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Conjectural invariance with respect to the fusion system of an almost-source algebra

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Abstract. We show that, given an almost-source algebra A of a p-block of a finite group G, then the unit group of A contains a basis stabilized by the left and right multiplicative action of the defect group if and only if, in a sense to be made precise, certain relative multiplicities of local pointed groups are invariant with respect to the fusion system. We also show that, when G is p-solvable, those two equivalent conditions hold for some almost-source algebra of the given p-block. One motive lies in the fact that, by a theorem of Linckelmann, if the two equivalent conditions hold for A, then any stable basis for A is semicharacteristic for the fusion system.

1 Introduction

In p-local representation theory of finite groups, when dealing with an object P of a fusion system, we often consider attributes of P that are not invariants of P. That is, we often consider conditions on P, constructions determined by P, numbers associated with P that are not isomorphism invariants with respect to the fusion system. The two conjectures below imply that certain attributes of P are, despite appearances, invariants after all.

To better explain what we mean, we must first set the scene. Throughout, we let $\mathcal{O}$ be a complete local Noetherian commutative unital ring with an algebraically closed residue field $\mathbb{F}$ of prime characteristic p. The hypothesis on $\mathcal{O}$ implies that either $\mathcal{O} = \mathbb{F}$ or else $\mathcal{O}$ is a complete discrete valuation ring. We understand any $\mathcal{O}$-module, and in particular any algebra over $\mathcal{O}$, to be finitely generated and free over $\mathcal{O}$ or over $\mathbb{F}$. We understand a *basis* for an $\mathcal{O}$-module to be an $\mathcal{O}$-basis or an $\mathbb{F}$-basis. A basis Ω for an algebra over $\mathcal{O}$ is said to be *unital* provided every element of $\mathcal{O}$ is a unit.

Throughout, we let G be a finite group. We deem G-algebras to be over $\mathcal{O}$. We presume familiarity with the theory of G-algebras, Brauer pairs, almost-source algebras, fusion systems, as discussed by Linckelmann [5, Chapters 5, 6, 8]. We shall freely make use of much notation and terminology from there. However, adopting two definitions by Craven [1, Definitions 4.1, 4.11], we understand the term *fusion system* in the broad sense, as distinct from *saturated fusion system*.

For a finite $G \times G$-set Γ, the image of $c \in \Gamma$, under the action of $(f, g) \in G \times G$, can be written as $(f, g)c$ or as fcg^{-1}. When the latter notation prevails, we call Γ a (G, G)-*biset*. Regard G as a (G, G)-biset by left and right multiplication. Extending $\mathcal{O}$-linearly, $\mathcal{O}G$ becomes a permutation $\mathcal{O}(G \times G)$-module.

The scenario of main concern in this paper is as follows. Let b be a block of $\mathcal{O}G$. Let D be a defect group of the block algebra $\mathcal{O}Gb$. Let A be an almost-source D-algebra of $\mathcal{O}Gb$. Let $\mathcal{F}$ be the fusion system on D associated with A. By Green's Indecomposability Criterion, A is a permutation $\mathcal{O}(D \times D)$-submodule of $\mathcal{O}G$. Let Ω be a $D \times D$-stable basis for A. By the Krull–Schmidt Theorem, the (D, D)-biset Ω is well-defined up to isomorphism.

A result of Linckelmann, Theorem 7.2 below, describes how $\mathcal{F}$ is determined by Ω. One clause of the description involves a condition that is not $\mathcal{F}$-isomorphism invariant. Theorem 7.5 asserts that the condition can be omitted when b is the principal block of $\mathcal{O}G$. Most of this paper, though, is concerned with some apparently stronger properties of A and Ω.

The notion of an $\mathcal{F}$-semicharacteristic biset is defined by Gelvin–Reeh [3]. We shall recall the definition in Section 6. The next result is presented in [5, Proposition 8.7.11] for source algebras, but the proof carries over immediately to almost-source algebras.

Theorem 1.1 (Linckelmann). *Let b be a block of $\mathcal{O}G$. Let D be a defect group of the block algebra $\mathcal{O}Gb$. Let A be an almost-source D-algebra of $\mathcal{O}Gb$. Write $\mathcal{F}$ for the fusion system on D associated with A. If A has a unital $D \times D$-stable basis Ω, then the (D, D)-biset Ω is $\mathcal{F}$-semicharacteristic. If, furthermore, A is a source algebra, then Ω is $\mathcal{F}$-characteristic.*

The weaker of the two conjectures below asserts that some almost-source algebra of $\mathcal{O}Gb$ do have a stable basis. The stronger asserts that the source algebras of $\mathcal{O}Gb$ have stable bases. The next theorem gives two necessary and sufficient criteria for the stability hypothesis.

We understand an *idempotent decomposition* in a ring to be a sum of finitely many mutually orthogonal idempotents. When the summands are primitive idempotents, we call the sum a *primitive idempotent decomposition*. Given a pointed group U_μ on any G-algebra A, we define the *multiplicity* of U_μ, denoted $m_A(U_\mu)$, to be the number of elements of μ appearing in a primitive idempotent decomposition of the unity element of the U-fixed subalgebra A^U.

Now suppose A is an interior G-algebra. Let U_μ and V_ν be pointed groups on A. Suppose $U \cong V$, and let $\phi : V \to U$ be a group isomorphism. We call ϕ an *iso-*

fusion $V_\nu \to U_\mu$ provided, in the sense of Puig [6, Section 2], ϕ is an "A-fusion" $V_\nu \to U_\mu$. In Section 4, we shall be reviewing that notion, and we shall also be noting some further characterizations of isofusions.

Theorem 1.2. *Let b be a block of $\mathcal{O}G$ with defect group D and almost-source D-algebra A. Write $\mathcal{F}$ for the fusion system on D associated with A. Then the following three conditions are equivalent.*

(a) *For every $\mathcal{F}$-isomorphism $\phi : Q \to P$, there is a bijective correspondence between the local points γ of P on A and the local points δ of Q on A whereby $\gamma \leftrightarrow \delta$ if and only if ϕ is an isofusion $Q_\delta \to P_\gamma$. Furthermore, when $\gamma \leftrightarrow \delta$, we have $m_A(P_\gamma) = m_A(Q_\delta)$.*

(b) *For every $\mathcal{F}$-isomorphism $\phi : Q \to P$, there is a bijective correspondence between the points α of P on A and the points β of Q on A whereby $\alpha \leftrightarrow \beta$ if and only if ϕ is an isofusion $Q_\beta \to P_\alpha$. Furthermore, when $\alpha \leftrightarrow \beta$, we have $m_A(P_\alpha) = m_A(Q_\beta)$*

(c) *There exists a unital $D \times D$-stable basis for A.*

We shall prove that theorem in Section 8. When the equivalent conditions (a), (b), (c) hold, we call A a *uniform almost-source D-algebra* of $\mathcal{O}Gb$.

Condition (b) can be mnemonically summarized as follows: multiplicities of pointed groups are $\mathcal{F}$-invariant, condition (a) likewise. Let us explain another way of interpreting condition (a) as an assertion of $\mathcal{F}$-invariance. Recall that, for each $\mathcal{F}$-centric subgroup P of D, there is a unique local point λ_P^A of P on A. Given $\mathcal{F}$-centric subgroups P and Q of D and an $\mathcal{F}$-isomorphism $\phi : Q \to P$, then ϕ is an isofusion $Q_{\lambda_Q^A} \to P_{\lambda_P^A}$. Condition (a), confined to the centric objects of $\mathcal{F}$, says that the function $P \mapsto m_A(P_{\lambda_P^A})$ is constant on $\mathcal{F}$-isomorphism classes.

Conjecture 1.3. *Let b be a block of $\mathcal{O}G$. Then the block algebra $\mathcal{O}Gb$ has a uniform almost-source algebra.*

Part of our evidence for the conjecture is the following result, proved in Section 8.

Theorem 1.4. *Suppose G is p-solvable. Given a block b of $\mathcal{O}G$, then the block algebra $\mathcal{O}Gb$ has a uniform almost-source algebra.*

By the uniqueness of source algebras up to conjugation, if some source algebra of $\mathcal{O}Gb$ is uniform, then every source algebra of $\mathcal{O}Gb$ is uniform.

Conjecture 1.5. *For any block b of $\mathcal{O}G$, the source algebras of $\mathcal{O}Gb$ are uniform.*

In Section 8, we shall show that Conjecture 1.5 holds in the following special case. Recall a fusion system $\mathcal{F}$ on a finite p-group D satisfies $\mathcal{F} = N_{\mathcal{F}}(D)$ if and only if every $\mathcal{F}$-isomorphism is a restriction of an $\mathcal{F}$-automorphism of D.

