bounded unit interpolation problem to the classical Nevanlinna-Pick interpolation problem by utilizing the given form. Sufficient conditions are derived using the associated Pick matrix of the transformed problem on top of the well known necessary conditions for bounded unit interpolation in $\mathcal{H}_\infty$ (e.g. P.I.P.).

The performance of the proposed approach is compared to another method from literature over two different examples; a two point and a three point bounded unit interpolation problems. This method from literature addresses exactly the same problem and it was observed that the proposed method is able to find lower degree interpolating functions compared to this approach.

The conservatism caused by the proposed method is discussed on four different strong stabilization problems. Example 1 is a simple modification of the problem studied in [59]. The same example was also studied in [65] which suggests an interpolation method to interpolate only real interpolation data. The method of this chapter performs as good as the method in [65] and additionally it has the ability to handle complex interpolation data. Example 2 is created in order to discuss the performance of the proposed method when the complex interpolation data is involved. It is clear from this example that the proposed method can handle complex interpolation data as well. We can also observe that the proposed method approximates the performance of the infinite dimensional interpolating function better as the allowable degree of the final interpolating function increases, as expected. Example 3 and 4 are considered in order to compare the performance of the proposed method through known examples from strong stabilization literature. The degree of the interpolating function increases rapidly as the problem data comes closer to violate the necessary condition, i.e. parity interlacing property, as expected. This behavior conforms to the discussions in the relevant papers. The proposed method is also able to find a controller in the infeasible region of [73] with the expanse of the increase in the controller degree. The proposed method is also capable of finding lower degree controllers than [26] despite the degradation in $\mathcal{H}_\infty$ performance.

One main disadvantage of the proposed interpolation algorithm is that it can find interpolating functions having order at the integer multiples of the number one less than the number of interpolation points. For some small size problems (having 2-3 interpolation points) like the examples in this chapter, this is not a big problem. However, this can be a major disadvantage when the size of the problem increases. In order to handle this, some future work can be conducted to find conditions under which F will stay bounded and unit when ρ in (5.6) is a unit function in $\mathcal{H}_\infty$ instead of being scalar.

Chapter 6

Stable and Robust Controller Synthesis for Unstable Time Delay Systems via Interpolation and Approximation

Publication Notice: The materials of this section are at least partially covered in the publication [69] which was published by the author and his advisor during the study time of this thesis dissertation.

In this chapter, we study the robust stabilization of single input single output systems, which have finitely many unstable zeros in the open right half plane, by stable controllers.

In this chapter, first we concentrate on a simplified case in which we assume that the time delay system has finitely many unstable poles in the open right half plane. We propose a method to approximate the infinite dimensional and invertible part of the system by a finite dimensional transfer function. After that, using the error associated with this approximation, we introduce a sufficient condition under which it is possible to design a stable controller robustly stabilizing the time delay system. We additionally explain how to design the desired stable and finite dimensional controller when the problem is feasible. In the second part of the study, we deal with a more complicated case in which the time delay system has infinitely many unstable poles in the open right half plane. Similar to first part, by using the approximation error and the approximation itself, we introduce a sufficient condition under which the problem is feasible and outline how to design stable and finite dimensional controllers.

The rest of the chapter is organized as follows: Section 6.1 defines the main problem of this section together with the assumptions. In Section 6.2, we briefly point out the method defined in [59] for the sake of completeness in addition to a basic result about the feasibility of the mNPIP. Section 6.3 is about robust stabilization of time delay systems having finitely many unstable poles in the open right half plane. Section 6.4 considers the case where the plant has infinitely many unstable poles. Section 6.5 compares the effectiveness of the method of [59] and the methods given in Section 6.3 and 6.4 via numerical examples in order to present the conservatism of the proposed methods. Finally, Section 6.6 concludes the chapter by some remarks.

6.1 Problem Statement

Following assumption holds throughout the chapter:

Assumption 1. *Let us assume that the time delay system P is a ratio of two quasi polynomials, i.e. $P(s) = q_n(s)/q_d(s)$ where $q_n(s)$ is retarded type with no direct I/O delay. The denominator quasi polynomial $q_d(s)$ can be retarded or neutral type. Then, in this case, it has been shown that P has finitely many zeros in $\mathbb{C}_+$ and can be written in the form*

$$P = \frac{M_n}{M_d} N_o \tag{6.1}$$

where M_n and M_d are inner and they hold zeros and poles of P in $\mathbb{C}_+$, respectively. Readers are directed to [10] and its references for further details on the analysis *of delay systems of retarded and neutral type. We further assume that $q_n(s)$ and $q_d(s)$ are coprime in the sense that they do not have common roots in $\mathbb{C}_+$. Since the plant has finitely many zeros in $\mathbb{C}_+$, M_n is a finite dimensional transfer function. We also assume that the zeros of M_n are distinct and they are denoted by $z_1, \ldots, z_n$. Note that $N_o = PM_d/M_n$ is infinite dimensional and outer, for the sake of simplicity we assume that the relative degree of the plant is zero, in this case $N_o, N_o^{-1} \in \mathcal{H}_\infty$.*

