SHIPMENT CONSOLIDATION UNDER DIFFERENT DELIVERY DATE OPTIONS FOR E-TAILING

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By

Tuğçe Vural

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this thesis, we consider a shipment consolidation problem for an e-retailer company which has two type of services for its customers: "regular" and "premium". In the regular service, the e-retailer guarantees a delivery time to its customers. However, in the premium service, customers get their items in negligible or zero amount of time, such as same-day delivery, supplied physical inventories located sufficiently close. When a shipment decision is made, it serves both customers of the regular service and small inventories for the premium service. In our study, we analyze shipment consolidation operation given these two services for both deterministic and stochastic demand structure. In the deterministic demand problem, our average profit maximizing model decides the optimal service choice; we provide optimality conditions, an algorithm to find optimal solution, structural analyses and numerical results. In the stochastic demand setting, we evaluate the problem for the regular service which has Poisson demand. Then, we expand the problem by including the premium service which has deterministic demand. For this problem, we present an approximate model for a modified version of the policy used for regular-service-only problem and compare the performance of the approximation with a simulation.

Keywords: E-retailing, Shipment Consolidation, Promised Delivery Time.

ÖZET

FARKLI SERVİS SÜRELERİ ALTINDA E-TİCARET ŞİRKETLERİ İÇİN SEVKİYAT OPERASYONU

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Bu tezde, müşteriye malı gönderme özelliklerinin "normal" ve "özel" servis adı altında farklılaştıran bir e-ticaret sirketi için sevkiyat konsolidasyonu problemi ele alınmıştır. Normal serviste e-ticaret şirketi müşterilerine belli bir teslimat süresini garantilemektedir. Özel serviste ise şirket satın alınan ürünleri anında sayılabilecek bir hızla müşterilerine yeterince yakın olan bir envanterden tedarik edip teslimat gerçekleştirilmektedir. Herhangi bir sevkiyat kararı verildiğinde, sevkiyat aracı hem normal servisi kullanan müşterilere satın aldıkları ürünleri taşımakta, hem de anlık tedariği mümkün kılacak müşterilere yakın envantere ürün taşıyabilmektedir. Çalışmamızda hem bilinen, hem de rassal talep için sevkiyat konsolidasyonu operasyonu iki servis çeşidi de göz önünde bulundurularak analiz edilmiştir. Bilinen talep probleminde, modelimiz en iyi servis çeşidine karar vererek ortalama karın maksimumunu bulmaktadır. Bu problem için, eniyilik şartları, en iyi çözümü bulan algoritma, yapısal analizler ve numerik sonuçlar sunulmuştur. Rassal talep modelinde ise, normal servisin müşterilerinin Poisson dağılımından geldiği varsayılmış ve problem bu varsayımın üzerinden sadece normal servis için değerlendirilmiştir. Sonrasında, probleme özel servis, bilinen bir talep yarattığı varsayılarak dahil edilmiştir. Genişletilmiş problem için ise, normal servis için kullanılan politikanın bu duruma uyarlanmış bir biçimi altında çalışacak yaklaşık bir analitik model önerilmiştir. Yaklaşık modelin performansını gözlemlemek için bir benzetim modeliyle karşılaştırılmıştır.

Anahtar sözcükler: E-Ticaret, Sevkiyat Konsolidasyonu, Teslimat Süresi.

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Lastly, I would like to dedicate this thesis to memory of my father, Mustafa Vural. His words have been a source of strength, and guidance in my life.

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Chapter 1

Introduction

As the Internet has brought a new channel for communication, many traditional businesses got benefit from it to reach their current and potential customers. Using this new channel not only provided a link to better communication for businesses but also cultivated new ideas about alternative ways of operating them. At the end of 1990s, one of the emerging new business ideas on the Internet was observed in converting traditional retailers into online shops. The basic idea is to carry brick-and-mortar retailing into the Internet by presenting products, making agreements and completing transactions online while keeping some assets such as inventories and distribution systems for physical operations. As a result, the concept of e-retailers (electronic retailers) came into play.

Today, Amazon.com, Alibaba, eBay, Peapod, Etsy, Dell, Walmart.com are some of leading e-retailers. Their customers visit e-shops, choose their items, make transactions via internet and their orders are shipped to them. According to the Forrester Research eCommerce Forecast, online sales are expected to yield $334 billion in 2015 and $480 billion by 2019 in US [1]. Additionally, this pattern for expected increase of sales in e-retailers is also foreseen for many countries. When this is the case, to get bigger slice in the market, e-retailers not only should use the Internet efficiently to provide what customers may want to purchase but also should manage their distribution channels effectively to complete their agreements with their customers. Therefore, today taking a place near the top in the competition of e-retailers depends on both putting high customer expectations and fulfilling them in terms of physical attributes in their agreements.

One of the key areas of this competition where the e-retailers are battling to beat each other, and perhaps the most significant one, is the promised delivery time. Almost all world wide e-retailers, informs their customers about expected delivery dates and sometimes provides exclusive services such as same-day or two-day shipping. This is such a competitive area that, in 2014, Amazon.com introduced their new patent "Method for Anticipatory Package Shipping" which is basically a method to start delivering items that are anticipated to be purchased by customers [2]. It is a very futuristic idea based on forecasting and has not being applied yet. Of course, one would expect that a powerful inventory and distribution system will be the backbone for its realization. Furthermore, an initial step for such kind of applications is announced in May 2015 and Amazon.com introduced their free same-day delivery option for only a group of items that is exclusively available for the members of "prime service". When this is the case for Amazon.com, to be able to compete, retailers try to optimize their supply chain operations by opening new warehouses, finding ways to place inventories as close as possible to potential customers, and make use of efficient logistics alternatives.

Directed shipments for an order, though, creates better service levels in terms of the customer satisfaction, they come with very high costs. Moreover, these costs may not be reflected to customers as the benchmark service level in the market already has high standards. Therefore, managing distribution channels under better utilization of delivery options may result with huge savings in operational costs of e-retailers. We consider a stylized environment with a single product distributed via a single distribution point, say a fulfillment center. "Regular Service" specifies those customer orders within a specified delivery time. The fulfillment center may ship some products, without an actual order realized, to an inventory location, closer to the location of potential customers and may satisfy a group of customers with a shorter (usually negligible) delivery time. We call this as "Premium Service", and the customers of the Premium Service may be a part of the regular customer set, as well as additional customers may be attracted with the more favorable delivery option. The issue of shipment consolidation enters the picture here: How one should plan for a shipment, which consists of products shipped to Regular Service customers, and products shipped to inventory for Premium Service customers? We have two constraints: specified promised delivery dates for Regular Service customers should be obeyed, as well as the capacity limitation of the truck. We introduce other details at the end of the chapter. In this study, we are interested in the cost saving opportunities in terms of shipment activities without exceeding the customer service level which is the maximum length of delivery time.

1.1 Literature Review

In this section, we briefly mention about the related preceding studies in the literature in terms of both their settings and ways of approaching to their solutions.

1.1.1 Background in E-tailing

Basu and Muylle [3] separates an e-commerce activity into five steps: *search* which is the activity to find products in the web, *valuation* which is the defining step of possible prices and deciding one of them to buy the item, *authentication* which is the agreement part between the buyer and the seller, *payment* where the seller gets the payment, *logistics* which is the delivering the item physically and lastly *support* which is the supporting the buyer after sale. The scope of our study is included only in the operations of logistics for e-commerce activities.

Any operation regarding logistics requires a strong distribution network to be able to fulfill expectations of customers. Li and Muckstadt [4] defines an e-retailer's distribution environment as multi-echelon distribution system. In detail, from central warehouses to regional centers, they may have various types of inventories, all can be considered as customer fulfillment centers. These warehouses also carry various types of products. Therefore, there is not only a direct stream from inventory to customers but also in between warehouses to keep the balance between inventories and be prepared to any possible future demand. Rabinovich and Evers [5] also discusses this problem and tries to provide an insight. Hence, considering inventory balancing in terms of being responsive operationally to any order is also major problem itself in logistics of e-retailers.

Another dimension of interest regarding distribution system of e-retailers are lead time of order deliveries to customers. De Koster [6] states that it is one of the substantial parts about meeting the expectations of customers. Rabinovich [7] identifies one of the performance measure for inventory management as the ability to satisfying customer expectation with preventing high level of accumulations in inventories. This can be realized with an intelligent warehouse distribution system which provides direct fulfillment of orders to avoid from longer delivery times [6]. Therefore, general attitude towards deciding the right fulfillment center among warehouses for an order is to choose the closest available warehouse to the order destination. Furthermore, today delivery lead times not only has to fulfill the customer expectations, but also be able to compete with other firms. Even if the market still has promised delivery times in days or weeks, e-tailing evolves to promise their customers same-day deliveries. Hence, the importance of quick response to demand increases. However, this short delivery time constraint triggers high shipment costs. Order deliveries can be operated by e-retailers' own fleet or by third party logistic firms such as UPS, FedEx etc. In either case, the main concern for the e-retailer is to keep its delivery time promise to be able to protect and increase its market share. Thus, sometimes at the expense of high shipment costs, companies are willing to deliver orders in without exceeding the promised delivery time that they indicated to their customers.

Recent studies on e-retailers generally concentrate on inventory allocations to find ways to prevent high shipment costs. Since there is generally an uncorrelated geographical demand and supply distribution in the real world, a wise item allocation to warehouses, at least, may prevent supplying an item from a very distant warehouse with high costs. Xu et al. [8] discusses the order fulfillment decision of an e-retailer company by considering periodic evaluation of customer and fulfillment center assignment and provide a heuristic to minimize costs. Furthermore, Acimovic and Graves [9] points out the shipment decision problem of an e-retailer company by considering inventories located in different regions. They present a heuristic method that considers possible future costs and minimizes current outbound shipment costs. Also, Acimovic and Graves [10] emphasizes replenishment policies and right product allocation problem in inventories or fulfillment centers of an e-retailer company to minimize outbound shipment costs while considering geographical mismatch of supply and demand. On the other hand, Aksen and Altinkemer [11] analyses hybrid brick-and-mortar retailing business in which they have both walk-in and online customers. They consider the problem under classic vehicle routing concept between warehouses and customers. Yanık et al. [12] carries this approach one step ahead and examines premium products in e-tailers such as groceries that should be delivered within the same day under a multi-vendor environment.

As the literature of e-retailer distribution system concentrates right and practical inventory allocation decisions, there is a lack of study on shipment operations to customers. Our interest in logistic operations of e-retailers is the last stage of the distribution system. Specifically, in our study, we focus on the shipment operation between a fulfillment center and customers.

1.1.2 Idea of Shipment Consolidation

Another stream of research that touches our study uses the idea of consolidation. Consolidation is a strategy in logistics to attain economies of scale in costs per unit carried. The main aim behind the consolidation idea is lowering the cost of individual dispatching by holding items until a predefined threshold. Therefore, consolidation policies may have various characteristics.

Hall [13] classifies consolidation operations in three major groups in terms of physical location: terminal consolidation, vehicle consolidation and inventory consolidation. In the terminal consolidation, items are brought from different locations to be loaded in a temporary terminal and sent via new vehicles to different locations such as cross-docking. On the other hand, vehicle consolidation is defined as a type of *milk run* and the consolidation is based on collecting and distributing items. Inventory consolidation is basically locating different items in the same place, and dispatching them via a single vehicle. From Hall's [13]'s definition, *shipment consolidation* is the combination of concepts regarding inventory and vehicle consolidation: It uses truck's capacity as an inventory, and releases shipments in the same truck when a threshold is reached. Furthermore, two major groups for cargo types are identified in consolidation strategies by Higginson [14] as *common* and *private carriages*. In private carriage, transportation vehicles are operated within the same company, and costs are generally fixed per dispatching operations. On the other hand, if the producer company does not have vehicles and shipment operations are managed by an outside company, the carriage type is considered as common type. In that case, costs my have discount rates according to the agreements between the outside and the producer companies.

The main incentive behind the shipment consolidation idea is to get benefits from economies of scale in per items carried. Higginson [14] states that shipment consolidation also increases the control on transportations by reducing handling operations of items and resulting more dedicated and direct deliveries. Ülkü [15] provides an insight that another benefit of consolidation can be observed in environmental issues considering carbon emission. On the other hand, more direct and fast deliveries are achieved generally by individual and expensive shipments. That is why, a shipment consolidation should always consider customer service level. Moreover, some shipment consolidation policies do not only result longer waiting time but also may cause uncertain shipment durations, which is also an another drawback. Additionally, consolidation effects inventory levels and holding costs negatively. Managers may need to find extra spaces or even have to keep larger safety stock caused by uncertain shipment times. As a result, planning and managing a good shipment consolidation policy requires elaborated administrative effort.

Even though, defining a consolidation policy has many dimensions, according to Çetinkaya [16] they are separated into two branches in terms of implementation: *pure* and *integrated consolidation policies*. The pure consolidation policies include only consolidation decisions without including managerial concerns of other operations. If the extend of a consolidation policy contains a coordination between other operations such as inventory, it is called integrated consolidation policy. Specifically, integrated inventory and shipment consolidation policies are popular as they are applied by coordinating firms for supply chain operations.

In the literature, there are vast amount of works combining retail activities with shipment consolidation ideas. Especially, vendor managed inventories (VMI), which is an agreed operation between a supplier and a business to keep inventory in certain level by providing information about business' inventory levels, attracts attention in recent years. Çetinkaya et al. [17], Axsäter [18], Çetinkaya and Lee [19], Ching et al. [20], Çetinkaya et al. [21], Kutanoglu and Lohiya [22], Mutlu and Çetinkaya [23], Çapar [24], Kaya et al. [25], analyze shipment consolidation problem under different VMI concepts. Additionally, there exist other retail activities which are studied under the shipment consolidation structure. Hong and Lee [26], Hong et. al. [27] study price dependent demands, where demand is a function of price, under integrated inventory and shipment consolidation activities. Besides, another concept in retail activities that got attention recently is the promised delivery time. It is the length of delivery time that the company guarantees to their customers as promise, and in some studies it is mentioned as *quoted delivery time to customers*. Ülkü and Bookbinder[28], Ülkü and Bookbinder [29] focus customer sensitivity to promised delivery time in shipment consolidation problems.

Many practical examples regarding shipment consolidation policies exist in literature (Hall [13], Higginson and Bookbinder [30]) Most popular operational rules are quantity-based, time-based and time-and-quantity based policies. *Quantity-based policy* implies that a shipment should be realized when the indicated accumulation of items is reached in terms of quantity or weight. As the load in a shipment is one of the cost determining factor, quantity-based policy have long been take a place in the literature. (as examples see: Jackson [31], Gupta and Bagchi [32], Çetinkaya and Bookbinder [33], Hong et al. [27]) On the other hand, *time-based policy* limits the waiting time to wait the accumulation of items and when the specified duration ends the shipment is realized. (as examples see: Jackson [31], Çetinkaya et al. [17], Çapar [24]) By limiting the consolidation time length service levels in terms of time are controlled. Lastly, *time-and-quantity based* or *hybrid policy* enforces the realization of shipment either when the target load or the specified time limit is reached. (Bookbinder and Higginson [34], Mutlu and Çetinkaya [23]) In a hybrid policy, both acceptable service levels and scale economies are considered.

Consolidating shipments received considerable an attention since 1990's to search for an effective way of operating cargo trucks. Early works generally focus on to find practical justification for shipment consolidation operations. Jackson [31] uses simulation to understand effects of consolidation on holding and fix costs in terms of order volumes and gives an insight about consolidation cycles under time-based, quantity-based and hybrid policies. Burns et al. [35] compares direct shipping operation for each order with the strategy of dispatching orders in a truck in an environment where there is a single supplier and multiple customers located in different regions. They develop an analytical method to minimize the distribution cost that is caused by inventory carrying and transportations. Gupta and Bagchi [32] computes an optimal lot size in a shipment consolidation to minimize cost. All these studies advocate the effectiveness of shipment consolidation in terms of cost minimization. Earlier studies are reviewed by Çetinkaya in [16].

Consolidation literature can be reviewed in terms of the demand characteristics considered. There are two main distributions used to describe demand: a distribution that describes the time between two consecutive customer orders, and another distribution that describes the intensity of a customer order (in other words number of units demanded per order). Markovian structures are employed by several studies [36]. (see examples: Minkoff [37], Higginson and Bookbinder [38], Bookbinder and Higginson [34], Cai et al. [39], Çetinkaya et al. [17], Çetinkaya et al. [40], Mutlu et al. [41]) A discrete time Markov Chain represents the demand encountered at each discrete time unit (time between two consecutive customer demand is constant, and intensity of demand at each point is represented by a discrete distribution). A continuous time Markov Process may represent the demand. In that case, the time between two consecutive orders are exponentially distributed random variable. The intensity of demand can be unit (in that case we have Poisson distributed demand) or can be represented by a discrete distribution (in that case we have a compound Poisson distribution with the discrete distribution being the compounding distribution). Specifically, Poisson distributed demand is considered in many studies including: Çetinkaya et al. [17], Ching et al. [20], Kutanoglu and Lohiya [22], Hong et al. [27], Çapar [24], Mutlu and Çetinkaya [23], Mutlu et al. [41], Çetinkaya et al. [42], Marklund [43], Çetinkaya and Bookbinder [33]. On the other hand, Bookbinder and Higginson [34], Higginson and Bookbinder [38], Çetinkaya and Bookbinder [33] and Çetinkaya et al. [40] are examples who constructed their problems with compound Poisson distribution.

For dynamic and stochastic shipment consolidation problem, Minkoff [37] considers Markovian approach to model of serving inventories of different customers with various size of vehicles. Higginson and Bookbinder [38] approaches the shipment consolidation problem with discrete-time Markovian Decision Process (MDP) model. In their model, with the arrival of each customer the shipper has to make a decision regarding realizing the shipment or not. Their aim is to minimize the cost of per shipment in pure shipment consolidation structure by considering the problem for both private and common carrier. Another study of Bookbinder and Higginson [34] evaluates hybrid policy by using stochastic clearance system under private carrier assumption. Bookbinder et al. [44] considers the problem for private carrier under discrete-time batch Markov arrival process. For an arbitrary time accumulated weight of arrived orders in the system and total consolidated weight in a shipment cycle are analyzed and they present a computational method for such performance measures under time-bases, quantity-based and hybrid policies. Cai et al. [39] also employs discrete time batch Markovian arrival process in shipment consolidation problem. In this study dispatching decision is left as a function in the model. Lastly, Kaya et al. [25] applies MDP in a stochastic environment where a single supplier and a retailer exist by analyzing quantity-based, time-based and hybrid policies.

Çetinkaya et al. [17] employs a shipment consolidation problem under renewal theoretical model for supplier operating VMI's. In its setting, the objective is to minimize the expected long run average cost of inventory replenishments under time-based policy by deciding both the optimal shipment release interval and the optimal replenishment quantity. An algorithm to calculate exact optimal values for the problem of Çetinkaya et al. [17] is provided by Axsäter [18]. Later, Çetinkaya et al. [40] addresses the problem of [17] with quantity-based consolidation policy having the construction of general compound renewal process for demands. Moreover, Çetinkaya et al. [21] compares numerically time-based and quantity-based policies mentioned in Çetinkaya et al. [17] and validate that quantity-based policy outperforms in their cost minimization setting. They also propose hybrid policy and provide comparisons with time-based and quantity-based via simulation. Another integrated inventory and transportation problem with similar demand setting to Çetinkaya et al. [17] is modeled in multi-facility and single-echelon environment by Kutanoglu and Lohiya [22] under time-based consolidation policy.

Under VMI context in Ching et al. [20] for time-and-quantity-based consolidation policy. It gives closed form version of the optimal solution regarding the sum of the dispatching, the transportation, the inventory and the re-order costs. Furthermore, Çetinkaya and Bookbinder [33] also considers to apply both time-based and quantity-based shipment consolidation policies by covering cost minimization objectives for common and private carrier settings separately. They present analytical results for the optimal waiting time for time-based policy and the optimal dispatching quantity for the quantity-based policy. Even if they assume exponential order weight distribution, their results are applicable for renewal theory setting. Mutlu and Çetinkaya [23] focus on finding a solution approach regarding optimal inventory level and outbound shipment scheduling policy parameters under the existence of common carrier costs. They employ time-based and quantity-based dispatching policies and provide a search algorithm for policy parameter for given bounds.

One of the closest study to ours is presented by Mutlu et al. [41] considering pure consolidation problem with hybrid shipment consolidation policy by having Poisson distributed demand. Their objective is to minimize the total cost which includes expected cost per shipment and expected waiting cost of shipment load. The expected shipment cost constitutes fix cost and a loading cost per demand. The capacity of shipment per truck is not incorporated. Furthermore, Mutlu et al. [41] derives an analytical expression for the optimal value of the objective function. They also present an analytical comparison of three shipment policies regarding their performances. On the other hand, Çetinkaya et al. [42] focus on the service based comparison of different shipment policies. Order arrivals are distributed as Poisson and they consider maximum waiting time and average order delay as service measures. Their results show that under fixed policy parameters, hybrid policy outperforms the time-based policy in terms of maximum waiting time. Considering the average order delay, again hybrid policy outperforms both quantity based and time based policy separately.

Another stream of VMI concept studies on nonidentical retailers. Marklund [43] analyze an inventory replenishment and shipment consolidation between single supplier and multiple nonidentical retailers. The demand is assumed to be Poisson distributed under time-based dispatching policy. Marklund [43] provides a two heuristic method to to identify expected backorder and holding costs. Furthermore, Çapar [24] also use time-based policy to operate shipment activities between an outside supplier, a distribution center and nonidentical retailers where each retailer again has Poisson demand. An optimization method to obtain an optimal order-up-to level for retailers and optimal replenishment quantity for the distribution center is provided by them.

Recently, the literature with Poisson demand setting in retail activities is expanded for more complex demand functions: price effecting the demand; quoted delivery time effecting the demand. Hong et al. [27] considered integrated inventory and transportation problem with pricing concern under quantity-based policy. They constructed the model with Renewal theory and analyze the case where demand is a linear function of time. Then, a more general cases such as demand is a concave and convex function of price is presented. Moreover, Hong and Lee [26] employ analysis of optimal time-based consolidation policy by assuming the same relation between demand and price. They present an algorithm to obtain the optimal price, consolidation cycle and replenishment quantity. Their study also includes extensions regarding quantity discounts for dispatching costs and hybrid policy version of the problem. Another case is where demand is affected by the quoted delivery time. Ülkü and Bookbinder [28] consider time and price sensitive demand in third party logistics scheme with time based policy, and extend the study for different pricing schemes. Furthermore, Ülkü et al. [29] employ the customer demand that is sensitive to both price and service as delivery time-guarantee. By maximizing the vendors profit, they validate that optimal price is concave for the capacity of the shipment vehicle.

Ultimately, when the shipment consolidation problem has the stochastic demand setting, it can be easily said that one of the main concerns is to wait and accumulate orders. The general attitude towards avoiding the extreme drawbacks from this situation is either incorporating the waiting costs into a cost minimizing objective function or comparing the effectiveness of different shipment policies in terms of expected waiting times or costs as the studies in literature shows.

The second demand class in shipment consolidation problem is the deterministic demand. Even though, most of the recent studies cover the problem for time-based, quantity-based and hybrid policies with stochastic demand settings, there are some recent studies which assumes deterministic demand in retail activities for popular shipment consolidation policies. In Çetinkaya and Lee [19], consolidation problem is addressed in third party logistics with deterministic demand. Their main concerns are about frequency of shipments and the critic quantity to replenishing the third party inventories. Thus, Çetinkaya and Lee [19] aim is to minimize the total cost and decide the optimal consolidation cycle length within an inventory replenishment cycle. The objective function constitutes four type of costs: inventory replenishment, inventory carrying, customer waiting and outbound transportation costs. The capacitated and uncapacitated cargo versions are presented and Karush-Kuhn-Tucker conditions [45] is used to construct enumeration method to reach optimal solution. Later, Moon et al. [46] carry the problem of Çetinkaya and Lee [19] a one step further and employ the joint replenishment with multiple items. They introduce two time-based policy and algorithms to reach near optimal parameters for specified policies. Another study using deterministic demand assumption is used by Hwang [47]. In this study, economic lot sizing for transportation, production and inventory decisions is considered with the assumption for stepwise cargo costs.

1.2 Problem Definition

In our study, we combine e-retailer's environment with shipment consolidation problem under hybrid policy. We assume that the e-retailer has an option to offer two types of delivery services which differ according to their promised delivery times, as well as prices. The offered options are available on the web page when customers navigate in a product. Specifically, when the e-retailer decides to operate a service type for a specific product, customers are able to see only the type of service with promised latest delivery time and price. The first type of service is named as the "Regular Service". If an item is operated under this service, on the web page customers see a positive maximum length of time that is guaranteed for the item's delivery. The second type of service is the "Premium Service". If the e-retailer chooses to operate under the premium service for an item, customers are guaranteed to have immediate delivery when they purchase. In other words, customers are served in negligible amount of time such as shipping in a few hours or at most in the same-day which can be interpreted as zero promised delivery time. Note that, both regular and premium services can be offered at the same time, as well. As a result of the difference in promised delivery times in the premium and the regular services, the demand rates for each service type in each service combination can be different. Thus, the demand rates in different services are represented separately.

Another important issue arising with the idea of zero promised delivery time is the existence of inventories. We assume that the e-retailer has small inventories very close to its demand points such as having small "lockers" in neighborhoods or even a body of shipment vehicle that can serve in zero promised delivery time. The main idea having small inventories is that when the company decides to apply the premium service for an item, they make an anticipated shipment of that item to the vicinity of the demand location and keep inventory to satisfy possible future demand. From the customers' point of view, if the e-retailer applies this service to an item, customers who are in the same region with such inventories see an immediate delivery option. Hence, this inventory can be considered as a demand point, which actually is a fulfillment center for the premium service customers.

Our problem considers a single echelon structure for the consolidation operation which is between a single fulfillment center and a single customer region. In Figure 1.1 consolidation activity is represented. In this region, there exist customers and a single inventory point for the premium service. Note that the region is the local area that makes a shipment vehicle serves within a negligible amount of time. When orders come from this region under the regular service, the shipment vehicle waits to consolidate and then ships to the customers in that region. We assume that the time spend within the region for delivery processes and the transportation time between the fulfillment center and the region are both zero without loss of generality. Hence, the shipment vehicle waits in the fulfillment center at most the promised delivery time. On the other hand, it may also carry items which will be offered under the premium service. Therefore, the vehicle consolidates items for both services in a single shipment.