Proposition 1.6. *Let b be a block of $\mathcal{O}G$ with defect group D and source D-algebra A. Write $\mathcal{F}$ for the fusion system on D associated with A. If $\mathcal{F} = N_{\mathcal{F}}(D)$, then A is uniform. In particular, A is uniform whenever D is abelian or $D \trianglelefteq G$.*

Let us outline the organization of the paper. The next three sections may be of use in applications quite different from our main concerns here. Section 2 supplies a necessary and sufficient criterion for an interior G-algebra to have a $G \times G$-stable basis. In Section 3, we show that, for any G-algebra, distinct points of a p-subgroup cannot share a defect pointed subgroup. In Section 4, we present a new approach to fusions in interior G-algebras. The remaining four sections are focused on proofs of the new results stated in this introduction. Sections 5 and 6 establish the crux of Theorem 1.2 but, for simplicity, in a more abstract setting.

A (D, D)-biset Ω is said to be $\mathcal{F}$*-divisible* provided $D \times 1$ and $1 \times D$ act freely on Ω and, given any isomorphism ϕ between subgroups of D, then ϕ is an $\mathcal{F}$-isomorphism if and only if there exists $w \in \Omega$ satisfying $\phi(y)wy^{-1} = w$ for all y in the domain of ϕ. In Section 7, we shall prove the following theorem.

Theorem 1.7. *Let b, D, A, $\mathcal{F}$ be as in Theorem 1.1. Let Ω be a $D \times D$-stable basis for A. If $\mathcal{F}$ is p-constrained or G is p-solvable or b is the principal block of $\mathcal{O}G$, then Ω is $\mathcal{F}$-divisible.*

In the final section, we shall complete the proofs of Theorems 1.2, 1.4 and Proposition 1.6.

2 Unital stable bases

Our hypothesis on $\mathcal{O}$ implies that every algebra over $\mathcal{O}$ has a unital basis. More generally, for any finite p-group D, any permutation D-algebra has a unital D-stable basis; indeed, a further generalization, with a weaker condition on the coefficient ring, can be found in [5, Proposition 5.8.13] by Linckelmann. The purpose of this section is to give a criterion for an interior G-algebra to have a unital $G \times G$-stable basis.

As a reminder of a convention announced in Section 1, let us repeat that any $\mathcal{O}$-module X is deemed to have a finite basis (over $\mathcal{O}$ or over $\mathbb{F}$); we mean a finite subset Ω such that Ω is an $\mathcal{O}$-basis when X is $\mathcal{O}$-free, an $\mathbb{F}$-basis when X is annihilated by $J(\mathcal{O})$. We understand the *rank* of X to be the well-defined natural number $|\Omega|$. We define $\underline{X} = X/J(\mathcal{O})X$ as an $\mathbb{F}$-module. We write $\underline{w}$ and $\underline{\Omega}$ for the images in $\underline{X}$ of an element w of X and a subset Ω of X.

Lemma 2.1. *Given an $\mathcal{O}$-free $\mathcal{O}$-module X and $\Omega \subseteq X$ such that the reduction map $X \to \underline{X}$ restricts to a bijection $\Omega \to \underline{\Omega}$, then Ω is an $\mathcal{O}$-basis for X if and only if $\underline{\Omega}$ is an $\mathbb{F}$-basis for $\underline{X}$.*

Proof. This easy exercise demands, at most, just a hint: when $\underline{\Omega}$ is an $\mathbb{F}$-basis, express each element of Ω as an $\mathcal{O}$-linear combination of elements of an $\mathcal{O}$-basis for X; then consider the determinant of the square matrix formed by the coefficients. □

Lemma 2.2. *Let Ω be a basis for an $\mathcal{O}$-module X. For each $w \in \Omega$, let $v_w \in X$. Taking λ to run over the elements of $\mathcal{O}$, then, for all except at most $|\Omega|$ values of the reduction $\underline{\lambda} \in \mathbb{F}$, the set $\{w + \lambda v_w : w \in \Omega\}$ is a basis for X.*

Proof. By the previous lemma, we may assume that X is annihilated by $J(\mathcal{O})$. Hence, in fact, we may assume that $\mathcal{O} = \mathbb{F}$. Let $w(\lambda) = w + \lambda v_w$. Let $M(\lambda)$ be the matrix, with rows and columns indexed by Ω, such that the (w', w)-entry $M_{w',w}(\lambda)$ of $M(\lambda)$ is given by $w(\lambda) = \sum_{w'} M_{w',w}(\lambda)w'$. The set $\{w(\lambda) : w \in \Omega\}$ is an $\mathbb{F}$-basis for M if and only if $M(\lambda)$ is invertible. The function $\lambda \mapsto \det(M(\lambda))$ is a polynomial function over $\mathbb{F}$ with degree at most Ω and with non-zero constant term. Therefore, $\det(M(\lambda)) = 0$ for at most $|\Omega|$ values of λ. □

Lemma 2.3. *Let A be an algebra over $\mathcal{O}$, let n be the rank of A, and let $a \in A$ and $u \in A^\times$. Taking λ to run over the elements of $\mathcal{O}$, then, for all except at most n values of the reduction $\underline{\lambda} \in \mathbb{F}$, we have $a + \lambda u \in A^\times$.*

Proof. Since units of $\underline{A}$ lift only to units in A, we may assume that $\mathcal{O} = \mathbb{F}$. Let $\rho : A \to \mathrm{End}_{\mathcal{O}}(A)$ be the regular representation. We have $a + \lambda u \in A^\times$ if and only if $\rho(a + \lambda u) \in \mathrm{End}_{\mathcal{O}}(A)^\times$, equivalently, $\rho(-u^{-1}a) - \lambda.\mathrm{id}_A \in \mathrm{End}_{\mathcal{O}}(A)^\times$; in other words, λ is not an eigenvalue of $\rho(-u^{-1}a)$. □

When discussing interior G-algebras, we shall freely use notation and terminology from Linckelmann [5, Chapter 5], but let us reiterate a few conventions. Let A be an interior G-algebra. For $g \in G$ and $a \in A$, we write $ga = \sigma_A(g)a$, similarly for ag, where $\sigma_A : G \to A^\times$ is the structural homomorphism of A. As usual, we regard A as an $\mathcal{O}G$-module via σ_A. That is, g sends a to ${}^g a = gag^{-1}$. We also regard A as an $\mathcal{O}(G \times G)$-module, with $(f, g) \in G \times G$ sending a to fag^{-1}.

Theorem 2.4. *Let A be an interior G-algebra. Then A has a unital $G \times G$-stable basis if and only if A has a $G \times G$-stable basis Ω such that, for all $w \in \Omega$, the stabilizer $N_{G\times G}(w)$ fixes a unit of A.*

Proof. One direction is obvious. Conversely, suppose there exists Ω as specified. For $a \in A$, we write $N(a) = N_{G\times G}(a)$. Letting w run over representatives of the $G \times G$-orbits of Ω, we choose elements $u(w) \in A^\times \cap A^{N(w)}$. Now, letting w run over all the elements of Ω, we let $w \mapsto u(w)$ be the unique function such that $u(fwg^{-1}) = fu(w)g^{-1}$ for all w and all $f, g \in G$. We have

$$N(u(fwg^{-1})) = {}^{(f,g)}N(u(w)).$$

So $N(w) = N(u(w))$ for all $w \in \Omega$.

Let $n = |\Omega|$. Lemma 2.3 implies that, letting λ run over the elements of $\mathcal{O}$, then, for each $w \in \Omega$, the element $w_\lambda - w + \lambda u(w)$ is a unit for all except at most n values of $\overline{\lambda}$. Lemma 2.2 implies that A has basis $\Omega_\lambda = \{w_\lambda : w \in \Omega\}$ for all except at most n values of $\overline{\lambda}$. Therefore, Ω_λ is a unital basis for all except at most n^{n+1} values of λ. □

Corollary 2.5. *Let e be an idempotent of $Z(\mathcal{O}G)$. Let S be a Sylow p-subgroup of G. Then the algebra $\mathcal{O}Ge$ has a unital $S \times S$-stable basis.*

Proof. Let $A = \mathcal{O}Ge$. We can regard A as a direct summand of the permutation $\mathcal{O}(S \times S)$-module $\mathcal{O}G$. So A has an $S \times S$-stable basis Ω; moreover, any such Ω is isomorphic to an $S \times S$-subset of the basis G of $\mathcal{O}G$. Given $w \in \Omega$, then $N_{S\times S}(w) = N_{S\times S}(g)$ for some $g \in G$. On the other hand, viewing A as an algebra, the unit $ge \in A$ also has stabilizer $N_{S\times S}(ge) = N_{S\times S}(g)$. The required conclusion now follows by applying the latest theorem to A as an interior S-algebra. □

3 A uniqueness theorem

The little theorem in this section, and the subsequent remark, will be needed in Section 5.