Assumption 1 does not declare the number of poles of the plant P in $\mathbb{C}_+$. If $q_d(s)$ is retarded type, or neutral type with all the asymptotic chains in the open left half plane, then P has finitely many poles in $\mathbb{C}_+$ (as it will be the case in Section 4), hence M_d is a finite dimensional transfer function and all the infinite dimensionality of the plant is captured by invertible N_o. However, if $q_d(s)$ is neutral type with at least one asymptotic root chain in the open right half plane, then, the plant has infinitely many unstable poles in $\mathbb{C}_+$ (as it will be the case in Section 5), and M_d is infinite dimensional.

Let us further assume that P is the nominal model and the actual plant belongs to a set $\mathfrak{P}(P)$ as given in (2.10).

The following assumption about the uncertainty weight W holds throughout the chapter:

Assumption 2. *Uncertainty weight W is a unit in $\mathcal{H}_\infty$, i.e. W, $W^{-1} \in \mathcal{H}_\infty$; moreover, it satisfies $\|W\|_\infty < 1$.*

Now, we can define the main problem as follows:

Problem 6. *Find a **finite dimensional** controller $C \in \mathcal{H}_\infty$ which internally stabilizes the nominal plant P and satisfies* (2.11) *under Assumptions 1 and 2.*

Problem 6 is called the **R**obust **S**tabilization of Infinite Dimensional Plants by **S**table and **F**inite Dimensional **C**ontrollers (RSSFC).

6.2 Relevant Literature

In [59], a relaxed version of Problem 6 is considered where the controller is allowed to be infinite dimensional. According to [59], this relaxed problem has a solution if it is possible to find a function U in $\mathcal{H}_\infty$ such that

- $U, U^{-1} \in \mathcal{H}_\infty$
- $U(z_i) = 1/M_d(z_i)$ for $i = 1, \ldots, n$ where $M_n(z_i) = 0$
- $\|W_s^{-1}U\|_\infty < 1$

where W_s is also a unit in $\mathcal{H}_\infty$ whose frequency response satisfies the following relation

$$|W_s(j\omega)| \leq \frac{1 - |W(j\omega)|}{|W(j\omega)|}, \quad \forall \omega \in \mathbb{R}. \tag{6.2}$$

If such U exists than the robustly stabilizing stable controller is given as

$$C = \frac{1 - M_d U}{M_n N_o U}. \tag{6.3}$$

As it is discussed in the previous section, N_o and possibly M_d are the infinite dimensional parts of the controller. Additionally, design of U may also lead to infinite dimensional transfer functions as it is described in [27] and [40].

6.3 Solution for the Case of Finitely Many Unstable Poles

When the plant has finitely many unstable poles in $\mathbb{C}_+$, the only infinite dimensional part of the controller is N_o. Following design method is based on finite dimensional approximation of N_o.

Proposition 1. *RSSFC has a solution if there exists a rational transfer function R such that*

- $R, R^{-1} \in \mathcal{H}_\infty$
- $R(z_i) = 1/M_d(z_i)$ *for all* $i = 1, \ldots, n$
- $\|K^{-1}R\|_\infty < 1$

for some $K, K^{-1} \in \mathcal{H}_\infty$ *satisfying*

$$|K(j\omega)| \leq \frac{1 - |W(j\omega)|}{|W(j\omega)| + |E(j\omega)|}, \quad \forall \omega \in \mathbb{R} \tag{6.4}$$

where $E = \hat{N}_o N_o^{-1} - 1$ *is the error introduced by the approximation and* $\hat{N}_o$ *is a finite dimensional approximation of* N_o.

Proof. Let us consider a **finite dimensional** controller of the form

$$C = \frac{1 - M_d R}{M_n \hat{N}_o R} \tag{6.5}$$

where $\hat{N}_o, \hat{N}_o^{-1} \in \mathcal{H}_\infty$ is a finite dimensional approximation of N_o. Note that if it is possible to find a rational transfer function $R \in \mathcal{H}_\infty$ such that $R^{-1} \in \mathcal{H}_\infty$ and R satisfies the following interpolation conditions for $z_i \in \mathbb{C}_+$ and $\forall i$

$$R(z_i) = 1/M_d(z_i)$$

where $M_n(z_i) = 0$ then $C \in \mathcal{H}_\infty$ and **Strong Stability** condition of RSSFC is satisfied.