We assume the problem for unlimited fleet availability. However, the shipment vehicle has a constant capacity, another constraint for the problem in addition to the promised delivery time constraint. The problem considers infinite time horizon, hence shipments repeat themselves. Finally, for sake of simplicity, we assume the e-retailer offers a single item to their customers.

The main concerns are to decide how much time a shipment vehicle should wait in a fulfillment center. Note that, by this decision, we implicitly affect two measures: cycle length as the time between two consecutive shipment realizations, and realized capacity of the truck.

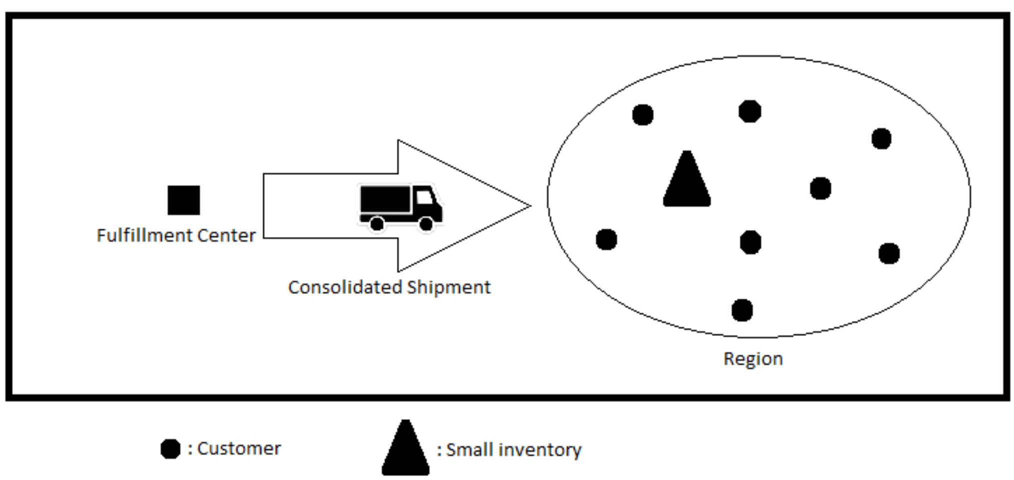

Figure 1.1: Shipment Consolidation

We differentiate orders coming in each service in terms of their revenues. Furthermore, each shipment realization has a fixed cost regardless of where it serves: the inventory or customers of the regular service. However, items carried to the inventory causes holding cost per unit per time. Different from literature, we do not include waiting costs to the problem as it is an agreed time length to wait between a customer and the e-retailer. We provide two type of settings for the e-retailer's consolidation problem. In the first setting, the problem is constructed upon deterministic demand assumption for all type of services, and the model maximizes the average profit. In the deterministic setting, we decide the optimal cycle length by considering different services' profitabilities. Note that under deterministic demand, finding optimal cycle length corresponds to finding which service (or services) to use within the cycle and their duration. In other words, we allow for one service (or both services together) to stop and another service (or both services together) to start within the cycle, if it is profitable. In the second setting, we first assume that the e-retailer excludes the premium service option and operates only the regular service which has Poisson distributed demand under hybrid consolidation policy, similar with Mutlu et al. [41]. However, different then Mutlu et al. [41] we have only fixed cost per shipment as pure consolidation problem, and the model maximizes expected profit. Then, we extend the problem by including the premium service option having a constant deterministic demand rate, and provide an approximate model. In the extended problem,the model maximizes the expected profit by deciding optimal amount to send the inventory for the premium service, serving the regular customers under hybrid consolidation policy.

The rest of the thesis includes is as follows. In the next chapter, we present the shipment consolidation problem for an e-retailer under deterministic demand. We present analytical results and a solution methodology to determine an optimal solution. In Chapter 3, a continuum of Chapter 2, we provide numerical results and analyze special cases. In Chapter 4, we address the shipment consolidation problem of e-retailer under stochastic demand structure, specifically we assume that the regular services' demand follows a Poisson process and we analyze two cases. The first case, when there is no possibility for the premium service, we evaluate hybrid (time-and-quantity) shipment consolidation policy for the regular service operation. In the second case, we assume that additional to the regular service, we have the premium service possibility to serve customers with deterministic demand. We propose a simple policy.. policy. Note that when demand is stochastic, we can only optimize the decisions with respect to a stated policy, unlike like deterministic demand case, where we were able to find the optimal policy (which allows the use a combination of services within a cycle, so long objective function is maximized). We evaluate an approximate analytical model to optimize the policy proposed and evaluate the approximation with simulation. In the last chapter, we conclude our study by summarizing our results and offering possible extensions.

Chapter 2

Shipment Consolidation Problem of E-tailing with Deterministic Demand

2.1 Model Description

In this chapter we consider a deterministic constant demand environment. Particularly, we assume the demand rate for the regular service customers to be constant and known, as well as the premium service. We assume an infinite horizon problem, resulting in finding shipment cycles which will repeat. Hence, the objective function can be specified as maximization of average contribution, contribution meaning as revenues minus costs. For simplicity, we assume that each order comes as a single unit. Under above constructions, we investigate different operational strategies for the e-retailer.

Note that shipment consolidation for this case means that we would like to include the orders for the regular service customers, as well as the products that are moved to closer inventory location for the prospective premium service customers. We allow for possible lost sales for both types of customers. Hence the decision problem is to find a shipment cycle. Note that within a shipment cycle, there will be a single shipment, part of the products in this shipment will be used to satisfy orders (regular service) and the remaining will be inventoried to satisfy premium service customers. The costs associated with this decision are a fixed cost of transportation and inventory carrying cost for the inventoried products. The revenue associated with the decision is the revenue obtained from sales of both type of customers. Note that loss of revenue can be defined as cost of lost sales, as well. The contribution obtained for the duration of the cycle is then divided by the cycle length to find the average contribution.

Finally, we define the cycle length in terms of three decision variables: T_0, total duration within the cycle where we only aim to satisfy regular service customers; T_P, total duration within the cycle where we only satisfy premium service customers; T_J, total duration within the cycle where we satisfy regular service, as well as premium service customers at the same time. These three durations can be considered to represent three different operational strategies. We analyze these strategies in detail.

The first operational strategy is to apply "regular policy". In this policy, the company quotes a delivery time to its customers and delivers them in the quoted amount of time. Hence, TQ based policy has a defining role on the cycle length of shipments. The relation between the regular policy's demand rate and time length of two consecutive shipments is shown in Figure 2.1.

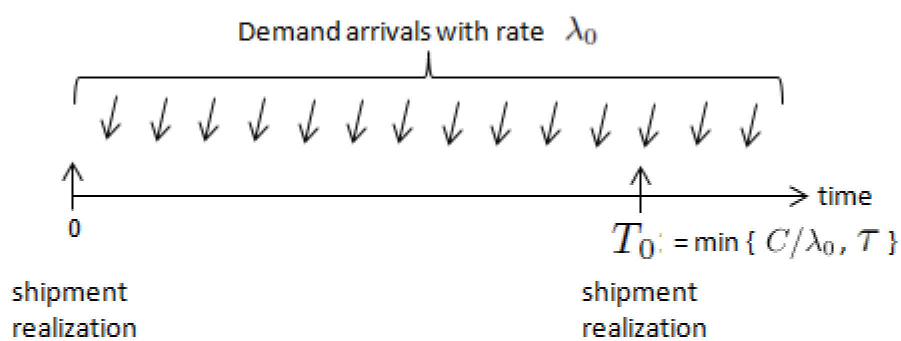

Figure 2.1: The Regular Policy

where

λ_0: the demand rate of regular service

τ: the promised delivery time

C : capacity of the shipment truck

T_0: total duration within the cycle length where only the regular service exists.

The second operational strategy for deliveries of the e-retailer company is what we call "premium policy". Being different from the regular policy, in this operation rule, the company keeps a close inventory for its customers. This inventory ensures that the company can deliver customers' orders in a very short amount of time such as within the same day or even within a few hours after customers give their orders. In that case, the promised delivery time is not the concern of the company's service quality. Thus, this policy can be considered as an exclusive service for committed or loyal customers, as well. As all orders are fulfilled from this "closest" inventory, a shipment, in this case is to the location of inventory before demand is realized. Hence, the only limitation for the cycle length of a shipment is the capacity of vehicle. In Figure 2.2, the relation between the inventory level and the time between two consecutive shipment is demonstrated. Notice that each shipment can be considered to feed next cycle's demand as customers are assumed to be satisfied from the inventory. Here, we show the demand rate for the premium service as an inventory that decreases with demand.

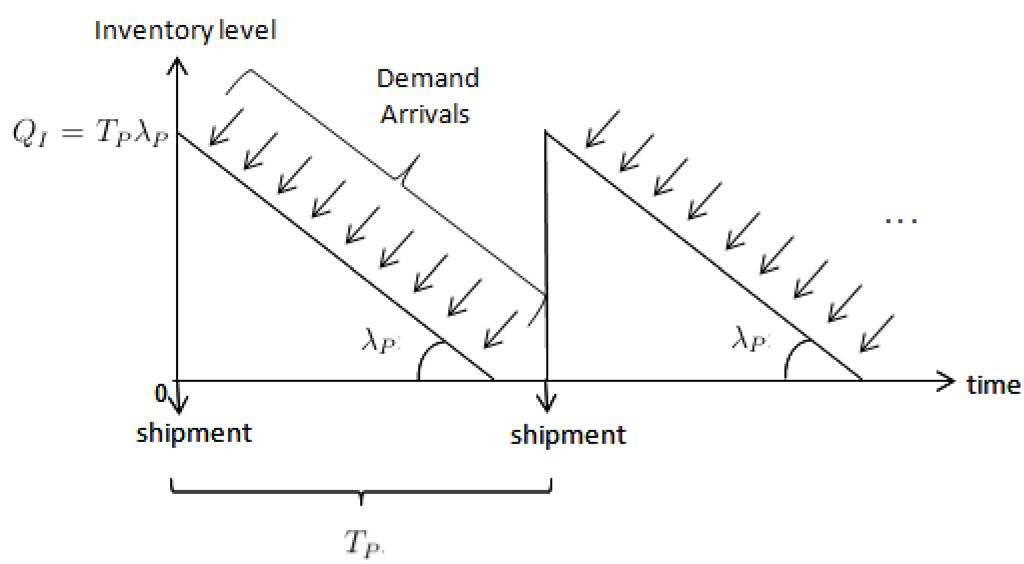

Figure 2.2: The Premium Policy

where

λ_P: demand rate of the premium service

T_P: total duration within the cycle length where only the premium service exists

Q_I: the beginning inventory level (assumed to be delivered at the end of each cycle).

As a final operational strategy, the regular and the premium policies are considered together and called the joint policy. In this operation option, the company leaves the delivery choice to its customers by offering the regular and the premium service options at the same time. In other words, customers see two type of delivery options with different promised delivery times for the item. Since the decision is made by customers, the total demand of the joint policy are fed by two streams according to customers' choices. At this point, it is assumed that having two types of service option at the same time will affect the demand rate coming from regular customers. Hence, the demand rate of regular service users in the joint policy is different from the demand rate of the regular policy, λ_0. The intuition would be a decreasing demand rate for regular service, when both are offered. However, we allow for more general structures and do not restrict the demand rate in this case. Furthermore, as the joint policy has the premium service option, it also uses the inventory. In the same manner, for customers who choose the regular service option, the joint policy takes the promised delivery time into account. Notice that, the capacity limitation of a single shipment which serves regular customers and the inventory is still valid. Additionally, under the joint policy, we do not allow lost sales from neither the regular nor the premium delivery service as both services are available to the customers. In Figure 2.3, the relation of inventory levels and the cycle length considering demand rates can be seen.

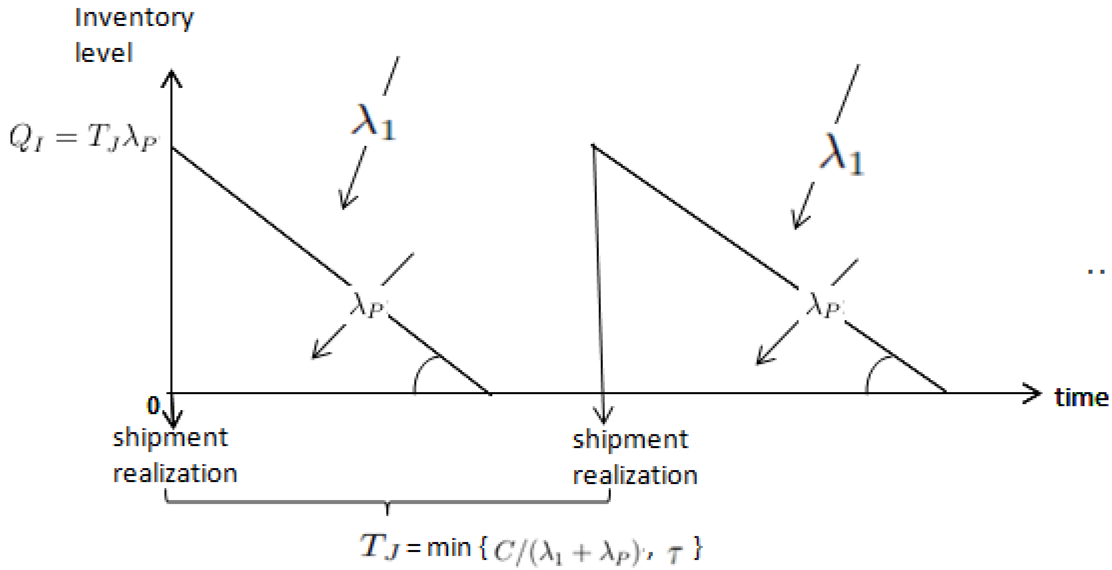

Figure 2.3: The Joint Policy

where

λ_1: demand rate of regular service customers in the joint policy

T_J: total duration within the cycle length where only the joint policy exists.

Ultimately, to be able to make a comparison between all policies, we consider all of them at the same time. As it can be seen in Figure 2.4, the cycle length is constituted by the sum of time lengths devoted to each policy. Therefore, deciding contributions of T_0, T_P and T_J to the cycle length leads us to the optimal policy selection if we consider all policies together in the same deterministic setting problem. From the customers point of the meaning of the cycle length is the following: When the shipment is realized and T_P is positive, customers can see only the premium service option to purchase the item with zero promised delivery time which is provided by keeping the items in the close inventory. In the same manner, when customers visit the web page of the item in T_J parts of the shipment cycle, they see both types of delivery option available and they decide according to their wishes. Lastly, if the web page of the item is visited in T_0 portion of the cycle, customers has only regular service option for the item's delivery. Notice that, if the cycle length has only positive T_P, we lose from customers who are strictly want to use the regular service. Likewise, having only T_0 in the cycle length means lost sale from customers of the premium service.

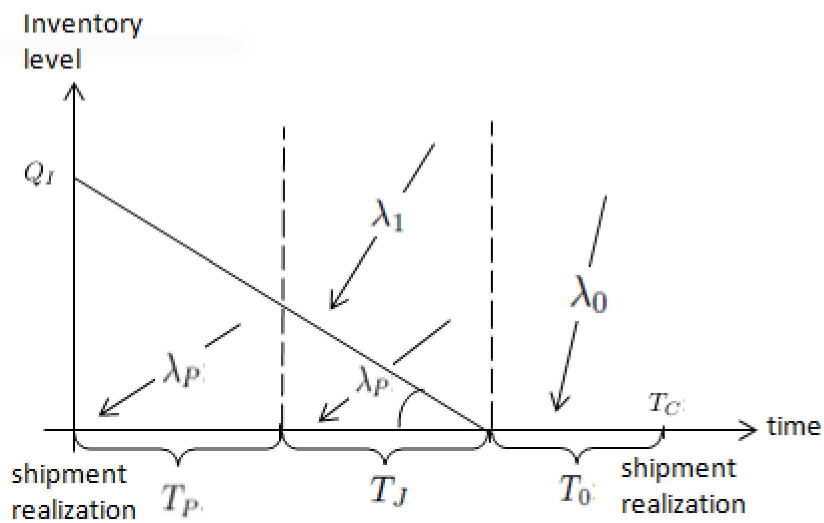

Figure 2.4: The Regular, The Premium and The Joint Policies

where
T_C: the cycle length.

Regarding different delivery strategies that the company can operate, the revenue per unit is denoted separately for different demand types. Also, the total revenue coming from each policy option is calculated and divided by the cycle time as the problem takes place in the infinite horizon. First of all, the total revenue of the regular policy per cycle time is expressed.

$$\frac{T_0 * r_0 * \lambda_0}{T_P + T_J + T_0}$$

where r_0 is the revenue in regular service policy for per unit.

Accordingly, the revenue per cycle time generated by the premium service policy is shown as follows.

$$\frac{T_P * r_P * \lambda_P}{T_P + T_J + T_0}$$

where r_P is the revenue per unit of the premium service.

The joint policy leaves the delivery decision to customers of the company by offering both the regular and the premium delivery options at the same time. For customers who prefer the regular delivery service in the joint policy, the revenue per unit is taken as the same with the regular policy's revenue per unit. In the same manner, customers who prefer the premium delivery option when the company operates according to the joint policy, the revenue per unit of premium service is taken same with the revenue per unit of extend policy. To calculate the joint policy's revenue per cycle time, the revenue coming from each stream of shipment type is accounted and represented as can be shown below.

$$\frac{T_J * (r_0 * \lambda_1 + r_P * \lambda_P)}{T_P + T_J + T_0}$$

In the model, there are two types of operational costs: the inventory and shipment related costs. First of all, it is assumed that for each shipment and respectively for each cycle time, there is a fixed cost to pay. In our problem, the fix cost is the same for each shipment cycle and expressed as follows.

$$\frac{K}{T_P + T_J + T_0}$$

where K is the fixed cost per shipment.

The second cost component is inventory related costs. The inventory related cost is taken as the holding costs for each unit per time. Since both the joint and the premium service policies use the inventory, the inventory cost function takes the time spend on premium service and the time spend on joint service as its two variables. Hence, the inventory cost per cycle time can be shown as:

$$\frac{H(T_P, T_J)}{T_P + T_J + T_0}$$

where $H(T_P, T_J)$ is the inventory cost function.

The inventory cost function, $H(T_P, T_J)$, is taken as a linear function of time. h' is considered as the holding cost per unit per time and the expression for the inventory cost per cycle time becomes:

$$\frac{h * \lambda_P * (T_P + T_J)^2}{T_P + T_J + T_0}$$

where $h = h'/2$.

In the problem of deciding the right operation rule for the e-commerce company, our objective is to maximize the total contribution to profit. As a result, the objective function of the problem is taken as the total revenue per cycle time subtracted from the total revenue per cycle time. The final form of the objective function is expressed as follows.

$$\frac{T_P * r_P * \lambda_P + T_J * (r_P * \lambda_P + r_0 * \lambda_1) + T_0 * r_0 * \lambda_0 - K - H(T_P, T_J)}{T_P + T_J + T_0}$$

where

$$H(T_P, T_J) = h * \lambda_P * (T_P + T_J)^2.$$

However, there are two important considerations for the admissibility of model regarding the capacity of a shipment and the promised delivery time. The constraint that ensures serving to customers in the promised delivery time considers only customers who use regular delivery option as it is assumed that customers of the premium delivery get their orders immediately from the inventory. Therefore, the promised delivery time is a bound for the regular delivery service which exists in the operation rule of both the joint and the regular policy. Explicitly, the sum of time durations of these two policies should be less than or equal to the promised delivery time and it can be shown below.

$$T_0 + T_J \leq \tau$$

where τ is the promised delivery time.

Secondly, in each cycle, a shipping vehicle serves with its constant capacity and carries all customer demands regardless of their types. Therefore, the capacity limit of the vehicle is filled in the regular policy duration with the regular service demand, in the premium policy duration with the premium service demand and in the joint policy duration with its premium and regular services' demand. The constraint belonging to the capacity consideration of the shipment vehicle is expressed as follows.

$$T_P * \lambda_P + T_J * (\lambda_P + \lambda_1) + T_0 * \lambda_0 \leq C$$

where C is the capacity of a shipment.

Finally, variables of the problem cannot be negative since time lengths cannot be negative.

$$T_P, T_J, T_0 \geq 0$$

2.2 The Model

The complete form of the model is as follows.

Parameters:

r_P: revenue per unit of the premium service

r_0: revenue per unit of the regular service

λ_P: demand rate of the premium service

λ_1: demand rate of regular service customers in the joint policy

λ_0: demand rate of regular service

h: inventory holding cost

K: fixed cost of a shipment

τ: promised delivery time

C: capacity

Variables:

T_P: time length used in the premium policy

T_J: time length used in the joint policy

T_0: time length used in the regular policy

$$\textit{Maximize} \qquad \frac{T_P * r_P * \lambda_P + T_J * (r_P * \lambda_P + r_0 * \lambda_1) + T_0 * r_0 * \lambda_0 - K - H(T_P, T_J)}{(T_P + T_J + T_0)} \tag{2.1}$$

$$\textit{subject to} \qquad T_0 + T_J \leq \tau \tag{2.2}$$

$$T_P * \lambda_P + T_J * (\lambda_P + \lambda_1) + T_0 * \lambda_0 \leq C \tag{2.3}$$

$$T_P, T_J, T_0 \geq 0 \tag{2.4}$$

where

$$H(T_P, T_J) = h * \lambda_P * (T_P + T_J)^2 \tag{2.5}$$

2.3 Analysis of the Model and the Optimality Conditions

The objective function, (2.1), has a nonlinear structure. It is neither concave nor convex. See the examples in Appendix A. Hence, it is not possible to use sufficient conditions of *Karush-Khun-Tucker* (KKT) to reach the global optimal solution [45]. On the other hand, constraints and the objective function of the model satisfy requirements for KKT to be used as necessary conditions. Hence, it is possible to explicitly evaluate all solutions of the problem indicated the necessary conditions and choose the one that yields the maximum value for the objective function. Notice that, the necessary KKT conditions do not guarantee to get a positive objective value in the optimal solution. As values of cost variables change, the sign of the objective value may change. Hence, the necessary KKT conditions yield positive or negative solutions as the optimal value.

To obtain the necessary KKT conditions, we need the Lagrangian function form of the problem. We define μ_1 and μ_2 as the Lagrangian multipliers of the promised delivery time constraint (2.2) and the capacity constraint (2.3), respectively. Under these constructions, the Lagrangian function of the problem is formed.

$$L(T_P, T_J, T_0, \mu_1, \mu_2) = f(T_P, T_J, T_0) + \mu_1 * g_1(T_J, T_0) + \mu_2 * g_2(T_P, T_J, T_0) \tag{2.6}$$

where

$$\begin{aligned}
f(T_P, T_J, T_0) &= \frac{T_P r_P \lambda_P + T_J(r_P \lambda_P + r_0 \lambda_1) + T_0 r_0 \lambda_0 - K - h\lambda_P (T_P + T_J)^2}{T_P + T_J + T_0} && (2.7)\\
g_1(T_J, T_0) &= T_0 + T_J - \tau && (2.8)\\
g_2(T_P, T_J, T_0) &= T_P \lambda_P + T_J(\lambda_P + \lambda_1) + T_0 \lambda_0 - C && (2.9)\\
\mu_1 &\geq 0 && (2.10)\\
\mu_2 &\geq 0. && (2.11)
\end{aligned}$$

Finally, we are able to construct KKT conditions to find each possible solution of the problem.

Stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \quad (or \leq 0 \quad if \quad T_0^* = 0) \tag{2.12}$$

$$\frac{\partial L}{\partial T_P} = 0 \quad (or \leq 0 \quad if \quad T_P^* = 0) \tag{2.13}$$

$$\frac{\partial L}{\partial T_J} = 0 \quad (or \leq 0 \quad if \quad T_J^* = 0) \tag{2.14}$$

Primal Feasibility:

$$g_1(T_J, T_0) \leq 0 \tag{2.15}$$

$$g_2(T_P, T_J, T_0) \leq 0 \tag{2.16}$$

$$T_P, T_J, T_0 \geq 0 \tag{2.17}$$

Dual Feasibility:

$$\mu_1 \geq 0 \tag{2.18}$$

$$\mu_2 \geq 0 \tag{2.19}$$

Complementary Slackness:

$$\mu_1 * g_1(T_J, T_0) = 0 \tag{2.20}$$

$$\mu_2 * g_2(T_P, T_J, T_0) = 0 \tag{2.21}$$

After defining the necessary KKT conditions of the problem, we evaluate each of them to be able to generate all possible solutions. Cases are created according to the domain of each variable of the Lagrange function (2.6). More explicitly, a case can take a value of a variable either equal to zero or greater than zero. In the model, two possible options for subsets of three original variables and two Lagrange multipliers are shown in (2.22), (2.23), (2.24), (2.25) and (2.26) and they generate $2^5 = 32$ different cases for our model.

$$T_0 = 0 \quad or \quad T_0 > 0 \tag{2.22}$$

$$T_J = 0 \quad or \quad T_J > 0 \tag{2.23}$$

$$T_P = 0 \quad or \quad T_P > 0 \tag{2.24}$$

$$\mu_1 = 0 \quad or \quad \mu_1 > 0 \tag{2.25}$$

$$\mu_2 = 0 \quad or \quad \mu_2 > 0 \tag{2.26}$$

2.4 Analysis of KKT Conditions

By obtaining 32 cases of the main problem, we have all required equalities for each case that the necessary KKT conditions yield. Each case is solved with corresponding KKT conditions valid for the case. Solutions and related derivations are in Appendix B. Our analyses on 32 cases reveal that some case are not meaningful in terms of the problem considered. The following observations and a proposition are used to identify those situations.

Notice that cases that yield "no operation" option are eliminated from further consideration. Of course, if the problem parameters require no economical operation (this means that we have negative contribution to profit if operation is forced) these cases would be important. Nevertheless we can always depict such situations once the problem solution is obtained.

Observation 1: *There exist some cases with solutions requiring problem parameters to attain unrealistic values. Hence, these cases are eliminated from further consideration. Please refer Appendix B for indicated cases.*

1. *Case 4.1 is infeasible since its solution,* $(T_0, T_J, T_P) = (\infty, 0, 0)$*, is only possible when* $K = \infty$*.*

2. *Cases 1.3 and 2.3 that are constructed by* $\mu_1 > 0$ *with* $T_0 = 0$ *and* $T_J = 0$ *are not admissible as this structure forces the first inequality to be zero. However, the promised delivery time,* τ*, cannot be zero.*

Observation 2: *There exist some cases with solutions requiring trivial relations to hold among parameters. Hence, these cases are eliminated from further consideration unless the parameter values are such that the mentioned trivial relations hold. Please see Appendix B.*

1. *Since Case 1.1 is in the intersection of Case 2.1 and Case 3.1, Case 1.1 is feasible if and only if* $C = \lambda_0 \tau$.