Let us first establish some notation concerning relative multiplicities. Given a G-algebra A and pointed groups T_τ and U_μ on A with $T \le U$, we define the *relative multiplicity* of T_τ in U_μ, denoted $m_A(T_\tau, U_\mu)$, to be the number of elements of τ that appear in a primitive idempotent decomposition of i in the algebra A^T, where i is any element of μ. Observe that, given $U \le V \le G$, then, by considering a primitive idempotent decomposition in A^V and refining to a primitive idempotent decomposition in A^U, we obtain

$$m_A(U_\mu) = \sum_\nu m_A(U_\mu, V_\nu) m_A(V_\nu),$$

where ν runs over the points of V on A. Similarly, given pointed groups U_μ and W_ω on A and $U \le V \le W$, then

$$m_A(U_\mu, W_\omega) = \sum_\nu m_A(U_\mu, V_\nu) m_A(V_\nu, W_\omega)$$

with ν running as before.

Now let D be a finite p-group, and suppose A a D-algebra. A theorem of Puig, given by Linckelmann in [5, Theorem 5.12.20], implies that, given a point α of D on A and a defect pointed subgroup P_γ of D_α, then any element of α can be expressed in the form $\mathrm{tr}_P^D(i)$, where $i \in \gamma$ and the idempotents ${}^g i$ of A are mutually orthogonal as gP runs over the left cosets of P in D. In [5, Section 5.12], it is explained how the theorem can be seen as a generalization of Green's Indecomposability Criterion.

Theorem 3.1. *Given a D-algebra A, a local pointed group P_γ and points α and β of D on A such that P_γ is a defect pointed subgroup of D_α and D_β, then $\alpha = \beta$.*

Proof. Let $a \in \alpha$ and $b \in \beta$. By Puig's generalization of Green's Indecomposability Theorem, there exist $i, j \in \gamma$ such that $a = \mathrm{tr}_P^D(i)$ and $b = \mathrm{tr}_P^D(j)$, with ${}^g i.i = 0 = {}^g j.j$ for all $g \in D - P$. Let $r \in (A^P)^\times$ such that $i = {}^r j$. Define $u = \mathrm{tr}_P^D(irj)$ and $v = \mathrm{tr}_P^D(jr^{-1}i)$. By direct calculation, $uv = i$ and $vu = j$. So the idempotents i and j of A^P are associate, hence conjugate. □

Proof of the following remark is a straightforward application of Mackey decomposition.

Remark 3.2. Given a D-algebra A, a point α of D on A and a defect pointed subgroup P_γ of D_α, then $m_A(P_\gamma, D_\alpha) = |N_D(P_\gamma) : P|$.

4 Fusions in interior G-algebras

We shall discuss Puig's notion of fusion between pointed groups. Some of the criteria we shall give for the existence of morphisms do not appear to be widely known. Throughout this section, we let A be an interior G-algebra.

Given a group isomorphism θ, we write $\mathrm{cod}(\theta)$ for the codomain, $\mathrm{dom}(\theta)$ for the domain, and we define

$$\Delta(\theta) = \{(\theta(v), v) : v \in \mathrm{dom}(\theta)\}$$

as a subgroup of $\mathrm{cod}(\theta) \times \mathrm{dom}(\theta)$. Note that, given $U, V \leq G$ and a group isomorphism $\phi : V \to U$, then the $\Delta(\phi)$-fixed $\mathcal{O}$-submodule $A^\phi = A^{\Delta(\phi)}$ is the $\mathcal{O}$-submodule of elements $a \in A$ such that $\phi(v)a = av$ for all $v \in V$. Condition (a) in the next lemma was considered by Puig [6, 2.5, 2.12]. He used it in his definition of an "A-fusion".

Lemma 4.1. *Let U_μ and V_ν be pointed groups on A. Let $\phi : V \to U$ be a group isomorphism. Choose $i \in \mu$ and $j \in \nu$. Then the following two conditions are equivalent:*

(a) *there exists $r \in A^\times$ such that $\phi(v)i = {}^r(vj)$;*

(b) *there exist $s \in iA^\phi j$ and $s' \in jA^{\phi^{-1}}i$ such that $i = ss'$ and $j = s's$.*

Moreover, the equivalent conditions (a) *and* (b) *are independent of the choices of i and j.*

Proof. Assuming (a) and putting $s = irj$ and $s' = jr^{-1}i$, we deduce (b). Conversely, assume (b). Since associate idempotents of a semiperfect ring are conjugate, there exists $q \in A^\times$ such that $i = qjq^{-1}$. Putting $r = s + (1-i)q(1-j)$ and $r' = s' + (1-j)q^{-1}(1-i)$, then $rr' = 1 = r'r$. So $r \in A^\times$. A straightforward manipulation yields the equality in (a). Confirmation of the rider is routine. □

When conditions (a) and (b) hold, we call ϕ an *isofusion* $V_\nu \to U_\mu$ and we call (s, s') a *ϕ-witness* $j \mapsto i$. Witnesses can be combined in the following way. Let U_μ, V_ν, W_ω be pointed groups on A. Let $i \in \mu$, $j \in \nu$, $k \in \omega$. Let $(s, s') : j \mapsto i$, $(t, t') : k \mapsto j$ be witnesses for group isomorphisms $\phi : V \to U$, $\psi : W \to V$, respectively. Then $(st, t's') : k \mapsto i$ is a witness for $\phi\psi$. Also, (s', s) is a witness for ϕ^{-1}.

The next two lemmas are implicit in [6, 2.5, 2.6, 2.7]. They can be proved easily and routinely using condition (a) in Lemma 4.1. Alternatively, they can be proved very quickly using the above comments about witnesses.

Lemma 4.2 (Puig). *Let $U, V \leq G$, and let $\phi : V \to U$ be a group isomorphism.*

(1) *For each point ν of V on A, there is at most one point μ of U on A such that ϕ is an isofusion $V_\nu \to U_\mu$.*

(2) *For each point μ of U on A, there is at most one point ν of V on A such that ϕ is an isofusion $V_\nu \to U_\mu$.*

(3) *Given points μ of U and ν of V on A such that ϕ is an isofusion $V_\nu \to U_\mu$, then ϕ^{-1} is an isofusion $U_\mu \to V_\nu$.*

Lemma 4.3 (Puig). *There is a groupoid whose objects are the pointed groups on A and whose isomorphisms are the isofusions, the composition being the usual composition of group isomorphisms. In other words, given a pointed group U_μ on A, then the identity automorphism id_U is an isofusion $U_\mu \to U_\mu$; moreover, the isofusions between the pointed groups on A are closed under composition and inversion.*

We shall be making frequent use of the latest lemma, without further mention.

Consideration of arbitrary points of a p-group, not just the local points, will be crucial in the proof of Theorem 5.2 below. That should not be surprising since all the points of V on A appear in the formula for $m_A(U_\mu, W_\omega)$ in Section 3. We mention, without proof, that when A is the principal 2-block algebra of the symmetric group S_5, there exist local pointed groups $U_\mu \leq W_\omega$ and $U < V < W$ such that all the points ν of V on A satisfying $U_\mu < V_\nu < W_\omega$ are non-local.

We shall be needing a lemma describing how isofusions restrict to pointed subgroups.

Lemma 4.4. *Let U_μ and V_ν be pointed groups on A. Let $\phi : V_\nu \to U_\mu$ be an isofusion. Then the pointed subgroups S_σ of U_μ and the pointed subgroups T_τ of V_ν are in a bijective correspondence such that $S_\sigma \leftrightarrow T_\tau$ if and only if ϕ restricts to an isofusion $T_\tau \to S_\sigma$. Furthermore, when $S_\sigma \leftrightarrow T_\tau$, we have*

$$m_A(S_\sigma, U_\mu) = m_A(T_\tau, V_\nu),$$

and ϕ restricts to an isomorphism $N_V(T_\tau) \to N_U(S_\sigma)$.

Proof. Fix a pointed subgroup $T_\tau \leq V_\nu$. Define $S = \phi(T)$. We shall show that there exists a point σ of S on A such that $m_A(S_\sigma, U_\mu) \geq m_A(T_\tau, V_\nu)$ and ϕ restricts to an isofusion $T_\tau \to S_\sigma$ and to a monomorphism $N_V(T_\tau) \to N_U(S_\sigma)$. That will suffice because the inequality will imply that $S_\sigma \leq U_\mu$, whereupon replacement of ϕ by ϕ^{-1} and an appeal to Lemma 4.2 will yield the non-strictness of the inequality, the bijectivity of the injection $T_\tau \mapsto S_\sigma$ and the bijectivity of the monomorphism $N_V(T_\tau) \to N_U(S_\sigma)$.

Let $i \in \mu$ and $j \in \nu$. Let (c, d) be a ϕ-witness $j \mapsto i$. Choose $k \in \tau$ such that $k \leq j$. It is easily checked that ckd is a primitive idempotent of A^S and (ck, kd) is a ϕ-witness $k \mapsto ckd$. Letting σ be the point of S on A owning ckd (the point to which ckd belongs), then $S_\sigma \leq U_\mu$. By the rider of Lemma 4.1, σ is independent of the choice of k. So, given any set $\{k_1, \dots, k_m\}$ of mutually orthogonal elements of $\tau \cap jA^T j$, then $\{ck_1 d, \dots, ck_m d\}$ is a set of mutually orthogonal elements of $\sigma \cap iA^S i$. Taking m to be as large as possible, we deduce the required inequality of multiplicities.

