CONCRETE SHEAVES AND CONTINUOUS SPACES

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In algebraic topology and differential geometry, most categories lack some good "convenient" properties like being cartesian closed, having pullbacks, pushouts, limits, colimits... We will introduce the notion of *continuous spaces* which is more general than the concept of topological manifolds but more specific when compared to topological spaces. After that, it will be shown that the category of continuous spaces have "convenient" properties we seek. For this, we first define concrete sites, concrete sheaves and say that a generalized space is a concrete sheaf over a given concrete site. Then it will be proved that a category of generalized spaces (for a given concrete site) has all limits and colimits. At the end, it will be proved that the category of continuous spaces is actually equivalent to the category of generalized spaces for a specific concrete site.

Keywords: Site and Sheaves, Generalized spaces.

ÖZET

SOMUT DEMETLER VE SÜREKLİ UZAYLAR

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Cebirsel topoloji ve diferansiyel geometride bulunan çoğu kategori kartezyen kapalı olma, geri çekişe sahip olma, dışarı itmeye sahip olma, limit ve eşlimite sahip olma gibi uygun özellikleri barındırmıyor. Topolojik manifoldlardan daha genel fakat topolojik uzaylardan daha özel olan sürekli uzaylar kavramını tanıtacağız. Sonra ise sürekli uzaylar kategorisinin aradığımız uygun özellikleri barındırdığını ispatlayacağız. Bunun için ilk olarak somut siteler, somut demetlerin tanımını vereceğiz ve bir genelleştirilmiş uzayın aslında verilen bir somut site üzerinde bir somut demet olduğunu söyleyeceğiz. Sonra ise bir genelletirilmiş uzay kategorisinin (verilen bir somut site için) limit ve eşlimitlere sahip olduğu ispatlanacak. Sonunda ise sürekli uzaylar kategorisinin aslında bir genelletirilmiş uzay kategorisine (belli bir somut site için) denk olduğu ispatlanacak.

Anahtar sözcükler: Site ve Demetler, Genelleştirilmiş uzaylar.

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Contents

Chapter 1

Introduction

This thesis introduces some basic category theory, the notion of sites and sheaves, generalized spaces and then an important proposition about continuous spaces. Nowadays category theory has become an inevitable tool for algebraic topologists and differential geometricians. Because of this reason, they have started to use a lot of category theory and as a result they are trying to work on such categories of spaces which has some convenient properties. So, first of all we have to give some basic notions of category theory which will be useful in Chapter 3 and 4. Then in Chapter 3 we will give the definitions of sites and sheaves, then talk about their concreteness and at the end we will give the definition of a generalized space. Finally, in Chapter 4 we will give continuous spaces and show that it is equivalent to a generalized space. As a conclusion, it will be proved that the category of continuous spaces is actually equivalent to a category of generalized spaces and thanks to this relation continuous spaces will have lots of "convenient properties" which are very essential in the field of algebraic topology and differential geometry.

1.1 Categories,Functors and Natural Transformations

Definition 1.1.1. *[1] A* **category** *C consists of*

- *A collection $Obj(C)$ of objects*
- *For each pair of objects A and B, a set $hom(A, B)$ of morphisms from A to B (morphisms are also called maps or arrows)*
- *For each object A, an identity morphism $\mathrm{id}_A : A \to A$*
- *For A, B, C a composition function*

$$\mathrm{hom}(A, B) \times \mathrm{hom}(B, C) \to \mathrm{hom}(A, C)$$
$$\langle f, g \rangle \mapsto g \circ f$$

such that the following rules are satisfied

1. *Identity: for a given morphism $f : A \to B$, $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$*
2. *Associativity: for given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, $(h \circ g) \circ f = h \circ (g \circ f)$*

Note that from these axioms, one can prove that there is exactly one identity morphism for every object. One also says that a category is **small** if its collection of objects is a set.

Definition 1.1.2. *[2] For a category C, we can build the* **opposite** *category C^{op} whose objects are the same with the objects of the original category C and the morphisms from b to a are $\{f^{\mathrm{op}} \mid f : a \to b$ a morphism of $C\}$ for a, $b \in \mathrm{ob}(C)$.*

The composite in the category C^{op} is defined as $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$. With this construction it can be obviously seen that C^{op} is a category.

Definition 1.1.3. *[2] A functor $T : C \to B$ for given categories C and B consists of two assignments:*

- *The* **object assignment***: For a given $c \in \mathrm{ob}(C)$*

$$c \mapsto Tc$$

- *The* **morphism assignment***: For given c, $c^{'} \in \mathrm{ob}(C)$*

$$(f : c \to c^{'}) \mapsto (Tf : Tc \to Tc^{'})$$

in such a way that for given composable morphisms f, g in C

$$T(\mathrm{id}_c) = \mathrm{id}_{Tc}, \qquad T(g \circ f) = Tg \circ Tf$$

Note that a functor is a morphism in the category of categories.

Now we will give the hom-functor as an example to a functor which will be used in many notions later.

Example 1.1.4. *[2] Let C be a category. Then we define* **hom − functor** *for $a \in \mathrm{ob}(C)$ as follows:*

$$\begin{aligned} \hom_C(a, -) : C &\longrightarrow \mathbf{Set} \\ b &\mapsto \hom_C(a, b) \\ (b \xrightarrow{k} b^{'}) &\mapsto (\hom_C(a, b) \xrightarrow{\hom_C(a,k)(f) = k \circ f} \hom_C(a, b^{'})) \end{aligned}$$

Note that the hom-functor defined on a category C for given $a \in \mathrm{ob}(C)$ can also denoted by $C(a, -)$.

Definition 1.1.5. *[2] A functor $T : C \to B$ is an isomorphism if there exists a functor $S : B \to C$ such that $S \circ T = \mathrm{id}_C$ and $T \circ S = \mathrm{id}_B$. Let C and B are both small. Then we can say that a functor $T : C \to B$ is called an* **isomorphism** *of categories, if it is bijective for both object and morphism assignments.*

Definition 1.1.6. *[2] A functor $T : C \to B$ is called* **full** *if for given objects c, $c^{'} \in C)$ and a morphism $g : Tc \to Tc^{'}$ of B, we can find a morphism $f : c \to c^{'}$ of C such that $g = Tf$.*

And of course note that the composite of two full functors is a full functor.

Definition 1.1.7. *[2] A functor $T : C \to B$ is called* **faithful** *if for each pair c, $c^{'} \in \mathrm{ob}(C)$ and $f_1, f_2 : c \to c^{'}$ of parallel morphisms of C, $Tf_1 = Tf_2 : Tc \to Tc^{'}$ implies $f_1 = f_2$.*

Composites of faithful functors are again faithful.

Remark 1.1.8. *[2] We can summarize these two properties by using hom-sets. Explicitly, given c, $c^{'} \in \mathrm{ob}(C)$, the morphism function of T defines a function*

$$T_{c,c'} : \hom(c, c^{'}) \to \hom(Tc, Tc^{'})$$
$$f \mapsto Tf$$

T is full if every such function is surjective and faithful if every such function is injective.

If a functor is both full and faithful then we call it **fully faithful**. In terms of hom-sets, a functor is fully faithful if for every objects in C, the function defined in above remark is a bijection. But readers should be careful about the fact that this does not mean the functor itself is an isomorphism.