2. *Since Case 1.2 is in the intersection of Case 2.2 and Case 3.2, Case 1.2 is feasible if and only if* $C = (\lambda_P + \lambda_1)\tau$.

3. *Case 4.3, Case 4.6 and Case 4.7 are feasible if and only if* $r_0\lambda_1 = 0$.

4. *Case 4.4 is a special version of Case 4.2 and it is feasible if and only if* $K = \frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)^2}{4h\lambda_P}$ *with having* $T_0 = 0$.

5. *Case 4.5 is a special version of Case 4.3 and it is feasible if and only if* $K = \frac{(r_P\lambda_P - r_0\lambda_0)^2}{4h\lambda_P}$ *with having* $T_J = 0$.

Observation 3: *There exists some cases with solutions forcing the decision variables to take values that are also admissible within the range of another case. Hence these cases are eliminated from further consideration. For such cases please see following cases in Appendix B.*

1. *Case 3.7 converges Case 3.4 as its solution forces* T_P *to be zero.*

2. *Case 4.6 converges Case 4.3 as its solution forces* T_J *to be zero.*

3. *Case 4.7 converges Case 4.5 as its solution forces* T_J *and* T_0 *to be zero.*

Proposition 1. *If two cases take their values of parameters from the same set, then the case with a larger feasible region yields the same or a better objective value. Hence, if two such cases are feasible given a parameter set, the case having a larger feasible area dominates* *the other. In the following list, we show some of such cases and for the others please refer Appendix C. Let, O_i be the objective value of case i and F_i be the feasible region of case i.*

i. $O_{1.1} \leq O_{2.1}$ *since* $F_{1.1} \subseteq F_{2.1}$

ii. $O_{1.1} \leq O_{3.1}$ *since* $F_{1.1} \subseteq F_{3.1}$

iii. $O_{1.2} \leq O_{2.2}$ *since* $F_{1.2} \subseteq F_{2.2}$

iv. $O_{1.2} \leq O_{3.2}$ *since* $F_{1.2} \subseteq F_{3.2}$

v. $O_{1.3} \leq O_{2.3}$ *since* $F_{1.3} \subseteq F_{2.3}$

vi. $O_{1.3} \leq O_{3.3}$ *since* $F_{1.3} \subseteq F_{3.3}$.

2.5 Solution

After solving the problem with necessary KKT conditions and eliminating redundant cases, we further investigate each case with respect to the cost parameters K and h. For most of the cases, we are able to detect bounds for K and h from the inequalities originated by (2.12), (2.13) and (2.14). On the other hand, for other cases finding neat expressions is impossible due to complexities of inequalities. For generated bounds of K and h please refer to Appendix D. These bounds are interpreted as the value limits of related cost parameters and outside these bounds, cases are not admissible.

Observation 4: *There exist economical limitations for cases to operate, which are different from the original feasible conditions. These limitations are generated from their stationary KKT conditions. If for a case with given parameter set, its stationary conditions do not hold, it is eliminated from the possible solutions. For some cases, these economical conditions are simplified to obtain neat cost bounds for K and h. Such cases and their cost bounds are presented in Appendix D.*

2.5.1 The Solution Framework

We observe that bounds generated for K and h contain similar expressions for marginal revenues and demand rates in almost all remaining cases. Therefore, we further expose these cases by looking at expressions of marginal revenues and demand rates that they have in their K or h bounds. We use these expressions as a framework which enables us to understand the environment and the behavior of the problem in terms of demand and revenue relations between different policies. Therefore, according to the demand relations, we divide the problem into three mutually exclusive subsets in terms of demand rates.

- X: $\lambda_P \geq \lambda_0$ where the demand rate of regular service is not greater than the demand rate of the premium service.
- Y: $\lambda_P + \lambda_1 \geq \lambda_0$ and $\lambda_P \leq \lambda_0$ where the demand of only premium service is less than the demand of regular service. Yet, the total demand rate of joint policy is greater than or equal to the demand of regular service.
- Z: $\lambda_0 \geq \lambda_P + \lambda_1$ where the demand of regular service is greater than or equal to the total demand rate of the joint policy.

To address more plausible division of subsets we also consider relations of different demand rates with respect to their revenues generated per unit time. As a result, we reach three subsets.

- *A:* $r_P\lambda_P \geq r_0\lambda_0$ where the total revenue of the premium policy is greater than or equal to the revenue coming from the regular policy.
- B: $r_P\lambda_P + r_0\lambda_1 \geq r_0\lambda_0$ and $r_P\lambda_P \leq r_0\lambda_0$ where the revenue generated from joint policy is greater than or equal to the revenue originated from the regular policy. Yet, the

revenue of premium policy is not greater than the revenue of regular policy.

- C: $r_0\lambda_0 \geq r_P\lambda_P + r_0\lambda_1$ where the total revenue coming from regular policy is greater than or equal to the revenue originated from the joint policy.

2.5.2 The Analysis of Model Subsets

We have three subsets for demand rates and three subsets for total revenues per unit time. With the combination of A,B,C and X,Y,Z, the problem environment is divided into nine distinct subsets: AX,AY,AZ, BX,BY,BZ, CX,CY,CZ. To find the subset(s) where each of remaining 17 cases is able to survive, both feasibility of Lagrange multipliers and economical conditions mentioned in Observation 4 are considered in terms of bounds that we find for K and h. Notice that cases which are feasible only when a special relation holds between parameters or cases which converge to other cases are not included. Consequently, the categorization of each case is shown in Table 2.1. For details please refer to Appendix E.1.

	X	Y	Z
A	1.4, 1.5, 1.6, 1.7, 2.2, 2.4, 2.5, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 4.2	1.4, 1.5, 1.6, 1.7, 2.2, 2.4, 2.5, 2.6, 2.7, 3.2, 3.3, 3.4, 3.5, 3.6, 4.2	1.4, 1.5, 1.6, 1.7, 2.2, 2.4, 2.5, 2.6, 2.7, 3.2, 3.3, 3.5, 3.6, 4.2
B	1.4, 1.6, 1.7, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, 3.2, 3.4, 3.6, 4.2	1.4, 1.5, 1.6, 1.7, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 4.2	1.4, 1.5, 1.6, 1.7, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, 3.2, 3.3, 3.5, 3.6, 4.2
C	1.6, 2.1, 2.5, 2.6, 3.4, 3.6	1.5, 1.6, 2.1, 2.5, 2.6, 3.3, 3.4, 3.5, 3.6	1.4, 1.5, 1.6, 1.7 2.1, 2.5, 2.6, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6

Table 2.1: The separation of cases according to 9 subsets

As each subset of the problem has different characterization regarding demand rates and total revenues per unit time, cases in a single subset are analyzed separately from cases of other subsets. There exist intervals for K or h such that some cases are not admissible at the same time and these value intervals can be ordered consecutively. For explicit relations please refer to Appendix E.2. Therefore, values of K or h alter the case which yield feasible solution to us. Unfortunately, we cannot order relations for regarding K or h for all subsets. However, these ordered relations help us to decrease the number of possibilities which we search for the optimal solution.

Observation 5: *There exist global orders of cases that are valid in all nine sets in terms of their K intervals which are defined by lower and upper bounds of K. Please see Appendix E for the detailed ordering. Let $\underline{K}_i$ be the minimum K value that makes the case i feasible and respectively $\overline{K}_i$ is the maximum value of K which makes case i feasible. We define K_i as the interval defined by $[\underline{K}_i, \overline{K}_i]$ for case i. An order for cases i,j $K_i \leq K_j$ means that $\overline{K}_i \leq \underline{K}_j$. Note that same notation for cost parameter h is also valid, such ordered cases are as follows.*

1. $K_{2.1} \leq K_{2.5}$

2. $K_{4.2} \leq K_{2.2} \leq K_{2.6}$

3. $K_{4.2} \leq K_{3.2} \leq K_{3.6} \leq K_{3.3} \leq K_{3.5}$

4. $h_{1.4} \leq h_{2.4}$ *- except AZ and BZ subsets*

The significance of these divisions of the problem set comes from its ability to differentiate the products. For example, if a product has aspects that match one of the nine combinations, then there are actually less candidates for the optimal policy. Hence, the solution structure may change with products which have different demand and revenue per unit characteristics.

2.5.3 Solution Algorithm

The model satisfies only necessary conditions of KKT. Hence, the solution algorithm should be based on finding all feasible cases and selecting the feasible case that yields the highest optimal value. Our solution algorithm is constructed on nine sets and ordered intervals of K and h as mentioned in Observation 5.

Initially, the algorithm checks each special case which have special equations to be admissible as in Observation 2 and 3. If feasibility conditions hold, the solution algorithm adds it to the Candidate List. Else, it writes down the condition(s) which violates its feasibility.

When the algorithm finishes the check of special cases, it finds the subset that the problem belongs by using values of demand and revenue per unit parameters. Subsequently, the algorithm directs itself only this subset and considers only cases of this subset. At this point, the algorithm starts to check which interval K and h fall by using the information given in Observation 5. Each ordered relation provides at most one candidate and if the algorithm has one candidate, it checks its feasibility. By controlling all ordered relation lists of this subset and cases that are not belong such ordering in the subset, the algorithm updates the Candidate List. When the check finishes, the algorithm calculates optimal values of each member of the Candidate List. Consequently, it gives the highest value as the optimal solution. Of course, it is easy to check if there are any alternative optimal solutions. In the following Algorithm 1, the pseudo code of algorithm is presented.

Algorithm 1 Algorithm

```
procedure FindTheOptimalSolution(item : case)
    for all Case : SpecialCase do
        if Case is feasible then
            Add it to the Candidate List
        else
            Inform the user about the violating constraint(s)
        end if
    end for
    function Find the Surviving Set
        Step 1: Find relations of demands in terms of X,Y, Z
        Step 2: Find relations of total revenues in terms of A, B, C
        Step 3: Combine Step 1 & 2 to find surviving set
        Step 4: Call cases of the surviving set
    end function
    for all List of K Intervals do
        if K belongs to the feasible region of case then
            Add the case to the candidate list
        end if
    end for
    for all List of h Intervals do
        if h belongs to the feasible region of the case then
            Add the case to the candidate list
        end if
    end for
    for all Individual Cases do
        Check feasibility
        if The is feasible then
            Add it to the candidate list
        end if
    end for
    for all Cases in the candidate list do
        Calculate optimall values
    end for
return The case with the highest optimal value
end procedure
```

2.5.4 Verification of the Solution Algorithm

The solution algorithm presented in the previous subsection is verified by a total enumeration scheme. In this scheme, for a given problem we solve all 32 cases, and then applying constraints (including non-negativity) we check if the solution is feasible. We select the feasible solution that yields the best objective value. The related MATLAB code is in Appendix F.

Chapter 3

Analysis of the Shipment Consolidation Problem of E-tailing with Deterministic Demand

In this chapter, we present numerical results to better understand the problem introduced in Chapter 2. Then, a number of special cases of shipment consolidation problem with deterministic demand is addressed.

3.1 Numerical Results

To observe how values of parameters affect the optimal case solution, numerical experiments are conducted. As we think that BY region is operationally the most interesting one, we define our parameter set according to limits indicated by BY. In Table 3.1, choices regarding parameter values are presented.

First, we observe the effect of fixed cost K values for our parameter set. Starting from the zero fixed cost, we increase the value of K and solve the problem for each iteration. Figure 3.1 depicts the value of optimal objective value versus the fix cost. It also indicates the specific case for the optimal solution and the associated positive variables of the case.

The first observation regarding to Figure 3.1 is that when fixed cost is very small or zero, variables operate as if there is no constraint. The main reason behind this result is that when the shipment cost is small enough, shipments do not have a tendency to wait until the full capacity. Furthermore, in Case 4.2 the optimal operation uses customers of both premium and regular services. As the fixed cost increases, the optimal objective value decreases as expected. In addition to it, to compensate high fixed cost, shipments wait until the full capacity. Finally, high K shifts the optimal operation away from only joint policy to utilizing both the regular and joint policies to control inventory carrying costs.

Paramter	r_P	r_0	λ_P	λ_1	λ_0	K	h	τ	C
Value	6	5	15	10	20	100	2	8	150

Table 3.1: The experimental parameter set fits BY region.

Note that one can find the threshold value for K where the optimal policy is indifferent between Case 4.2 and Case 3.4, by equating the objective functions of those cases at their respective optimum.

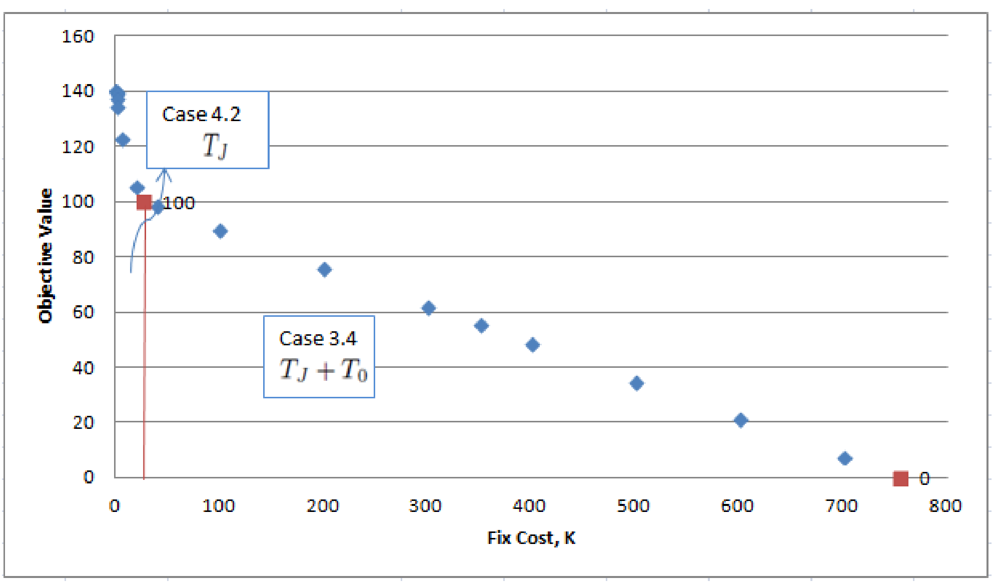

Figure 3.1: Fixed Cost vs Optimal Profit

In the second experiment, we iteratively change the value of capacity given the parameter set of Table 3.1 and illustrate it in Figure 3.2. Our first observation is about the change of objective value. It is clear to see that if C is small enough to make the objective value negative then the decision will be not to operate as it is not profitable. We also observe that operating the regular and the joint policies at the same time is always profitable in our parameter set as in this experiment cost and demand parameters stay same. For the small C values that allows operating, it is easily seen that the limitation for our problem is the capacity. When C is big enough, the promised delivery time becomes the binding constraint as Case 2.4 becomes the optimal case. Notice that the shift from Case 3.4 to Case 2.4, the optimal value is also observed in the Case 1.4 since Case 1.4 is the subset of both Case 3.4 and Case 2.4.

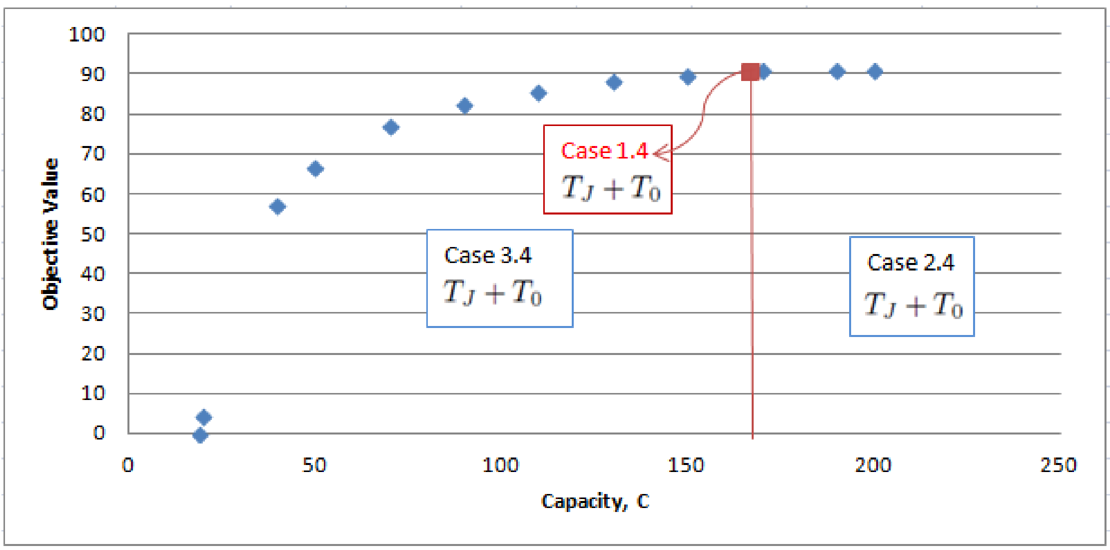

Figure 3.2: Capacity vs Optimal Profit

We change the value of promised delivery time in the given parameter set. Our findings are similar with the second experiment except for extremely small values of the promised delivery time. In Figure 3.3, the optimal case starts with Case 2.5 for every small amount of τ. The main reason behind this choice lies in hybrid consolidation policy. When τ is small enough to realize shipments very frequently, to balance the cost coming from K, the model wants to use unused slots in the capacity to feed the inventory. Hence, the cycle time is combined by both the regular and the premium policies. In Figure 3.3, for small values of

τ, our binding constraint is the promised delivery time constraint. On the other hand, as τ increases, the next optimal appears as Case 1.4. Later, the capacity constraint becomes the binding constraint and there is no further improvement possible.

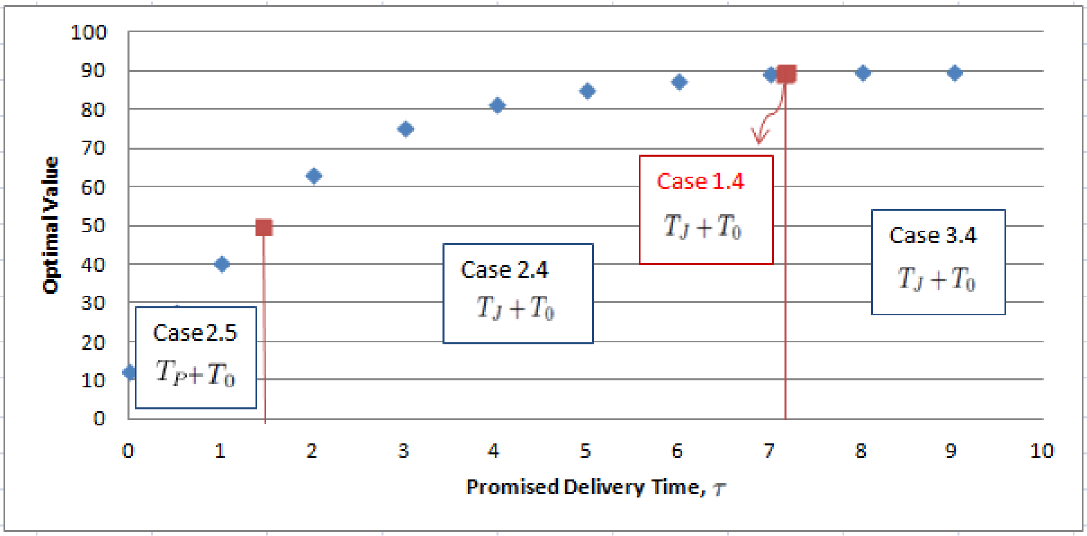

Figure 3.3: Promised Delivery Time vs Optimal Profit

After changing the parameters related to operational limitations, we intend to illustrate the effect of the premium service's demand rate with respect to different costs. In the first analysis, we keep the fixed cost small as 15 and change the value of λ_P, which can be seen in Figure 3.4. Note that the axis scales are not linear in order to show the effects of large parameters. Case 3.1 is seen as the optimal case for small values of λ_P since operating in the regular policy is much more profitable related to the premium service. It is also the case for moderate values of K as in Figure 3.5 and for high values of K as in Figure 3.6. Increasing trend of the λ_P carries the optimal operation from the regular policy to the joint policy for all values of K since higher demand rate brings more revenue. Thus, we observe the transition from T_0 to $T_0 + T_J$ and lastly T_J.

Furthermore, if Figure 3.4, 3.5 and 3.6 are considered together, as K increases, the region of Case 3.4 expands and Case 3.1 shrinks. The reason behind this trend is the tendency to get away from lost sale when the fixed cost is high. In addition, for higher values of K, to compensate the cost of it, the model gives the optimal by using full capacity utilization.

That is why, we do not observe Case 4.2 in Figure 3.6. In the same manner, the area of Case 3.2 increases as K increases. Notice that Case 3.2 does not allow lost sale from any kind of demand.

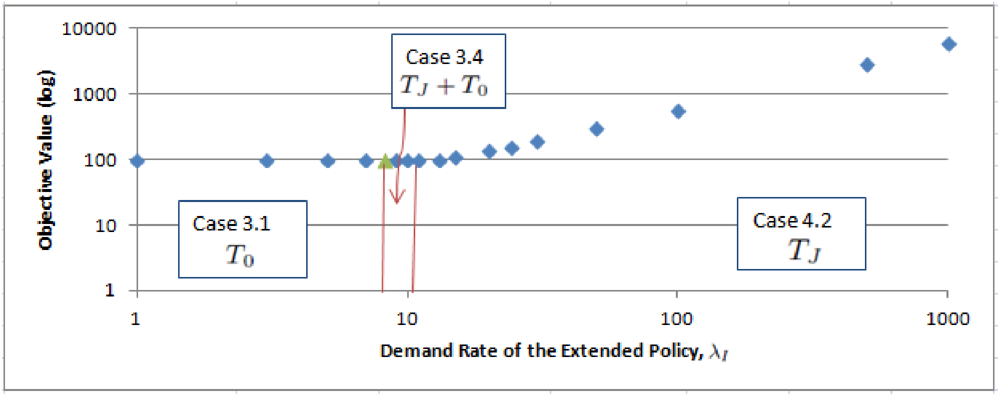

Figure 3.4: λ_P vs Objective Profit when K=*15*

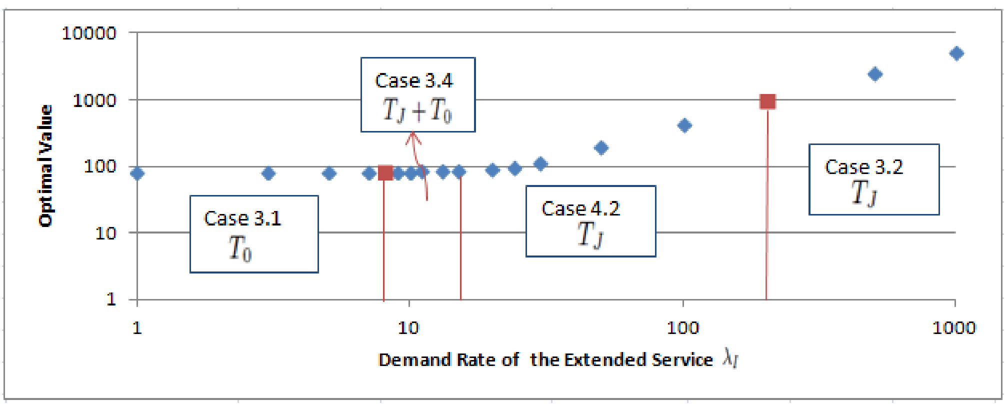

Figure 3.5: λ_P vs Objective Profit when K=*100*

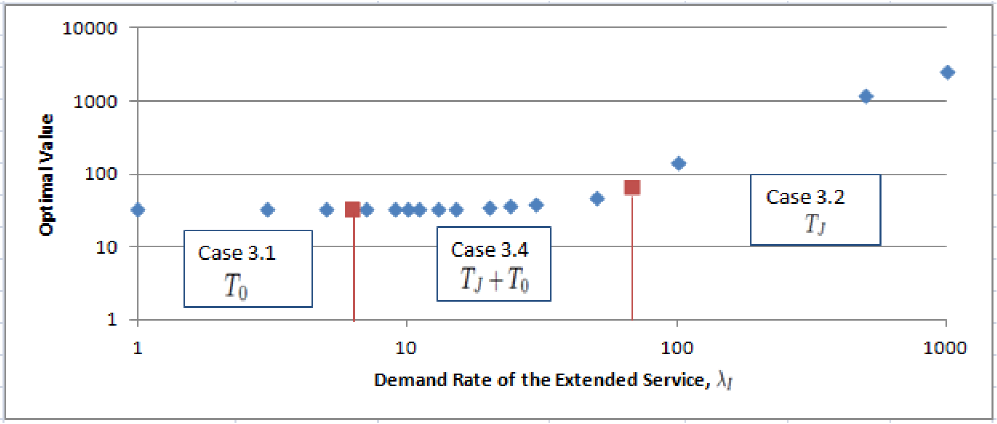

Figure 3.6: λ_P vs Objective Profit when K=*500*

3.2 Special Cases

In this section, we address what will happen if the company has additional concerns and policies besides the optimal policy selection. Therefore, we consider special cases where the problem is reduced according to special operational concerns of the company. We consider the case where $r_P\lambda_P + r_0\lambda_1 \geq r_0\lambda_0$.

3.2.1 Ranking Method Proposed For Multi-Item Systems

Let C^* denotes the contribution of the optimal solution and C^N denotes the optimal contribution of the nominal case, which is "operating under only the regular service". The largest objective value for the problem considered, in general, is when we are allowed to make instantaneous shipments. Therefore, K is rather small and h effect is not apparent. Hence, for C^* one can replace this term with $r_P\lambda_P + r_0\lambda_1$ without loss of generality.

$$C^* = r_P\lambda_P + r_0\lambda_1$$

$$C^N = r_0\lambda_0 - \frac{K}{min\{\frac{C}{\lambda_0}, \tau\}}$$

By using C^* and C^N, conveniences to be served under premium service of products can be expressed. Hence, an indicator for potential to improve total contribution for a single product if it is served under the premium service is constructed as

$$\frac{C^* - C^N}{C^N}.$$

Ranking items according to the above indicator in decreasing order provides a comparison technique regarding which items are more suitable for premium service. Having a comparison method has high significance since e-retailer companies today have to deal with over a million SKU. Even if they are willing to operate to provide high quality customer service such as zero promised delivery times, physical constraints make them to decide wisely among their products as all of them cannot be served immediate delivery techniques for all their potential customers. Hence, above indicator provides a method to decide by giving an insight about products' profitability in case of being served under premium service.