It remains only to show that, given $g \in N_V(T_\tau)$, then $\phi(g) \in N_U(S_\sigma)$. We have ${}^{\phi(g)}(ckd) = c.{}^g k.d$ and ${}^g k \in \tau$, so, arguing as before, ${}^{\phi(g)}(ckd) \in \sigma$, as required. □

The rest of this section is concerned with a characterization of isofusions that pertains only to local points. We shall be making use of Brauer maps, as defined in [5, Definitions 5.4.2, 5.4.10] for $\mathcal{O}G$-modules and, in particular, for G-algebras. For a p-subgroup $P \leq G$ and $\mathcal{O}G$-module M, we write the P-relative Brauer map on M as $\mathrm{br}_P : M^P \to M(P)$. When an element of the Brauer quotient $M(P)$ is written in the form $\overline{x}$, it is to be understood that $\overline{x} = \mathrm{br}_P(x)$ and $x \in M^P$. We shall sometimes employ the overbar just as a reminder that $\overline{x}$ is an element of a Brauer quotient, and we shall not always need to consider the choice of lift x.

Let P and Q be p-subgroups of G, and let ϕ be a group isomorphism $Q \to P$. Viewing A as an $\mathcal{O}(G \times G)$-module, we write the $\Delta(\phi)$-relative Brauer map on A as $\mathrm{br}_\phi : A^\phi \to A(\phi)$. Let R be another p-subgroup of G and let $\psi : R \to Q$ be an isomorphism. The multiplication operation $A \times A \to A$ restricts to an $\mathcal{O}$-bilinear map $A^\phi \times A^\psi \to A^{\phi\psi}$ which induces an $\mathbb{F}$-bilinear map $A(\phi) \times A(\psi) \to A(\phi\psi)$, written $(\overline{u}, \overline{v}) \mapsto \overline{u} * \overline{v}$, where we define $\overline{u} * \overline{v} = \overline{uv}$. To see that $*$ is well-defined, observe that, by an application of the Frobenius relations, $uv \in \ker(\mathrm{br}_{\phi\psi})$ whenever $u \in \ker(\mathrm{br}_\phi)$ or $v \in \ker(\mathrm{br}_\psi)$. We call $*$ a *localized multiplication*. Some similar bilinear maps appear in [7, 3.2] by Puig–Zhou.

The localized multiplications, taken together, inherit an associativity property: given yet another p-subgroup S of G and a group isomorphism $\chi : S \to R$, then the expression $\overline{u} * \overline{v} * \overline{w}$ is unambiguous for any $\overline{u} \in A(\phi), \overline{v} \in A(\psi), \overline{w} \in A(\chi)$; in fact, $\overline{u} * \overline{v} * \overline{w} = \overline{uvw}$. Also note that, putting $\phi = \mathrm{id}_P$, the localized multiplication $A(P) \times A(P) \to A(P)$ is the usual multiplication on the Brauer quotient $A(P)$.

Lemma 4.5. *Let P_γ and Q_δ be local pointed groups on A. Let $i \in \gamma$ and $j \in \delta$. Let $\phi : Q \to P$ be a group isomorphism. Then ϕ is an isofusion $Q_\delta \to P_\gamma$ if and only if there exist $\overline{s} \in \overline{i} * A(\phi) * \overline{j}$ and $\overline{t} \in \overline{j} * A(\phi^{-1}) * \overline{i}$ such that $\overline{i} = \overline{s} * \overline{t}$ and $\overline{j} = \overline{t} * \overline{s}$.*

Proof. One direction is clear. Conversely, suppose such $\overline{s}$ and $\overline{t}$ exist. Let $b = isj$ and $y = jti$, which belong to $iA^\phi j$ and $jA^{\phi^{-1}}i$, respectively. Then $\overline{by} = \overline{i}$, and hence

$$i - by \in iA^P i \cap \ker(\mathrm{br}_P) \subseteq J(iA^P i).$$

But $iA^P i$ is a local ring, so by has an inverse a in $iA^P i$. Putting $x = ab$, then $x \in iA^\phi j$ and $i = xy$. We have $(yx)^2 = yix = yx$, which belongs to the local ring $jA^Q j$. So $j = yx$. We have shown that (x, y) is a ϕ-witness $j \mapsto i$. □

5 Uniformity

For any interior G-algebra A, we shall introduce a category $\mathscr{F}^{\text{uni}}(A)$ whose objects are the p-subgroups of G and whose morphisms are some group monomorphisms. In subsequent sections, strong further hypotheses will be imposed. Our motive for the present generality is not any anticipated breadth of application. The removal of irrelevant conditions is simply for the sake of clarity.

Theorem 5.1. *Let A be an interior G-algebra. Let P and Q be p-subgroups of G, and let $\phi : Q \to P$ be a group isomorphism. Then the following three conditions are equivalent.*

(a) *The localized multiplications*

$$* : A(\phi) \times A(\phi^{-1}) \to A(P) \quad \textit{and} \quad * : A(\phi^{-1}) \times A(\phi) \to A(Q)$$

are surjective.

(b) *There exist $\overline{u} \in A(\phi)$ and $\overline{v} \in A(\phi^{-1})$ such that*

$$1_{A(P)} = \overline{u} * \overline{v} \quad \textit{and} \quad 1_{A(Q)} = \overline{v} * \overline{u}.$$

(c) *There is a bijective correspondence between the local points γ of P on A and the local points δ of Q on A whereby $\gamma \leftrightarrow \delta$ provided ϕ is an isofusion $Q_\delta \to P_\gamma$. Furthermore, $m_A(P_\gamma) = m_A(Q_\delta)$ when $\gamma \leftrightarrow \delta$.*

Proof. We first deal with a degenerate case. Suppose $A(P) = 0$ or $A(Q) = 0$. Given $u \in A^\phi$, we then have $\mathrm{br}_\phi(u) = \mathrm{br}_P(1) * \mathrm{br}_\phi(u) * \mathrm{br}_Q(1)$. So $A(\phi) = 0$ and $A(\phi^{-1}) = 0$. It is now easy to deduce that, if at least one of (a), (b), (c) holds, then both $A(P)$ and $A(Q)$ vanish; hence all three of (a), (b), (c) hold. So we may suppose that $A(P)$ and $A(Q)$ are non-zero.

Assuming (a), we shall deduce (b). By the surjectivity of one of the local multiplication operations, $1_{A(P)} = \overline{u} * \overline{v}$ for some $\overline{u} \in A(\phi)$, $\overline{v} \in A(\phi^{-1})$. We shall show that $1_{A(Q)} = \overline{v} * \overline{u}$ for any such $\overline{u}$ and $\overline{v}$. Given $\overline{a} \in A(P)$, then $\overline{a} = \overline{a} * \overline{u} * \overline{v}$ and $\overline{a} * \overline{u} \in A(\phi)$. So the $\mathbb{F}$-linear map $- * \overline{v} : A(\phi) \to A(P)$ is surjective. Similarly, $\overline{u} * - : A(Q) \to A(\phi)$ is surjective. Therefore,

$$\dim_{\mathbb{F}}(A(P)) \le \dim_{\mathbb{F}}(A(\phi)) \le \dim_{\mathbb{F}}(A(Q)).$$

Replacing ϕ with ϕ^{-1} yields two further inequalities; hence

$$\dim_{\mathbb{F}}(A(P)) = \dim_{\mathbb{F}}(A(\phi)) = \dim_{\mathbb{F}}(A(Q)) = \dim_{\mathbb{F}}(A(\phi^{-1})).$$

Since $1_{A(P)} = \overline{u} * \overline{v} * \overline{u} * \overline{v}$, the composite $\mathbb{F}$-map

$$(\overline{u} * -) \circ (\overline{v} * -) \circ (\overline{u} * -) \circ (\overline{v} * -) : A(\phi) \leftarrow A(Q) \leftarrow A(\phi) \leftarrow A(Q) \leftarrow A(\phi)$$

is the identity on $A(\phi)$. By considering dimensions, the composite

$$\overline{v} * \overline{u} * - = (\overline{v} * -) \circ (\overline{u} * -) : A(Q) \leftarrow A(\phi) \leftarrow A(Q)$$

must be an $\mathbb{F}$-isomorphism. But $\overline{v} * \overline{u}$ is easily seen to be an idempotent of $A(Q)$. Therefore, $1_{A(Q)} = \overline{v} * \overline{u}$. We have deduced (b).

Now assuming (b), we shall deduce (c). Consider a primitive idempotent decomposition $1 = \sum_{j \in J} j$ in A^Q. For each $j \in J$, write $\overline{j} = \mathrm{br}_Q(j)$, and let $J_0 = \{j \in J : \overline{j} \neq 0\}$. For each $j \in J_0$, the element $\overline{u} * \overline{j} * \overline{v} \in A(P)$ is a primitive idempotent, which we lift to a primitive idempotent $i_j \in A^P$. We have

$$1 = \sum_{j \in J_0} \overline{i_j} \quad \text{and} \quad 1 = \sum_{j \in J_0} \overline{j}$$

as primitive idempotent decompositions in $A(P)$ and $A(Q)$, respectively.