Definition 1.1.9. *[2] Let $S, T : C \to B$ be functors. A* **natural transformation** *from S to T is an assignment which assigns a morphism $\tau_c = \tau$ from Sc to Tc in B to every object c in S.*

$$S \xrightarrow{\tau} T$$
$$c \mapsto \tau_c = \tau c$$

such that

$$\begin{tikzcd}
Sc \arrow[r, "\tau c"] \arrow[d, "Sf"'] & Tc \arrow[d, "Tf"] \\
Sc' \arrow[r, "\tau c'"] & Tc'
\end{tikzcd}$$

commutes for each morphism $f : c \to c'$

When that holds we say $\tau_c : Sc \to Tc$ *is* **natural** *in* c.
In addition, we call $\tau a, \tau b, \tau c, \ldots$ *the components of the natural transformation* τ.

Remark 1.1.10. *[2] A natural transformation can be considered as a morphism of functors*

A natural transformation between functors from C to B will be called a **natural equivalence** or a **natural isomorphism** when every component is invertible in B. ($\tau : S \cong T$).

Definition 1.1.11. *[3] Given* $c \in \mathrm{ob}(C)$, *if there exists a unique morphism* $c \dashrightarrow d$ *for any* $d \in \mathrm{ob}(D)$, *then we say that* c *is an* **initial object** *in the category* C.

Definition 1.1.12. *[3] Given* $c \in \mathrm{ob}(C)$, *if there exists a unique morphism* $d \dashrightarrow c$ *for any* $d \in \mathrm{ob}(D)$, *then we say that* c *is an* **terminal object** *in the category* C.

Initial and terminal objects are dual notions. If an object is both initial and terminal, then we call it a **zero object** (also called **null object**).

1.2 Functor Categories

Given categories C and B, we consider all functors from C to B. If $R : C \to B$, $S : C \to B$, $T : C \to B$ are functors and $\sigma : R \to S$, $\tau : S \to T$ are two natural transformations, their components for each $c \in C$ define composite arrows $(\tau\sigma)_c = \tau_c \circ \sigma_c$ which are the components of a transformation $\tau\sigma : R \to T$. Now we want to show that $\tau\sigma$ is natural. Thus,for any $f : c \to c^{'}$ we need the following diagram commutes:

$$\begin{tikzcd} Rc \arrow[r, "Rf"] \arrow[d, "\sigma c"'] & Rc^{'} \arrow[d, "\sigma c^{'}"] \\ Sc \arrow[r, "Sf"] \arrow[d, "\tau c"'] & Sc^{'} \arrow[d, "\tau c"] \\ Tc \arrow[r, "Tf"] & Tc^{'} \end{tikzcd}$$

Since σ and τ are natural, both small squares are commutative. So the rectangle commutes. Besides, we have defined above $(\tau\sigma)_c$ as equal to $\tau_c \circ \sigma_c$. Therefore the composite $\tau\sigma$ is natural.

This composition of transformations is associative, it has for each functor T an identity ($\mathrm{Id}_T : T \to T$ with components $\mathrm{Id}_T\, c = \mathrm{Id}_{Tc}$).
After all these work now we can introduce a very important notion which is used extensively in the category theory.

Definition 1.2.1. *[2] Given categories B and C, a* **functor category**, *denoted by $B^C = \{T \mid T : C \to B functor\}$, has the functors $T : C \to B$ as its objects and natural transformations between functors as its morphisms.*

For any S, $T \in \mathrm{ob}(B^C)$, "hom-set" of this category is

$$B^C(S,T) = Nat(S,T) = \{\tau \mid S \to T \text{ natural}\}.$$

Example 1.2.2. *Given a category B, $B^{[\mathbf{0}]}$ is isomorphic to B itself.*
$B^{[\mathbf{1}]}$ is the category of morphisms of B; its objects are morphisms $f : a \to b$ of B, and its morphisms $f \to f^{'}$ are those pairs $\langle h, k\rangle$ of morphisms in B for which the square

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "f"'] & a^{'} \arrow[d, "f^{'}"] \\ b \arrow[r, "k"'] & b^{'} \end{tikzcd}$$

commutes. Another example is that if B and C are sets, then B^C is also a set; namely, the usual "function-set" from C to B.

Chapter 2

Universals and Limits

Since most of notions in category theory basically depends on **universals**, we will try to explain what these "universals" are. Thus in this chapter we will examine some of these universals which will be used later.

2.1 Universal Arrows and the Yoneda Lemma

Definition 2.1.1. *[2] Let $S : D \to C$ be a functor and $c \in \mathrm{ob}(C)$, then we say that a pair $\langle r, u : c \to Sr\rangle$ where $r \in \mathrm{ob}(D)$ and $u : c \to Sr$ a morphism in C, is a* **universal morphism** *from c to S if for every pair $\langle d, f : c \to Sd\rangle$ where $d \in \mathrm{ob}(D)$ and $f : c \to Sd$ a morphism in C, there exists a unique morphism $f' : r \to d$ in D such that $Sf' \circ u = f$. Diagrammatically,*

$$\begin{tikzcd} c \arrow[r, "u"] \arrow[d, equal] & Sr \arrow[d, dotted, "Sf'"] \\ c \arrow[r, "f"] & Sd \end{tikzcd}$$

commutes.

Informally every morphism f to S factors through the universal morphism u.

The universality can be formulated with hom-sets, as follows by omitting the proof:

Proposition 2.1.2. *[2] Let $S : D \to C$ be a functor.Then $\langle r, u : c \to Sr \rangle$ is a universal morphism from c to S if and only if the function*

$$D(r, d) \xrightarrow{\cong} C(c, Sd)$$
$$f^{'} \mapsto Sf^{'} \circ u$$

is bijective for every d. This bijection is natural in d.

Conversely, given r and c, any natural isomorphism is determined in this way by a unique morphism $u : c \to Sr$ such that $\langle r, u \rangle$ is universal from c to S.

Definition 2.1.3. *[2] Let D be a category. A a pair $\langle r, \psi \rangle$ where $r \in \mathrm{ob}(D)$ and $\psi : D(r, -) \cong K$ a natural isomorphism is defined to be a* **representation** *of a functor $K : D \to$* **Set** *.*

Lemma 2.1.4. *[2] (Yoneda)Let $K : D \to$* **Set** *be a functor and $r \in \mathrm{ob}(D)$, then there exists a bijection*

$$y : Nat(D(r, -), K) \xrightarrow{\cong} Kr$$
$$\alpha \mapsto \alpha_r \,\mathrm{id}_r$$

Proof. See page 61 in [2] □

2.2 Coproducts, Colimits and Pushouts

Before starting to define coproducts, let us first give the definition of a very important functor, **diagonal functor**, which will be used in a lot of definition.

Definition 2.2.1. *[2] Let C and J be categories (J for index category, usually small and often finite) and C^J be the functor category. The* **diagonal functor** *$\Delta : C \to C^J$ is defined to be the functor which sends*

- *each* $c \in \mathrm{ob}(C)$ *to* Δc
  - *here* Δc *is the constant functor which sends every* $i \in \mathrm{ob}(J)$ *to* c *and every morphism in* J *to the identity* id_c.
- *each morphism* $f : c \to c^{'} \in C$ *to* $\Delta f : \Delta c \to \Delta c^{'}$ *which has the same value* f *at each* $i \in J$.

Now let us give an example to a diagonal functor by specifying the index category J.