Next, let us derive the conditions under which the internal stability of the feedback loop is satisfied. To do so, we need to find the conditions which satisfy

$$S, PS, CS \in \mathcal{H}_\infty.$$

We can write S as

$$S = \frac{1}{1 + PC} = \frac{RM_d\hat{N}_o}{N_o\left(1 + \frac{RM_d(\hat{N}_o - N_o)}{N_o}\right)}. \tag{6.6}$$

Note that, if $\|ER\|_\infty < 1$ than $S \in \mathcal{H}_\infty$ by small gain theorem where

$$E = \frac{\hat{N}_o - N_o}{N_o}. \tag{6.7}$$

It is also easy to show that the aforementioned condition is sufficient to show $PS,\ CS \in \mathcal{H}_\infty$, hence **Internal Stability** for RSSFC is satisfied.

In order to derive a condition for robust stability, let us first write T as

$$T = \frac{PC}{1+PC} = \frac{1-RM_d}{1+RE}. \tag{6.8}$$

For robust stability due to multiplicative uncertainty, we need to satisfy (2.11). Since $\|W\|_\infty < 1$ then it is sufficient to simplify the condition as

$$|R(j\omega)| < \frac{1-|W(j\omega)|}{|W(j\omega)|+|E(j\omega)|} \tag{6.9}$$

for all ω. Let us assume that there exists an outer function K such that

$$|K(j\omega)| \leq \frac{1-|W(j\omega)|}{|W(j\omega)|+|E(j\omega)|}$$

and $K, K^{-1} \in \mathcal{H}_\infty$. With this assumption, we can simplify (6.9) to $\|K^{-1}R\|_\infty < 1$. If this is satisfied than **Robust Stability** condition of RSSFC is also satisfied. It is easy to show that $\|K^{-1}R\|_\infty < 1$ implies $\|ER\|_\infty < 1$. $\square$

6.4 Solution for the Case of Infinitely Many Unstable Poles

When the plant has infinitely many unstable poles, M_d becomes infinite dimensional, in addition to N_o. We need to incorporate a finite dimensional approximation of M_d into the controller in order to design a finite dimensional one. Following proposition quantifies the effect of the error of this approximation on the controller design process when the plant has infinitely many unstable poles in $\mathbb{C}_+$.

Proposition 2. *Consider Problem 6 under Assumptions 1 and 2. Additionally assume that the plant has infinitely many unstable poles, i.e. M_d is infinite dimensional. RSSFC has a solution if there exists a finite dimensional and rational transfer function H such that*

- $H, H^{-1} \in \mathcal{H}_\infty$
- $H(z_i) = 1/\hat{M}_d(z_i)$ *for all* $i = 1, \ldots, n$
- $\|L^{-1}H\|_\infty < 1$

for some $L, L^{-1} \in \mathcal{H}_\infty$ *satisfying*

$$|L(j\omega)| \leq \frac{1 - |W(j\omega)|}{|W(j\omega)\hat{M}_d(j\omega)| + |E(j\omega)|}, \quad \forall \omega \in \mathbb{R} \tag{6.10}$$

where $\hat{N}_o$ *and* $\hat{M}_d$ *are finite dimensional approximations of* N_o *and* M_d*, respectively. Note that, differently from Proposition 1,* $E = \hat{M}_d - M_d \hat{N}_o N_o^{-1}$ *is the error introduced by the finite dimensional approximations of both* M_d *and* N_o*.*

Proof. Proof is omitted since it is very similar to the previous case, provided that the stable controller is taken to be

$$C = \frac{1 - \hat{M}_d H}{M_n \hat{N}_o H}. \tag{6.11}$$

□

Let us compare (6.2), (6.9) and (6.10): (6.2) is the bound on the interpolating unit function when the controller is assumed to be infinite dimensional. Note that (6.9) has an additional term in its denominator which is associated with the error of the finite dimensional approximation of N_o. As the approximation error increases the maximum allowable norm of the interpolating unit decreases, and the problem becomes harder to solve, as expected. In (6.10), we again observe the additional error term as the approximation error which is associated with the finite dimensional approximation of both N_o and M_d. However, additionally the finite dimensional approximation of M_d takes place in the denominator next to the plant's uncertainty bound W. As a result of (6.10), we can say that the deviation of the approximation of M_d from being inner is modelled within Proposition 2 as an extra uncertainty in the plant.

6.5 Examples

In this section, we compare the methods proposed in this study and the method proposed in [59] to present the conservatism caused by the finite dimensional approximation approach. We make use of three different numerical examples. First two examples are systems with time delay having finitely many unstable poles. Such plants are suitable to be analysed by the method defined in Proposition 1. Third one will also be a system with time delay, however, this time the plant has infinitely many unstable poles and is suitable for Proposition 2.