We claim that, for every local point δ of Q on A, there exists a local point γ of P on A such that $\phi : Q_\delta \to P_\gamma$. Plainly, δ intersects with J_0. Choose $j \in \delta \cap J_0$, write $i = i_j$, and let γ be the local point of P owning i. Defining $s = iuj$ and $t = jvi$, then $\overline{s}$ and $\overline{t}$ satisfy the hypothesis of Lemma 4.5. The claim is now established.

Replacing ϕ with ϕ^{-1} and applying Lemma 4.2, we deduce that the condition $\phi : Q_\delta \to P_\gamma$ characterizes a bijective correspondence between the local points γ of P on A and the local points δ of Q on A. Suppose $\gamma \leftrightarrow \delta$. We complete the deduction of (c) by observing that

$$m_A(P_\gamma) = |\{j \in J_0 : \overline{i_j} \in \mathrm{br}_P(\gamma)\}| = |\{j \in J_0 : \overline{j} \in \mathrm{br}_Q(\delta)\}| = m_A(Q_\delta).$$

Assuming (c), we shall deduce (b). Let $1 = \sum_{i \in I} i$ and $j = \sum_{j \in J} j$ be primitive idempotent decompositions in A^P and A^Q, respectively. Define J_0 as before and I_0 similarly. By the assumption, there is a bijection $J_0 \ni j \mapsto i_j \in I_0$ such that, for each $j \in J_0$, there exists a ϕ-witness $(u_j, v_j) : j \mapsto i_j$. Let $u = \sum_{j \in J_0} u_j$ and $v = \sum_{j \in J_0} v_j$. Then $uv = \sum_{j \in J_0} i_j$ and $vu = \sum_{j \in J_0} j$. So

$$\overline{u} * \overline{v} = \sum_{j \in J_0} \overline{i_j} = \sum_{i \in I_0} \overline{i} = 1_{A(P)}$$

and $\overline{v} * \overline{u} = 1_{A(Q)}$. We have deduced (b).

Finally, assuming (b), then any $\overline{t} \in A(P)$ satisfies $\overline{t} = \overline{t} * 1_{A(P)} = \overline{tu} * \overline{v}$, so the localized multiplication $A(\phi) \times A(\phi^{-1}) \to A(P)$ is surjective, likewise the localized multiplication $A(\phi^{-1}) \times A(\phi) \to A(Q)$. We have deduced (a). $\square$

When the three equivalent conditions in Theorem 5.1 hold, we call ϕ *locally A-uniform*.

Theorem 5.2. *Let A and $\phi : Q \to P$ be as in the previous theorem. Then the following three conditions are equivalent.*

(a) *Every isomorphism restricted from ϕ is locally A-uniform.*

(b) *There is a bijective correspondence between the points α of P on A and the points β of Q on A whereby $\alpha \leftrightarrow \beta$ provided ϕ is an isofusion $Q_\beta \to P_\alpha$. Furthermore, $m_A(P_\alpha) = m_A(Q_\beta)$ when $\alpha \leftrightarrow \beta$.*

(c) *We have $A^\times \cap A^\phi \neq \emptyset$.*

Proof. We shall show that each condition implies the next and the last implies the first.

Assume (a). Let β be a point of Q on A. Consider a local pointed subgroup $T_\tau \leq Q_\beta$. Let $S = \phi(T)$. By the assumption, there exists a unique local point σ of S on A such that ϕ restricts to an isofusion $T_\tau \to S_\sigma$. Let $\mathcal{V}$ be the set of points ν of Q on A such that $T_\tau \leq Q_\nu$. Let $\mathcal{U}$ be the set of points μ of P on A such that $S_\sigma \leq P_\mu$. We claim that there is a bijection $\mathcal{U} \leftrightarrow \mathcal{V}$ such that $\mu \leftrightarrow \nu$ if and only if ϕ is an isofusion $Q_\nu \to P_\mu$; moreover, when those equivalent conditions hold, $m_A(P_\mu) = m_A(Q_\nu)$. Since $\beta \in \mathcal{V}$, the claim implies the existence of a point α of P on A such that ϕ is an isofusion $Q_\beta \to P_\alpha$ and $m_A(P_\alpha) = m_A(Q_\beta)$. Again replacing ϕ with ϕ^{-1} and applying Lemma 4.2, part (d) will follow. (Note that, in the application of the claim, T_τ does not appear in the conclusion. The role of T_τ is to supply an opportunity for an inductive proof of the claim.)

To demonstrate the claim, we argue by induction on $|Q : T|$. By the assumption, $m_A(S_\sigma) = m_A(T_\tau)$. That is,

$$\sum_{\mu \in \mathcal{U}} m_A(S_\sigma, P_\mu) m_A(P_\mu) = \sum_{\nu \in \mathcal{V}} m_A(T_\tau, Q_\nu) m_A(Q_\nu).$$

Let $\mathcal{V}'$ be the set of $\nu \in \mathcal{V}$ such that T_τ is not a defect pointed subgroup of V_ν. Define $\mathcal{U}' \subseteq \mathcal{U}$ similarly for S_σ. Given $\nu \in \mathcal{V}'$, then $Q_{\nu'}$ has a defect pointed subgroup $T'_{\tau'}$ strictly containing T_τ. Let $S' = \phi(T')$. By the assumption, there exists a unique local point σ' of S' on A such that ϕ restricts to an isofusion $T'_{\tau'} \to S'_{\sigma'}$. By the inductive hypothesis, there exists a point μ' of P on A such that ϕ is an isofusion $Q_{\nu'} \to P_{\mu'}$ and $m_A(P_{\mu'}) = m_A(Q_{\nu'})$. By Lemma 4.4, $S_\sigma < S'_{\sigma'} \leq P_{\mu'}$. So $\mu' \in \mathcal{U}'$. The same lemma also implies that $m_A(S_\sigma, P_{\mu'}) = m_A(T_\tau, Q_{\nu'})$. Replacing ϕ with ϕ^{-1}, there is a bijective correspondence $\mathcal{U}' \ni \mu' \leftrightarrow \nu' \in \mathcal{V}'$ characterized by the condition that ϕ is an isofusion $Q_{\nu'} \to P_{\mu'}$. Therefore,

$$\sum_{\mu' \in \mathcal{U}'} m_A(S_\sigma, P_{\mu'}) m_A(P_{\mu'}) = \sum_{\nu' \in \mathcal{V}'} m_A(T_\tau, Q_{\nu'}) m_A(Q_{\nu'}).$$

Comparing with the similar equality established earlier, we deduce that $\mathcal{U} = \mathcal{U}'$ if and only if $\mathcal{V} = \mathcal{V}'$. In that case, the claim is now clear. So we may assume that $\mathcal{U}' \subset \mathcal{U}$ and $\mathcal{V}' \subset \mathcal{V}$. By Theorem 3.1, the differences are singleton, say, $\mathcal{U} - \mathcal{U}' = \{\mu\}$ and $\mathcal{V} - \mathcal{V}' = \{\nu\}$. So

$$m_A(S_\sigma, P_\mu) m_A(P_\mu) = m_A(T_\tau, Q_\nu) m_A(Q_\nu).$$

By Lemma 4.4, $|N_P(S_\sigma)| = |N_Q(T_\tau)|$. Since S_σ and T_τ are defect pointed subgroups of P_μ and Q_ν, respectively, Remark 3.2 yields

$$m_A(S_\sigma, P_\mu) = |N_P(S_\sigma) : S| = |N_Q(T_\tau) : T| = m_A(T_\tau, Q_\nu).$$

Therefore, $m_A(P_\mu) = m_A(Q_\nu)$. The claim is demonstrated. The deduction of (b) is complete.

Assume (b). Again, let $1 = \sum_{j \in J} j$ be a primitive idempotent decomposition in A^Q. By the assumption, there is a primitive idempotent decomposition $1 = \sum_{j \in J} i_j$ in A^P such that, writing α and β for the points P and Q owning i_j and j, respectively, then ϕ is an isofusion $Q_\beta \to P_\alpha$. For each $j \in J$, let (s_j, t_j) be a ϕ-witness $j \mapsto i_j$. Let $s = \sum_{j \in J} s_j$ and $t = \sum_{j \in J} t_j$. A short calculation yields $st = 1 = ts$. In particular, $s \in A^\times \cap A^\phi$. We have deduced (c).