Example 2.2.2. *If we take the index category* J *as the discrete category* $J = \{1, 2\}$, *then our diagonal functor will be as follows*

$$C \longrightarrow C^{\{1,2\}} = C \times C$$

$$\begin{tikzcd} c \arrow[r, mapsto] \arrow[d, "f"] & \langle c, c\rangle \arrow[d, "\langle f,f\rangle"] \\ c^{'} \arrow[r, mapsto] & \langle c^{'}, c^{'}\rangle \end{tikzcd}$$

Definition 2.2.3. *[2] A universal morphism from* $\langle a, b\rangle \in \mathrm{ob}(C \times C)$ *to the diagonal functor* Δ *on* C *as in the previous example, is called a* **coproduct** *diagram.*

A coproduct diagram consists of an object $c \in \mathrm{ob}(C)$ and a morphism $\langle a, b\rangle \xrightarrow{\langle i,j\rangle} \langle c, c\rangle$ as in the diagram below

$$a \xrightarrow{i} c \xleftarrow{j} b$$

Here c can be denoted by $a \coprod b$ as the coproduct of a and b.
The arrows i and j in the coproduct diagram are called the **injections** of the coproduct $a \coprod b$.
The pair $\langle i, j\rangle$ has the universal property, that is, for any $\langle f : a \to d, g : b \to d\rangle$ there exists a unique morphism $h : c \to d$ with $\langle f, g\rangle = \langle h \circ i, h \circ j\rangle$. Diagrammatically

$$\begin{tikzcd} a \arrow[r, "i"] \arrow[dr, "f"'] & a \coprod b \arrow[d, dotted, "h"] & b \arrow[l, "j"'] \arrow[dl, "g"] \\ & d & \end{tikzcd}$$

is commutative. Actually we can observe that in many familiar categories there exists the coproduct of any two objects. Here are two basic examples:

- **Set**: the coproduct of any two sets is just their disjoint unions.
- **Top**: the coproduct of any two topological spaces is the disjoint union of these spaces.

Definition 2.2.4. *[4] Given a pair of morphisms $\langle f : a \to, g : a \to b \rangle$ in C, a* **coequalizer** *of this pair is a morphism $u : b \to e$ such that $uf = ug$ and the following universal property is satisfied:*

- *if for a morphism $h : b \to c$ has the property $hf = hg$ then, there exist a unique morphism $h^{'} : e \to c$ such that $h = h^{'}u$.*

Let us state it with the diagram below:

$$\begin{tikzcd}
a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "u"] \arrow[dr, "h"'] & e \arrow[d, dotted, "h^{'}"] & uf = ug, \\
 & & c & hf = hg
\end{tikzcd}$$

Note that a coequalizer is nothing but a universal morphism from an object of C^J to Δ where the index category J is defined as below:

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet
\end{tikzcd}$$

Coproduct and coequalizer are just the special cases of the colimit obtained by just changing the index category J. Now let us define colimit:

Definition 2.2.5. *[2] A* ***colimit*** *diagram is a universal morphism from a functor $F \in \mathrm{ob}(C^J)$ to Δ.*

The colimit diagram consists of an object $r \in C$, usually denoted by $r = \varinjlim F$ or $r = \mathrm{Colim}\, F$, together with a natural transformation $u : F \to \Delta r$ which is universal among other natural transformations from F to Δ

Let τ be a morphism in C^J from F to Δc. Since Δc is the constant functor for every $c \in C$, τ consists of morphisms $\tau_i : F_i \to c$ of C, one for each object $i \in J$, with $\tau_j \circ Fv = \tau_i$ for each morphism $v : i \to j$ of J. Thus, a natural transformation $\tau : F \to \Delta c$, usually written as $\tau : F \to c$ since Δc sends every object in J to c, is called a **cone** from F to c. Pictorially,

$$\begin{tikzcd} F_i \arrow[r, "F_v"] \arrow[dr, "\tau_i"'] & F_j \arrow[r, "F_w"] \arrow[d, "\tau_j"] & F_k \arrow[dl, "\tau_k"] \\ & c & \end{tikzcd}$$

Now, in terms of the definiton of a cone, it can be obviously said that a colimit of $F : J \to C$ consists of an object $\operatorname{Colim} F \in C$ and a cone $\mu : F \to \operatorname{Colim} F$ which is universal, i.e. for any cone $\tau : F \to c$ there exists a unique morphism $t^{'} : \operatorname{Colim} F \to c$ with $\tau_i = t^{'} \circ \mu_i$, for every i in the index category J. Here μ is called the **limiting cone** or the **universal cone**. Diagrammatically

$$\begin{tikzcd} F \arrow[r, "\mu"] \arrow[dr, "\tau"'] & \operatorname{Colim} F \arrow[d, dotted, "t^{'}"] \\ & c \end{tikzcd}$$

Earlier we have mentioned that coproducts and coequalizers are actually the special cases of the colimit obtained by just changing index category J. It obviously means that coproducts and colimits are some examples of colmit. Now we present another important example to colimit.

Definition 2.2.6. *[5] Given $\langle f : a \to b, g : a \to c \rangle$ in C, a* **pushout** *of $\langle f, g \rangle$ is a commutative square*

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[d, "u"] \\ c \arrow[r, "v"] & r \end{tikzcd}$$

such that for every other commutative square built on f, g there exists a unique morphism $t : r \to s$ with $t \circ u = h$ and $t \circ v = k$. Let us display this fact with a *diagram like this*

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "u"] \arrow[ddr, bend left, "h"] & \\
c \arrow[r, "v"] \arrow[drr, bend right, "k"'] & r \arrow[dr, dotted, "t"] & \\
& & s
\end{tikzcd}
$$

Note that a pushout is just a colimit where the index category J is the category

$$
\bullet \longleftarrow \bullet \longrightarrow \bullet
$$

2.3 Products, Limits and Pullbacks

The limit, product, equalizer and pullback notions are just dual to that of colimit, coproduct, coequalizer and pushout, respectively. Since product, equalizer and pullback are the special cases of the limit notion, let us start with the definition of limit. But first let us give a remark.

Remark 2.3.1. *Let $S : D \to C$ be a functor and $c \in \mathrm{ob}(C)$. Earlier we have defined a universal morphism from c to S in the Definition 2.1.1. Likewise in that definition, we can define the dual notion of it, i.e., a universal morphism from S to c. Let us show both notions with a diagram*

$$
\begin{tikzcd}
Sr' \arrow[r, "u'"] & c \arrow[r, "u"] \arrow[dr] & Sr \arrow[d, dotted] \\
Sd' \arrow[u, dotted] \arrow[ur] & & Sd
\end{tikzcd}
$$

Here the pair $\langle r', u' \rangle$ is the universal morphism from S to c.

Definition 2.3.2. *[2] Given categories C, J, and $\Delta : C \to C^J$, then a* **limit** *of a functor $F : J \to C$ is defined to be a universal morphism from Δ to F.*

Note that limit and colimit are dual notions.

Now let us picturize them in one diagram:

$$\begin{tikzcd}
\varprojlim F \arrow[r] & F \arrow[r] \arrow[dr] & \operatorname{Colim} F \arrow[d, dotted] \\
c \arrow[u, dotted] \arrow[ur] & & c
\end{tikzcd}$$

Definition 2.3.3. *[2] A universal morphism from $\Delta : C \to C^J$ where J is the discrete category $\{1, 2\}$ to an object $\langle a, b\rangle \in C^J$ is called a* **product** *diagram.*

A product diagram is shown by diagrammatically as below:

$$a \xleftarrow{p} a \times b \xrightarrow{q} b$$

where p and q are called the **projections** of the product.

Definition 2.3.4. *[2] If the index category J is the same as defined in Definition 3.2.4., then a limit object d is called an* **equalizer***. The limit diagram is as follows*

$$\begin{tikzcd}
d \arrow[r, "e"] & b \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & a
\end{tikzcd} \qquad fe = ge$$

The universal property for this definition is that for any $h : c \to b$ with $fh = gh$ there exists a unique morphism $h^{'} : c \to d$ with $eh^{'} = h$.