6.5.1 Example 1

Let us consider the plant $P = M_n N_o / M_d$, given as

$$\begin{aligned}
P &= \frac{(e^{-s} + 0.1s - 2)(s+1)(s-z_1)}{(e^{-s} + 0.3s + 0.2)(s-0.6)(s-1.5)} \\
M_n &= \frac{(s-z_1)(s-z_2)}{(s+z_1)(s+z_2)} \\
M_d &= \frac{(s-0.6)(s-1.5)(s^2 - 0.7488s + 4.3109)}{(s+0.6)(s+1.5)(s^2 + 0.7488s + 4.3109)} \qquad (6.12) \\
N_o &= P M_d / M_n \\
W &= K \frac{s+1}{s+10}
\end{aligned}$$

where $K > 0$ and $z_2 \approx 20$ is the only root of the quasi-polynomial $(e^{-s} + 0.1s - 2)$ in $\mathbb{C}_+$. Figure 6.1 illustrates the maximum allowable uncertainty level K for which a solution can be found for Problem 1, for the values of z_1 between 1.5 and 7. Note that, when $z_1 < 1.5$, the plant P does not satisfy PIP, hence it is not possible to stabilize it by a stable controller. As z_1 becomes larger than 1.5, the plant relaxes (i.e. it becomes far from violating PIP) and according to [51], it becomes easier to find a finite dimensional and stable controller to stabilize the plant. This effect is clear in Figure 6.1 as the maximum allowable uncertainty bound (i.e. K) under which RSSFC is feasible gets larger as z_1 gets larger for all methods. Figure 6.1 also shows the effect of the conservatism caused by the finite dimensional approximation of N_o. Matlab built-in function `pade` is used to approximate N_o by finite dimensional functions of 13 and 21 degrees and results in Proposition 1 are used to derive the bounds in Figure 6.1. Throughout this study, all finite dimensional approximations of each N_o is conducted via Pade, however, it is not compulsory to use Pade. Any approximation method can be used to generate $\hat{N}_o$ provided that the resulting transfer function is a unit in $\mathcal{H}_\infty$. To satisfy this requirement, each delay element in N_o is replaced by its Pade approximation and an approximate right half plane pole-zero cancellation is used to have a unit approximation in $\mathcal{H}_\infty$.

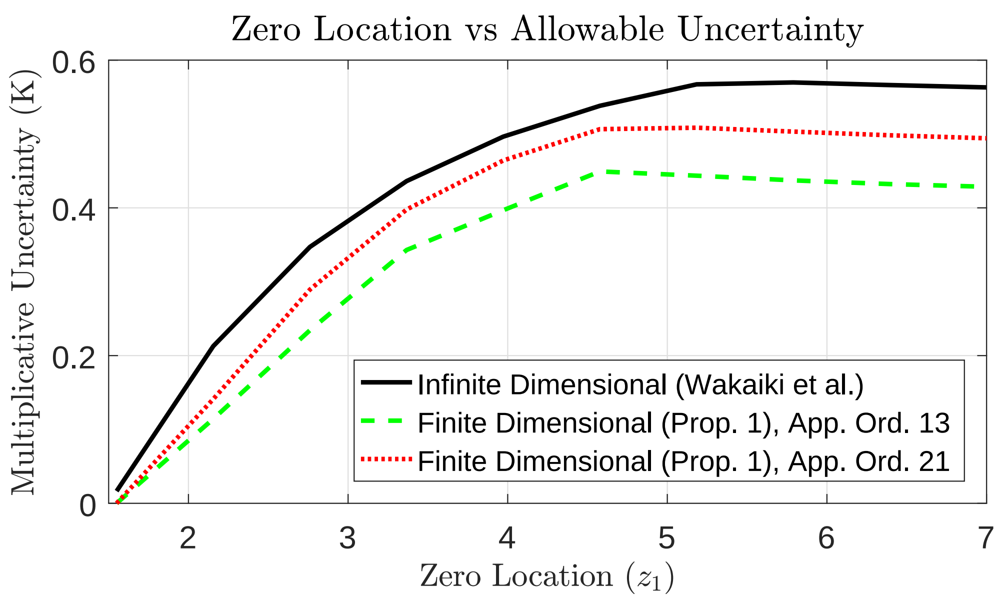


Figure 6.1: Maximum allowable multiplicative uncertainty level with respect to the location of the unstable zero z_1 in Example 1

Figure 6.2 represents an example case where $z_1 = 7$ and the approximation order is 13. In the figure, the pole-zero map of the approximating finite dimensional transfer function ($\hat{N}_o$) is shown.