Finally, assume (c). Let $u \in A^\times \cap A^\phi$. Define $v = u^{-1}$. Then $v \in A^\times \cap A^{\phi^{-1}}$, and the elements $\underline{u} \in A(\phi)$, $\underline{v} \in A(\phi^{-1})$ satisfy $1_{A(P)} = \underline{u} * \underline{v}$ and $1_{A(Q)} = \underline{v} * \underline{u}$. We have deduced (a). □

When the three equivalent conditions in Theorem 5.2 hold, we say that ϕ is *A-uniform*. We form a category $\mathcal{F}^{\mathrm{uni}}(A)$ such that the objects are the p-subgroups of G and the morphisms are the composites of the inclusions and the A-uniform isomorphisms. For any Sylow p-subgroup S of G, we let $\mathcal{F}_S^{\mathrm{uni}}(A)$ denote the full subcategory of $\mathcal{F}^{\mathrm{uni}}(A)$ such that the objects of $\mathcal{F}_S^{\mathrm{uni}}(A)$ are the subgroups of S. Plainly, $\mathcal{F}_S^{\mathrm{uni}}(A)$ is a fusion system on S, and $\mathcal{F}_S^{\mathrm{uni}}(A)$ is well-defined up to isomorphism, in fact, well-defined up to G-conjugation, independently of the choice of S. We call $\mathcal{F}_S^{\mathrm{uni}}(A)$ the *uniform fusion system* of A on S.

6 Bifree bipermutation algebras

Let D be a finite p-group. We shall discuss the uniform fusion system $\mathcal{F}^{\mathrm{uni}}(A)$ for various kinds of interior D-algebra A. We shall be imposing hypotheses as we need them. Again, the generality is in the service of clarity. We shall find, in Section 8, that whenever $\mathcal{F}^{\mathrm{uni}}(A)$ comes to be of concern in our contexts of application, it will coincide with the usual fusion system. For the time being, though, the logic of our discussion requires us to treat $\mathcal{F}^{\mathrm{uni}}(A)$ as a potential innovation.

That having been said, Question 6.2 does raise a possibility that the generality may be of interest in itself.

We write $\mathcal{I}(D)$ to denote the set of group isomorphisms ϕ such that $\mathrm{cod}(\phi)$ and $\mathrm{dom}(\phi)$ are subgroups of D. Observe that the condition $\Delta(\phi) = N$ characterizes a bijective correspondence $\phi \leftrightarrow N$ between the isomorphisms ϕ in $\mathcal{I}(D)$ and those subgroups $N \leq D \times D$ that intersect trivially with $D \times 1$ and $1 \times D$. Let Ω be a (D, D)-biset. We call Ω *bifree* provided $D \times 1$ and $1 \times D$ act freely on Ω. Thus, Ω is bifree if and only if, for all $w \in \Omega$, there exists $\phi \in \mathcal{I}(D)$ such that $N_{D \times D}(w) = \Delta(\phi)$.

We define a *bipermutation D-algebra* to be an interior D-algebra A such that A has a $D \times D$-stable basis. Suppose A is a bipermutation D-algebra, and let Ω be a $D \times D$-stable basis. Generalizing a comment made in Section 1, we note that, by Green's Indecomposability Theorem and the Krull–Schmidt Theorem, Ω is well-defined as a (D, D)-biset up to isomorphism. When the left action of D and the right action of D on Ω are free, we say that A is *bifree*.

Theorem 6.1. *Let A be a bifree bipermutation D-algebra. Then the following two conditions are equivalent.*

(a) *Given $\phi \in \mathcal{I}(D)$ such that $A(\phi) \neq 0$, then ϕ is A-uniform.*

(b) *A has a unital $D \times D$-stable basis.*

Proof. Let Ω be a $D \times D$-stable basis for A. Given $\phi \in \mathcal{I}(D)$, then $A(\phi) \neq 0$ if and only if $\Omega^{\Delta(\phi)} \neq \emptyset$. So (b) implies (a). Conversely, suppose (a) holds. By Theorem 2.4, (b) will follow when we have shown that, given $w \in \Omega$, then $N_{D \times D}(w)$ fixes a unit of A. By the bifreeness, $N_{D \times D}(w) = \Delta(\phi)$ for some $\phi \in \mathcal{I}(D)$. We have $A(\phi) \neq 0$, so, by the assumption, ϕ is A-uniform and $\Delta(\phi)$ fixes a unit of A. We have deduced (b). □

When the two equivalent conditions in the latest theorem hold, we say that the bifree bipermutation D-algebra A is *uniform*. The term *uniform bifree bipermutation G-algebra* is rather long, but a decision on how to improve it might best await resolution of the following question.

Question 6.2. Is every bifree bipermutation D-algebra uniform?

We see no reason to expect the affirmative answer, but we have been unable to find a counter-example. Consider a finite D-set Γ. It is an easy exercise to show that, if D is non-cyclic, then, for some Γ, the bipermutation D-algebra $\mathrm{End}_{\mathcal{O}}(\mathcal{O}\Gamma)$ does not have a unital $D \times D$-stable basis. However, it is also easy to show that, given D and Γ such that $\mathrm{End}_{\mathcal{O}}(\mathcal{O}\Gamma)$ does not have a unital $D \times D$-stable basis, then $\mathrm{End}_{\mathcal{O}}(\mathcal{O}\Gamma)$ is not bifree.

Proposition 6.3. *Let e be an idempotent of $Z(\mathcal{O}G)$. Let S be a Sylow p-subgroup of G. Let $A = \mathcal{O}Ge$ as an interior S-algebra by restriction. Then A is a uniform bifree bipermutation S-algebra and $\mathcal{F}^{\mathrm{uni}}(A)$ coincides with the fusion system $\mathcal{F}_S(G)$ of G on S.*

Proof. Corollary 2.5 says that A is a uniform bifree bipermutation S-algebra. The proof of the corollary shows the required equality of fusion systems. □

In particular, every group fusion system can be realized as the uniform fusion system of a uniform bifree bipermutation algebra. At the time of writing, the authors know nothing about whether some or all exotic fusion systems can be realized in that way.

As in [3, Definition 3.4] by Gelvin–Reeh, a bifree (D, D)-biset Ω is said to be $\mathcal{F}$*-semicharacteristic* provided the following condition holds: given an isomorphism ϕ in $\mathcal{I}(D)$, then ϕ is an $\mathcal{F}$-isomorphism if and only if $\Omega^{\Delta(\phi)} \neq \emptyset$; furthermore, in that case,

$$|\Omega^{\Delta(\mathrm{cod}(\phi))}| = |\Omega^{\Delta(\phi)}| = |\Omega^{\Delta(\mathrm{dom}(\phi))}|.$$

It is easy to see that, given an $\mathcal{F}$-semicharacteristic (D, D)-biset Ω, then the integer $|\Omega|/|D|$ is coprime to p if and only if Ω is $\mathcal{F}$-characteristic as defined by Linckelmann [5, Definition 8.7.9] or Ragnarsson–Stancu [8].

Theorem 6.4. *Let A be a uniform bifree bipermutation D-algebra. Then every $D \times D$-stable basis for A is an $\mathcal{F}^{\mathrm{uni}}(A)$-semicharacteristic (D, D)-biset.*

Proof. Let Ω be any $D \times D$-stable basis for A, and let $\phi : Q \to P$ be an $\mathcal{F}^{\mathrm{uni}}(A)$-isomorphism. By the hypothesis on A, there exists $u \in A^{\times} \cap A^{\phi}$. The bijection $u\Omega \ni uw \leftrightarrow w \in \Omega$ restricts to a bijection $(u\Omega)^{\Delta(\phi)} \leftrightarrow \Omega^{Q}$. So

$$|(u\Omega)^{\Delta(\phi)}| = |\Omega^{Q}|.$$

But $u\Omega \cong \Omega$ as (P, Q)-bisets. So $|\Omega^{\Delta(\phi)}| = |\Omega^{Q}|$, and similarly for $|\Omega^{P}|$. □

Theorem 6.4, together with Proposition 6.3, recovers the main result of Gelvin [2], which asserts that, in the notation of the proposition, any $S \times S$-stable basis of $\mathcal{O}Gb$ is $\mathcal{F}_S(G)$-semicharacteristic.

In Proposition 8.1, we shall confirm that, for uniform almost-source algebras, the uniform fusion system discussed in the present section coincides with the fusion system discussed in Section 1. Incidentally, that will realize Theorem 1.1 as a special case of Theorem 6.4.

7 Divisibility of some stable bases

We shall be discussing the fusion system of an almost-source algebra and Linckelmann's characterization of the fusion system in terms of a stable basis of the almost-source algebra. The characterization mentions a condition that is not an isomorphism invariant, namely, the condition that a given object is fully centralized. We shall show that, in some cases, that condition can be omitted.

For the moment, let D be any finite p-group, let $\mathcal{F}$ be a fusion system on D, and let Ω be a (D, D)-biset. Recall, from Section 1, that Ω is *$\mathcal{F}$-divisible* provided Ω is bifree and, given any isomorphism ϕ in $\mathcal{I}(D)$, then ϕ is an $\mathcal{F}$-isomorphism if and only if $\Delta(\phi)$ fixes an element of Ω.