3.2.2 Performance Comparison Method For Different Cases

Let C^S denotes the optimal contribution of the "special" case considered. A number of measures can be defined to assess the effect of using the special case versus the optimal solution. They are listed below:

- PC: % contribution lost when special solution is applied.

$$PC = \frac{C^* - C^S}{C^*} * 100\%$$

  Note that PC will be in between 0-100%.

- IPC: % contribution improvement over the special case if optimal solution was applied.

$$IPC = \frac{C^* - C^S}{C^S} * 100\%$$

Note that IPC will be any value.

- PC^N: % contribution lost over the nominal solution when special case is applied.

$$PC^N = \frac{(C^* - C^N) - (C^S - C^N)}{(C^* - C^N)} * 100\%$$

- IPC^N: % contribution improvement over (special solution - nominal solution) if optimal solution was applied.

$$IPC^N = \frac{(C^* - C^N) - (C^S - C^N)}{(C^S - C^N)} * 100\%$$

Of course, $C^S \neq C^N$ for this case.

Our aim in this subsection is to find the worst case analytical bounds for the measures computed. Denote $\overline{PC}, \overline{IPC}, \overline{PC^N}$ and $\overline{IPC^N}$ as the worst case.

3.2.2.1 No Premium Service Allowed / Only Regular Service

In the case where premium service is not allowed, we want to see the effect of excluding premium service from possible operational options. The company's problem becomes:

$$\begin{aligned}
&\textit{Maximize} \qquad \frac{T_0 r_0 \lambda_0 - K}{T_0} && (3.1)\\
&\textit{subject to} \qquad T_0 \leq \tau && (3.2)\\
&\qquad\qquad\qquad T_0 \lambda_0 \leq C && (3.3)\\
&\qquad\qquad\qquad T_0 \geq 0 && (3.4)
\end{aligned}$$

Accordingly, the optimal solution is either $T_0^* = \tau$ as in Case 2.1 or $T_0^* = C/\lambda_0$ as in Case 3.1 or the intersection of the two cases $T_0^* = C/\lambda_0 = \tau$ which is the solution of Case 1.1. Clearly, if C and τ are infinite, the solution of Case 4.1, $T_0^* = \infty$, can also be valid. However, it is not an applicable solution.

In fact, this is the nominal case. Hence, we can compute $\overline{PC}$, contribution lost when we do not allow premium service.

$$\overline{PC} = \frac{r_P \lambda_P + r_0 \lambda_1 - r_0 \lambda_0 + \frac{K}{min\{\tau, \frac{C}{\lambda_0}\}}}{r_0 \lambda_1 + r_P \lambda_P} * 100\%.$$

or if we rewrite,

$$\overline{PC} = \left(1 - \frac{r_0 \lambda_0}{r_0 \lambda_1 + r_P \lambda_P} + \frac{K}{(r_0 \lambda_1 + r_P \lambda_P) min\{\tau, \frac{C}{\lambda_0}\}}\right) * 100\%$$

For given λ_0, C, τ, K; $\overline{PC}$ becomes

$$\overline{PC} = \left(1 - \frac{r_0 \lambda_0}{r_0 \lambda_1 + r_P \lambda_P} + \frac{Constant}{r_0 \lambda_1 + r_P \lambda_P}\right) * 100\%.$$

Note that *Constant* should be less than $r_0 \lambda_0$.

Let p_1 denote the profitability proportion of the regular service, where

$$p_1 = \frac{r_0\lambda_0 - \frac{K}{min\{\tau, \frac{C}{\lambda_0}\}}}{r_0\lambda_0}$$

$p_1 \leq 1$ and

$$\overline{PC} = \left(1 - p_1 \frac{r_0\lambda_0}{r_0\lambda_1 + r_P\lambda_P}\right) * 100\%.$$

In the same manner, we calculate $\overline{IPC}$.

$$\overline{IPC} = \frac{r_0\lambda_1 + r_P\lambda_P - p_1 r_0\lambda_0}{p_1 r_0\lambda_0} * 100\%.$$

$$\overline{IPC} = \left(\frac{r_0\lambda_1 + r_P\lambda_P}{p_1 r_0\lambda_0} - 1\right) * 100\%.$$

3.2.2.2 Only Premium Service

In this special case, we consider the only existed policy is the premium policy and construct its model according to it.

$$\begin{aligned} &\textit{Maximize} \quad && \frac{T_P r_P \lambda_P - K - h\lambda_P T_P^2}{T_P} && (3.5) \\ &\textit{subject to} \quad && T_P\lambda_P \leq C && (3.6) \\ & && T_P \geq 0 && (3.7) \end{aligned}$$

Referring to KKT cases having the solution is to use only the premium policy, the above model has two possible optimal solutions: $T_P^* = \sqrt{\frac{K}{h\lambda_P}}$ and $T_P^* = C/\lambda_P$. As $T_P^* = \sqrt{\frac{K}{h\lambda_P}}$ is the economic order quantity, the worst possible objective function is attained when $T_P^* = C/\lambda_P$. Hence, the worst objective value for this special case is

$$r_P\lambda_P - \frac{\lambda_P K}{C} - hC.$$

To see the % contribution lost when the worst objective value is attained, we calculate $\overline{PC}$.

$$\overline{PC} = \frac{r_0\lambda_1 + r_P\lambda_P - r_P\lambda_P + \frac{\lambda_P K}{C} + hC}{r_0\lambda_1 + r_P\lambda_P} * 100\%.$$

$$\overline{PC} = \frac{r_0\lambda_1 + \frac{\lambda_P K}{C} + hC}{r_0\lambda_1 + r_P\lambda_P} * 100\%.$$

Let

$$p_2 = \frac{r_P\lambda_P - \frac{\lambda_P K}{C} - hC}{r_P\lambda_P}$$

Thus $p_2 \leq 1$. Accordingly $\overline{PC}$ can be written as

$$\overline{PC} = \left(1 - p_2\frac{r_P\lambda_P}{r_0\lambda_1 + r_P\lambda_P}\right) * 100\%.$$

And $\overline{IPC}$ becomes

$$\overline{IPC} = \frac{r_0\lambda_1 + r_P\lambda_P - p_2 r_P\lambda_P}{p_2 r_P\lambda_P} * 100\%.$$

$$\overline{IPC} = \left(\frac{r_0\lambda_1 + r_P\lambda_P}{p_2 r_P\lambda_P} - 1\right) * 100\%.$$

3.2.2.3 No Lost Sale For Regular Service

We further analyze the special case that the company does not want any lost any sales from its regular customers when it considers to operate the premium service. Therefore, we evaluate the model for the regular and the joint policy separately. Notice that cases with ".1" extension have only the regular policy and cases with ".2" extension have only the joint policy and ".4" extension have both the joint policy and regular policy as a solution. According to our initial assumption, we only consider the subset of B. Therefore, we reach Table 3.2.

	X	Y	Z
B	2.1, 2.2, 3.2, 4.2 1.4, 2.4, 3.4	3.1 2.1, 2.2, 3.2, 4.2 1.4, 2.4, 3.4	2.1, 2.2, 3.2, 4.2 1.4, 2.4

Table 3.2: Sets which have only the regular and the joint policy exist

Table 3.2 shrinks our search area significantly. If the problem parameters do not satisfy one of the special cases such as *Case 1.1*, *Case 1.2* and *Case 2.1*, one can directly eliminate regular policy in BX and BZ and continue with joint policy only or joint and regular policy according to the parameter set.

3.2.2.4 $r_P = r_0$ & $\lambda_0 \leq \lambda_1 + \lambda_P$

One of the special cases for our problem is having same amount of revenue per unit for all type of services, $r_P = r_0$. Therefore, with the help of the assumption that we made for special cases, we reach the demand relation as $\lambda_0 \leq \lambda_P + \lambda_1$. This construction for the special case decreases the number of subsets that we have in Table 2.1 by deleting A and C as a result of our initial assumption, and the column of Z. Hence, new problem framework is as follows.

	X	Y
A	1.4, 1.5, 1.6, 1.7, 2.2, 2.4, 2.5, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 4.2	1.4, 1.5, 1.6, 1.7, 2.2, 2.4, 2.5, 2.6, 2.7, 3.2, 3.3, 3.4, 3.5, 3.6, 4.2
B	1.4, 1.6, 1.7, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, 3.2, 3.3, 3.4, 3.6, 4.2	1.4, 1.5, 1.6, 1.7, 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 4.2

Table 3.3: Subset of problems when $r_P = r_0$ and $\lambda_0 \leq \lambda_P + \lambda_1$

Consider the same problem for an item, which has a constant revenue for all services but yields more demand in the extend of premium service, and the e-retailer does not want to draw back the item from regular service as it is their basis operation. In that case, the solution space shrinks significantly as it can be seen in Table 3.4.

	X	Y
A	1.4, 1.7, 2.2, 2.4, 2.7, 3.1, 3.2, 3.4, 4.2	1.4, 1.7, 2.2, 2.4, 2.7, 3.2, 3.4, 4.2
B	1.4, 1.7, 2.1, 2.2, 2.4, 2.7, 3.2, 3.4, 4.2	1.4, 1.7, 2.1, 2.2, 2.4, 2.7, 3.1, 3.2, 3.4, 4.2

Table 3.4: Subset of problems when $r_P = r_0$ and $\lambda_0 \leq \lambda_P + \lambda_1$ with no lost sale for regular service.

Chapter 4

Shipment Consolidation Problem of E-tailing with Stochastic Demand

In this chapter, we construct a shipment consolidation model with stochastic demand setting for the e-retailer company. Firstly, we assume a single type of basic service for the customers of the e-retailer: the Regular Service. We evaluate a shipment consolidation model and a policy to operate it. Later, we extend the model by integrating a new type of service with its constant demand: the Premium Service. For this premium model, we present a modified policy for the shipment consolidation operation and provide an approximate model with simulation results.

4.1 Shipment Consolidation Problem for Regular Service

In this section, we assume the e-retailer company offers a single type of service for items' delivery operation: *the regular service*. In the regular service, the e-retailer quotes a maximum delivery time to its customers, namely *promised delivery time*. When a customer agrees to purchase an item from the e-retailer, his/her order is delivered latest at the end of promised delivery time duration. This is a strict time limit to make a delivery since it is an indicator for the service level of the e-retailer in such a competitive market. For simplicity, we assume that the company sells a single type of item and the quoted promised delivery time is constant. In this problem, the demand of the e-retailer is stochastic and follows a Poisson distribution with coming a single unit per order.

After an order is received by the e-retailer, the decision regarding which fulfillment center will send the order is made. For simplicity, we assume again a single fulfillment center and a single region where customers are very close to each other so that within the region the shipment vehicle visits customers in a negligible amount of time to deliver their orders. A representative figure is illustrated in Figure 4.1. To avoid the cost of individual shipment for each order, the e-retailer plans to consolidate orders as much as possible, and then the shipment vehicle delivers orders to the customers without exceeding any promised time length of carried orders in a vehicle. We assume that even if there is no restriction for the number of trucks that the company uses in our setting, there is a single constant vehicle capacity for all shipment operations.

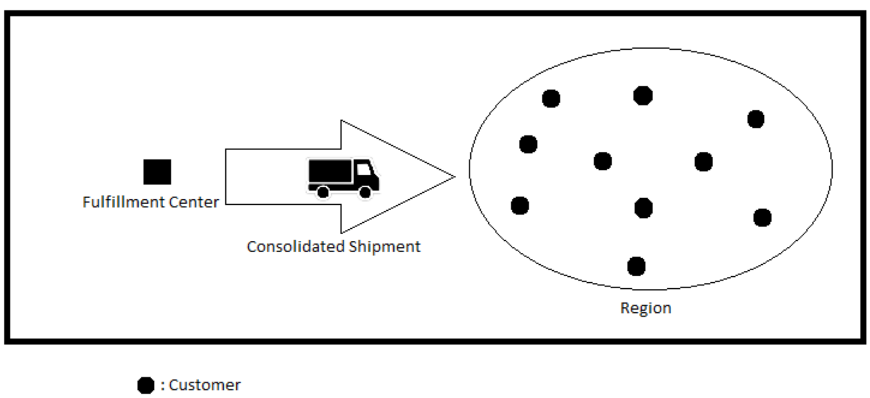

Figure 4.1: Shipment consolidation operation for only regular service.

The e-retailer operates shipment consolidation activities under hybrid (time-and-quantity based) policy. Thus, a shipment is realized either when the capacity of a truck/vehicle or the promised delivery time for the earliest demand is reached. Note that transportation times regarding from the fulfillment center to the customer region, and within the region are not included to the model. However,the promised delivery time can be considered as the difference between the quoted delivery time to the customer on the web page and transportation times.

In the problem of shipment consolidation of the e-retailer company, the focus is on analyzing only the shipment consolidation operation. Therefore, we do not include decisions or settings regarding choosing the right fulfillment center, item levels in warehouses, transportation times or generalized fleet operations. As a result, costs in the model include only shipment related costs and in this case we assume a fix cost per shipment operation. The e-retailer's delivery activities are strictly guaranteed in terms of time, customers willing to wait until the quoted delivery time. Hence, we exclude waiting costs. At this point, our interest is different from the closest study to ours in the literature which is Mutlu et al. [41]. Additionally, Mutlu et al. [41] aim to minimize the cost but again we differ with profit maximization approach.

We assume that the company's demand for regular customers follows Poisson distribution with rate λ_0, and each customer order comes as a single unit. The inter-shipment time (cycle length) is the time length between two consecutive shipment realization from the fulfillment center. The promised delivery time, τ is defined as the maximum promised delivery duration, and it is valid for all orders carried in the shipment vehicle.Hence, after a shipment has been send, the arrival of the first customer initiates the "promised-delivery-time-clock" for the next shipment. Note that, with exponential times between customer arrivals, the arrival for this first customer initiates a renewal cycle; i.e. the cycles are identically distributed. Thus Renewal Theory applies, and hence we only need to compute expected revenues and costs for a cycle, and expected cycle length [48].

$$\frac{e^{-\lambda_0\tau}(\lambda_0\tau)^n}{n!}$$

As the company operates according to the hybrid policy, there are two possible thresholds to realize each shipment: τ; the total waiting time for the first coming order and C; the capacity of a shipment vehicle. The time length between two shipment realization is named as *inter-shipment time* and it depends on both of these parameters.

Let IS_k be the inter-shipment time between $(k-1)^{th}$ and k^{th} shipments,

$$IS_k = t_1 + \begin{cases} \tau, & \text{if } t_{C-1} > \tau \\ t_{C-1}, & \text{otherwise} \end{cases}$$

where t_i is defined as the time of i^{th} arrival unit.

To find the expected inter-shipment time, $E[IS]$, we evaluate expected probabilities of both cases. The first case is the realization of a shipment when the waiting time for the earliest coming demand reaches τ. It is the situation where waiting time for the full truck load is greater than τ.

$$P\{t_{C-1} > \tau\} = P\{t_{C-2} \geq \tau\} = \sum_{n=0}^{C-2} \frac{e^{-\lambda_0 \tau}(\lambda_0 \tau)^n}{n!}$$

In the same manner, the probability of filling *C-1* unit capacity on τ is the Erlang distribution for the $(C-1)^{th}$ event [49]. The cumulative probability for this to happen on or before τ is defined as

$$F(t) = \int_0^{\tau} t * \frac{e^{-\lambda_0 t}\lambda_0^{C-1} t^{C-2}}{\Gamma(C-2)} dt.$$

Hence, the expression for the expected inter-shipment time is as follows.

$$E[IS] = \frac{1}{\lambda_0} + \tau * \sum_{n=0}^{C-2} \frac{e^{-\lambda_0 \tau}(\lambda_0 \tau)^n}{n!} + \int_0^{\tau} t * \frac{e^{-\lambda_0 t}\lambda_0^{C-1} t^{C-2}}{\Gamma(C-2)} dt$$

or in more compact form:

$$E[IS] = \frac{C}{\lambda_0} + e^{-\lambda_0 \tau}\left(\sum_{n=0}^{C-3} \frac{\lambda_0^n \tau^{n+1}}{n!} + \frac{(1-C)}{\lambda_0} \sum_{n=0}^{C-2} \frac{\lambda_0^n \tau^n}{n!} \right). \tag{4.1}$$

Another quantity of interest is the expected number of units per shipment. It is constructed easily from the probabilities evaluated above and it is expressed as

$$E[\text{number of units per shipment}] = 1 + n * \sum_{n=0}^{C-2} \frac{e^{-\lambda_0 \tau}(\lambda_0 \tau)^n}{n!} + (C-1) * \int_0^{\tau} \frac{e^{-\lambda_0 t}\lambda_0^{C-1} t^{C-2}}{\Gamma(C-2)} dt. \tag{4.2}$$

Note that equations (4.1) and (4.2) are related to each by a factor of λ_0. This is verified in Appendix G.

In this problem, we also assume that each regular customer yields a revenue r_0. As it is mentioned before, the only cost for the regular service in shipment consolidation is a fix cost per shipment decision, K. Our aim is to maximize the expected profit. After the evaluation of probability measures for each shipment realization condition, we construct the objective function to get average profit. The final structure for the e-retailer's shipment problem is

$$E[Profit]= \qquad r_0\lambda_0 - \frac{K}{\frac{C}{\lambda_0} + e^{-\lambda_0\tau}\left(\sum_{n=0}^{C-3}\frac{\lambda_0^n\tau^{n+1}}{n!} + \frac{(1-C)}{\lambda_0}\sum_{n=0}^{C-2}\frac{\lambda_0^n\tau^n}{n!}\right)} \tag{4.3}$$

where

K : the fix cost per shipment

r_0 : revenue per unit for regular demand.

Now, we analyze two extremes of the consolidation policy: where we have unlimited capacity and where we have no promised shipment time.

- **Unlimited Capacity**

  For this case we took the limit as C goes to infinity for E[IS] and the result is expressed as:

  $$\lim_{C\to\infty}\frac{C}{\lambda_0} + e^{-\lambda_0\tau}\left(\sum_{n=0}^{C-3}\frac{\lambda_0^n\tau^{n+1}}{n!} + \frac{1-C}{\lambda_0}\sum_{n=0}^{C-2}\frac{\lambda_0^n\tau^n}{n!}\right) = \frac{1}{\lambda_0} + \tau$$

  Hence, the objective value is as follows.

  $$r_0\lambda_0 - \frac{K}{\frac{1}{\lambda_0} + \tau}$$

  This is the situation where we have only the promised delivery time as a threshold for our shipment policy. Therefore, this version of the problem becomes time-based consolidation policy and it provides us an upper bound for the objective function.

Proposition 2. *An upper bound for the equation (4.3) is given by* $r_0\lambda_0 - \frac{K}{\frac{1}{\lambda_0}+\tau}$. *Proof: Follows unlimited capacity argument for the truck.*

- **Unlimited Promised Delivery Time**

  If there is no promised delivery time consideration of the company, the operation strategy act according to quantity-based consolidation policy. Thus, a shipment is realized only when the capacity of the shipment vehicle is full without concerning service time for customers. It is equivalent to taking the limit of expected inter-shipment time as τ goes to infinity and the new expected inter-shipment time becomes

$$\lim_{\tau\to\infty} \frac{1}{\lambda_0} + \int_0^\infty t * \frac{e^{-\lambda_0 t}\lambda_0^{C-1}t^{C-2}}{\Gamma(C-2)} dt = \frac{C}{\lambda_0}.$$

  Accordingly, another upper bound for the objective function is

$$r_0\lambda_0 - \frac{\lambda_0 K}{C}.$$

Proposition 3. *An upper bound for the equation (4.3) is given by* $r_0\lambda_0 - \frac{\lambda_0 K}{C}$. *Proof: Follows unlimited promised delivery time argument for the truck.*

4.2 Shipment Consolidation Problem Including Premium Service

In the previous section, we assume the e-retailer company offers only the regular service to its customers. Now, we integrate the *premium service* option to the problem. However, we also assume that the e-retailer does not want to lose any customers preferring to use the regular service. The premium service provides an immediate delivery to its customers compared to the regular service. This immediate delivery can be considered as same day shipping or delivery in a few hours. To enable such a service, we assume very close inventories to customers or demand points. When the e-retailer physically keeps items be ready in those inventories, customers within regions of those inventories see the premium service with immediate delivery option on the web page for physically available items. For simplicity, we assume a single small inventory in a single customer region. The inventory is fed by the fulfillment center for prospective demands. The e-retailer wants to decide how much to send to the inventory in each shipment realization from the fulfillment center to the region by carrying both the orders of the regular service and items for the inventory.

4.2.1 Approximate Model of The Problem with Premium Service

The e-retailer company brings the premium service option in addition to its regular service. We assume deterministic demand for the premium service with rate λ_P. As the e-retailer does not want any lost sales from the regular service, inter-shipment times are still stochastic and depend on both the promised delivery time and the capacity dedicated to the regular service in a shipment truck. However, stochasticity in realizations of the hybrid policy makes inter-shipment times inconsistent while feeding the premium service's inventories which has constant demand rate. Additionally, it is hard to identify the exact distribution of the inter-shipment times due to its complicated structure, and this makes it harder to detect the exact state of inventory position. Even if having a constant demand rate for the premium service makes the problem slightly easier, we still have to consider possible accumulations in the inventory with respect to high holding costs. Therefore, the crucial point in this problem is to decide right amount to send small inventories per shipment by deciding the optimal combination of capacities dedicated to each services in the shipment vehicle.

To avoid extreme holding costs caused by accumulations of items in the inventory, we allow lost sale from the premium service by assuming items send to the inventory are always sold. The main idea behind it is that if each shipment realization finds an inventory empty, the accumulations in small inventories are prevented. Hence, in each cycle inventory clears itself and loses the remaining demand which come until the next shipment arrival. In Figure 4.2, an illustration for our model assumption is depicted. Under this assumption, the problem is to decide how much to send the inventory for the premium demand while also sending the regular service's demand in the same shipment vehicle.

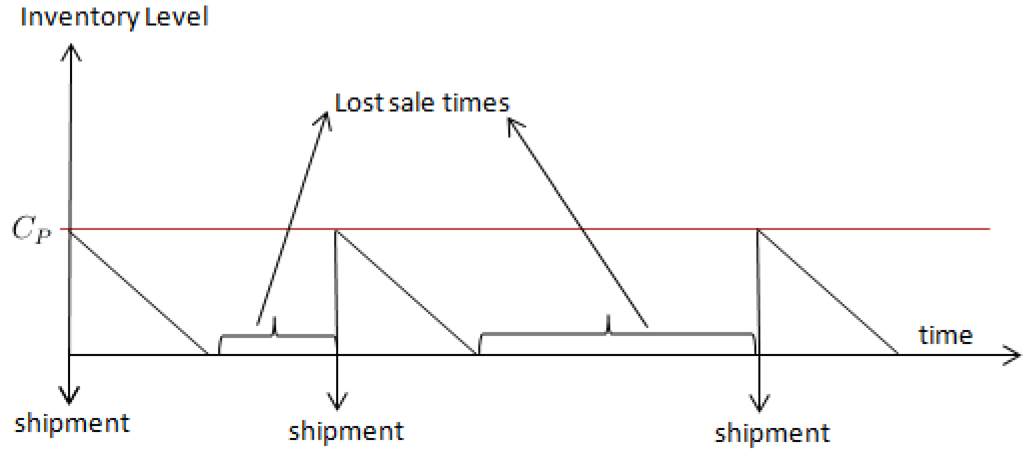

Figure 4.2: Model Assumption

where C_P is the amount send to the inventory with each shipment realization.

Since, we include the existence of the inventory to the problem due to the premium service expansion, the new objective function additionally has inventory carrying costs. The discrete holding cost function is expressed as

$$h\frac{C_P(C_P+1)}{2\lambda_P}$$

where
h: holding cost per item per time
λ_P: demand rate of the premium service
C_P: amount send to the small inventory.

Each premium demand comes with a revenue r_P. As a result, we construct expected average profit maximizing model for the shipment consolidation problem.

$$\begin{aligned}
&\textit{Maximize} \qquad r_0\lambda_1 + \frac{r_P C_P - h\frac{C_P(C_P+1)}{2\lambda_P} - K}{\frac{C_0}{\lambda_1} + e^{-\lambda_1\tau}\left(\sum_{n=0}^{C_0-3}\frac{\lambda_1^n\tau^{n+1}}{n!} + \frac{(1-C_0)}{\lambda_1}\sum_{n=0}^{C_0-2}\frac{\lambda_1^n\tau^n}{n!}\right)} && (4.4)\\
&\textit{subject to} \qquad C_0 + C_P \le C && (4.5)\\
&\qquad\qquad\qquad C_P, C_0 \in \mathbb{Z}_+ && (4.6)
\end{aligned}$$

where

C_0: amount dedicated to the regular service per shipment

C_P: amount dedicated to the inventory per shipment for the premium service

C: the total capacity of a shipment vehicle

h: holding costs of premium service per unit per time

r_0: revenue per item bought from the regular service

r_P: revenue per item bought from the premium service

λ_1: demand rate of the regular service when the premium service also offered

λ_P: demand rate of the premium service.

In the approximate model of the problem including premium service, our aim is to decide right amount of capacity to dedicate both of services given the shipment truck's capacity constraint by maximizing the expected profit. Different from the hybrid policy, which requires the information of the promised delivery time and the capacity per shipment (τ, C), the new modified policy requires information regarding (τ, C_0, C_P) to initiate a shipment and keep items ready for the premium services' prospective customers.

4.2.2 Analysis of The Approximate Model

In reality, stochasticity in inter-shipment times may lead too frequent shipments. Considering an infinite horizon, consecutive shipments to the inventory of premium service can cause accumulations of the inventory level due to not cleared inventory from the previous cycle. In Figure 4.3, an example of realization is illustrated.