Definition 2.3.5. *[5] Given $\langle f : b \to a, g : c \to a\rangle$ in C, a* **pullback** *of $\langle f, g\rangle$ is a commutative square*

$$\begin{tikzcd}
r \arrow[r, "u"] \arrow[d, "v"] & c \arrow[d, "g"] \\
b \arrow[r, "f"] & a
\end{tikzcd}$$

such that for every other commutative square built on f, g there exists a unique morphism $t : r \to s$ with $u \circ t = k$ and $v \circ t = h$. Diagrammatically:

$$\begin{tikzcd}
s \arrow[dr, dotted, "t"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] & & \\
& r \arrow[r, "u"] \arrow[d, "v"] & c \arrow[d, "g"] \\
& b \arrow[r, "f"] & a
\end{tikzcd}$$

Note that a pullback is just a colimit where the index category J is the category

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

Chapter 3

Generalized Spaces

The main goal in this chapter is to define **generalized spaces** after giving some basic definitions and explanations about **sites** and **sheaves**. But before doing all these works, we should tell the story of why we are trying to understand and develop all the concepts which will be studied along this chapter. With this aim, let us look at some motivations.

Many mathematicians, especially algebraic topologists and differential geometricians, come up with many problems about the categories in which they work mostly. Some of these problems are the followings:

- The category of topological spaces is not cartesian closed,
- In the category of manifolds, mapping space $C^{\infty}(X, Y)$ of finite dimensional smooth manifolds X and Y may not be finite-dimensional.
- A quotient subspace or subspace of a topological manifold may not be a manifold,
- The category of manifolds lacks having limits and colimits.

Because of these lacking properties encountered frequently, researchers have started to investigate a 'convenient category' of spaces. Some studies have shown that generalized spaces may help to find the desired categories.

In conclusion, our job here will be to construct a new category of spaces that has all good convenient properties.

3.1 Sites and Sheaves

In this section we will give some basic notions about sites and sheaves which will help us to construct generalized spaces.

Definition 3.1.1. *[6] Given a category $\mathfrak{D}$, a function sending every $D \in \mathrm{ob}(\mathfrak{D})$ to a collection of covering families $R(D) = (f_i : D_i \to D \mid i \in I)$ is defined to be a* **coverage** *on $\mathfrak{D}$, if for a given covering family $(f_i : D_i \to D \mid i \in I)$ and a morphism $g : C \to D$, there is a covering family $(h_j : C_j \to C \mid j \in J)$ such that for every $j \in J$ there exists $i \in I$, there exists $k : C_j \to D_i$ such that $g \circ h_j = f_i \circ k$.*

Now by means of the notion of coverage we will simply define **site** as follows :

Definition 3.1.2. *[6] A category $\mathfrak{D}$ equipped with a coverage is called a* **site** *and every object $D \in \mathfrak{D}$ is called a* **domain**.

Definition 3.1.3. *[6] A functor $X : \mathfrak{D}^{\mathrm{op}} \to \mathrm{Set}$ on a category $\mathfrak{D}$ is called a* **presheaf**. *Given a domain $D \in \mathrm{ob}(\mathfrak{D})$, the elements of $X(D)$ are called* **plots** *in X.*

Definition 3.1.4. *[6] Let $(f_i : D_i \to D \mid i \in I)$ be a covering family of $D \in \mathrm{ob}(\mathfrak{D})$ and $X : \mathfrak{D}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf. Then a collection $\{\varphi_i \in X(D_i) \mid i \in I\}$ is said to be* **compatible** *if the diagram*

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "g"'] & D_j \arrow[d, "f_j"] \\ D_i \arrow[r, "f_i"'] & D \end{tikzcd}$$

commutes then $X(g)(\varphi_i) = X(h)(\varphi_j)$ *for any* $g : C \to D_i$ *and* $h : C \to D_j$. *Notice that when we have a commutative diagram as follows*

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "g"'] & D_j \arrow[d, "f_j"] \\ D_i \arrow[r, "f_i"'] & D \end{tikzcd}$$

then we have a commutative diagram as follows

$$\begin{tikzcd} X(C) & X(D_j) \arrow[l, "X(h)"'] \\ X(D_i) \arrow[u, "X(g)"] & X(D) \arrow[l, "X(f_i)"] \arrow[u, "X(f_j)"'] \end{tikzcd}$$

Definition 3.1.5. *[6] A* **sheaf** *is defined to be a presheaf* $X : \mathfrak{D}^{\mathrm{op}} \to \mathbf{Set}$ *on a given site* $\mathfrak{D}$ *with an extra condition:*

- *Given a covering family* $(f_i : D_i \to D \mid i \in I)$ *of* D *and a compatible collection* $\{\varphi_i \in X(D_i) \mid i \in I\}$, *then for each plot* $\varphi_i \in X(D_i)$ *there is a unique plot* $\varphi \in X(D)$ *such that* $X(f_i)(\varphi) = \varphi_i$.

3.2 Concreteness of Sites and Sheaves

In this chapter we will give some important definitions like concrete sheaves and concrete sites by which we will construct generalized spaces.

Remark 3.2.1. *[6] Given a presheaf* $X : \mathfrak{D}^{\mathrm{op}} \to \mathbf{Set}$, *if for a given* $D \in \mathrm{ob}(\mathfrak{D})$, $X \cong \mathrm{hom}(-, D)$, *then* X *is called* **representable** *as in definition 2.1.3.*

Definition 3.2.2. *[6] If each representable presehaf on a site* $\mathfrak{D}$ *is a sheaf, then* $\mathfrak{D}$ *is called* **subcanonical**.

Definition 3.2.3. *[6] If a subcanonical site has a terminal object* 1 *and satisfies the following conditions, then we call it a* **concrete site**:

1. $\mathrm{hom}(1, -) : \mathfrak{D} \to \mathbf{Set}$ *is a faithful.*

2. *Given a covering family $(f_i : D_i \to D \mid i \in I)$,for each $i \in I$ the following functions are jointly surjective*

$$\begin{aligned} \hom(1, f_i) : \hom(1, D_i) &\to \hom(1, D) \\ (1 \xrightarrow{h} D_i) &\mapsto (1 \xrightarrow{f_i \circ h} D) \end{aligned}$$

in the sense that the union of their images is all of $\hom(1, D)$

This definition helps us to think the objects of the category $\mathfrak{D}$ basically as sets with extra structure. What we are tying to say here is that objects of $\mathfrak{D}$ can actually be considered with their own underlying sets and morphisms of $\mathfrak{D}$ with their underlying functions between sets.

For a given object $D \in \mathrm{ob}(\mathfrak{D})$, $\hom(1, D)$ (also denoted as $\underline{D}$) can be considered as the **underlying set** of D and called the **set of points** of D. Given morphism $f : D_1 \to D_2$ in $\mathfrak{D}$, then the underlying function will be $\underline{f} = \hom(1, f) : \underline{D_1} \to \underline{D_2}$. So, the first condition in the previous definition actually implies that for given $f, g : C \to D$ in $\mathfrak{D}$, $f = g$ if $\hom(1, f) = \hom(1, g)$. Furthermore the second condition actually implies that if we have a covering for an object $D \in \mathrm{ob}(D)$ then its underlying family of functions is a covering as well.