Now let D be a defect group of a block b of $\mathcal{O}G$. Let A be an almost-source D-algebra of $\mathcal{O}Gb$. Write ℓ for the unity element of A. For each $P \leq D$, we make the usual identification $(\mathcal{O}G)(P) = \mathbb{F}C_G(P)$. Let us recall some facts that can be found, for instance, in [5, Sections 6.4, 8.7] by Linckelmann. We have an inclusion of idempotents $\mathrm{br}_P(\ell) \leq b_P$ for a unique block b_P of $\mathbb{F}C_G(P)$, and we have an inclusion of Brauer pairs $(P, b_P) \leq (D, b_D)$. For $P, Q \leq D$, the $\mathcal{F}$-isomorphisms $Q \to P$ are the conjugation isomorphisms $Q \ni y \mapsto {}^g y$, with $g \in G$ satisfying $(P, b_P) = {}^g(Q, b_Q)$.

The next remark is well known.

Remark 7.1. Let b be a block of $\mathcal{O}G$ with defect group D and almost-source algebra A. Let Ω be a $D \times D$-stable basis for A. Then Ω is isomorphic to its opposite biset and $|\Omega^{\Delta(\phi)}| = |\Omega^{\Delta(\phi^{-1})}|$ for any $\phi \in \mathcal{I}(D)$.

Proof. This follows from Gelvin [2, Proposition 6, Lemma 7]. □

The next result is due to Linckelmann [5, Theorem 8.7.1].

Theorem 7.2 (Linckelmann). *Let b be a block of $\mathcal{O}G$ with defect group D and almost-source algebra A. Let $\mathcal{F}$ be the fusion system on D associated with A. Let Ω be a $D \times D$-stable basis for A. Given an isomorphism ϕ in $\mathcal{I}(D)$, if $\Delta(\phi)$ fixes a point of Ω, then ϕ is an $\mathcal{F}$-isomorphism; furthermore, the converse holds when* $\mathrm{dom}(\phi)$ *or* $\mathrm{cod}(\phi)$ *is fully $\mathcal{F}$-centralized.*

Can we omit the clause on the domain or codomain being fully $\mathcal{F}$-centralized? More precisely, given b, must Ω be divisible for some almost-source algebra A of $\mathcal{O}Gb$? Conjecture 1.3 implies the affirmative. Given b, must Ω be divisible for some source algebra and hence for every almost-source algebra of $\mathcal{O}Gb$? To that, Conjecture 1.5 implies the affirmative.

In the next section, we shall be discussing some cases where Ω has the stronger property of being $\mathcal{F}$-semicharacteristic. In the rest of the present section, we shall be proving the $\mathcal{F}$-divisibility of Ω in some cases where we have been unable to prove the stronger condition.

Following Craven [1, Definition 3.69], we call $\mathcal{F}$ *constrained* provided

$$C_D(O_p(\mathcal{F})) = Z(O_p(\mathcal{F})).$$

By [1, Proposition 4.46], if $\mathcal{F}$ is constrained, then $O_p(\mathcal{F})$ is the minimum among the $\mathcal{F}$-radical $\mathcal{F}$-centric subgroups of D.

Proposition 7.3. *In the notation of Theorem 7.2, if $\mathcal{F}$ is p-constrained, then Ω is $\mathcal{F}$-divisible.*

Proof. We must show that, given $P, Q \leq D$ and an $\mathcal{F}$-isomorphism $\phi : Q \to P$, then $\Delta(\phi)$ fixes an element of Ω. Let $R = O_p(\mathcal{F})$. Then ϕ extends to an $\mathcal{F}$-isomorphism $\psi : QR \to PR$. Since R is $\mathcal{F}$-centric, PR and QR are $\mathcal{F}$-centric and hence fully $\mathcal{F}$-centralized. So $\Omega^{\Delta(\psi)} \neq \emptyset$. In particular, $\Omega^{\Delta(\phi)} \neq \emptyset$. □

By [1, Corollary 5.95], the group fusion system of a p-solvable group is solvable. Part of [1, Lemma 5.90] says that all saturated subsystems of a solvable fusion system are solvable. So, if G is p-solvable, then $\mathcal{F}$ is solvable. A theorem of Aschbacher [1, Theorem 5.91] asserts that every solvable fusion system is constrained. Therefore, Proposition 7.3 has the following corollary.

Corollary 7.4. *In the notation of Theorem 7.2, if G is p-solvable, then Ω is $\mathcal{F}$-divisible.*

Alternatively, the corollary can be deduced from Proposition 7.3 by using [5, Theorem 8.1.8] to reduce to the case where $O_{p'}(G) = 1$, then applying a theorem of Hall–Higman [1, Lemma 5.93].

To present one more case where we can guarantee divisibility, we need some preliminaries. Consider a p-subgroup P of G. Recall the condition $\mathrm{br}_P(\gamma)V \neq 0$ characterizes a bijective correspondence $\gamma \leftrightarrow [V]$ between the local points γ of P on $\mathcal{O}G$ and the isomorphism classes $[V]$ of simple $\mathbb{F}C_G(P)$-modules V. Let σ_P^G denote the local point of P on $\mathcal{O}G$ corresponding to the trivial $\mathbb{F}C_G(P)$-modules. The next result, well-known, can be viewed as a version of Brauer's Third Main Theorem. A proof of it is implicit in [5, Theorem 6.3.14] by Linckelmann. For convenience, we extract the argument.

Theorem 7.5. *Let Q and P be p-subgroups of G. Then $Q \leq P$ if and only if $Q_{\sigma_Q^G} \leq P_{\sigma_P^G}$.*

Proof. We may assume that $\mathcal{O} = \mathbb{F}$ and $Q \leq P$. Let $\eta_G : \mathbb{F}G \to \mathbb{F}$ be the augmentation map. Given an idempotent e of $\mathbb{F}G$, then $\eta_G(e) = 1$ if and only if e acts as the identity on a trivial $\mathbb{F}G$-module. That is equivalent to the condition that $k \leq e$ for some $k \in \sigma_1^G$. So, letting $i \in \sigma_P^G$ and $j \in \sigma_Q^G$, we have

$$\eta_{C_G(P)}(i) = 1 = \eta_{C_G(Q)}(j).$$

For all $x \in (\mathbb{F}G)^P$, we have $\eta_{C_G(P)}(\mathrm{br}_P(x)) = \eta_G(x)$ and similarly with Q in place of P. Hence

$$\eta_{C_G(Q)}(\mathrm{br}_Q(i)) = \eta_G(i) = \eta_{C_G(P)}(\mathrm{br}_P(i)).$$

Therefore,

$$\eta_{C_G(Q)}(\mathrm{br}_Q(ij)) = \eta_{C_G(Q)}(\mathrm{br}_Q(i)).\eta_{C_G(Q)}(\mathrm{br}_Q(j)) = 1.$$

So $ij \notin J((\mathbb{F}G)^Q)$. It follows that $i \geq j'$ for some conjugate j' of j in $(\mathbb{F}G)^Q$. □

Note that, from Theorem 7.5, we immediately recover the version of Brauer's Third Main Theorem that is explicitly stated in [5, Theorem 6.3.14]: if b is the principal block of $\mathcal{O}G$, then b_P is the principal block of $\mathbb{F}C_G(P)$ for each $P \leq D$. In particular, if b is the principal block, then D is a Sylow p-subgroup of G and $\mathcal{F} = \mathcal{F}_D(G)$, the fusion system of G on D.

Theorem 7.6. *In the notation of Theorem 7.2, if b is the principal block of $\mathcal{O}G$, then Ω is $\mathcal{F}$-divisible.*

Proof. Given $P, Q \leq D$ and $g \in G$ such that $P = {}^gQ$, then $P_{\sigma_P^G} = {}^g(Q_{\sigma_Q^G})$. So any $\mathcal{F}$-isomorphism $\phi : Q \to P$ is an isofusion $Q_{\sigma_Q^G} \to P_{\sigma_P^G}$. By Theorem 7.5, we have $P_{\sigma_P^G} \leq D_{\sigma_D^G}$. But $A \cap \sigma_D^G \neq \emptyset$. So we can define $\sigma_P^A = A \cap \sigma_P^G$ as a local point of P on A, likewise σ_Q^A. As ϕ is an isofusion $Q_{\sigma_Q^A} \to P_{\sigma_P^A}$, Lemma 4.5 implies that $A(\phi)$ and $A(\phi^{-1})$ are non-zero. □

The proof of Theorem 1.7 is complete.

8 Uniformity of some almost-source algebras

Again, we let D be a defect group of a block b of $\mathcal{O}G$. We shall prove Theorems 1.2, 1.4 and Proposition 1.6.

First, though, we are now ready to note that Conjecture 1.3 implies another characterization of the fusion system of an almost-source algebra.