6.5.2 Example 2

Let us consider the plant $P = M_n N_o / M_d$, given as

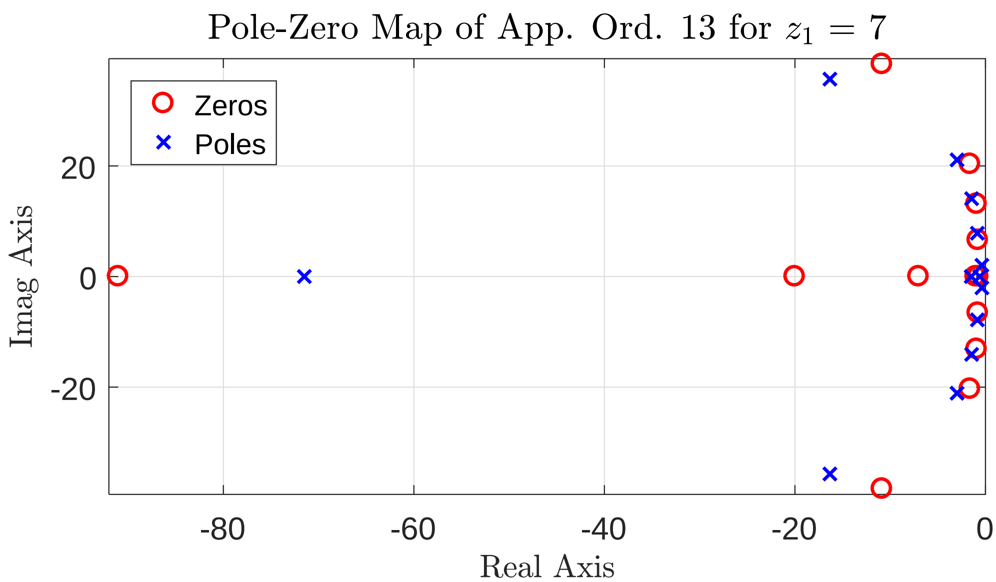


Figure 6.2: Pole-zero map of the finite dimensional approximation of $\hat{N}_o$ given in (6.12). Maximum approximation error $(\max_{\omega\in\mathbb{R}} |N_o(j\omega) - \hat{N}_o(j\omega)|)$ is -14.15 dB.

$$
\begin{aligned}
P &= \frac{(e^{-0.1s} + 0.1s - 1.25)(s^2 - 2s + (1+\omega_1))}{(e^{-s} + 0.3s + 0.2)(s-2)(s+1)} \\
M_n &= \frac{(s-p)(s^2 - 2s + (1+\omega_1))}{(s+p)(s^2 + 2s + (1+\omega_1))} \\
M_d &= \frac{(s-2)(s^2 - 0.7488s + 4.3109)}{(s+2)(s^2 + 0.7488s + 4.3109)} \\
N_o &= PM_d/M_n \\
W &= K\frac{s+1}{s+10}
\end{aligned}
\tag{6.13}
$$

where $K > 0$ and $p \approx 8.0122$ is the only root of the quasi-polynomial $(e^{-0.1s} + 0.1s - 1.25)$ in $\mathbb{C}_+$.

Note that, as $\omega_1 \to 0$, the plant P gets closer to violating PIP since when $\omega_1 = 0$ PIP does not hold because of the pole at 2 staying in between the zeros at 1 and p. Similar to discussions in Example 1, according to [51], the strong stabilization problem becomes harder and requires higher degrees of interpolating functions as the plant comes closer to violate PIP. Because of this phenomena, problem relaxes and becomes feasible for larger uncertainty levels as ω_1 gets larger.