Now we will define 'concrete sheaves' , but first we obtain a set by considering $X(1) = \underline{X}$ from a sheaf $X : \mathfrak{D}^{\mathrm{op}} \to \mathbf{Set}$ on a concrete site. Then, what we need is to turn a plot $\varphi \in X(D)$ into a function $\underline{\varphi}$ (called **underlying function**). For this, we set

$$\begin{aligned} \underline{\varphi} : \hom(1, D) &\to X(1) \\ (1 \xrightarrow{d} D) &\mapsto (X(d)(\varphi)) \end{aligned}$$

Here the morphism $1 \xrightarrow{d} D$ gives us a morphism

$$\begin{aligned} X(d) : X(D) &\to X(1) \\ \varphi &\mapsto X(d)(\varphi) \end{aligned}$$

Definition 3.2.4. *[6] Let $\mathfrak{D}$ be a concrete site and $\mathfrak{X} : \mathfrak{D}^{\mathrm{op}} \to$ **Set** be a sheaf. Given $D \in \mathrm{ob}(\mathfrak{D})$, if the function $\varphi \mapsto \underline{\varphi}$ is injective, then $\mathfrak{X}$ is called a* **concrete sheaf***.*

Notation 3.2.5. *From now on, when we say X is a concrete sheaf over a concrete site $\mathfrak{D}$, we will denote it by mathfrak letter notation $\mathfrak{X}$.*

Definition 3.2.6. *[6] A* **generalized space** *(also called $\mathfrak{D}$* **space***) is nothing but a concrete sheaf $\mathfrak{X} : \mathfrak{D}^{\text{op}} \to \textbf{Set}$ on a given concrete site $\mathfrak{D}$.*

Note that since each $\mathfrak{D}$ space is a functor it is obvious that a map between $\mathfrak{D}$ spaces $\mathfrak{X}$ and $\mathfrak{Y}$ is a natural transformation $\mathfrak{F} : \mathfrak{X} \to \mathfrak{Y}$.

Now we construct a category $\mathfrak{D}$space to be the category of $\mathfrak{D}$ spaces and maps between these.

Remark 3.2.7. *It should not to be confused that a $\mathfrak{D}$ space is just a space given in Definition 4.2.5. and* $\mathfrak{D}$space *is the category whose objects are $\mathfrak{D}$ spaces and morphisms are natural transformations between them.*

The reason of why we call them as 'generalized spaces' is that an object $\mathfrak{D}$ space in the category $\mathfrak{D}$space has all general "convenient" properties we seek for(see Theorem 52 in [6]). Besides, 'generalized spaces' can be thought as concrete sheaves .

Lemma 3.2.8. *[7] The category of sheaves have all (small) limits.*

Proof. See page 15 Lemma 10.1. in [7] □

Theorem 3.2.9. *[6] Given a concrete site $\mathfrak{D}$,* $\mathfrak{D}$space *has all (small) limits.*

Proof. [6] Suppose that we are given a functor $F : J \to \mathfrak{D}\text{space}$. We claim that limits in $\mathfrak{D}$space are limits of the underlying diagram of sheaves. For any $D \in \text{ob}(\mathfrak{D})$, consider the two diagrams of sets below

$$L : \mathfrak{D}\text{space} \to \textbf{Set}, \qquad \underline{L} : \mathfrak{D}\text{space} \to \textbf{Set}$$
$$\mathfrak{X} \mapsto \mathfrak{X}(D) \qquad \mathfrak{X} \mapsto \underline{\mathfrak{X}}^{\underline{D}}$$

There is a natural transformation $U : L \to \underline{L}$ which is defined for a given $\mathfrak{D}$ space $\mathfrak{X}$ as follows

$$U_{\mathfrak{X}} : L(\mathfrak{X}) = \mathfrak{X}(D) \to \underline{L}(\mathfrak{X}) = \underline{\mathfrak{X}}^{\underline{D}}$$
$$\varphi \mapsto (\underline{\varphi} : \hom(1, D) \to \mathfrak{X}(1))$$

Note that φ s are plots of the concrete sheaf $\mathfrak{X}$ and $\underline{\varphi}$ s are the underlying functions of it. Recall that for a concrete sheaf the map $\varphi \to \underline{\varphi}$ is injective. So every component of U is injective. Thus, for given any $F(j) \in \mathrm{ob}(\mathfrak{D}\mathrm{space})$ we get the following injective function

$$\lim_{j\in J} F(j)(D) \mapsto \lim_{j\in J}(\underline{F(j)})^{\underline{D}}$$

By using the properties of limit, we can write this function as

$$\lim_{j\in J} F(j)(D) \mapsto (\lim_{j\in J} \underline{F(j)})^{\underline{D}}$$

$\underline{F(j)} = F(j)(1)$, so $\lim \underline{F(j)} = \lim F(j)(1) = \underline{\lim F(j)}$ and limits of sheaves can be computed pointwise. (Previous Lemma). Therefore the following function is also injective.

$$U_{F(j)} : (\lim F(j))(D) \mapsto (\underline{\lim F(j)})^{\underline{D}}$$

Here we actually proved that the limit of F is concrete by showing the previous injectivity. Thus, it is in $\mathfrak{D}$space □

Theorem 3.2.10. *[6] For a given concrete site $\mathfrak{D}$, $\mathfrak{D}$space has all (small) colimits.*

Proof. [6] Suppose that we are given a functor $\tilde{F} : J \to \mathbf{Set}^{\mathfrak{D}^{\mathrm{op}}}$. The colimit, say L, of $\tilde{F}$ can be obtained pointwisely. For any presheaf in the category $\mathbf{Set}^{\mathfrak{D}^{\mathrm{op}}}$ can be turned into a concrete sheaf in order to obtain a $\mathfrak{D}$ space and this process preserves colimits ([5]). So, the $\mathfrak{D}$ space obtained from the presheaf L is the colimit of F. □

From these two theorems we obtain the following corollary.

Corollary 3.2.11. *[6] The category of $\mathfrak{D}$ spaces, that is* $\mathfrak{D}\mathrm{space}$*, has all (small) limits and colimits.*

Proof. See Theorem 3.2.8. and Theorem 3.2.9. or (See [6], page 40.) $\square$

Chapter 4

Continuous Spaces

In this chapter we will construct the notion of **continuous space** which is defined by modifying the first axiom of diffeological spaces (see page 5 in [6]) and later it will be shown that the category possessed these continuous spaces as objects is actually equivalent a $\mathfrak{D}$space.

Along all this chapter an open set is considered as an open set of $\mathbb{R}^n$, therefore a function $f : U \to U'$ for open sets U and U' is considered continuous in the usual sense.

Definition 4.0.12. *A set X equipped with some functions $\{\varphi : U \to X\}$ which we call as* **plots in** *X, is defined to be a* **continuous space** *, if the three following axioms are satisfied:*

1. *For given a plot φ in X and a continuous function $f : U' \to U$, their composition $\varphi \circ f$ is always plot in X.*
2. *Given an open cover $U_j \xrightarrow{i_j} U$ where i_j s are inclusion morphisms , if $\varphi \circ i_j$ is a plot in X for each j, then φ is also a plot in X.*
3. *Every morphism from the one point of $\mathbb{R}^0$ to X is a plot in X.*

Notation 4.0.13. Continuous *denotes the category whose objects are continuous spaces and morphisms are continuous maps which we define them as follows*

Definition 4.0.14. *$f : X \to Y$ is a continuous morphism if for every plot φ in X $f \circ \varphi$ is a plot in Y.*

Notation 4.0.15. *$\mathfrak{Cont}$ denotes the category such that open subsets of $\mathbb{R}^n$ are its objects and continuous functions are its morphisms.*

Now we will see that the category $\mathfrak{Cont}$ is a concrete site:

Firstly, we need to define a coverage on $\mathfrak{Cont}$ in order to make it a site. Explicitly, we need to define a covering family for each object of $\mathfrak{Cont}$ and show tat it is a coverage.
We build a coverage on the category $\mathfrak{Cont}$ as follows:

- $(i_j : D_j \to D \mid j \in J)$ is a covering family where $i_j : D_j \to D$ are the inclusion maps iff $D_j \subseteq D$ form an open covering for $D \subseteq \mathbb{R}^n$ with its usual subspace topology.