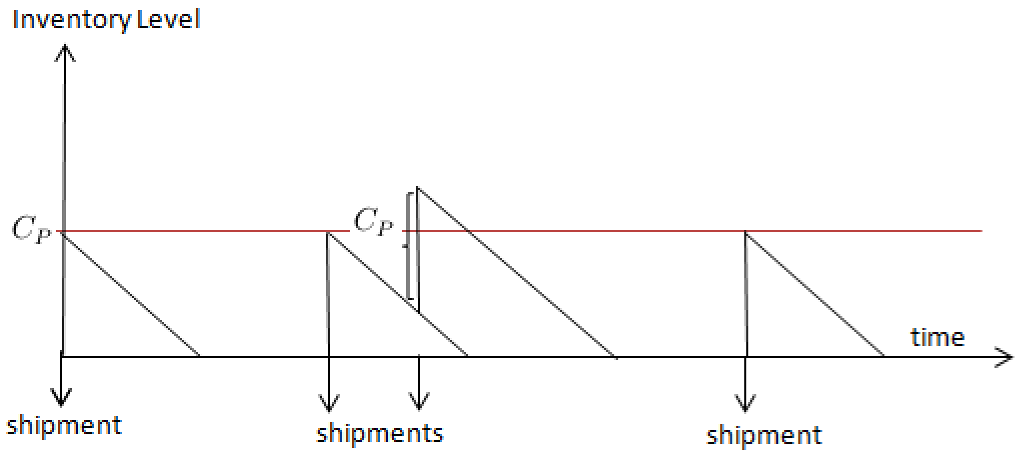

Figure 4.3: Realization

As a result, our assumption to avoid extreme inventory accumulations underestimates holding costs in the expected profit function, (4.4). Furthermore, another important outcome arising from our assumption in the approximate model is overestimating revenue generated by the premium service. In each shipment cycle duration we assume that amount sent to the inventory in the beginning of each cycle will be sold until the next shipment's realization. This causes an overestimation in revenue per shipment. As a result of underestimated holding costs and over estimated revenues, the expected profit is overestimated in the approximate model. Hence, our assumption's accuracy is high when we have longer lost sale duration in realized inter-shipment times.

To interpret our assumption and strengthen the approximate model, we introduce the parameter α which is the fraction of premium demand to satisfy. In other words, each shipment arrival finds the inventory empty by allowing $(1 - \alpha) * 100\%$ lost sale in the premium service. Therefore, in each cycle the inventory clears itself and loses the demand of premium service until next shipment's arrival. Even if we do not have the distribution of inter-shipment times, we have the expected value of it, and it can be used as an upper limit to the the amount shipped for the premium service as a caution. Our assumption states that in each shipment the inventory level is found empty as it is illustrated in Figure 4.2. Therefore, to make our approximate model strengthen in the application, we construct a natural bound for the amount to be shipped for the premium service by using the E[IS] and

α. This natural bound of the approximation in the model is expressed as follows.

$$C_P \leq \alpha\lambda_P E[IS] \tag{4.7}$$

where $\alpha \in [0, 1]$
C_P: the amount shipped to for the premium service per shipment
λ_P: demand rate of the premium service.

On the other hand, there is an another bound for C_P to operate in an economical manner. By considering the first order condition coming from the objective function (4.4), we construct a new bound for C_P. Therefore, there exists a global C_P value such that the model prevents to send more than that amount as it would be illogical in economical manner. manner. Please refer to Appendix H for the derivation of the condition. This global upper bound is expressed in an interval and C_P takes the integer value from this interval.

$$\frac{r_P\lambda_P}{h} - 1 \leq \overline{C_P}^* \leq \frac{r_P\lambda_P}{h} \tag{4.8}$$

where $\overline{C_P}^*$ is the global integer bound for C_P.

By using upper bounds that we constructed, we are able to provide an operational new constraint for the number of items that can be send for premium service. For the sake of simplicity, we take the bound coming from (4.8) as $\frac{r_P\lambda_P}{h}$ and expressed the operational upper bound as

$$C_P \leq min\Big\{\frac{r_P\lambda_P}{h}, \alpha\lambda_P E[IS], C - C_0\Big\}. \tag{4.9}$$

The first upper bound comes from economical benefit of the premium service. The second is to strengthen the application of our assumption and the last one is the feasibility condition.

In the modified policy, our interest is to identify (τ, C_0, C_P). We provide a simple search algorithm to find best combination of (C_0, C_P) given the total available capacity and the promised delivery time. Algorithm 2 is presented as follows.

Algorithm 2 Maximize The Expected Profit for Given Parameter Set

procedure COMPUTE THE BEST COMBINATION FOR(C_0, C_P)
  for $C_0 = 1 : C$ **do**
    $CalculateE[IS]$
    for each $C_P = 0 : min\left\{\frac{r_P \lambda_P}{h}, \alpha\lambda_P E[IS], C - C_0\right\}$ **do**
      Calculate the expected profit.
    end for
    Return C_P that yields maximum expected profit.
  end for
  Return C_0 and C_P that yields maximum expected profit.
end procedure

By using Algorithm 2 and the parameter set given in Table 4.1, we generate a numerical example.

Parameters	r_0	r_P	λ_1	λ_P	h	K	C	τ	α
Values	5	6	20	15	1	100	150	8	0.8

Table 4.1: Parameter set to find best combination between C_0 and C_P

After we run Algorithm 2, the best (C_0, C_P) combination is found as (92, 55), and the total used capacity is 147. The main reason behind this observation is C_P always hits the natural bound (4.7) for small C_0 which causes small E[IS]. However, increasing C_0 one by one does not necessarily increase this natural bound incrementally in each iteration. Therefore, we observe wavy shapes in the left hand side of the figure. In Figure 4.4, it can be seen that the horizontal line is the place where the smallest upper bound of C_P changes. Since C_0 increases, E[IS] increases so the remaining capacity becomes more more strict bound for C_P.

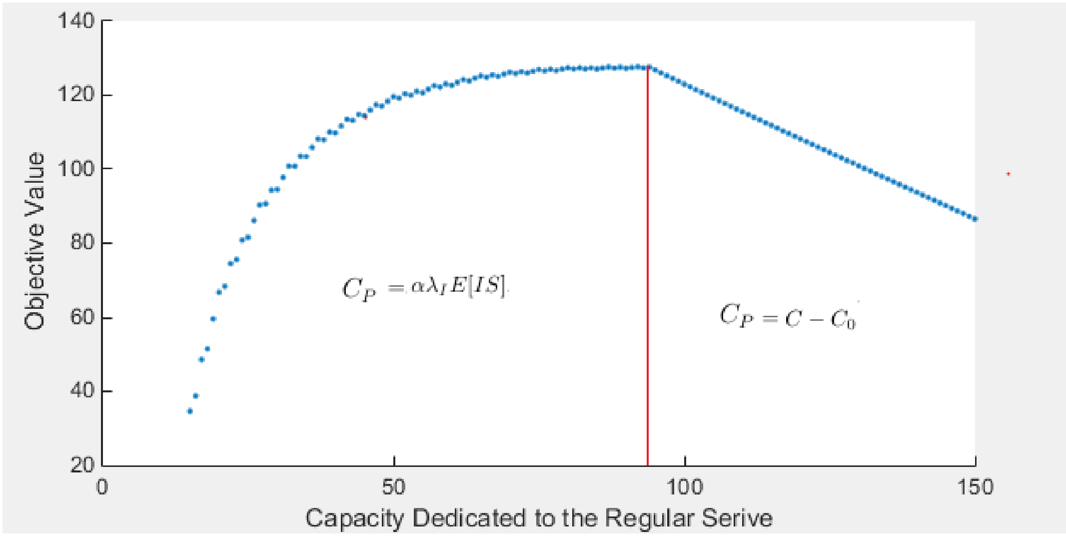

Figure 4.4: Results of the best combination of (C_0, C_P)

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4.3 Simulation for The Approximate Model

Our main concern is to understand the extend of overestimation in the expected profit function given the problem in Subsection 4.2. Therefore, we conduct a simulation experiment in MATLAB to asses the accuracy of the approximate model proposed.

One of drawbacks of the approximate model is its underestimation of holding costs. Hence, in the simulation, we aim to see the difference between the analytical model and the realization regarding the average inventory carried. In the approximate model, the average inventory carried is calculated according to the following expression.

$$\alpha \frac{C_P(C_P+1)}{2\lambda_P E[IS]} \tag{4.10}$$

where $\alpha \in [0,1]$ is the fraction to satisfy the premium demand per shipment.

By using parameters in Table 4.2, we calculate the expected inter-shipment time with C_0 and τ and it is found as E[IS]=5.

Parameter	λ_P	h	K	τ	C_0
Value	15	1	100	8	100

Table 4.2: Parameter set for the comparison between the simulation and the approximate analytical model.

Finding the exact distribution for the inter-shipment time is hard because of its complicated probabilistic structure. Hence, in the simulation we generate inter-shipment times from Gamma distribution for different coefficient of variations (CV= 0.5, 1 and 2) by taking the mean value of the distribution same as we found in the analytical model, E[IS]=5. Under the above parameter settings, we generate 100 consecutive cycle lengths as a single stream and repeat it 200 times to reach sample size n=200. In other words, we have 200 streams of 100 consecutive shipment realization and each stream is started freshly with C_P amount of inventory level.

The other significant point in the simulation is to decide C_P level to use both in the approximate analytical model and in the simulation to see the effect of underestimation of items hold in the inventory. We know from (4.9), C_P can be chosen as its global upper bound, the remaining capacity or the expected number of demand in per cycle time. To be able to see the effect of our assumption in the model, we arrange parameters to have the expected number of demand per cycle as minimum among other bounds and take total capacity C as infinity. Hence, $\lambda_P E[IS]$ yields the minimum upper bound as having the value of 75.

For different α and CV combinations, we make a simulation and keep the average lost sale and the average inventory carried for both the approximate analytical model and realizations of inter-shipment times and their results are as follows.

		α		
		0.5	0.80	0.95
CV	0.5	9.3633 vs 9.3733	25.3298 vs 23.60	57.5120 vs 34.08
	1	9.8497 vs 9.3733	33.9462 vs 23.60	99.0078 vs 34.08
	2	11.3478 vs 9.3733	49.3201 vs 23.60	146.8524 vs 34.08

Table 4.3: Average Inventory Carried (Simulation vs Analytical Model)

		α		
		0.5	0.80	0.95
CV	0.5	50.76 %	21.51 %	5.87 %
	1	50.79 %	21.71 %	6.96 %
	2	50.86 %	22.06 %	7.86 %

Table 4.4: Average Lost Sale in the Simulation

In Table 4.3, we present average inventory carried from the simulation versus the analytical model as pairs. Note that results of the analytical model are independent of CV values. However, under our assumption, C_P value of the analytical model changes regarding different α values. As it is shown in Table 4.2, the analytical model always underestimates carried inventory and its estimation is better when we have low α values, as expected. On the other hand, since high CV values create more variation in the realizations of inter-shipment times, we carry more inventory as CV increases.

It is expected that when α gets bigger, more demand is satisfied as we send more item to the inventory. Hence, we lose less demand as it is shown in Table 4.4. However, if we decide to send very close to the expected premium demand per shipment, for high coefficient of variation we loose more than we assume. That is why in the last column of Table 4.4, an increase in the lost sale is observed as CV gets bigger.

Regarding the CV level to select, more analyses can be conducted. Here, our aim is to show that the accuracy of the expected profit function is well affected by the actual CV and α value selected.

Chapter 5

Conclusion

To the extent of our knowledge, this is the first work combines shipment consolidation problem with e-retailer activities under different customer characterizations. We assume an e-retailer that has two types of services to offer their customers possibly with different prices: the regular and the premium. In the regular service, the company guarantees a promised delivery time to its customers and realizes delivery of orders without exceeding the promised time length. In the premium service, we assume zero promised delivery time, and the e-retailer has inventories located very close to its prospective customers to provide practically zero promised delivery time service. The problem considers a single fulfillment center to send items and a single customer region, which includes a single inventory location for the premium service. When an item is in such inventory, customers who navigate the item's web page observe the premium service option. Otherwise, items are served under the regular service.

Each shipment realization from the fulfillment center to the region serves to customers of the regular service, who already purchased items and wait for their delivery, and/or to the inventory to be kept for prospective premium service customers. As our interest is shipment consolidation operation, the problem becomes determining conditions to consolidate items send for both of services from the fulfillment center under different services.

Shipments are carried out by trucks with limited capacity. We assume a single constant capacity for each shipment. Moreover, we are not concerned with fleet capacity or truck availability, and hence assume that we have unlimited number of trucks or cargo vehicles. We consider an infinite horizon for the problem as demand is taken to have a constant rate. As a result, an optimal solution for deterministic demand case will have a constant inter-shipment time for the trucks. Similarly, for the stochastic demand case where demand follows a Poisson distribution, we discuss that renewal theorem is applicable for the analysis under the policies considered.

The e-retailer company is assumed to operate under hybrid policy. Under this policy, a single shipment is realized either when the promised delivery time or the capacity of the truck is reached. For simplicity, we assume the e-retailer offers a single type of an item, and each demand comes as a single unit.

We differentiate demand rates coming in each service type. Similarly, demand for regular service will be different if premium service is offered at the same time. In the same manner, revenues generated by orders from each service are allowed to be different. We consider a fixed cost per shipment realization, and holding cost for items kept in the inventory. The objective function is to maximize the average contribution to profit (or expected profit), which is average revenues minus average costs. We employ two settings for this shipment consolidation problem.

In the first setting, we assume that demand of each service is deterministic, and there is a third service option for the company: operate these two services together - we call it joint operation. In the deterministic setting, we have truck capacity and promised delivery time constraints. There is no restriction in the policy applied, and hence each service type's (individual and joint) contribution to the cycle length (inter-shipment time) are the decisions. As the objective function is non-linear, we apply Karush-Kuhn-Tucker conditions and provide analytical results to find an optimal solution. Even if the objective function is shown to be neither concave nor convex, a number of properties on the solution are shown to be active, decreasing the solution time. Our results show that items can be categorized according to their revenue generating abilities, of course depending on the problem parameters.

In the second setting, we assume stochastic demand for the regular service. Demand is assumed to follow a Poisson distribution. We also assume that the e-retailer does not allow any lost customer from its regular service. First, we evaluate the problem where we have only the regular service. Similar to the deterministic setting we assume constant revenue for each demand and a fix cost per shipment. As a result, we aim to maximize average expected contribution to profit. Next, the problem is extended by integrating the premium service with constant and deterministic demand rate. We present an approximate model that will utilize the following policy: Trigger a shipment if the promised delivery time is reached, or if the capacity limit for regular customers is reached. Additionally, ship a constant amount for the use of premium service. Hence, capacity limit for the regular customers and the constant amount sent as inventory are decision variables of the proposed policy. The analytical model is built with the premise that when truck reaches the region, there will be no units left for the inventory. However, this may not be the case, as we have stochastic demand. As a result of this assumption, we show that the approximate model overestimates the average expected profit. A simulation is used to show when the margin of error caused by the approximate model is high.

The current work is valuable as it proposes benchmarks for the more complicated, real problem. In the deterministic demand problem, there is no restriction with respect to the policies one can use, making the solution very general. The categorization proposed in Chapter 3 is an important contribution for practical version of the problem.

Stochastic demand problem can be extended in a number of ways. Different policies can be proposed and modeled. Policies that depend on the current inventory level allocated for the premium service can be devised as a future work.

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Appendix A

An Example For The Objective Function

To form the Hessian matrix of objective function, (1), required derivatives are constructed in the following. Let

$$O(T_0, T_J, T_P) = \frac{T_P r_P \lambda_P + T_J(r_P \lambda_P + r_0 \lambda_1) + T_0 r_0 \lambda_0 - K - H(T_P, T_J)}{(T_P + T_J + T_0)} \tag{A.0.1}$$

And elements in the diagonal of the Hessian matrix are:

$$\begin{aligned}
\frac{\partial^2 O}{\partial^2 T_0} &= -\frac{2(K + \lambda_P(T_P + T_J)(h(T_P + T_J) - r_P)) + r_0(\lambda_0(T_P + T_J) - \lambda_1 T_J)}{(T_P + T_J + T_0)^3} \\
\frac{\partial^2 O}{\partial^2 T_J} &= -\frac{2(K + \lambda_1 r_0(T_P + T_0) + T_0(\lambda_P r_P - \lambda_0 r_0 + h\lambda_P T_0))}{((T_P + T_J + T_0)^3} \\
\frac{\partial^2 O}{\partial^2 T_P} &= -\frac{2(K - \lambda_1 r_0 T_J + T_0(\lambda_P r_P - \lambda_0 r_0 + h\lambda_0 T_0))}{(T_P + T_J + T_0)^3}
\end{aligned}$$

If we choose $K = 10, h = 0.1, \lambda_0 = 10, \lambda_1 = 8, \lambda_P = 5, T_P = 1, T_J = 1, T_0 = 1, r_0 = 4, r_P =$

8,

$$
\begin{aligned}
\frac{\partial^2 O}{\partial^2 T_0} &= 1.48148 \\
\frac{\partial^2 O}{\partial^2 T_J} &= -5.51852 \\
\frac{\partial^2 O}{\partial^2 T_P} &= 1.59259
\end{aligned}
$$

and related eigenvalues are: -6.28318, -0.167412, 4.00615. Hence, in this combination of parameter set, the objective function is neither convex or concave.

Appendix B

32 KKT Cases

Possible case types are structured as follows.

Case 1: $\mu_1 > 0$ and $\mu_2 > 0$

Case 2: $\mu_1 = 0$ and $\mu_2 > 0$

Case 3: $\mu_1 > 0$ and $\mu_2 = 0$

Case 4: $\mu_1 = 0$ and $\mu_2 = 0$

Moreover, each of 4 cases is then divided into 8 cases according to (2.22), (2.23) and (2.24). The associated case numbers are expressed as shown below $\forall i = 1, 2, 3, 4$.

Case i.1: $T_0 > 0$, $T_J = 0$, $T_P = 0$

Case i.2: $T_0 = 0$, $T_J > 0$, $T_P = 0$

Case i.3: $T_0 = 0$, $T_J = 0$, $T_P > 0$

Case i.4: $T_0 > 0$, $T_J > 0$, $T_P = 0$

Case i.5: $T_0 > 0$, $T_J = 0$, $T_P > 0$

Case i.6: $T_0 = 0$, $T_J > 0$, $T_P > 0$

Case i.7: $T_0 > 0$, $T_J > 0$, $T_P > 0$

Case i.8: $T_0 = 0$, $T_J = 0$, $T_P = 0$

Notice that as Case i.8 $\forall i = 1, 2, 3, 4$ refers no operation.

B.1 Case 1

B.1.1 Case 1.1

- $\mu_1 > 0$ and $\mu_2 > 0$
- $T_0 > 0$, $T_J = 0$, $T_P = 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies T_0 = \tau$$
$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_P + \lambda_1) + T_0\lambda_0 = C$$
$$\implies T_0 = C/\lambda_0$$

By combining above solutions we get:

$$C/\lambda_0 = \tau$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies K = T_0^2(\mu_1 + \mu_2\lambda_0) \tag{B.1.1}$$
$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{T_0} - \mu_1 - \mu_2(\lambda_P + \lambda_1 - \lambda_0) \leq 0 \tag{B.1.2}$$
$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{T_0(r_P\lambda_P - r_0\lambda_0) - K}{T_0^2} - \mu_2\lambda_P \leq 0 \tag{B.1.3}$$

Hence, the solution of Case 1.1 is

$$(T_0, T_J, T_P) = (\tau, 0, 0) = (C/\lambda_0, 0, 0) \tag{B.1.4}$$

B.1.2 Case 1.2

- $\mu_1 > 0$ and $\mu_2 > 0$

- $T_0 = 0$, $T_J > 0$, $T_P = 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies T_J = \tau$$
$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_P + \lambda_1) + T_0\lambda_0 = C$$
$$T_J(\lambda_P + \lambda_1) = C \implies T_J = C/(\lambda_1 + \lambda_P)$$

By combining above solutions:

$$C/(\lambda_1 + \lambda_P) = \tau$$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{T_J(r_0\lambda_0 - r_P\lambda_P - r_0\lambda_1) - K - h\lambda_P T_J^2}{T_J^2} - \mu_1 - \mu_2\lambda_0 \leq 0 \tag{B.1.5}$$

$$\frac{\partial L}{\partial T_J} = 0 \implies \frac{T_J(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P T_J^2)}{T_J^2}$$
$$- \mu_1 - \mu_2(\lambda_1 + \lambda_P) = 0 \tag{B.1.6}$$

$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{T_J(r_P\lambda_P - 2h\lambda_P TJ) - (T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P T_J^2)}{T_J^2} - \mu_2\lambda_P \leq 0 \tag{B.1.7}$$

Hence, the solution of Case 1.2 is

$$(T_0, T_J, T_P) = (0, \tau, 0) = (0, C/(\lambda_1 + \lambda_P), 0) \tag{B.1.8}$$

B.1.3 Case 1.3

- $\mu_1 > 0$ and $\mu_2 > 0$

- $T_0 = 0$, $T_J = 0$, $T_P > 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies \tau = 0$$
$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_P + \lambda_1) + T_0\lambda_0 = C$$
$$\implies T_P = C/\lambda_P fca$$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{T_P r_0 \lambda_0 - (T_P r_P \lambda_P - K - h\lambda_P T_P^2)}{T_P^2} - \mu_1 - \mu_2\lambda_0 \leq 0 \tag{B.1.9}$$

$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{T_P(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_P) - (T_P r_P \lambda_P - K - h\lambda_P T_P^2)}{T_P^2} - \mu_1 - \mu_2(\lambda_1 + \lambda_P) \leq 0 \tag{B.1.10}$$

$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{T_P(r_P\lambda_P - 2h\lambda_P T_P) - (T_P r_P \lambda_P - K - h\lambda_P T_P^2)}{T_P^2} - \mu_2\lambda_P = 0 \tag{B.1.11}$$

Hence, the solution of Case 1.3 is

$$(T_0, T_J, T_P) = (0, 0, C/\lambda_P) \tag{B.1.12}$$

B.1.4 Case 1.4

- $\mu_1 > 0$ and $\mu_2 > 0$
- $T_0 > 0$, $T_J > 0$, $T_P = 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau$$
$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_J(\lambda_P + \lambda_1) + T_0\lambda_0 = C$$

By combining above solutions:

$$T_0 = \tau - \frac{C - \tau\lambda_0}{\lambda_P + \lambda_1 - \lambda_0}$$
$$T_J = \frac{C - \tau\lambda_0}{\lambda_P + \lambda_1 - \lambda_0}$$

Notice that Case 1.4 does not exist when $\lambda_P + \lambda_1 - \lambda_0 = 0$.

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies \frac{(T_0 + T_J)r_0\lambda_0 - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K - h\lambda_P T_J^2)}{(T_0 + T_J)^2} - \mu_1 - \mu_2\lambda_0 = 0 \tag{B.1.13}$$

$$\frac{\partial L}{\partial T_J} = 0 \implies \frac{(T_0 + T_J)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0}{(T_0 + T_J)^2}$$
$$\frac{-K - h\lambda_P T_J^2)}{(T_0 + T_J)^2} - \mu_1 - \mu_2(\lambda_1 + \lambda_P) = 0 \tag{B.1.14}$$

$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{(T_0 + T_J)(r_P\lambda_P - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K - h\lambda_P T_J^2)}{(T_0 + T_J)^2}$$
$$- \mu_2\lambda_P \leq 0 \tag{B.1.15}$$

Hence, the solution of Case 1.4 is

$$(T_0, T_J, T_P) = (\tau - \frac{C - \tau\lambda_0}{\lambda_P + \lambda_1 - \lambda_0}, \frac{C - \tau\lambda_0}{\lambda_P + \lambda_1 - \lambda_0}, 0) \tag{B.1.16}$$

B.1.5 Case 1.5

- $\mu_1 > 0$ and $\mu_2 > 0$
- $T_0 > 0$, $T_J = 0$, $T_P > 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies T_0 = \tau$$
$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_0\lambda_0 = C \implies T_P = \frac{C - \tau\lambda_0}{\lambda_P}$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies \frac{(T_0+T_P)r_0\lambda_0 - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_P^2)}{(T_0+T_P)^2} - \mu_1 - \mu_2\lambda_0 = 0 \tag{B.1.17}$$

$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{(T_0+T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_P) - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_P^2)}{(T_0+T_P)^2}$$
$$- \mu_1 - \mu_2(\lambda_1 + \lambda_P) \leq 0 \tag{B.1.18}$$

$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{(T_0+T_P)(r_P\lambda_P - 2h\lambda_P T_P) - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_J^2)}{(T_0+T_P)^2} - \mu_2\lambda_P = 0 \tag{B.1.19}$$

Hence, the solution of Case 1.5 is

$$(T_0, T_J, T_P) = (\tau, 0, \frac{C - \tau\lambda_0}{\lambda_P}). \tag{B.1.20}$$

B.1.6 Case 1.6

- $\mu_1 > 0$ and $\mu_2 > 0$
- $T_0 = 0$, $T_J > 0$, $T_P > 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies T_J = \tau$$
$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_1 + \lambda_P) = C \implies$$
$$T_P = \frac{C - \tau(\lambda_1 + \lambda_P)}{\lambda_P}$$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{(T_J+T_P)r_0\lambda_0 - (T_P r_P \lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P(T_J+T_P)^2)}{(T_J+T_P)^2}$$
$$- \mu_1 - \mu_2\lambda_0 \leq 0 \tag{B.1.21}$$
$$\frac{\partial L}{\partial T_J} = 0 \implies \frac{(T_J+T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_J+T_P)^2}$$
$$\frac{-K - h\lambda_P(T_J+T_P)^2)}{(T_J+T_P)^2} - \mu_1 - \mu_2(\lambda_1+\lambda_P) = 0 \tag{B.1.22}$$
$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{(T_J+T_P)(r_P\lambda_P - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_J+T_P)^2}$$
$$\frac{-K - h\lambda_P(T_J+T_P)^2)}{(T_J+T_P)^2} - \mu_2\lambda_P = 0 \tag{B.1.23}$$

Hence, the solution of Case 1.6 is the following

$$(T_0, T_J, T_P) = (0, \tau, \frac{C - \tau(\lambda_1+\lambda_P)}{\lambda_P}). \tag{B.1.24}$$

B.1.7 Case 1.7

- $\mu_1 > 0$ and $\mu_2 > 0$
- $T_0 > 0$, $T_J > 0$, $T_P > 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau$$
$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_1+\lambda_P) + T_0\lambda_0 = C$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = \frac{(T_0+T_J+T_P)r_0\lambda_0 - (T_P r_P \lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0 \lambda_0 - K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2}$$

$$- \mu_1 - \mu_2\lambda_0 = 0 \tag{B.1.25}$$

$$\frac{\partial L}{\partial T_J} = \frac{(T_0+T_J+T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_0+T_J+T_P)^2}$$

$$\frac{+T_0 r_0\lambda_0 - K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2} - \mu_1 - \mu_2(\lambda_1+\lambda_P) = 0 \tag{B.1.26}$$

$$\frac{\partial L}{\partial T_P} = \frac{(T_0+T_J+T_P)(r_P\lambda_P - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0}{(T_0+T_J+T_P)^2}$$

$$\frac{-K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2} - \mu_2\lambda_P = 0 \tag{B.1.27}$$