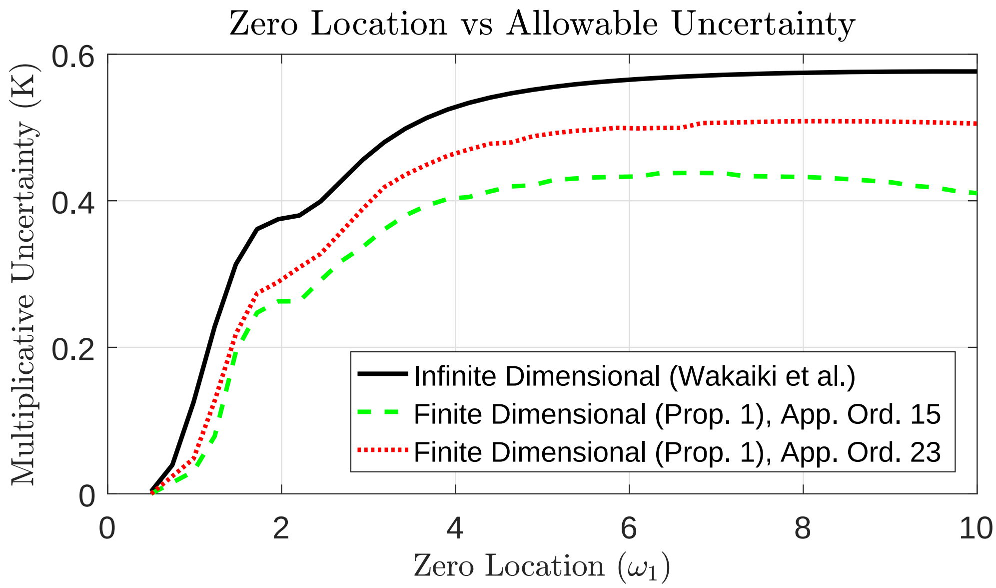


Figure 6.3: Maximum allowable multiplicative uncertainty level with respect to the location of the real part of the unstable zero (ω_1) in Example 2

As an example, the pole-zero map of the 15^{th} order finite dimensional approximation ($\hat{N}_o$) is given in Figure 6.4 for $\omega_1 = 10$.

It is important to note that in Figures 6.1 and 6.3, the multiplicative uncertainty bounds under which RSSFC is feasible (i.e. red and green dotted lines) are the unattainable upper bounds, i.e. it is not possible to achieve these bounds by finite dimensional controllers because it is not possible to solve the bounded unit interpolation problem by finite dimensional interpolating functions at that level. However, as described in detail in [70], it is always possible to get closer to these bounds by increasing the order of the finite dimensional unit interpolating function. These bounds are calculated by utilizing $\hat{N}_o$, the finite dimensional approximation of N_o, and than solving the infinite dimensional mNPIP as described in [27] and [40].

6.5.3 Example 3

Let us consider the infinite dimensional system example from [59] as follows:

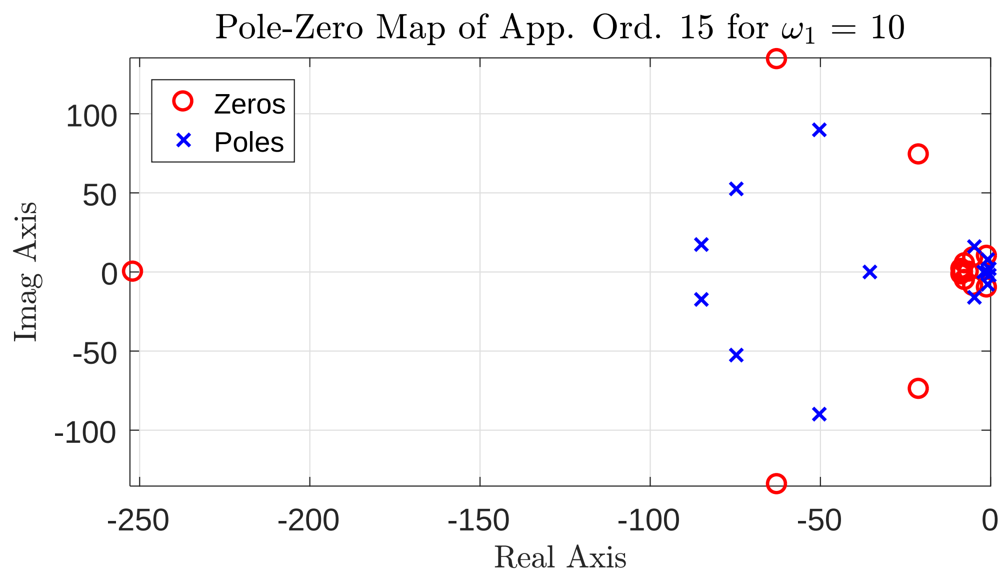


Figure 6.4: Pole-zero map of the finite dimensional approximation of $\hat{N}_o$ given in (6.13). Maximum approximation error $(\max_{\omega\in\mathbb{R}} |N_o(j\omega) - \hat{N}_o(j\omega)|)$ is -21.69 dB.

$$\begin{aligned} P &= \frac{(s-2)(s-4e^{-s}+1)}{(s-10)(s-15)(2e^{-s}+1)} \\ M_n &= \frac{(s-2)(s-p)}{(s+2)(s+p)} \\ M_d &= \frac{(s-10)(s-15)(2e^{-s}+1)}{(s+10)(s+15)(e^{-s}+2)} \\ N_o &= PM_d/M_n \\ W &= K\frac{s+1}{s+10} \end{aligned} \tag{6.14}$$

where $K > 0$ and $p \approx 0.799$ is the only root of the quasi-polynomial $(s-4e^{-s}+1)$ in $\mathbb{C}_+$. It is shown in [59] that for $K < 0.47$ it is possible to find an infinite dimensional and stable controller to robustly stabilize the given plant P in (6.14). They have additionally designed a controller when $K = 0.468$.