Lemma 4.0.16. *The category $\mathfrak{Cont}$ is a site with this coverage.*

Proof. Given covering family $(i_j : D_j \to D \mid j \in J)$ ($\cup i_j(D_j) = D$) and $g : C \to D$ in $\mathfrak{Cont}$, then $(g^{-1}(i_j(D_j)) \mid j \in J)$ covers C, which means $\cup_{j \in J} g^{-1}(i_j(D_j)) = C$. Let k_j denotes the inclusion $g^{-1}(i_j(D_j))$ to C. Since $g \circ k_j = g_j i_j$, it is a coverage for $\mathfrak{Cont}$.
Therefore, the category $\mathfrak{Cont}$ is a site. □

Lemma 4.0.17. *$\mathfrak{Cont}$ is subcanonical.*

Proof. Let X be a representable presheaf on $\mathfrak{Cont}$ and D_X be its representing object, i.e.,

$$X : \mathfrak{Cont}^{\mathrm{op}} \to \mathbf{Set}$$
$$D \mapsto \hom(D, D_X)$$

For a given compatible collection of plots $\{\varphi_j \in X(D_j) \mid j \in J\}$, we need to find a unique plot φ in $\mathfrak{X}(D)$ such that $X(i_j)(\varphi) = \varphi_j$. To do this we define $\varphi : D \to D_X$

as $\varphi(z) = \varphi_j(z)$ if $z \in D_j$. Let z be in D_j and $D_{j'}$ then $D_j \cap D_{j'}$ is an open subset of D it is equal to $D_{j''}$ for some $j'' \in J$, this means $\varphi_j(z) = \varphi_{j''}(z) = \varphi_j'(z)$. So φ is well-defined. Therefore $\mathfrak{Cont}$ is subcanonical. $\square$

Since one-point open set is a terminal object, the two conditions in the definition 3.2.3. is automatically satisfied.
In conclusion, $\mathfrak{Cont}$ is a concrete site.

4.1 Building Generalized Spaces From Continuous Spaces

We have a concrete site $\mathfrak{Cont}$ and now we will construct a concrete sheaf over this site by using the objects of the category **Continuous**, that is continuous spaces.

For a given object $X \in$ **Continuous**, there exists a concrete sheaf $\mathfrak{X}$: $\mathfrak{Cont}^{\mathrm{op}} \to$ **Set** :

Given $C \in \mathrm{ob}(\mathfrak{Cont})$, $\mathfrak{X}(C)$ is defined as the set of plots $\{\varphi : C \to X\}$.
Given $f : C' \to C$,we set $\mathfrak{X}(f) : \mathfrak{X}(C) \to X(C')$, $\varphi \mapsto \varphi \circ f$

Let us explain it with a simple diagram like this:

$$\mathfrak{X} : \mathfrak{Cont}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

$$\begin{tikzcd}
C \arrow[r, mapsto] & \mathfrak{X}(C) = \{\varphi : C \to X\} \arrow[d, "\mathfrak{X}(f)(\varphi)=\varphi f"] \\
C' \arrow[u, "f"] \arrow[r, mapsto] & \mathfrak{X}(C') = \{\varphi : C' \to X\}
\end{tikzcd}$$

Thanks to the axiom 1 in Definition 4.0.8. we ensure that φf lies in $\mathfrak{X}(C')$.
Next we will finish the proof in 3 steps.

- Firstly, we have to show that $\mathfrak{X}$ is a presheaf, more precisely $\mathfrak{X} : \mathfrak{Cont}^{\text{op}} \to$ **Set** must be a functor:

  - Given arrows $f^{'} : C^{''} \to C^{'}$, $f : C^{'} \to C$ in $\mathfrak{Cont}$ and a plot $\varphi : C \to X$ in $\mathfrak{X}(C)$ we have

    $$\mathfrak{X}(f \circ f^{'})(\varphi) = \varphi \circ f \circ f^{'}$$

    $$\mathfrak{X}(f^{'}) \circ \mathfrak{X}(f) = \mathfrak{X}(f^{'})(\varphi \circ f) = \varphi \circ f \circ f^{'}$$

    Thus $\mathfrak{X}(f \circ f^{'}) = \mathfrak{X}(f^{'}) \circ \mathfrak{X}(f)$

  - Given $C \in \text{ob}(\textbf{Continuous})$ and a plot $\varphi : C \to X$. Then we have $\mathfrak{X}(\text{Id}_C)(\varphi) = \varphi \circ \text{Id}_C = \varphi$ and $(\text{Id}_{\mathfrak{X}(C)})(\varphi) = \varphi$.

    Thus $\mathfrak{X}(\text{Id}_C) = \text{Id}_{\mathfrak{X}(C)}$

  Therefore $\mathfrak{X} : \mathfrak{Cont}^{\text{op}} \to \textbf{Set}$ is a presheaf on the given concrete site $\mathfrak{Cont}$.

- Secondly, we need to show that the presheaf $\mathfrak{X}$ is actually a sheaf:

  Suppose that we are given a covering family $(i_j : D_j \to D)$ where i_j s are inclusion maps and a compatible collection $\{\varphi_j \in \mathfrak{X}(D_j) \mid j \in J\}$. Then for each plot φ_j we must find a unique plot $\varphi \in \mathfrak{X}(D)$ such that $\mathfrak{X}(i_j)(\varphi) = \varphi_j$. Thanks to the compatibility of $\{\varphi_j \in \mathfrak{X}(D_j) \mid j \in J\}$, for any $g : C \to D_j$ and $h : C \to D_z$ the diagram below commutes

  $$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "g"'] & D_z \arrow[d, "i_z"] \\ D_j \arrow[r, "i_j"'] & D \end{tikzcd}$$

  So,

  $$\begin{tikzcd} \mathfrak{X}(C) & \mathfrak{X}(D_z) \arrow[l, "\mathfrak{X}(h)"'] \\ \mathfrak{X}(D_j) \arrow[u, "\mathfrak{X}(g)"] & \mathfrak{X}(D) \arrow[l, "\mathfrak{X}(i_j)"] \arrow[u, "\mathfrak{X}(i_z)"'] \end{tikzcd}$$

  also commutes and $\mathfrak{X}(g)(\varphi_j) = \mathfrak{X}(h)(\varphi_z)$ Existence of the plot $\varphi \in \mathfrak{X}(D)$ such that $\mathfrak{X}(i_j)(\varphi) = \varphi_j$ comes from the axiom 2 in Definition 4.0.8.

  Now we will show that this φ is unique:
  Suppose that there exists another $\varphi^{'} \in \mathfrak{X}(D)$ such that $\mathfrak{X}(i_j)(\varphi^{'}) = \varphi_j$. Then we have

$$\mathfrak{X}(i_j)(\varphi') = \varphi' \circ i_j = \varphi_j$$

$$\mathfrak{X}(i_j)(\varphi) = \varphi \circ i_j = \varphi_j$$

So $\varphi \circ i_j = \varphi' \circ i_j$ and since i_js are inclusion maps $\varphi = \varphi'$. Thus the uniqueness is proved.

Therefore the presheaf $\mathfrak{X} : \mathfrak{Cont} \to \mathbf{Set}$ is a sheaf.