By solving above equations together, we have the solution of Case 1.7. The variables used in Figure B.1 contain the subscripts as a part of their variable name. Also subscript P is represented by i in the figure.

$$\textit{Solution} \tag{B.1.28}$$

$$\{\{\mathrm{TI} \to \frac{1}{h\,li\,(l1-lo)^2}\,(-h\,l1^2\,li\,T + 2\,h\,l1\,li\,lo\,T - h\,li\,lo^2\,T + \sqrt{(h\,li\,(l1-lo)^2\,(c^2\,h\,li + k\,(l1+li-lo)^2 - li\,T\,((li-lo)^2\,ri + l1\,(li-lo)\,(ri+ro) + l1^2\,(ro-h\,T)) - c\,(l1^2\,ro + (li-lo)\,(li\,ri - lo\,ro) + l1\,(-2\,lo\,ro + li\,(ri+ro+2\,h\,T)))))}),$$

$$\mathrm{TJ} \to \frac{1}{h\,(l1-lo)^2\,(l1+li-lo)}\,(c\,h\,l1^2 - 2\,c\,h\,l1\,lo + c\,h\,lo^2 + h\,l1^2\,li\,T - h\,l1^2\,lo\,T - 2\,h\,l1\,li\,lo\,T + 2\,h\,l1\,lo^2\,T + h\,li\,lo^2\,T - h\,lo^3\,T - \sqrt{(h\,li\,(l1-lo)^2\,(c^2\,h\,li + k\,(l1+li-lo)^2 - li\,T\,((li-lo)^2\,ri + l1\,(li-lo)\,(ri+ro) + l1^2\,(ro-h\,T)) - c\,(l1^2\,ro + (li-lo)\,(li\,ri - lo\,ro) + l1\,(-2\,lo\,ro + li\,(ri+ro+2\,h\,T)))))}),$$

$$\mathrm{TO} \to \frac{1}{h\,(l1-lo)^2\,(l1+li-lo)}\,(-c\,h\,l1^2 + 2\,c\,h\,l1\,lo - c\,h\,lo^2 + h\,l1^3\,T - 2\,h\,l1^2\,lo\,T + h\,l1\,lo^2\,T + \sqrt{(h\,li\,(l1-lo)^2\,(c^2\,h\,li + k\,(l1+li-lo)^2 - li\,T\,((li-lo)^2\,ri + l1\,(li-lo)\,(ri+ro) + l1^2\,(ro-h\,T)) - c\,(l1^2\,ro + (li-lo)\,(li\,ri - lo\,ro) + l1\,(-2\,lo\,ro + li\,(ri+ro+2\,h\,T)))))})\},$$

Figure B.1: Solution of Case 1.7

B.2 Case 2

B.2.1 Case 2.1

- $\mu_1 > 0$ and $\mu_2 = 0$
- $T_0 > 0$, $T_J = 0$, $T_P = 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies T_0 = \tau$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies K = T_0^2 \mu_1 \tag{B.2.1}$$

$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{(r_P \lambda_P + r_0 \lambda_1 - r_0 \lambda_0)}{T_0} - \mu_1 \leq 0 \tag{B.2.2}$$

$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{T_0(r_P \lambda_P - r_0 \lambda_0) - K}{T_0^2} \leq 0 \tag{B.2.3}$$

Hence, the solution of Case 2.1 is

$$(T_0, T_J, T_P) = (\tau, 0, 0) \tag{B.2.4}$$

B.2.2 Case 2.2

- $\mu_1 > 0$ and $\mu_2 = 0$
- $T_0 = 0$, $T_J > 0$, $T_P = 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies T_J = \tau$$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{T_J(r_0\lambda_0 - r_P\lambda_P - r_0\lambda_1) - K - h\lambda_P T_J^2}{T_J^2} - \mu_1 \leq 0 \tag{B.2.5}$$

$$\frac{\partial L}{\partial T_J} = 0 \implies \frac{T_J(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P T_J^2)}{T_J^2} - \mu_1 = 0 \tag{B.2.6}$$

$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{T_J(r_P\lambda_P - 2h\lambda_P TJ) - (T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P T_J^2)}{T_J^2} \leq 0 \tag{B.2.7}$$

Hence, the solution of Case 2.2 is

$$(T_0, T_J, T_P) = (0, \tau, 0) \tag{B.2.8}$$

B.2.3 Case 2.3

- $\mu_1 > 0$ and $\mu_2 = 0$
- $T_0 = 0$, $T_J = 0$, $T_P > 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies \tau = 0$$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{T_P r_0\lambda_0 - (T_P r_P\lambda_P - K - h\lambda_P T_P^2)}{T_P^2} - \mu_1 \leq 0 \tag{B.2.9}$$

$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{T_P(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_P) - (T_P r_P\lambda_P - K - h\lambda_P T_P^2)}{T_P^2} - \mu_1 \leq 0 \tag{B.2.10}$$

$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{T_P(r_P\lambda_P - 2h\lambda_P T_P) - (T_P r_P\lambda_P - K - h\lambda_P T_P^2)}{T_P^2} = 0 \tag{B.2.11}$$

By solving $\frac{\partial L}{\partial T_P} = 0$, T_P's solution is reached. Hence, the solution of Case 2.3 is

$$(T_0, T_J, T_P) = (0, 0, \sqrt{K/h\lambda_P}). \tag{B.2.12}$$

B.2.4 Case 2.4

- $\mu_1 > 0$ and $\mu_2 = 0$
- $T_0 > 0$, $T_J > 0$, $T_P = 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies \frac{(T_0 + T_J)r_0\lambda_0 - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K - h\lambda_P T_J^2)}{(T_0 + T_J)^2} - \mu_1 = 0 \tag{B.2.13}$$

$$\frac{\partial L}{\partial T_J} = 0 \implies \frac{(T_0 + T_J)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K}{(T_0 + T_J)^2}$$

$$\frac{-h\lambda_P T_J^2)}{(T_0 + T_J)^2} - \mu_1 = 0 \tag{B.2.14}$$

$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{(T_0 + T_J)(r_P\lambda_P - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K - h\lambda_P T_J^2)}{(T_0 + T_J)^2} \leq 0 \tag{B.2.15}$$

By solving (B.2.13), (B.2.14) and (B.2.15) simultaneously:

$$\begin{aligned} T_0 =& \tau - \frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2h\lambda_P} \\ T_J =& \frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2h\lambda_P} \end{aligned}$$

Hence, the solution of Case 2.4 is

$$(T_0, T_J, T_P) = (\tau - \frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2h\lambda_P}, \frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2h\lambda_P}, 0). \tag{B.2.16}$$

B.2.5 Case 2.5

- $\mu_1 > 0$ and $\mu_2 = 0$
- $T_0 > 0$, $T_J = 0$, $T_P > 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies T_0 = \tau$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies \frac{(T_0 + T_P)r_0\lambda_0 - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_P^2)}{(T_0 + T_P)^2} - \mu_1 = 0 \tag{B.2.17}$$

$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{(T_0 + T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_P) - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_P^2)}{(T_0 + T_P)^2}$$
$$- \mu_1 \leq 0 \tag{B.2.18}$$

$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{(T_0 + T_P)(r_P\lambda_P - 2h\lambda_P T_P) - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_J^2)}{(T_0 + T_P)^2} = 0 \tag{B.2.19}$$

By solving (B.2.17) and (B.2.19) simultaneously, the expression for T_P is reached. Hence, the solution of Case 2.5 is

$$(T_0, T_J, T_P) = (\tau, 0, \sqrt{\tau^2 + \frac{K - \tau(r_0\lambda_0 - r_P\lambda_P)}{h\lambda_P}} - \tau). \tag{B.2.20}$$

B.2.6 Case 2.6

- $\mu_1 > 0$ and $\mu_2 = 0$
- $T_0 = 0$, $T_J > 0$, $T_P > 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau \implies T_J = \tau$$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{(T_J + T_P)r_0\lambda_0 - (T_P r_P \lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P(T_J + T_P)^2}{(T_J + T_P)^2} - \mu_1 \leq 0 \tag{B.2.21}$$

$$\frac{\partial L}{\partial T_J} = \frac{(T_J + T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P(T_P + T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_J + T_P)^2} \frac{-K - h\lambda_P(T_J + T_P)^2)}{(T_J + T_P)^2} - \mu_1 = 0 \tag{B.2.22}$$

$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{(T_J + T_P)(r_P\lambda_P - 2h\lambda_P(T_P + T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_J + T_P)^2} \frac{-K - h\lambda_P(T_J + T_P)^2)}{(T_J + T_P)^2} = 0 \tag{B.2.23}$$

By solving above equations the expression for T_P is obtained. Hence, the solution of Case 2.6 is

$$(T_0, T_J, T_P) = (0, \tau, \sqrt{\frac{K - \tau r_0\lambda_1}{h\lambda_P}} - \tau). \tag{B.2.24}$$

B.2.7 Case 2.7

- $\mu_1 > 0$ and $\mu_2 = 0$
- $T_0 > 0$, $T_J > 0$, $T_P > 0$

From complementary slackness:

$$\mu_1 g_1(T_J, T_0) = 0 \implies g_1(T_J, T_0) = 0 \implies T_0 + T_J = \tau$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = \frac{(T_0+T_J+T_P)r_0\lambda_0 - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2}$$
$$- \mu_1 = 0 \tag{B.2.25}$$

$$\frac{\partial L}{\partial T_J} = \frac{(T_0+T_J+T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_0+T_J+T_P)^2}$$
$$\frac{+T_0 r_0\lambda_0 - K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2} - \mu_1 = 0 \tag{B.2.26}$$

$$\frac{\partial L}{\partial T_P} = \frac{(T_0+T_J+T_P)(r_P\lambda_P - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0}{(T_0+T_J+T_P)^2}$$
$$\frac{-K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2} = 0 \tag{B.2.27}$$

$$(T_P, T_J, T_0) = (\frac{-(\lambda_1-\lambda_0)r_0)^2 + 4h\lambda_P(K - \lambda_P r_0\tau}{4h(\lambda_1-\lambda_P)\lambda_P r_0},$$
$$\frac{(\lambda_P r_P + (\lambda_1-\lambda_0)r_0)^2 + 4h\lambda_P(K - \lambda_1 r_0\tau}{4h(\lambda_1-\lambda_P)\lambda_P r_0},$$
$$\frac{((\lambda_1^2-\lambda_0^2)r_0^2 - \lambda_P^2 r_P(r_P+2r_0) + 2\lambda_P r_0(-\lambda_1 r_0 + \lambda_0(r_P+r_0)) + 4h\lambda_P(K - \lambda_1 r_0\tau))}{4h(\lambda_1-\lambda_P)\lambda_P r_0}) \tag{B.2.28}$$

B.3 Case 3

B.3.1 Case 3.1

- $\mu_1 = 0$ and $\mu_2 > 0$
- $T_0 > 0$, $T_J = 0$, $T_P = 0$

From complementary slackness:

$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_P + \lambda_1) + T_0\lambda_0 = C$$
$$\implies T_0 = C/\lambda_0$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies K = T_0^2 \mu_2 \lambda_0 \tag{B.3.1}$$

$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{T_0} - \mu_2(\lambda_P + \lambda_1 - \lambda_0) \leq 0 \tag{B.3.2}$$

$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{T_0(r_P\lambda_P - r_0\lambda_0) - K}{T_0^2} - \mu_2\lambda_P \leq 0 \tag{B.3.3}$$

Hence, the solution of Case 3.1 is

$$(T_0, T_J, T_P) = (C/\lambda_0, 0, 0). \tag{B.3.4}$$

B.3.2 Case 3.2

- $\mu_1 = 0$ and $\mu_2 > 0$
- $T_0 = 0$, $T_J > 0$, $T_P = 0$

From complementary slackness:

$$\begin{aligned} \mu_2 g_2(T_P, T_J, T_0) = 0 &\implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_P + \lambda_1) + T_0\lambda_0 = C \\ &\implies T_J = C/(\lambda_1 + \lambda_P) \end{aligned}$$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{T_J(r_0\lambda_0 - r_P\lambda_P - r_0\lambda_1) - K - h\lambda_P T_J^2}{T_J^2} - \mu_2\lambda_0 \leq 0 \tag{B.3.5}$$

$$\begin{aligned} \frac{\partial L}{\partial T_J} = 0 \implies & \frac{T_J(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P T_J^2)}{T_J^2} \\ & - \mu_2(\lambda_1 + \lambda_P) = 0 \end{aligned} \tag{B.3.6}$$

$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{T_J(r_P\lambda_P - 2h\lambda_P TJ) - (T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P T_J^2)}{T_J^2} - \mu_2\lambda_P \leq 0 \tag{B.3.7}$$

Hence, the solution of Case 3.2 is

$$(T_0, T_J, T_P) = (0, C/(\lambda_1 + \lambda_P), 0). \tag{B.3.8}$$

B.3.3 Case 3.3

- $\mu_1 = 0$ and $\mu_2 > 0$
- $T_0 = 0$, $T_J = 0$, $T_P > 0$

From complementary slackness:

$$\begin{aligned}\mu_2 g_2(T_P, T_J, T_0) = 0 &\implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_P + \lambda_1) + T_0\lambda_0 = C \\ &\implies T_P\lambda_P = C \implies T_P = C/\lambda_P\end{aligned}$$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{T_P r_0 \lambda_0 - (T_P r_P \lambda_P - K - h\lambda_P T_P^2)}{T_P^2} - \mu_2\lambda_0 \leq 0 \tag{B.3.9}$$

$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{T_P(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_P) - (T_P r_P \lambda_P - K - h\lambda_P T_P^2)}{T_P^2} - \mu_2(\lambda_1 + \lambda_P) \leq 0 \tag{B.3.10}$$

$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{T_P(r_P\lambda_P - 2h\lambda_P T_P) - (T_P r_P \lambda_P - K - h\lambda_P T_P^2)}{T_P^2} - \mu_2\lambda_P = 0 \tag{B.3.11}$$

Hence, the solution of Case 3.3 is

$$(T_0, T_J, T_P) = (0, 0, C/\lambda_P). \tag{B.3.12}$$

B.3.4 Case 3.4

- $\mu_1 = 0$ and $\mu_2 > 0$
- $T_0 > 0$, $T_J > 0$, $T_P = 0$

From complementary slackness:

$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_J(\lambda_P + \lambda_1) + T_0\lambda_0 = C$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies \frac{(T_0+T_J)r_0\lambda_0 - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K - h\lambda_P T_J^2)}{(T_0+T_J)^2} - \mu_2\lambda_0 = 0 \tag{B.3.13}$$

$$\frac{\partial L}{\partial T_J} = 0 \implies \frac{(T_0+T_J)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K - h\lambda_P T_J^2)}{(T_0+T_J)^2} - \mu_2(\lambda_1+\lambda_P) = 0 \tag{B.3.14}$$

$$\frac{\partial L}{\partial T_P} \le 0 \implies \frac{(T_0+T_J)(r_P\lambda_P - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K - h\lambda_P T_J^2)}{(T_0+T_J)^2} - \mu_2\lambda_P \le 0 \tag{B.3.15}$$

The solution of Case 3.4 changes with respect to the relation between λ_0 and $(\lambda_1+\lambda_P)$. If $\lambda_0 = (\lambda_1+\lambda_P) \implies$ The solution of Case 3.4 is

$$(T_0, T_J, T_P) = (C/\lambda_0 - \frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2h\lambda_P}, \frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2h\lambda_P}, 0). \tag{B.3.16}$$

If $\lambda_0 \neq (\lambda_1+\lambda_P)$, the solution will be

$$\Big(\frac{Ch\lambda_P\lambda_0 + (\lambda_1+\lambda_P)\sqrt{h\lambda_P(C^2h\lambda_P + (\lambda_P+\lambda_1-\lambda_0)(K(\lambda_P+\lambda_1-\lambda_0) - C(r_P\lambda_P + (\lambda_1-\lambda_0)r_0)))}}{h\lambda_P\lambda_0(\lambda_P+\lambda_1-\lambda_0)}, \frac{Ch\lambda_P - \sqrt{h\lambda_P(C^2h\lambda_P + (\lambda_P+\lambda_1-\lambda_0)(K(\lambda_P+\lambda_1-\lambda_0) - C(r_P\lambda_P + (\lambda_1-\lambda_0)r_0)))}}{h\lambda_P(\lambda_P+\lambda_1-\lambda_0)}, 0). \tag{B.3.17}$$

B.3.5 Case 3.5

- $\mu_1 = 0$ and $\mu_2 > 0$
- $T_0 > 0$, $T_J = 0$, $T_P > 0$

From complementary slackness:

$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_0\lambda_0 = C$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies \frac{(T_0+T_P)r_0\lambda_0 - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_P^2)}{(T_0+T_P)^2} - \mu_2\lambda_0 = 0 \tag{B.3.18}$$

$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{(T_0+T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_P) - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_P^2)}{(T_0+T_P)^2}$$
$$- \mu_2(\lambda_1+\lambda_P) \leq 0 \tag{B.3.19}$$

$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{(T_0+T_P)(r_P\lambda_P - 2h\lambda_P T_P) - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_J^2)}{(T_0+T_P)^2} - \mu_2\lambda_P = 0 \tag{B.3.20}$$

Hence, the solution of Case 3.5 is

$$(T_0, T_J, T_P) = (\frac{Ch\lambda_0 - \sqrt{h\lambda_P(C^2h\lambda_P + K(\lambda_P-\lambda_0)^2 - C(\lambda_P-\lambda_0)(\lambda_P r_P - \lambda_0 r_0))}}{h\lambda_0(-\lambda_P+\lambda_0)},$$
$$0, \frac{Ch\lambda_P - \sqrt{h\lambda_P(C^2h\lambda_P + K(\lambda_P-\lambda_0)^2 - C(\lambda_P-\lambda_0)(\lambda_P r_P - \lambda_0 r_0))}}{h\lambda_0(\lambda_P-\lambda_0)}). \tag{B.3.21}$$

B.3.6 Case 3.6

- $\mu_1 = 0$ and $\mu_2 > 0$
- $T_0 = 0$, $T_J > 0$, $T_P > 0$

From complementary slackness:

$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_1+\lambda_P) = C$$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{(T_J + T_P)r_0\lambda_0 - (T_P r_P \lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P(T_J + T_P)^2)}{(T_J + T_P)^2}$$
$$- \mu_2\lambda_0 \leq 0 \tag{B.3.22}$$
$$\frac{\partial L}{\partial T_J} = \frac{(T_J + T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P(T_P + T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_J + T_P)^2}$$
$$\frac{-K - h\lambda_P(T_J + T_P)^2)}{(T_J + T_P)^2} - \mu_2(\lambda_1 + \lambda_P) = 0 \tag{B.3.23}$$
$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{(T_J + T_P)(r_P\lambda_P - 2h\lambda_P(T_P + T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_J + T_P)^2}$$
$$\frac{-K - h\lambda_P(T_J + T_P)^2)}{(T_J + T_P)^2} - \mu_2\lambda_P = 0 \tag{B.3.24}$$

Hence, the solution of Case 3.6 is

$$(T_0, T_J, T_P) = (0, \frac{Ch\lambda_1 - \sqrt{h\lambda_1^2\lambda_P(K - Cr_0)}}{h\lambda_1^2}, \frac{-Ch\lambda_1\lambda_P + (\lambda_1 + \lambda_P)\sqrt{h\lambda_1^2\lambda_P(K - Cr_0)}}{h\lambda_1^2\lambda_P}). \tag{B.3.25}$$

B.3.7 Case 3.7

- $\mu_1 = 0$ and $\mu_2 > 0$
- $T_0 > 0$, $T_J > 0$, $T_P > 0$

From complementary slackness:

$$\mu_2 g_2(T_P, T_J, T_0) = 0 \implies g_2(T_P, T_J, T_0) = 0 \implies T_P\lambda_P + T_J(\lambda_1 + \lambda_P) + T_0\lambda_0 = C$$

From stationarity:

$$\frac{\partial L}{\partial T_0} = \frac{(T_0+T_J+T_P)r_0\lambda_0 - (T_P r_P \lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2}$$
$$- \mu_2\lambda_0 = 0 \tag{B.3.26}$$
$$\frac{\partial L}{\partial T_J} = \frac{(T_0+T_J+T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_0+T_J+T_P)^2}$$
$$\frac{+T_0 r_0\lambda_0 - K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2} - \mu_2(\lambda_1+\lambda_P) = 0 \tag{B.3.27}$$
$$\frac{\partial L}{\partial T_P} = \frac{(T_0+T_J+T_P)(r_P\lambda_P - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K}{(T_0+T_J+T_P)^2}$$
$$\frac{-h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2} - \mu_2\lambda_P = 0 \tag{B.3.28}$$

The solution of this case exits only when $T_P = 0$. The main reason behind this solution is that $r_P\lambda_P$ can be produced by T_P and T_J separately. When both of them are positive, filling the cycle length with T_J is more reasonable as it additionally brings $r_0\lambda_1$. Consequently, it makes the Case 3.7 to has the same solution with Case 3.4.

B.4 Case 4

B.4.1 Case 4.1

- $\mu_1 = 0$ and $\mu_2 = 0$
- $T_0 > 0$, $T_J = 0$, $T_P = 0$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies K = T_0^2(\mu_1 + \mu_2\lambda_0) \implies K/0 = T_0^2 \implies T_0 = \infty \tag{B.4.1}$$
$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{T_0} - \mu_1 - \mu_2(\lambda_P + \lambda_1 - \lambda_0) \leq 0 \tag{B.4.2}$$
$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{T_0(r_P\lambda_P - r_0\lambda_0) - K}{T_0^2} - \mu_2\lambda_P \leq 0 \tag{B.4.3}$$

Hence, the solution of Case 4.1 is

$$(T_0, T_J, T_P) = (\infty, 0, 0) \tag{B.4.4}$$

B.4.2 Case 4.2

- $\mu_1 = 0$ and $\mu_2 = 0$
- $T_0 = 0$, $T_J > 0$, $T_P = 0$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{T_J(r_0\lambda_0 - r_P\lambda_P - r_0\lambda_1) + K + h\lambda_P T_J^2}{T_J^2} \leq 0 \tag{B.4.5}$$

$$\frac{\partial L}{\partial T_J} = 0 \implies \frac{T_J(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P T_J^2)}{T_J^2} = 0 \tag{B.4.6}$$

$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{T_J(r_P\lambda_P - 2h\lambda_P TJ) - (T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P T_J^2)}{T_J^2} \leq 0 \tag{B.4.7}$$

By solving $\frac{\partial L}{\partial T_J} = 0$ the expression for T_P is reached and, the solution of Case 4.2 is

$$(T_0, T_J, T_P) = (0, \sqrt{K/h\lambda_P}, 0). \tag{B.4.8}$$

B.4.3 Case 4.3

- $\mu_1 = 0$ and $\mu_2 = 0$
- $T_0 = 0$, $T_J = 0$, $T_P > 0$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{T_P r_0 \lambda_0 - (T_P r_P \lambda_P - K - h\lambda_P T_P^2)}{T_P^2} \leq 0$$

$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{T_P(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_P) - (T_P r_P \lambda_P - K - h\lambda_P T_P^2)}{T_P^2} \leq 0$$

$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{T_P(r_P\lambda_P - 2h\lambda_P T_P) - (T_P r_P \lambda_P - K - h\lambda_P T_P^2)}{T_P^2} = 0 \implies T_P = \sqrt{K/h\lambda_P}$$

Hence, the solution of Case 4.3 is $(T_0, T_J, T_P) = (0, 0, \sqrt{K/h\lambda_P})$. If we put the solution of Case 4.3 into the other stationarity inequalities $T_P r_0 \lambda_1 \leq 0$. To satisfy this inequality $r_0\lambda_1$ should be 0 as we $T_P > 0$.