In this study, we show that it is possible to design finite dimensional and stable controllers for the same plant in (6.14) when $K < 0.375$ by using Proposition 2. Additionally, as an example, we design a controller when $K < 0.25$. For this design, approximation of N_o is also obtained through its Pade approximation as it was described in prior examples. As it is given in (6.16), we designed a 7^{th}

order $\hat{N}_o$ to approximate N_o in (6.14) and the pole-zero map of $\hat{N}_o$ is depicted in Figure 6.5.

For the finite dimensional approximation of M_d, finitely many unstable zeros are utilized among its infinitely many zeros. Let us say that the zeros of M_d in $\mathbb{C}_+$ are $z_k = 0.6931 + j2\pi k$ and their complex conjugates (i.e. $\bar{z}_k$) for all $k \in \{1, 3, 5, \dots\}$ in addition to 10 and 15. In the light of this parameterization, we can generate N^{th} dimensional finite approximation of M_d for even $N > 2$ as follows

$$\hat{M}_d = \frac{(s-10)(s-15)}{(s+10)(s+15)} \prod_{k=1}^{\frac{N-2}{2}} \frac{(s-z_k)(s-\bar{z}_k)}{(s+z_k)(s+\bar{z}_k)}. \tag{6.15}$$

$$\hat{N}_o(s) = \frac{(s+30.01)(s+2)(s+0.7989)(s^2+0.423s+23.81)(s^2+5.362s+158.9)}{(s+86.47)(s+15)(s+10)(s^2+1.386s+10.35)(s^2+2.144s+101.4)} \tag{6.16}$$

$$L(s) = \frac{0.25787(s+86.95)(s^2+2.475s+110.3)}{(s+0.9844)(s^2+12.09s+77.58)}, \quad H(s) = \frac{0.98787(s+0.0002641)^{10}}{(s+0.2032)^{10}} L(s) \tag{6.17}$$

We used an approximation of M_d where $N = 26$ in (6.15) for the numerical example in (6.14). All other elements of the designed controller are given numerically in (6.17). Note that $L(s)$ in (6.17) is generated by Matlab built-in function `fitmagfrd` and the interpolating part of $H(s)$ is calculated by the method that is proposed in [70]. When all the elements are combined to form the controller in (6.11), a 44^{th} order finite dimensional and stable controller is obtained which robustly stabilizes the infinite dimensional plant given in (6.14) for $K < 0.25$.

6.6 Discussions

We considered the robust stabilization of a class of unstable time delay systems by finite dimensional and stable controllers. We divide the problem into two subclasses and derived similar sufficient conditions under which the associated

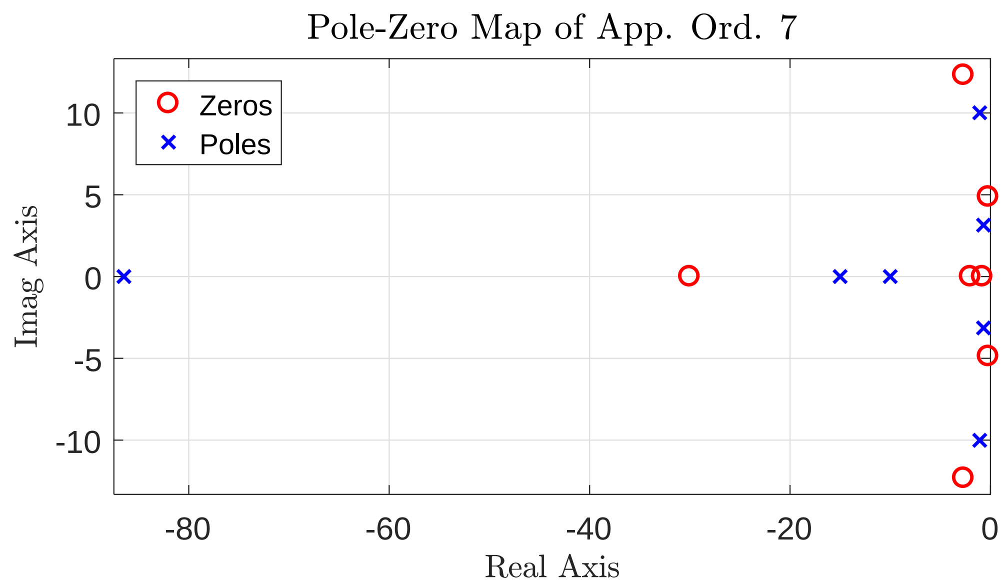


Figure 6.5: Pole-zero map of the finite dimensional approximation of $\hat{N}_o$ given in (6.14). Maximum approximation error $(\max_{\omega\in\mathbb{R}} |N_o(j\omega) - \hat{N}_o(j\omega)|)$ is -3.52 dB.