- Thirdly, we need to show that $\mathfrak{X}$ is a concrete sheaf over the concrete site $\mathfrak{Cont}$:

  Through the axiom 3 in Definition 4.0.12. we obtain a bijection between X and the set $\mathfrak{X}(1)$

$$X \xrightarrow{\cong} \mathfrak{X}(1) \cong \hom(1, \mathfrak{X})$$
$$x \mapsto \varphi_x \in \{\varphi : 1 \to X\}$$

  where $\varphi_x(1) = x$

  Then let $\varphi \in \mathfrak{X}(C)$ and define the underlying function $\underline{\varphi} : \hom(1, C) \to \hom(1, X) \cong \mathfrak{X}(1) \cong X$ as the map $\underline{\varphi}(c) = \mathfrak{X}(c)(\varphi) = \varphi \circ c = \varphi(c)$. Let us show it more explicitly with a diagram :

$$\mathfrak{X} : \mathfrak{Cont}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

$$\begin{tikzcd}
C \arrow[r, mapsto] & \mathfrak{X}(C) = \{\varphi : C \to X\} \arrow[d, "\mathfrak{X}(c)(\varphi)=\underline{\varphi}(c)=\varphi\circ c=\varphi(c)"] \\
1 \arrow[u, "c"] \arrow[r, mapsto] & \mathfrak{X}(1) = \{\varphi : 1 \to X\}
\end{tikzcd}$$

  where by c we denote the one-point map $c(1) = c$.

  Now we want the function U sending φ to the underlying function of itself $\underline{\varphi}$ be one-to-one in order to show that $\mathfrak{X}$ is concrete :

$$U : \mathfrak{X}(C) \to \underline{\mathfrak{X}(C)}$$
$$\varphi \mapsto \underline{\varphi} : \hom(1, C) \to \hom(1, X) \cong \mathfrak{X}(1) \cong X$$

where $\underline{\varphi}$ is the underlying function of φ defined above.

Now let us show that U is 1-1:

Suppose $U(\varphi) = U(\varphi^{'})$ for any $\varphi, \varphi^{'} \in \mathfrak{X}(C)$. Then

$$U(\varphi) = \underline{\varphi} = U(\varphi^{'}) = \underline{\varphi}^{'}. \qquad \text{Then for all } c \in C$$

$$\underline{\varphi}(c) = \mathfrak{X}(c)(\varphi) = \varphi(c) = \underline{\varphi}^{'}(c) = \mathfrak{X}(c)(\varphi^{'}) = \varphi^{'}(c).$$

So we conclude $\varphi = \varphi^{'}$.

Therefore $\mathfrak{X}$ is concrete.

. So we get a new concrete sheaf over the concrete site $\mathfrak{Cont}$, therefore we get a new $\mathfrak{D}$ space, that is $\mathfrak{Cont}$ space. By using this, we can build a new category $\mathfrak{Cont}$space.

So far we actually determined the objects of $\mathfrak{Cont}$space and now we need to determine morphisms of this category:

For a given continuous map $f : X \to Y \in$ **Continuous** , we will build a natural transformation $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ by defining

$$\begin{gathered} \mathfrak{f}_C : \mathfrak{X}(C) \to \mathfrak{Y}(C) \\ \varphi \mapsto f \circ \varphi \end{gathered}$$

Now we check that $\mathfrak{f}$ is natural. To do this check that the following square is commutative:

$$\begin{tikzcd} C & \mathfrak{X}(C) \arrow[r, "\mathfrak{f}_C"] \arrow[d, "\mathfrak{X}(g)"'] & \mathfrak{Y}(C) \arrow[d, "\mathfrak{Y}(g)"] \\ C^{'} \arrow[u, "g"] & \mathfrak{X}(C^{'}) \arrow[r, "\mathfrak{f}_{C^{'}}"'] & \mathfrak{Y}(C^{'}) \end{tikzcd}$$

We have the equalities

$$(\mathfrak{Y}(g) \circ \mathfrak{f}_C)(\varphi) = \mathfrak{Y}(g) \circ (f\varphi) = f\varphi g$$

$$(\mathfrak{f}_{C'} \circ \mathfrak{X}(g))(\varphi) = \mathfrak{f}_{C'}(\varphi \circ g) = f\varphi g$$

Thus from these equalities we can easily see that $\mathfrak{f}$ is natural.

Now we can construct the functor T:

$$T : \mathbf{Continuous} \longrightarrow \mathfrak{Cont}\text{space}$$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"] & \mathfrak{X} \arrow[d, "T(f)=\mathfrak{f}"] \\ X' \arrow[r, mapsto] & \mathfrak{X}' \end{tikzcd}$$

where $\mathfrak{X} : \mathfrak{Cont}^{\mathrm{op}} \to \mathbf{Set}$ and $\mathfrak{X}^{'} : \mathfrak{Cont}^{\mathrm{op}} \to \mathbf{Set}$ and $T(f)$ is defined as the natural transformation defined above, i.e., $T(f) := \mathfrak{f}$.
We can easily check that T is a functor by showing the following identities:

- $T((g \circ f)_C(\varphi)) = g \circ f \circ \varphi$ and $(T(g) \circ T(f))_C(\varphi) = (T(g))_C \circ (f \circ \varphi) = g \circ f \circ \varphi$ for each $f : X \to X'$, $g : X' \to X''$, $C \in \mathfrak{Cont}$, $\varphi : C \to X$.
- $T(\mathrm{Id}_X)_C(\varphi) = \varphi \circ \mathrm{Id}_X = \varphi$ and $(\mathrm{Id}_{T(X)})_C(\varphi) = \varphi$ for each $X \in \mathbf{Continuous}$, $C \in \mathfrak{Cont}$

4.2 Building Continuous Spaces From Generalized Spaces

Like we did in the previous section, we try to build continuous spaces for given generalized spaces in $\mathfrak{Cont}$space. So now e will construct a continuous space X from a given $\mathfrak{X} \in \mathrm{ob}(\mathfrak{Cont})$.
Take $X = \mathfrak{X}(1)$ and $\underline{\varphi}$ where $\varphi \in \mathfrak{X}(C)$. Now check the axioms for continuous space X defined earlier in Definition 4.0.8. :

- Axiom 1 is satisfed since $\mathfrak{X}$ is a presheaf.

- Axiom 2 is satisfied since $\mathfrak{X}$ is a sheaf.
- Axiom 3 is trivial because $X = \mathfrak{X}(1)$.

Next, we construct a function $f : X \to Y$ from a given natural transformation $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ in **Continuous** by setting

$$f = \mathfrak{f}_1 : \mathfrak{X}(1) \to \mathfrak{Y}(1)$$
$$\varphi \mapsto \mathfrak{f} \circ \varphi$$

Now we check that the construction

$$S : \mathfrak{Cont}\text{space} \longrightarrow \textbf{Continuous}$$

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, mapsto] \arrow[d, "\mathfrak{f}"'] & X = \mathfrak{X}(1) \arrow[d, "S(\mathfrak{f})=f=\mathfrak{f}_1"] \\
\mathfrak{X}' \arrow[r, mapsto] & X' = \mathfrak{X}'(1)
\end{tikzcd}$$

defines a functor :

- $S(\mathfrak{f} \circ \mathfrak{f}')(\varphi) = (f \circ f') \circ \varphi = f \circ (f' \circ \varphi) = f \circ (S(\mathfrak{f}')) = (S(\mathfrak{f}) \circ (S(\mathfrak{f}'))(\varphi)$ for each $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}'$, $\mathfrak{f}' : \mathfrak{X}' \to \mathfrak{X}''$, $\varphi : 1 \to X$
- $S(\mathrm{Id}_{\mathfrak{X}})(\varphi) = \mathrm{Id}_X \circ \varphi = \varphi$ and $(\mathrm{Id}_{S(\mathfrak{X})})(\varphi) = (\mathrm{Id}_{\mathfrak{X}(1)})(\varphi) = (\mathrm{Id}_X)(\varphi) = \varphi$ so $S(\mathrm{Id}_{\mathfrak{X}}) = \mathrm{Id}_{S(\mathfrak{X})}$

4.3 Equivalence Between Continuous and $\mathfrak{Cont}$space

Proposition 4.3.1. Continuous $\cong$ $\mathfrak{Cont}$space.