Proposition 8.1. *Let $\mathcal{F}$ be the fusion system of a uniform almost-source D-algebra A of $\mathcal{O}Gb$. Then $\mathcal{F} = \mathcal{F}^{\mathrm{uni}}(A)$.*

Proof. We need only show that $\mathcal{F}$ and $\mathcal{F}^{\mathrm{uni}}(A)$ have the same isomorphisms. By Theorem 7.2, every $\mathcal{F}^{\mathrm{uni}}(A)$-isomorphism is an $\mathcal{F}$-isomorphism. For the converse, consider an $\mathcal{F}$-isomorphism ϕ. Since $\mathcal{F}$ is a saturated fusion system, Alperin's Fusion Theorem guarantees that $\phi = \phi_1 \dots \phi_n$, where $\phi_1, \dots, \phi_n$, respectively, are isomorphisms restricted from $\mathcal{F}$-automorphisms $\psi_1, \dots, \psi_n$ of $\mathcal{F}$-centric subgroups of D. Letting $r_1 \in A^\times \cap A^{\psi_1}, \dots, r_n \in A^\times \cap A^{\psi_n}$, then

$$r_1 \dots r_n \in A^\times \cap A^\phi.$$

So ϕ is an $\mathcal{F}^{\mathrm{uni}}(A)$-isomorphism. □

Via the proposition, Theorem 1.1 is now recovered as a special case of Theorem 6.4.

Let us prove Theorem 1.2. Since the $\mathcal{F}$-isomorphisms are closed under restriction to group isomorphisms, (a) and (b) are equivalent, and they can be expressed as follows.

(y) Every $\mathcal{F}$-isomorphism is uniform.

On the other hand, by Theorem 6.1, condition (c) can be reformulated as follows.

(z) Given $\phi \in \mathcal{I}(D)$ such that $A(\phi) \neq 0$, then ϕ is uniform.

Theorem 7.2 immediately shows that (y) implies (z). Conversely, assume condition (z), and let ϕ be an $\mathcal{F}$-isomorphism. Invoking Alperin's Fusion Theorem, write $\phi = \phi_1 \dots \phi_m$, where each ϕ_t is an $\mathcal{F}$-isomorphism restricted from an $\mathcal{F}$-automorphism ψ_t of an $\mathcal{F}$-centric subgroup $P_t \leq D$. Each P_t is fully $\mathcal{F}$-centralized. So, by Theorem 7.2 again, $A(\psi_t) \neq 0$. By the assumption, ψ_t is uniform. Hence, each ϕ_t is uniform and ϕ is uniform. We have deduced (y). Theorem 1.2 is now proved.

To prove Theorem 1.4, some preparation is needed. Recall b is said to be of *principal type* provided $\mathrm{br}_P(b)$ is a block of $\mathbb{F}C_G(P)$ for all $P \leq D$. That is equivalent to the condition that the restriction ${}_D\mathrm{Res}_G(\mathcal{O}Gb)$ is an almost-source D-algebra of $\mathcal{O}Gb$. Corollary 2.5 immediately yields the next result.

Proposition 8.2. *If b is of principal type, the interior D-algebra ${}_D\mathrm{Res}_G(\mathcal{O}Gb)$ is a uniform almost-source D-algebra of $\mathcal{O}Gb$.*

The following theorem is part of [4, Proposition 3.1] by Harris–Linckelmann.

Theorem 8.3 (Harris–Linckelmann). *Suppose G is p-solvable. Then there exists a subgroup $H \leq G$ and a block c of $\mathcal{O}H$ such that*

(1) *D is a defect group of c and a Sylow p-subgroup of H,*

(2) *c is of principal type,*

(3) *$\mathcal{O}Gb = \bigoplus_{fH,gH \subseteq G} f\mathcal{O}Hcg^{-1}$ as $\mathcal{O}$-modules, $\mathcal{O}Gb \cong {}_G\mathrm{Ind}_H(\mathcal{O}Hc)$ as interior G-algebras and $b = \mathrm{tr}_F^G(c)$.*

We also need the following sufficient criterion for an idempotent of $(\mathcal{O}Gb)^D$ to be an almost-source idempotent.

Lemma 8.4. *Let b be a block of $\mathcal{O}G$ with defect group D. Let c be an idempotent of $(\mathcal{O}Gb)^D$ such that $\mathrm{br}_D(c) \neq 0$. Define $A = c\mathcal{O}Gc$ as an interior D-subalgebra of $\mathcal{O}Gb$. Suppose that, for all $P \leq D$, the Brauer quotient*

$$A(P) \cong \mathrm{br}_P(c)\mathbb{F}C_G(P)\,\mathrm{br}_P(c)$$

has a unique block. Then A is an almost source algebra of $\mathcal{O}Gb$.

Proof. Since A is a permutation D-algebra and $\mathrm{br}_D(c) \neq 0$, each $\mathrm{br}_P(c) \neq 0$. Let b_P be a block of $\mathbb{F}C_G(P)$ such that $\mathrm{br}_P(c)b_P \neq 0$. Then $\mathrm{br}_P(c)b_P$ is a block of $\mathrm{br}_P(c)\mathbb{F}C_G(P)\,\mathrm{br}_P(c)$. By the uniqueness hypothesis, $\mathrm{br}_P(c) \leq b_P$. □

We now prove Theorem 1.4. Let H and c be as in Theorem 8.3. Put $A = c\mathcal{O}Gc$ as an interior D-algebra. By part (3) of Theorem 8.3, $A = \mathcal{O}Hc$. By the other two parts of that theorem, together with Proposition 8.1, A is a uniform almost-source algebra of $\mathcal{O}Hc$. Given $P \leq D$, then $A(P)$ is isomorphic to the block algebra $\mathbb{F}C_G(P)\,\mathrm{br}_P(c)$. Hence, via Lemma 8.4, A is an almost-source D-algebra of $\mathcal{O}Gb$. The proof of Theorem 1.4 is complete.

The rest of this section is a discussion of the special case where A is a source algebra. The next result realizes the multiplicities in Theorem 1.2 as relative multiplicities.

Corollary 8.5. *Suppose that b is a block of $\mathcal{O}G$ with defect group D and source D-algebra A. Write $\mathcal{F}$ for the fusion system on D associated with A. Let λ_D be the unique local point of D on $\mathcal{O}G$ such that $\lambda_D \cap A \neq \emptyset$. Then the following three conditions are equivalent.*

(a) *For every $\mathcal{F}$-isomorphism $\phi : Q \to P$, there is a bijective correspondence between the local points γ of P on $\mathcal{O}G$ satisfying $P_\gamma \leq D_{\lambda_D}$, the local points δ of Q on $\mathcal{O}G$ satisfying $Q_\delta \leq D_{\lambda_D}$. The correspondence is such that $\gamma \leftrightarrow \delta$ if and only if ϕ is an isofusion $Q_\delta \to P_\gamma$. Furthermore, when $\gamma \leftrightarrow \delta$, we have $m_{\mathcal{O}G}(P_\gamma, D_{\lambda_D}) = m_{\mathcal{O}G}(Q_\delta, D_{\lambda_D})$.*

(b) *Upon removing the term* local *from* (a)*, the condition still holds.*

(c) *There exists a unital $D \times D$-stable basis for A.*

Proof. For $P \leq D$, the condition $\beta' = A \cap \beta$ characterizes a bijective correspondence between the points β' of P on A and the points β of P on $\mathcal{O}G$ satisfying $A \cap \beta \neq \emptyset$. Moreover, β' is local if and only if β is local. Noting that $\ell \in \lambda_D$, we have $m_A(P_{\beta'}) = m_{\mathcal{O}G}(P_\beta, D_{\lambda_D})$. It is now clear that the assertion to be proved is a special case of Theorem 1.2. □

By Theorem 1.1, when the equivalent conditions in the corollary hold, any unital $D \times D$-stable basis for A is $\mathcal{F}$-characteristic. We mention that some applications of characteristic bisets appear in [8] by Ragnarsson–Stancu.

To comment on the sense in which condition (a) of the corollary expresses an invariance property, let us note that, for any $\mathcal{F}$-centric $P \leq D$, there is a unique local point λ_P of P on $\mathcal{O}G$ such that $P_{\lambda_P} \leq D_{\lambda_D}$. Defining

$$m_A(P) = m_{\mathcal{O}G}(P_{\lambda_P}, D_{\lambda_D}),$$

uniformity implies that $m_A(P)$ is $\mathcal{F}$-invariant. We mean to say, if A is uniform, then $m_A(P)$ depends only on the $\mathcal{F}$-isomorphism class of P.

To fulfill the obligations taken on in Section 1, it remains only to prove Proposition 1.6. Still supposing that A is a source algebra of $\mathcal{O}Gb$, now supposing also that $\mathcal{F} = N_{\mathcal{F}}(D)$, let ϕ be an $\mathcal{F}$-isomorphism. Let ψ be an extension of ϕ to an $\mathcal{F}$-automorphism of D. Let $r \in A^\times$ such that (r, r^{-1}) is a ψ-witness $\ell \mapsto \ell$. Then $r \in A^{\Delta(\psi)} \subseteq A^{\Delta(\phi)}$. So, by Theorem 2.4, A has a unital $D \times D$-stable basis. The proof of Proposition 1.6 is complete.

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