B.4.4 Case 4.4

- $\mu_1 = 0$ and $\mu_2 = 0$
- $T_0 > 0$, $T_J > 0$, $T_P = 0$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies \frac{(T_0 + T_J) r_0 \lambda_0 - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0 \lambda_0 - K - h\lambda_P T_J^2)}{(T_0 + T_J)^2} = 0$$

$$\frac{\partial L}{\partial T_J} = 0 \implies \frac{(T_0 + T_J)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0 \lambda_0 - K}{(T_0 + T_J)^2}$$

$$\frac{-h\lambda_P T_J^2)}{(T_0 + T_J)^2} = 0$$

$$\frac{\partial L}{\partial T_P} \leq 0 \implies \frac{(T_0 + T_J)(r_P\lambda_P - 2h\lambda_P T_J) - (T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0 \lambda_0 - K - h\lambda_P T_J^2)}{(T_0 + T_J)^2} \leq 0$$

If $\frac{\partial L}{\partial T_0} = \frac{\partial L}{\partial T_J} = 0$ and solve it, the expression of $T_J = \frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2h\lambda_P}$ is reached. However, if we put the solution of T_J into $\frac{\partial L}{\partial T_0} = 0$ or $\frac{\partial L}{\partial T_J} = 0$, T_0 disappears from the solution and we get $K = \frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)^2}{4h\lambda_P}$ as special condition for existence of this case. Consequently, the solution of Case 4.4 is

$$(T_0, T_J, T_P) = (0, \frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2h\lambda_P}, 0). \tag{B.4.9}$$

B.4.5 Case 4.5

- $\mu_1 = 0$ and $\mu_2 = 0$
- $T_0 > 0$, $T_J = 0$, $T_P > 0$

From stationarity:

$$\frac{\partial L}{\partial T_0} = 0 \implies \frac{(T_0 + T_P)r_0\lambda_0 - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_P^2)}{(T_0 + T_P)^2} = 0$$
$$\frac{\partial L}{\partial T_J} \leq 0 \implies \frac{(T_0 + T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P T_P) - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_P^2)}{(T_0 + T_P)^2} \leq 0$$
$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{(T_0 + T_P)(r_P\lambda_P - 2h\lambda_P T_P) - (T_P r_P \lambda_P + T_0 r_0 \lambda_0 - K - h\lambda_P T_J^2)}{(T_0 + T_P)^2} = 0$$

Solving $\frac{\partial L}{\partial T_0}$ and $\frac{\partial L}{\partial T_P}$ together gives the solution of T_P as $\frac{r_P\lambda_P - r_0\lambda_0}{2h\lambda_P}$. Being similar with Case 4.4, putting the solution of T_P into equalities in the stationarity conditions leave us the feasibility condition for Case 4.5 which expressed as $K = \frac{(r_P\lambda_P - r_0\lambda_0)^2}{4h\lambda_P}$. Additionally, this condition pushes to T_0 be zero. Therefore, the solution of Case 4.5 is

$$(T_0, T_J, T_P) = (0, 0, \frac{r_P\lambda_P - r_0\lambda_0}{2h\lambda_P}). \tag{B.4.10}$$

B.4.6 Case 4.6

- $\mu_1 = 0$ and $\mu_2 = 0$
- $T_0 = 0$, $T_J > 0$, $T_P > 0$

From stationarity:

$$\frac{\partial L}{\partial T_0} \leq 0 \implies \frac{(T_J+T_P)r_0\lambda_0 - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) - K - h\lambda_P(T_J+T_P)^2)}{(T_J+T_P)^2} \leq 0$$

$$\frac{\partial L}{\partial T_J} = \frac{(T_J+T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) - K}{(T_J+T_P)^2}$$

$$\frac{-h\lambda_P(T_J+T_P)^2)}{(T_J+T_P)^2} = 0$$

$$\frac{\partial L}{\partial T_P} = 0 \implies \frac{(T_J+T_P)(r_P\lambda_P - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_J+T_P)^2}$$

$$\frac{-K - h\lambda_P(T_J+T_P)^2)}{(T_J+T_P)^2} = 0$$

Solving above equalities together gives the solution of $T_P = \sqrt{K/h\lambda_P}$. However, the only way to be feasible is having $r_0\lambda_1 = 0$ and it means no joint service, $T_J = 0$. Therefore, the solution of Case 4.6 becomes

$$(T_0, T_J, T_P) = (0, 0, \sqrt{K/h\lambda_P}). \tag{B.4.11}$$

B.4.7 Case 4.7

- $\mu_1 = 0$ and $\mu_2 = 0$
- $T_0 > 0$, $T_J > 0$, $T_P > 0$

From stationarity:

$$\frac{\partial L}{\partial T_0} = \frac{(T_0+T_J+T_P)r_0\lambda_0 - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0 - K}{(T_0+T_J+T_P)^2}$$
$$\frac{-h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2} = 0$$
$$\frac{\partial L}{\partial T_J} = \frac{(T_0+T_J+T_P)(r_P\lambda_P + r_0\lambda_1 - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1)}{(T_0+T_J+T_P)^2}$$
$$\frac{+T_0 r_0\lambda_0 - K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2} = 0$$
$$\frac{\partial L}{\partial T_P} = \frac{(T_0+T_J+T_P)(r_P\lambda_P - 2h\lambda_P(T_P+T_J)) - (T_P r_P\lambda_P + T_J(r_P\lambda_P + r_0\lambda_1) + T_0 r_0\lambda_0}{(T_0+T_J+T_P)^2}$$
$$\frac{-K - h\lambda_P(T_J+T_P)^2)}{(T_0+T_J+T_P)^2} = 0$$

By simultaneously solving three above equalities T_P is found as $\frac{r_P\lambda_P - r_0\lambda_0}{2h\lambda_P}$. However, feasibility is reached when $T_0 = 0$ and $r_0\lambda_1 = 0$. At this point, the case converges Case 4.5. However, it is proved that Case 4.5 has $T_J = 0$. Hence, the solution of Case 4.7 is

$$(T_0, T_J, T_P) = (0, 0, \frac{r_P\lambda_P - r_0\lambda_0}{2h\lambda_P}). \tag{B.4.12}$$

Appendix C

Proof of Proposition 1

Let, $f(T_0, T_J, T_P)$ be the objective function of the original model and $C_1(T_0, T_J)$ be the promised delivery time, $C_2(T_0, T_J, T_P)$ be the capacity constraint. The set for C_1 is defined as $C_1 = C_1^1 \cup C_1^2$ where $C_1^1 = \{T_0, T_J | T_0 + T_J = \tau\}$ and $C_1^2 = \{T_0, T_J | T_0 + T_J < \tau\}$. Similarly, $C_2 = C_2^1 \cup C_2^2$ where $C_2^1 = \{T_0, T_J, T_P | T_0\lambda_0 + T_J(\lambda_1 + \lambda_P) + T_P\lambda_P = C\}$ and $C_2^2 = \{T_0, T_J, T_P | T_0\lambda_0 + T_J(\lambda_1 + \lambda_P) + T_P\lambda_P < C\}$.

Now consider two models,

$$\begin{aligned} (1) \quad & \max \quad f(x) \\ & \text{st.} \quad\;\; x \in C_1^1 \end{aligned}$$

$$\begin{aligned} (2) \quad & \max \quad f(x) \\ & \text{st.} \quad\;\; x \in C_1^1 \cup C_1^2 \end{aligned}$$

As the feasible region of (2) is greater than (1) the objective value of (2) will be greater than or equal to the objective value of (1). We observe this relation between *Case 1.i* and *Case 2.i* for all $i = 1, .., 8$. The same relation also exits between models having feasible set C_2^1 or $C_2^1 \cup C_2^2$. Thus, for all $i = 1, .., 8$ *Case 1.i* and *Case 3.i* pairs also show this property. Nonetheless, we cannot compare *Case 2.i* and *Case 3.i* pairs as they have vice versa binding and unbinding constraints considering subsets of C_1 and C_2.

Lastly, as all *Case 4.i* have unbinding constraint comparing each of *Case 4.i* with *Case 1.i*, *Case 2.i* or *Case 3.i* is possible. Let, O_i be the objective value of case i and F_i be the feasible region of case i. Therefore, following statements are valid for for all $i = 1, .., 8$.

1. $O_{1.i} \leq O_{4.i}$ since $F_{1.i} \subseteq F_{4.i}$

2. $O_{2.i} \leq O_{4.i}$ since $F_{2.i} \subseteq F_{4.i}$

3. $O_{3.i} \leq O_{4.i}$ since $F_{3.i} \subseteq F_{4.i}$

Appendix D

K and h Bounds of Some Cases

By solving the equations of each case from Appendix B, we reach bounds for cost parameters K and h for some cases.

D.1 Case 1.4

With using the first constraint of the problem and solving (B.1.13) and (B.1.14) simultaneously, we reach the following equation:

$$\frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0 - 2\lambda_P h T_J}{\tau} - \mu_2(\lambda_P + \lambda_1 - \lambda_0) = 0$$

Notice that for Case 1.4 $\mu_2 \geq 0$. Moreover, solving (B.1.15) with either (B.1.13) or (B.1.14) gives

$$\frac{\lambda_1(r_P\lambda_P + r_0\lambda_P - 2\lambda_P h T_J)}{\tau(\lambda_P + \lambda_1 - \lambda_0)} \leq 0$$

We know that problem parameters cannot be zero. Thus we can make following conditions by using the solution of T_J. If

- $\lambda_P + \lambda_1 - \lambda_0 > 0$

$$\frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)(\lambda_P + \lambda_1 - \lambda_0)}{2\lambda_P(C - \tau\lambda_0)} > h \tag{D.1.1}$$

$$\frac{(r_P\lambda_P - r_0\lambda_P)(\lambda_P + \lambda_1 - \lambda_0)}{2\lambda_P(C - \tau\lambda_0)} < h \tag{D.1.2}$$

- $\lambda_P + \lambda_1 - \lambda_0 < 0$

$$\frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)(\lambda_P + \lambda_1 - \lambda_0)}{2\lambda_P(C - \tau\lambda_0)} < h \tag{D.1.3}$$

$$\frac{(r_P\lambda_P - r_0\lambda_P)(\lambda_P + \lambda_1 - \lambda_0)}{2\lambda_P(C - \tau\lambda_0)} > h. \tag{D.1.4}$$

D.2 Case 1.5

From (B.1.19), we have the expression equivalent to μ_2.

$$\mu_2 = \tau(r_P\lambda_P - r_0\lambda_0) - 2h\lambda_P T_P(\tau + T_P) + K + h\lambda_P T_P^2$$

If we put the above expression in (B.1.18), we reach an expression for μ_1.

$$\mu_1 = (T_P\lambda_P + \tau\lambda_0)(r_0\lambda_0 - r_P\lambda_P) - (\lambda_0 - \lambda_P)(K + h\lambda_P T_P^2) + \lambda_0 2h\lambda_P T_P(\tau + T_P)$$

In Case 1.5, we have $\mu_1 > 0$ and $\mu_2 > 0$. Thus, by using the solution of the Case 1.5, (B.1.20), in these conditions we reach bounds for K.

$$\tau(r_0\lambda_0 - r_P\lambda_P) + (C - \tau\lambda_0)^2 h/\lambda_P + 2h\tau(C - \tau\lambda_0) < K \tag{D.2.1}$$

$$\frac{C(r_0\lambda_0 - r_P\lambda_P) + (h\lambda_0 + h\lambda_P)(C - \tau\lambda_0)^2/\lambda_P + 2h\lambda_0\tau(C - \tau\lambda_0)}{\lambda_0 - \lambda_P} > K \tag{D.2.2}$$

D.3 Case 1.6

From (B.1.23),

$$\frac{\tau r_0\lambda_1 + K + h\lambda_P(\tau + T_P)^2 - 2h\lambda_P(\tau + T_P)^2}{(\tau + T_P)^2\lambda_P} = \mu_2$$

By solving (B.1.22) and (B.1.21) jointly, we are able to reach μ_1 also.

$$\frac{\lambda_1(r_0\lambda_P(\tau+T_P)-\tau r_0\lambda_1-K-h\lambda_P(\tau+T_P)^2+2h\lambda_P(\tau+T_P)^2)}{(\tau+T_P)^2\lambda_P}=\mu_1$$

We know that $\mu_1>0$ and $\mu_2>0$ for Case 1.6. If we condition above expressions to be non-negative and use the solution for T_P and T_J, we reach bounds for K.

$$Cr_0-2\tau r_0\lambda_1+h(C-\tau\lambda_1)^2/\lambda_P>K \tag{D.3.1}$$

$$h(C-\tau\lambda_1)^2/\lambda_P-\tau r_0\lambda_1<K \tag{D.3.2}$$

D.4 Case 1.7

By solving (B.1.26) and (B.1.27) simultaneously, we reach the following.

$$\mu_1+\mu_2\lambda_1=\frac{r_0\lambda_1}{T_P+T_J+T_0}$$

And similarly from (B.1.25) and (B.1.26), we have

$$\frac{r_P\lambda_P+r_0\lambda_1-r_0\lambda_0-2h\lambda_P(T_P+T_J)}{T_P+T_J+T_0}-\mu_2(\lambda_P+\lambda_1-\lambda_0)=0$$

From the above equations we are able to get expressions for μ_1 and μ_2.

$$\mu_1=\lambda_1\frac{r_0\lambda_P-r_P\lambda_P-2h\lambda_P(T_P+T_J)}{(T_P+T_J+T_0)(\lambda_P+\lambda_1-\lambda_0)}$$

$$\mu_2=\frac{r_P\lambda_P+r_0\lambda_1-r_0\lambda_0-2h\lambda_P(T_P+T_J)}{(T_P+T_J+T_0)(\lambda_P+\lambda_1-\lambda_0)}$$

Even though solving the equations with the exact solutions of T_P, T_J, T_0 gives too complicated expression that make simplifying impossible to get bounds for K or h, we are still able to reach following results. As parameters and decisions variables are positive, if

- $\lambda_P+\lambda_1-\lambda_0>0$

$$r_P\lambda_P+r_0\lambda_1-r_0\lambda_0-2h\lambda_P(T_P+T_J)>0 \tag{D.4.1}$$

$$r_0\lambda_P-r_P\lambda_P-2h\lambda_P(T_P+T_J)>0 \tag{D.4.2}$$

- $\lambda_P + \lambda_1 - \lambda_0 < 0$

$$r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0 - 2h\lambda_P(T_P + T_J) < 0 \tag{D.4.3}$$

$$r_0\lambda_P - r_P\lambda_P - 2h\lambda_P(T_P + T_J) < 0 \tag{D.4.4}$$

D.5 Case 2.1

By using the solution of Case 2.1, (B.2.4), and the inequality (B.2.3), the following bound for K is reached.

$$\tau(r_0\lambda_0 - r_P\lambda_P) \geq K \tag{D.5.1}$$

D.6 Case 2.2

In Case 2.2, the solution (B.2.8) and (B.2.6) gives the expression for μ_1 and the lower bound for K is reached.

$$h\lambda_P\tau^2 < K \tag{D.6.1}$$

Moreover, the inequality, (B.2.7), with the solution, (B.2.8), give us the upper bound for K.

$$h\lambda_P\tau^2 + r_0\lambda_1\tau \geq K \tag{D.6.2}$$

Lastly, by simplifying (B.2.5), a bound for h is reached.

$$\frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2\lambda_P} \geq h \tag{D.6.3}$$

D.7 Case 2.4

From equation (B.2.13), the expression for μ_1 is reached and simplified to get a lower bound for K.

$$\frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)^2}{4h\lambda_P} < K \tag{D.7.1}$$

Similarly, with the solution of Case 2.4, (B.2.16), (B.2.15) is simplified to get an upper bound for K.

$$\frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)^2}{4h\lambda_P} + \tau r_0\lambda_1 \geq K \tag{D.7.2}$$

We know that decision variables are non-negative. From (B.2.15), a lower bound for h is reached.

$$\frac{r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0}{2\tau\lambda_P} < h \tag{D.7.3}$$

Lastly, another lower bound for h is constructed by using the solution, (B.2.15) and the capacity constraint of the problem.

$$\frac{(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)(\lambda_P + \lambda_1 - \lambda_0)}{2\tau\lambda_P(C - \tau\lambda_0)} + \tau r_0\lambda_1 < h \tag{D.7.4}$$

D.8 Case 2.5

By solving (B.2.20) and (B.2.19) together, the expression for μ_1 is obtained. We know that for Case 2.5, μ_1 is positive. If we condition the expression for μ_1 to positivity, we are able to get a lower bound for K.

$$\frac{(r_P\lambda_P - r_0\lambda_0 + 2h\lambda_P\tau)^2}{4h\lambda_P} + \tau(r_0\lambda_0 - r_P\lambda_P) - \tau^2 h\lambda_P < K \tag{D.8.1}$$

Furthermore, we know that T_P is positive. Thus, an another lower bound is created from this condition.

$$\tau(r_0\lambda_0 - r_P\lambda_P) < K \tag{D.8.2}$$

If we put the solution (B.2.20) into the capacity constraint, an upper bound for K is expressed as follows.

$$h(C-\tau(\lambda_0-\lambda_P))^2/\lambda_P+\tau(r_0\lambda_0-r_P\lambda_P)-\tau^2 h\lambda_P > K \tag{D.8.3}$$

Lastly, we know that K is positive, so as its upper bound should be positive. If we condition (D.8.3) to be positive, we reach a lower bound for h.

$$\frac{r_0\lambda_0-r_P\lambda_P}{C-\tau(\lambda_0-\lambda_P)} < h \tag{D.8.4}$$

D.9 Case 2.6

We know that the decision variable T_P is positive. Hence, using the expression for it in (B.2.23), a lower bound for K is obtained.

$$\tau^2 h\lambda_P+\tau r_0\lambda_1 < K \tag{D.9.1}$$

On the other hand, an upper bound for K is created by using the solutions in the capacity constraint.

$$(C-\tau\lambda_1)^2 h/\lambda_1+\tau r_0\lambda_1 > K \tag{D.9.2}$$

D.10 Case 2.7

From (B.2.25) and (B.2.27), the following expression is obtained.

$$r_P\lambda_P+r_0\lambda_1-r_0\lambda_0 = 2h\lambda_P(T_P+T_J) \tag{D.10.1}$$

As the right hand side of the expression is the derivative of the inventory cost function, we know that it is positive. Hence, the expression in the left hand side also positive.

D.11 Case 3.1

We have the solution of Case 3.1 (B.3.4). If we solve (B.3.2) together with (B.3.1), an upper bound for K is obtained.

$$\frac{C(r_0\lambda_0 - r_0\lambda_1 - r_P\lambda_P)}{\lambda_0 - \lambda_1 - \lambda_P} \geq K \tag{D.11.1}$$

An another bound for K is also constructed by simplifying (B.3.3) with the solution of the case.

$$\frac{C(r_0\lambda_0 - r_P\lambda_P)}{\lambda_0 - \lambda_P} \geq K \tag{D.11.2}$$

D.12 Case 3.2

First, the solution of Case 3.2, (B.3.8), is used to get the expression for μ_2. If we condition this expression to be positive, we reach the following.

$$\frac{C^2 h\lambda_P}{(\lambda_P + \lambda_1)^2} < K \tag{D.12.1}$$

On the other hand, applying same simplification to (B.3.7) yield an upper bound for K.

$$\frac{C^2 h\lambda_P}{(\lambda_P + \lambda_1)^2} + Cr_0 \geq K \tag{D.12.2}$$

In the same manner, the inequality (B.3.5) gives

$$\frac{C(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{\lambda_P + \lambda_1 - \lambda_0} - \frac{C^2 h\lambda_P(\lambda_P + \lambda_0 + \lambda_1)}{(\lambda_P + \lambda_1)^2(\lambda_P + \lambda_1 - \lambda_0)} \geq K. \tag{D.12.3}$$

We know that K is positive, so its upper bound should be positive. Thus, by conditioning the left hand side of (D.12.3), we reach two possible bound for h. If $\lambda_P + \lambda_1 - \lambda_0 > 0$,

$$\frac{(\lambda_P + \lambda_1)^2(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{(\lambda_P + \lambda_0 + \lambda_1)C\lambda_P} > h, \tag{D.12.4}$$

else,

$$\frac{(\lambda_P + \lambda_1)^2(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{(\lambda_P + \lambda_0 + \lambda_1)C\lambda_P} < h. \tag{D.12.5}$$

D.13 Case 3.3

From the equation (B.3.11), the expression for μ_2 is obtained and simplifying the inequality (B.3.10) gives a lower bound for K.

$$C^2h/\lambda_P + Cr_0 \leq K \tag{D.13.1}$$

Similarly, an upper bound for K is derived from (B.3.9) by substituting the solution (B.3.10) and the expression for μ_2 from (B.3.11).

$$\frac{C(r_P\lambda_P - r_0\lambda_0) - C^2h(\lambda_P + \lambda_0)/\lambda_P}{\lambda_P - \lambda_0} \geq K \tag{D.13.2}$$

As K is positive, its upper bound should be positive. Thus, bounds for h are obtained. If $\lambda_P - \lambda_0 < 0$

$$\frac{(r_P\lambda_P - r_0\lambda_0)\lambda_P}{C(\lambda_P + \lambda_0)} < h \tag{D.13.3}$$

else

$$\frac{(r_P\lambda_P - r_0\lambda_0)\lambda_P}{C(\lambda_P + \lambda_0)} > h \tag{D.13.4}$$

D.14 Case 3.4

As the solution of Case 3.4 depends on $\lambda_P + \lambda_1 - \lambda_0$'s value, only bounds regarding K and h are generated when $\lambda_P + \lambda_1 - \lambda_0 \neq 0$.

We know that the expression being inside the square root in (B.3.17) should be positive. From this conditioning, we reach

$$\frac{C(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{\lambda_P + \lambda_1 - \lambda_0} - \frac{C^2h\lambda_P}{(\lambda_P + \lambda_1 - \lambda_0)^2} < K \tag{D.14.1}$$

On the other hand, by substituting the solution (B.3.17) into the promised delivery time constraint, the following upper bound is obtained.

$$\frac{C(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{\lambda_P + \lambda_1 - \lambda_0} + \frac{h\lambda_P(\tau^2\lambda_0^2 - C^2)}{(\lambda_P + \lambda_1 - \lambda_0)^2} > K \tag{D.14.2}$$

Let $\lambda_P + \lambda_1 - \lambda_0 > 0$,
from $T_J > 0$

$$\frac{C(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{\lambda_P + \lambda_1 - \lambda_0} < K \tag{D.14.3}$$

from $T_0 > 0$

$$\frac{C(r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0)}{\lambda_P + \lambda_1 - \lambda_0} - \frac{C^2h\lambda_P(\lambda_P + \lambda_1 + \lambda_0)}{(\lambda_P + \lambda_1 - \lambda_0)(\lambda_P + \lambda_1)^2} < K \tag{D.14.4}$$

If we have $\lambda_P + \lambda_1 - \lambda_0 < 0$ condition, above two inequalities will appear upper bounds for K.

D.15 Case 3.5

By making the the expression inside the square root in the solution (B.3.21), a lower bound for K is obtained.

$$\frac{C(r_P\lambda_P - r_0\lambda_0)}{\lambda_P - \lambda_0} - \frac{C^2h\lambda_P}{(\lambda_P - \lambda_0)^2} < K \tag{D.15.1}$$

Let $\lambda_P - \lambda_0 > 0$,
from $T_0 > 0$

$$\frac{C(r_P\lambda_P - r_0\lambda_0)}{\lambda_P - \lambda_0} - \frac{C^2h(\lambda_P + \lambda_0)}{(\lambda_P - \lambda_0)\lambda_P} < K \tag{D.15.2}$$

and from $T_P > 0$

$$\frac{C(r_P\lambda_P - r_0\lambda_0)}{\lambda_P - \lambda_0} > K \tag{D.15.3}$$

Now let $\lambda_P - \lambda_0$ be negative, then
from $T_0 > 0$

$$\frac{C(r_P\lambda_P - r_0\lambda_0)}{\lambda_P - \lambda_0} - \frac{C^2h(\lambda_P + \lambda_0)}{(\lambda_P - \lambda_0)\lambda_P} > K \tag{D.15.4}$$

and from $T_P > 0$

$$\frac{C(r_P\lambda_P - r_0\lambda_0)}{\lambda_P - \lambda_0} < K \tag{D.15.5}$$

D.16 Case 3.6

We know that for Case 3.6, $T_P, T_J > 0$. Therefore, for $T_P > 0$ a lower bound of K is constructed.

$$\frac{C^2h\lambda_P}{(\lambda_P + \lambda_1)^2} + Cr_0 < K \tag{D.16.1}$$

If we condition $T_J > 0$, an upper bound for K is reached.

$$\frac{C^2h}{\lambda_P} + Cr_0 > K \tag{D.16.2}$$

An another upper bound of K is constructed by substituting the solution (B.3.25) into the promised delivery time constraint.

$$\frac{(\tau\lambda_1 - C)^2h}{\lambda_P} + Cr_0 > K \tag{D.16.3}$$

D.17 Case 4.2

We know the solution of Case 4.2, (B.4.8). The promised delivery time constraint generates an upper bound for K.

$$\tau^2h\lambda_P \geq K \tag{D.17.1}$$

The capacity constraint also provides an upper bound for K.

$$\frac{C^2 h \lambda_P}{(\lambda_P + \lambda_1)^2} \geq K \tag{D.17.2}$$

Appendix E

Solution

E.1 Division of 9 Subsets

Notice that some cases are not valid to apply their K or h bounds when $r_P\lambda_P+r_0\lambda_1-r_0\lambda_0=0$. In the solution method, MATLAB code is included these special conditions for them. Note that in this Appendix we use the results of Appendix D, and hence refer the equations at that appendix as (D.x.y).

- Case 1.4: It is not feasible only when $r_P\lambda_P+r_0\lambda_1-r_0\lambda_0<0$ and $\lambda_P+\lambda_1-\lambda_0>0$ as (D.1.1) becomes negative. Hence, CX and CY are not applicable for this case.

- Case 1.5: From (D.2.2) we know that if $\lambda_P-\lambda_0<0$, then we should have $r_P\lambda_P-r_0\lambda_0<0$. However, when $r_P\lambda_P-r_0\lambda_0>0$, we cannot say anything about the regions A,B and C. Hence, only BX and CX are not applicable for it.

- Case 1.6: Since, none of the bounds created for Case 1.6 has relational expressions in it, it is assigned to all subsets.

- Case1.7: From (D.4.1) we know that $r_P\lambda_P+r_0\lambda_1-r_0\lambda_0$ should be greater than 0. However, it is not possible to define a condition when $\lambda_P+\lambda_1-\lambda_0<0$, it can be assigned A, B or C. Hence it is not feasible only when CX and CY.

- Case 2.1: Because of (D.5.1), this case only feasible in B and C.

- Case 2.2: As (D.6.3) should be positive, this case is not feasible when $r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0 < 0$.

- Case 2.4: As the decision variable T_J is positive, (B.2.16), this case feasible when $r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0 > 0$.

- Case 2.5: As we cannot find any relation between $r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0$ and $\lambda_P + \lambda_1 - \lambda_0$ from obtained bounds, this case is assigned all subsets.

- Case 2.6: Same situation with Case 2.5 is valid for the Case 2.6. Thus, it is in all 9 subsets.

- Case 2.7: Since $r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0$ should be positive. This case is not feasible only in C.

- Case 3.1: We know that K is positive. Hence, its upper bounds should be positive (D.11.1) and (D.11.2). Their valid relations are only observed in AX, BY and CZ.

- Case 3.2: From (D.12.4), we know that $r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0 > 0$ when $\lambda_P + \lambda_1 - \lambda_0 > 0$. However, we cannot say anything about the situation when $\lambda_P + \lambda_1 - \lambda_0 < 0$. Hence, it is not valid only in CX and CY.

- Case 3.3: From (D.13.3), we know that when $\lambda_P - \lambda_0 > 0$, $r_P\lambda_P - r_0\lambda_0$ should be positive. Thus, if X, then should be the region A. However, $\lambda_P - \lambda_0 < 0$, we cannot say anything just by looking the bounds of K or h.

- Case 3.4: In this case, because of the condition related to $\lambda_P + \lambda_1 - \lambda_0 < 0$, $r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0$ can only take negative values. However, again for the subsets containing $\lambda_P + \lambda_1 - \lambda_0 > 0$, cost bounds do not limit feasible subsets.

- Case 3.5: Since (D.15.3) only feasible when $r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0 > 0$, so in X it is feasible only in A. On the other hand, for the subsets Y and Z, it is not possible to find infeasible subset only by looking the cost bounds.

- Case 3.6: As this case has not implicit relation in its cost bounds to assign it subsets it is assigned all of them.

- Case 4.2: From the stationary inequality (B.4.5) and the positive decision variable T_J we know that $r_P\lambda_P + r_0\lambda_1 - r_0\lambda_0 > 0$ should hold. However, for the other subsets it is not possible to direct the case according to the K and h bounds.