Proof. In order to show that any two categories are equivalent to each other, we have to construct functors from one to another and prove the composite of these functors are naturally isomorphic to the identity. So our main job here is to define functors

$$\textbf{Continuous} \underset{S}{\overset{T}{\rightleftarrows}} \mathfrak{Cont}\text{space}$$

But earlier we have defined these two functors. So we just have to show that $S \circ T \cong \mathrm{Id}_{\textbf{Continuous}}$ and $T \circ S \cong \mathrm{Id}_{\mathfrak{Cont}\text{space}}$.
Let us recall our functors again:

$$\begin{aligned} T : \textbf{Continuous} &\to \mathfrak{Cont}\text{space} \\ X &\mapsto \mathfrak{X} \end{aligned} \qquad \begin{aligned} S : \mathfrak{Cont}\text{space} &\to \textbf{Continuous} \\ \mathfrak{X} &\mapsto X \end{aligned}$$

Now,

- Since $(S \circ T)(X) = S(\mathfrak{X}) = \mathfrak{X}(1) \cong X$ for each continuous space $X \in$ **Continuous** we get easily the conclusion $S \circ T \cong \mathrm{Id}_{\textbf{Continuous}}$.
- For the other side, first we take a concrete sheaf $\mathfrak{X}$ and turn it into a continuous space X. Then get the image of it under T, call it $\mathfrak{X}^{'}$ $\mathfrak{X}^{'}$, i.e.

  $$(T \circ S)(\mathfrak{X}) = T(\mathfrak{X}(1) \cong X) = \mathfrak{X}^{'}$$

  Then we have for a given $C \in \mathfrak{Cont}$,

  $$\mathfrak{X}^{'}(C) = \{\underline{\varphi} : C \to \mathfrak{X}(1)\}$$

  but, Since $\mathfrak{X}$ is concrete, there is a bijection

  $$\begin{aligned} \mathfrak{X}(C) &\xrightarrow{\cong} \mathfrak{X}^{'}(C) \\ \varphi &\mapsto \underline{\varphi} \end{aligned}$$

  Therefore $\mathfrak{X} = \mathfrak{X}^{'}$ and we now can say that $T \circ S \cong \mathrm{Id}_{\mathfrak{Cont}\text{space}}$.

$\square$

4.4 Convenient Properties for Continuous Spaces

The following theorem essentially gives an important clue about the subject that the category of continuous spaces, that is **Continuous**, satisfies almost all good formal properties mentioned at the beginning of Chapter 3.

Theorem 4.4.1. *[6] All (small) limits and colimits exist in the category* **Continuous***.*

.

Proof. Proof comes from the equivalence of the categories $\mathfrak{Cont}$space and **Continuous** proved in Proposition 4.1.4. and the fact that the category $\mathfrak{D}$space has all limits and colimits (see Theorem 3.2.10. in Chapter 3) □

At the beginning of Chapter 3 we talked about that our main purpose is to define new categories in which so many convenient properties is satisfied. Now, as a consequence of the previous theorem, we have a convenient category **Continuous**. Let us check some of the properties satisfied in **Continuous**

1. **Subspaces** Assume that X is a continuous space and take any subset $Y \subseteq X$. In order to make Y a continuous space, we consider that $\varphi : D \to Y$ is a plot in Y iff $i \circ \varphi$ is a plot in X where $i : Y \hookrightarrow X$.

   With this construction of plots, the inclusion $i : Y \to X$ is continuous.

2. **Quotient space** Suppose that we have a continuous space X and an equivalence relation $\sim$. We give a structure to the quotient space $Y = X/\sim$ by defining plots in Y as following :

   $\varphi : D \to Y$ is a plot in Y if there is an open cover $(D_j \mid j \in J)$ of D and a collection of plots $\{\varphi_j : D_j \to X\}_{j \in J}$ in X such that:

$$\begin{tikzcd} D_j \arrow[r, "\varphi_j"] \arrow[d, "i_j"'] & X \arrow[d, "p"] \\ D \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

commutes. Here $i_j : D_j \hookrightarrow D$ and $p : X \to Y$ is the induced function by $\sim$. This is called the **quotient space structure**.

With this construction of plots, the quotient space $p : X \to Y$ is continuous.

3. **Initial object**

   Now we consider that every map from every object to $\emptyset$ is a plot(This holds for only empty domain).This is the only way that we can make the $\emptyset$ a continuous space. this continuous space is the initial object of **Continuous**.

4. **Terminal object**

   There is only one way to make the one element set **1** a continuous space. This way is to consider that each function from each object to **1** is a plot. This setting makes **1** a terminal object.

5. **Locally cartesian closed**

   Continuous is locally cartesian since $\mathfrak{D}$space is locally cartesian closed (see page 33 in [6]). In addition, since a locally cartesian closed category which has a terminal object is automatically cartesian closed, **Continuous** is also cartesian closed.

6. **Products**

   Assume that we have two continuous spaces X and Y. We give a structure to the product $X \times Y$ of the underlying sets of X and Y by defining plots in $X \times Y$ like this :

   - $\varphi : D \to X \times Y$ is a plot iff $p_X \circ \varphi$, $p_Y \circ \varphi$ are plots in X and Y, where p_X and p_Y are projection maps.

   With this construction of plots, it can be said that p_X and p_Y are continuous. Besides, for given continuous space Q, continuous maps $f_X : Q \to X$ and $f_Y : Q \to Y$, there exists a unique continuous map $f : Q \to X \times Y$ such that

$$\begin{tikzcd}
 & Q \arrow[dl, "f_X"'] \arrow[d, "f"] \arrow[dr, "f_Y"] & \\
X & X \times Y \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y
\end{tikzcd}$$

is commutative. This clearly shows us that $X \times Y$ is the product of X and Y in **Continuous** .

7. **Equalizers**

   Assume that we have continuous maps $f, g : X \to Y$ between continuous spaces. Then,

$$Z = \{x \in X : f(x) = g(x)\} \subset X$$

   is also a continuous space. According to this fact, $i : Z \hookrightarrow X$ is the equalizer for f and g :

$$\begin{tikzcd} Z \arrow[r, "i"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \end{tikzcd}$$

Of course there are other convenient properties like *coproducts, coequalizers, pullbacks, pushouts ...* to check. But, since all these properties are just some special cases of *limits and colimits*, it is enough to check just these and by Theorem 4.2.1. limits and colimits exist in **Continuous** immediately.

Now we will the definition of a cartesian closed category.

Definition 4.4.2. *A category D is called* **cartesian closed** *if and only if it has finite finite products and exponentials, i.e. given Y, $Z \in \mathrm{ob}(D)$, there is an object Z^Y such that there exists $u : Z^Y \times Y \to Z$ for every $f : X \times Y \to Z$ and there exists $f^{'} : X \to Z^Y$ such that*

$$\begin{tikzcd} Z^Y & Z^Y \times Y \arrow[r, "u"] & Z \\ X \arrow[u, dotted, "f^{'}"] & X \times Y \arrow[u, dotted, "f^{'} \times \mathrm{id}_Y"] \arrow[ur, "f"'] & \end{tikzcd}$$

is a commutative diagram.

The category of topological spaces is not cartesian closed ([8]) and we have earlier said that (in page 30., property 5) **Continuous** is cartesian closed. Therefore we have the following remark.

Remark 4.4.3. Top $\ncong$ Continuous

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