problems are feasible. For the subclass of systems having finitely many unstable poles in $\mathbb{C}_+$, we propose a method to reduce the robust and strong stabilization problem to a mNPIP through the finite dimensional approximation of the infinite dimensional part of the plant, which is both stable and invertible. With this interpretation and via numerical examples, we show that as the dimension of the approximation increases, and as the error of the approximation decreases, it is possible to solve the problem for larger multiplicative uncertainty levels. We also compare the results of the proposed methods to the results of the method of [59] and concluded that we can design finite dimensional and stable controllers for satisfactory levels of uncertainty.

For the second subclass of systems having infinitely many unstable poles in $\mathbb{C}_+$, we propose another finite dimensional approximation scheme to reduce the original problem to a mNPIP. Since the infinite dimensional part of the plant is not invertible this time, we divide the approximation process into two parts. We approximate the inner part of the infinite dimensional plant by finitely many unstable roots. The approximation of the invertible part is done as it is explained in the first subclass. We use a numerical example from the literature in order to discuss the conservatism of the proposed method.

Chapter 7

Discussion and Conclusions

In this thesis, we mainly focus on strong and robust stabilization of infinite dimensional plants by finite dimensional controllers. It is known that the Nevanlinna-Pick interpolation problem is closely related to robust stabilization, whereas the unit interpolation problem is related to strong stabilization operation. Because of these close relations, we first focus on the finite dimensional solutions of these interpolation problems.

An alternative method is proposed for the optimal solution of the Nevanlinna-Pick interpolation. In order to apply the aforementioned method, interpolation data must contain distinct data points in the right half plane. The most important feature of the proposed method is to avoid any transformations and approximations through suboptimal solutions, it directly computes the optimal interpolating function. We illustrated the use of this optimal solution by some numerical robust stabilization examples.

There are two well known underactuated robots in the literature, Acrobot and Pendubot. In a recent contribution, it is shown that the linearized models of both robots around their upright equilibrium points can be stabilized by stable controllers. Using this property, we show that it is possible to design low order and stable controllers to robustly stabilize these robots around their upright equilibrium points. We utilize parameterization of all suboptimal solutions of the associated Nevanlinna-Pick interpolation problem and design the free parameter of the parameterization in order to satisfy the stability of the controller. We compare our controllers to the ones in the literature and show that we approximately obtain 25% improvements in terms of the closed loop system $\mathcal{H}_\infty$ norm. Moreover, we showed that the proposed controllers are more robust to parameter uncertainties and they increase the robustness to individual parameter perturbations for the locations of unstable poles and zeros.

Noting that the unit interpolation problem is equivalent to strong stabilization operation, it is shown in a recent contribution that the robust stabilization of a plant by a stable controller can be reduced to a bounded unit interpolation problem in $\mathcal{H}_\infty$. We propose an alternative method to solve finite dimensional bounded unit interpolation problem. We use a predetermined structure for the unit interpolating function and this structure transforms the bounded unit interpolation problem to classical Nevanlinna-Pick interpolation problem. We deduce sufficient conditions regarding the associated Pick matrix of the transformed problem. The optimal solution of the associated Nevanlinna-Pick interpolation which is introduced within this thesis in Chapter 4 is used to compute the desired interpolant. The conservatism of the proposed method is discussed on some different numerical examples from the literature. We conclude that the proposed method approximates the performance of the infinite dimensional interpolating function well, especially when the number of interpolation data points is relatively low. As the maximum allowable degree of the interpolation operation increases, the predetermined structure approaches the performance of the infinite dimensional case. We also compare the performance of the proposed method in some strong stabilization benchmark problems and observe that it is possible to have better performance in terms of (a) controller degree at the expense of $\mathcal{H}_\infty$ performance or (b) larger feasible regions at the expense of controller degree.

We divide the class of unstable time delay systems having finitely many right half plane zeros into two subclasses; ones having finitely poles in $\mathbb{C}_+$ and ones having infinitely many poles in $\mathbb{C}_+$. We derive sufficient conditions for two subclasses under which it is possible to design finite dimensional and stable controllers for a member of each subclass. Both conditions rely on the finite dimensional approximation of the infinite dimensional parts of the plant. Using these approximations, we show that the problem can be reduced to a finite dimensional bounded unit interpolation problem in $\mathcal{H}_\infty$. We use the finite dimensional solution method for the bounded unit interpolation problem which is introduced within this thesis in Chapter 5. Through numerical examples, we show that as the approximation error decreases and the degree of the finite dimensional interpolation increases, it is possible to find desired controllers for larger multiplicative uncertainties.

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