E.2 Ordered Cost Parameter Relations

Let $\underline{K}_i$ be the minimum K value that makes the case i feasible and respectively $\overline{K}_i$ is the maximum value of K which makes case i feasible. We define K_i as the interval defined by $[\underline{K}_i, \overline{K}_i]$ for case i. An order for cases i,j $K_i \leq K_j$ means that $\overline{K}_i \leq \underline{K}_j$. Same notation is valid for the cost parameter h.

E.2.1 AX

- $(D.1.2) \leq (D.1.1) \leq (D.7.4) \implies h_{1.4} \leq h_{2.4}$

- $0 < min\{(D.17.1), (D.17.2)\} \leq (D.6.1) \leq (D.6.2) \leq (D.9.1) \leq (D.9.2) \implies K_{4.2} \leq K_{2.2} \leq K_{2.6}$

- $0 < min\{(D.17.1), (D.17.2)\} \leq (D.12.1) \leq min\{(D.12.2), (D.12.3)\} \leq (D.16.1) \leq min\{(D.16.2), (D.16.3)\} \leq (D.13.1) \leq (D.13.2) \leq max\{(D.15.1), (D.15.2)\} \leq (D.15.3) \implies K_{4.2} \leq K_{3.2} \leq K_{3.6} \leq K_{3.3} \leq K_{3.5}$

- $(D.12.1) \leq min\{(D.12.2), (D.12.3)\} \leq max\{(D.14.1), (D.14.3), (D.14.4)\} \leq (D.14.2) \implies K_{3.2} \leq K_{3.4}$

- $(D.11.1) \leq (D.14.3) \implies K_{3.1} \leq K_{3.4}$

E.2.2 AY

- $(D.1.2) \leq (D.1.1) \leq (D.7.4) \implies h_{1.4} \leq h_{2.4}$

- $0 < min\{(D.17.1),(D.17.2)\} \leq (D.6.1) \leq (D.6.2) \leq (D.9.1) \leq (D.9.2) \implies K_{4.2} \leq K_{2.2} \leq K_{2.6}$

- $0 < min\{(D.17.1),(D.17.2)\} \leq (D.12.1) \leq min\{(D.12.2),(D.12.3)\} \leq (D.16.1) \leq min\{(D.16.2),(D.16.3)\} \leq (D.13.1) \leq (D.13.2) \leq max\{(D.15.1),(D.15.2)\} \leq (D.15.3) \implies K_{4.2} \leq K_{3.2} \leq K_{3.6} \leq K_{3.3} \leq K_{3.5}$

- $(D.12.1) \leq min\{(D.12.2),(D.12.3)\} \leq max\{(D.14.1),(D.14.3),(D.14.4)\} \leq (D.14.2) \implies K_{3.2} \leq K_{3.4}$

E.2.3 AZ

- $0 < min\{(D.17.1),(D.17.2)\} \leq (D.6.1) \leq (D.6.2) \leq (D.9.1) \leq (D.9.2) \implies K_{4.2} \leq K_{2.2} \leq K_{2.6}$

- $0 < min\{(D.17.1),(D.17.2)\} \leq (D.12.1) \leq min\{(D.12.2),(D.12.3)\} \leq (D.16.1) \leq min\{(D.16.2),(D.16.3)\} \leq (D.13.1) \leq (D.13.2) \leq max\{(D.15.1),(D.15.2)\} \leq (D.15.3) \implies K_{4.2} \leq K_{3.2} \leq K_{3.6} \leq K_{3.3} \leq K_{3.5}$

E.2.4 BX

- $(D.1.2) \leq (D.1.1) \leq (D.7.4) \implies h_{1.4} \leq h_{2.4}$

- $0 < min\{(D.17.1), (D.17.2)\} \leq (D.6.1) \leq (D.6.2) \leq (D.9.1) \leq (D.9.2) \implies K_{4.2} \leq K_{2.2} \leq K_{2.6}$

- $0 < min\{(D.17.1), (D.17.2)\} \leq (D.12.1) \leq min\{(D.12.2), (D.12.3)\} \leq (D.16.1) \leq min\{(D.16.2), (D.16.3)\} \implies K_{4.2} \leq K_{3.2} \leq K_{3.6}$

- $(D.12.1) \leq min\{(D.12.2), (D.12.3)\} \leq max\{(D.14.1), (D.14.3), (D.14.4)\} \leq (D.14.2) \implies K_{3.2} \leq K_{3.4}$

E.2.5 BY

- $(D.1.2) \leq (D.1.1) \leq (D.7.4) \implies h_{1.4} \leq h_{2.4}$

- $0 < min\{(D.17.1), (D.17.2)\} \leq (D.6.1) \leq (D.6.2) \leq (D.9.1) \leq (D.9.2) \implies K_{4.2} \leq K_{2.2} \leq K_{2.6}$

- $0 < min\{(D.17.1), (D.17.2)\} \leq (D.12.1) \leq min\{(D.12.2), (D.12.3)\} \leq (D.16.1) \leq min\{(D.16.2), (D.16.3)\} \leq (D.13.1) \leq (D.13.2) \leq max\{(D.15.1), (D.15.2)\} \leq (D.15.3) \implies K_{4.2} \leq K_{3.2} \leq K_{3.6} \leq K_{3.3} \leq K_{3.5}$

- $(D.11.1) \leq (D.14.3) \implies K_{3.1} \leq K_{3.4}$

- $(D.12.1) \leq min\{(D.12.2),(D.12.3)\} \leq max\{(D.14.1),(D.14.3),(D.14.4)\} \leq (D.14.2) \implies K_{3.2} \leq K_{3.4}$

- $(D.5.1) \leq max\{(D.8.1),(D.8.2)\} \leq (D.8.3) \implies K_{2.1} \leq K_{2.5}$

E.2.6 BZ

- $0 < min\{(D.17.1),(D.17.2)\} \leq (D.6.1) \leq (D.6.2) \leq (D.9.1) \leq (D.9.2) \implies K_{4.2} \leq K_{2.2} \leq K_{2.6}$

- $0 < min\{(D.17.1),(D.17.2)\} \leq (D.12.1) \leq min\{(D.12.2),(D.12.3)\} \leq (D.16.1) \leq min\{(D.16.2),(D.16.3)\} \leq (D.13.1) \leq (D.13.2) \leq max\{(D.15.1),(D.15.2)\} \leq (D.15.3) \implies K_{4.2} \leq K_{3.2} \leq K_{3.6} \leq K_{3.3} \leq K_{3.5}$

- $(D.5.1) \leq max\{(D.8.1),(D.8.2)\} \leq (D.8.3) \implies K_{2.1} \leq K_{2.5}$

E.2.7 CX

- $(D.5.1) \leq max\{(D.8.1),(D.8.2)\} \leq (D.8.3) \implies K_{2.1} \leq K_{2.5}$

E.2.8 CY

- $(D.5.1) \leq max\{(D.8.1),(D.8.2)\} \leq (D.8.3) \implies K_{2.1} \leq K_{2.5}$

- $(D.16.1) \leq min\{(D.16.2), (D.16.3)\} \leq (D.13.1) \leq (D.13.2) \leq max\{(D.15.1), (D.15.2)\} \leq (D.15.3) \implies K_{3.6} \leq K_{3.3} \leq K_{3.5}$

E.2.9 CZ

- $(D.5.1) \leq max\{(D.8.1), (D.8.2)\} \leq (D.8.3) \implies K_{2.1} \leq K_{2.5}$

- $(D.12.1) \leq min\{(D.12.2), (D.12.3)\} \leq (D.16.1) \leq min\{(D.16.2), (D.16.3)\} \leq (D.13.1) \leq (D.13.2) \leq max\{(D.15.1), (D.15.2)\} \leq (D.15.3) \implies K_{3.2} \leq K_{3.6} \leq K_{3.3} \leq K_{3.5}$

Appendix F

MATLAB Code for The Algorithm

```
clear
clc

%Parameters
ri = 6;
ro = 5;
li = 12;
l1 = 10;
lo = 20;
K = 15;
h = 1;
C = 150;
T = 8;

OBJcode=[]; %Array which includes names of cases
i=0; %Index initialization for OBJcode

%Bounds for parameters in each cases
%Case1.4
hUpperBound14XY=(((ri*li+ro*l1-ro*lo)*(li+l1-lo))/(2*li*(C-T*lo)));
hLowerBound14XY=(((ri*li-ro*li)*(li+l1-lo))/(2*li*(C-T*lo)));
hUpperBound14Z=(((ri*li-ro*li)*(li+l1-lo))/(2*li*(C-T*lo)));
hLowerBound14Z=(((ri*li+ro*l1-ro*lo)*(li+l1-lo))/(2*li*(C-T*lo)));

%Case1.5
KUpperBound15=(C*(ri*li-ro*lo)/(li-lo))- ...
                ((h*(li+lo)*(C-T*lo)^2)/(li*(li-lo)))-...
```

```matlab
%Case1.5
KUpperBound15=(C*(ri*li-ro*lo)/(li-lo))- ...
                ((h*(li+lo)*(C-T*lo)^2)/(li*(li-lo)))-...
                (2*h*lo*T*(C-T*lo)/(li-lo));
KLowerBound15=h*((C-T*lo)^2)/li+2*h*T*(C-T*lo)-T*(ri*li-ro*lo);

%Case1.6
KUpperBound16=(h*((C-T*l1)^2)/li)+C*ro-2*T*ro*l1;
KLowerBound16=(h*((C-T*l1)^2)/li)-T*ro*l1;

%Case2.1
KUpperBound21=T*(ro*lo-ri*li);

%Case2.2
KUpperBound22=(T^2)*h*li+(T*ro*l1);
KLowerBound22=(T^2)*h*li;
hUpperBound22=(ri*li-ro*l1-ro*lo)/(2*li);

%Case2.4
KUpperBound24=(((ri*li-ro*l1-ro*lo)^2)/(4*h*li))+T*ro*l1;
KLowerBound24=(((ri*li-ro*l1-ro*lo)^2)/(4*h*li));
hLowerBound24Array=[(((ri*li+ro*l1-ro*lo)*(li+l1-lo))/(2*li*(C-T*lo))),...
    (ri*li+ro*l1-ro*lo)/(2*li*T)];
hLowerBound24=max(hLowerBound24Array);



%Case2.5
KUpperBound25=T*(ro*lo-ri*li)+h*((C-T*(lo-li))^2)/li-h*li*T^2;
KLowerBound25Array=[T*(ro*lo-ri*li), ...
    T*(ro*lo-ri*li)+(((ri*li-ro*lo+2*h*li*T)^2)/(4*h*li))-h*li*T^2];
KLowerBound25=max(KLowerBound25Array);

%Case2.6
KUpperBound26=((((C-T*l1)^2)*h/li)+(T*ro*l1));
KLowerBound26=(T^2)*h*li+(T*ro*l1);

%Case3.1
KUpperBound31Array=[(C*(ri*li+ro*l1-ro*lo)/(li+l1-lo)),...
    (C*(ri*li-ro*lo)/(li-lo))];
KUpperBound31=min(KUpperBound31Array);
if (li+l1-lo)==0 || li==lo
    KUpperBound31=inf;
end

%Case3.2
KUpperBound32Array=[(C*ro+(((C^2)*h*li)/((li+l1)^2))),...
                (((C*(ri*li+ro*l1-ro*lo))/(li+l1-lo))...
                -(((li+l1+lo)*h*li*C^2)/(((li+l1)^2)*(li+l1-lo))))];
KUpperBound32=min(KUpperBound32Array);
KLowerBound32=(((C^2)*h*li)/((li+l1)^2));
if (li+l1-lo)==0
    KUpperBound32=inf;
end
```

```
%Case3.3
KUpperBound33=(C*(ri*li-ro*lo)-((li+lo)*h*C^2)/li)/(li-lo);
KLowerBound33=C*ro+((h*C^2)/li);
if li==lo
    KUpperBound33=inf;
end

%Case3.4
KUpperBound34XYArray=[((C*(ri*li+ro*l1-ro*lo))/(li+l1-lo)),...
    (((C*(ri*li+ro*l1-ro*lo))/(li+l1-lo))-...
    ((h*li*(C^2-T^2*lo^2))/((li+l1-lo)^2)))];
KUpperBound34XY=min(KUpperBound34XYArray);
KLowerBound34XYArray=[(((C*(ri*li+ro*l1-ro*lo))/(li+l1-lo))-...
    ((C^2*h*li)/((li+l1-lo)^2))),...
    ((C*(ri*li+ro*l1-ro*lo))/(li+l1-lo))-...
    ((C^2*h*li*(li+l1+lo))/(((li+l1)^2)*((li+l1-lo))))];
KLowerBound34XY=max(KLowerBound34XYArray);

KUpperBound34ZArray=[(((C*(ri*li+ro*l1-ro*lo))/(li+l1-lo))-...
    ((C^2*h*li)/((li+l1-lo)^2))), ...
    (((C*(ri*li+ro*l1-ro*lo))/(li+l1-lo))-...
    ((h*li*(C^2-T^2*lo^2))/((li+l1-lo)^2)))];
KUpperBound34Z=min(KUpperBound34ZArray);
KLowerBound34ZArray=[(((C*(ri*li+ro*l1-ro*lo))/(li+l1-lo))-...
    ((C^2*h*li)/((li+l1-lo)^2))),...
    ((C*(ri*li+ro*l1-ro*lo))/(li+l1-lo))];
KLowerBound34Z=max(KLowerBound34ZArray);
```

```
%Case3.5
KUpperBound35X=(C*(ri*li-ro*lo))/(li-lo);
KLowerBound35XArray=[((((C*(ri*li-ro*lo))/(li-lo))-((C^2)*h*li/((li-lo)^2))),...
    ((((C*(ri*li-ro*lo))/(li-lo))-((C^2)*h*(li+lo))/((li-lo)*li)))];
KLowerBound35X=max(KLowerBound35XArray);

KUpperBound35YZ=((((C*(ri*li-ro*lo))/(li-lo))-((C^2)*h*(li+lo))/((li-lo)*li)));
KLowerBound35YZArray=[(((C*(ri*li-ro*lo))/(li-lo))-((C^2)*h*li/((li-lo)^2))),...
    ((C*(ri*li-ro*lo))/(li-lo))];
KLowerBound35YZ=max(KLowerBound35YZArray);
if li==lo
    KUpperBound35X=inf;
    KUpperBound35YZ=inf;
    KLowerBound35X=0;
    KLowerBound35YZ=0;
end


%Case3.6
KUpperBound36Array=[(C*ro+((h*C^2)/li)), (C*ro+((C-T*l1)^2)*h/li)];
KUpperBound36=min(KUpperBound36Array);
KLowerBound36=C*ro+(((C^2)*h*li)/((li+l1)^2));

%Case4.2
KUpperBound42Array=[(T^2*h*li),(C^2*h*li)/((li+l1)^2)];
KUpperBound42=min(KUpperBound42Array);
```

```
%Special Cases
%Case1.1
if C==T*ro
    i=i+1;
    OBJcode(i,1:7)='Case1.1';

%Case1.2
elseif C==T*(lo+l1)
    i=i+1;
    OBJcode(i,1:7)='Case1.2';
%Case 1.3
elseif T==0
    i=i+1;
    OBJcode(i,1:7)='Case1.3';

%Case2.3
elseif T==0
    i=i+1;
    OBJcode(i,1:7)='Case2.3';

%Case4.3
elseif ro*l1==0
    i=i+1;
    OBJcode(i,1:7)='Case4.3';

%Case4.4
elseif K==((ri*li+ro*l1-ro*lo)^2)/(4*h*li)


%Case4.5
elseif K==((ri*li-ro*lo)^2)/(4*h*li)
    i=i+1;
    OBJcode(i,1:7)='Case4.5';

%Case4.6
elseif ro*l1==0
    i=i+1;
    OBJcode(i,1:7)='Case4.6';

%Case4.7
elseif ro*l1==0 && K==((ri*li-ro*lo)^2)/(4*h*li)
    i=i+1;
    OBJcode(i,1:7)='Case4.7';
end

%9 Classified cases

%CASE AX
if (ri*li-ro*lo)>=0 && (li-lo)>=0
    disp('CASE: AX')

%Feasilibity check of K bounds in order: 4.2 < 2.2 < 2.6
   if 0<K && K<=KUpperBound42
      i=i+1;
      OBJcode(i,1:7)='Case4.2';
```

```
%Feasilibity check of K bounds in order: 4.2 < 2.2 < 2.6
   if 0<K && K<=KUpperBound42
      i=i+1;
      OBJcode(i,1:7)='Case4.2';

   elseif KLowerBound22<=K && K<=KUpperBound22
      i=i+1;
      OBJcode(i,1:7)='Case2.2';

   elseif KLowerBound26<=K && K<=KUpperBound26
      i=i+1;
      OBJcode(i,1:7)='Case2.6';
   end

%%Feasilibity check of K bounds in order: 4.2 < 3.2 < 3.6 < 3.3 < 3.5
   if 0<K && K<=KUpperBound42
      i=i+1;
      OBJcode(i,1:7)='Case4.2';

   elseif KLowerBound32<=K && K<=KUpperBound32
      i=i+1;
      OBJcode(i,1:7)='Case3.2';

   elseif KLowerBound36<=K && K<=KUpperBound36
      i=i+1;
      OBJcode(i,1:7)='Case3.6';
```

```
    elseif KLowerBound36<=K && K<=KUpperBound36
       i=i+1;
       OBJcode(i,1:7)='Case3.6';

    elseif KLowerBound33<=K && K<=KUpperBound33
       i=i+1;
       OBJcode(i,1:7)='Case3.3';

    elseif KLowerBound35X<=K && K<=KUpperBound35X
       i=i+1;
       OBJcode(i,1:7)='Case3.5';
    end

 %Feasibility check K 3.2 < 3.4
     if KLowerBound32<=K && K<=KUpperBound32
       i=i+1;
       OBJcode(i,1:7)='Case3.2';

     elseif KLowerBound34XY<=K && K<=KUpperBound34XY
       i=i+1;
       OBJcode(i,1:7)='Case3.4';
     end

 %Feasibility check h 1.4 < 2.4
     if hLowerBound14XY<=h && h<= hUpperBound14XY && T*lo<C && C<T*(li+l1)
         i=i+1;
```

```
%Feasibility check h 1.4 < 2.4
    if hLowerBound14XY<=h && h<= hUpperBound14XY && T*lo<C && C<T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.4';

    elseif hLowerBound24<=h
        i=i+1;
        OBJcode(i,1:7)='Case2.4';
    end

 %Feasilibity check K of 1.5
    if KLowerBound15<=K && K<=KUpperBound15 && C>T*lo
        i=i+1;
        OBJcode(i,1:7)='Case1.5';
    end

 %Feasilibity check K of 1.6
    if KLowerBound16<=K && K<=KUpperBound16
        i=i+1;
        OBJcode(i,1:7)='Case1.6';
    end

 %Feasilibity check K of 3.1
    if 0<K && K<=KUpperBound31
        i=i+1;
        OBJcode(i,1:7)='Case3.1';
    end



%Feasiblity of 1.7
    i=i+1;
    OBJcode(i,1:7)='Case1.7';

%Feasiblity of 2.7
    i=i+1;
    OBJcode(i,1:7)='Case2.7';

%CASE AY
    elseif (ri*li-ro*lo)>=0 && (li+l1-lo)>=0 && (li-lo)<=0
        disp('CASE: AY')

    %Feasilibity Check K 4.2 < 2.2 < 2.6
    if 0<K && K<=KUpperBound42
        i=i+1;
        OBJcode(i,1:7)='Case4.2';

    elseif KLowerBound22<=K && K<=KUpperBound22
        i=i+1;
        OBJcode(i,1:7)='Case2.2';

    elseif KLowerBound26<=K && K<=KUpperBound26
        i=i+1;
        OBJcode(i,1:7)='Case2.6';
    end

  %Feasilibity Check K 4.2 < 3.2 < 3.6 < 3.3
```

```
%Feasilibity Check K 4.2 < 3.2 < 3.6 < 3.3
  if 0<K && K<=KUpperBound42
      i=i+1;
      OBJcode(i,1:7)='Case4.2';

  elseif KLowerBound32<=K && K<=KUpperBound32
      i=i+1;
      OBJcode(i,1:7)='Case3.2';

  elseif KLowerBound36<=K && K<=KUpperBound36
      i=i+1;
      OBJcode(i,1:7)='Case3.6';

  elseif KLowerBound33<=K && K<=KUpperBound33
      i=i+1;
      OBJcode(i,1:7)='Case3.3';
  end

Feasibility check K 3.2 < 3.4
  if KLowerBound32<=K && K<=KUpperBound32
      i=i+1;
      OBJcode(i,1:7)='Case3.2';

  elseif KLowerBound34XY<=K && K<=KUpperBound34XY
      i=i+1;
      OBJcode(i,1:7)='Case3.4';
  end
```

```
%Feasibility check h 1.4 < 2.4
    if hLowerBound14XY<=h && h<=hUpperBound14XY && T*lo<C && C<T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.4';

    elseif hLowerBound24 <=h
        i=i+1;
        OBJcode(i,1:7)='Case2.4';
    end

%Feasilibity check K of 1.6
    if KLowerBound16<=K && K<=KLowerBound16 && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.6';
    end

%Feasiblity of 1.7
    i=i+1;
    OBJcode(i,1:7)='Case1.7';

%Feasiblity of 2.7
    i=i+1;
    OBJcode(i,1:7)='Case2.7';

%CASE AZ
    elseif (ri*li-ro*lo)>=0 && (lo-(l1+li))>=0
        disp('CASE: AZ')
```

```
%Feasilibity Check K 4.2 < 2.2 < 2.6
    if 0<K && K<=KUpperBound42
        i=i+1;
        OBJcode(i,1:7)='Case4.2';

    elseif KLowerBound22<=K && K<=KUpperBound22
        i=i+1;
        OBJcode(i,1:7)='Case2.2';

    elseif KLowerBound26<=K && K<=KUpperBound26
        i=i+1;
        OBJcode(i,1:7)='Case2.6';
    end

%Feasilibity Check K 4.2 < 3.6 < 3.3
    if 0<K && K<=KUpperBound42
        i=i+1;
        OBJcode(i,1:7)='Case4.2';

    elseif KLowerBound36<=K && K<=KUpperBound36
        i=i+1;
        OBJcode(i,1:7)='Case3.6';
    end


%Feasibility check 1.4
    if hLowerBound14Z<=h && h<=hUpperBound14Z && T*lo>C && C>T*(li+l1)



%Feasibility check 1.4
    if hLowerBound14Z<=h && h<=hUpperBound14Z && T*lo>C && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.4';
    end

%Feasibility check 2.4
    if KLowerBound24<=K && K<=KUpperBound24
        i=i+1;
        OBJcode(i,1:7)='Case2.4';
    end

%Feasilibity check K of 1.6
    if KLowerBound16<=K && K<=KUpperBound16 && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.6';
    end

%Feasiblity of 1.7
    i=i+1;
    OBJcode(i,1:7)='Case1.7';

%Feasiblity of 2.7
    i=i+1;
    OBJcode(i,1:7)='Case2.7';
```

```
%CASE BX
elseif ri*li+ro*l1-ro*lo>=0 && ri*li-ro*lo<=0 && li-lo>=0
    disp('CASE: BX')

%Feasilibity Check K 4.2 < 2.2 < 2.6
    if 0<K && K<=KUpperBound42
        i=i+1;
        OBJcode(i,1:7)='Case4.2';

    elseif KLowerBound22<=K && K<=KUpperBound22
        i=i+1;
        OBJcode(i,1:7)='Case2.2';

    elseif KLowerBound26<=K && K<=KUpperBound26
        i=i+1;
        OBJcode(i,1:7)='Case2.6';
    end

%Feasilibity Check K 4.2 < 3.2 < 3.6 < 3.3
    if 0<K && K<=KUpperBound42
        i=i+1;
        OBJcode(i,1:7)='Case4.2';

    elseif KLowerBound32<=K && K<=KUpperBound32
        i=i+1;
        OBJcode(i,1:7)='Case3.2';



    elseif KLowerBound32<=K && K<=KUpperBound32
        i=i+1;
        OBJcode(i,1:7)='Case3.2';

    elseif KLowerBound36<=K && K<=KUpperBound36
        i=i+1;
        OBJcode(i,1:7)='Case3.6';

    elseif KLowerBound33<=K && K<=KUpperBound33
        i=i+1;
        OBJcode(i,1:7)='Case3.3';
    end

%Feasibility check K 3.2 < 3.4
    if KLowerBound32<=K && K<=KUpperBound32
        i=i+1;
        OBJcode(i,1:7)='Case3.2';

    elseif KLowerBound34XY<=K && K<=KUpperBound34XY
        i=i+1;
        OBJcode(i,1:7)='Case3.4';
    end

%Feasibility check h 1.4 < 2.4
    if hLowerBound14XY<=h && h<=hUpperBound14XY && T*lo<C && C<T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.4';
```

```
        OBJcode(i,1:7)='Case1.4';

    elseif hLowerBound24<=h
        i=i+1;
        OBJcode(i,1:7)='Case2.4';
    end

%Feasilibity check K of 1.6
    if KLowerBound16<=K && K<=KUpperBound16 && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.6';
    end

%Feasiblity of 1.7
    i=i+1;
    OBJcode(i,1:7)='Case1.7';

%Feasibility check K of 2.5
    if KLowerBound25<=K && K<=KUpperBound25
        i=i+1;
        OBJcode(i,1:7)='Case2.5';
    end

%Feasibility of 2.7
    i=i+1;
    OBJcode(i,1:7)='Case2.7';



%CASE BY
elseif ri*li+ro*l1-ro*lo>=0 && ri*li-ro*lo<=0 && li+l1-lo>=0 && li-lo<=0
    disp('CASE: BY')

%Feasilibity Check K 4.2 < 2.2 < 2.6
    if 0<K && K<=KUpperBound42
        i=i+1;
        OBJcode(i,1:7)='Case4.2';

    elseif KLowerBound22<=K && K<=KUpperBound22
        i=i+1;
        OBJcode(i,1:7)='Case2.2';

    elseif KLowerBound26<=K && K<=KUpperBound26
        i=i+1;
        OBJcode(i,1:7)='Case2.6';
    end

%Feasilibity Check K 4.2 < 3.2 < 3.6 < 3.3 < 3.5
    if 0<K && K<=KUpperBound42
        i=i+1;
        OBJcode(i,1:7)='Case4.2';

    elseif KLowerBound32<=K && K<=KUpperBound32
        i=i+1;
        OBJcode(i,1:7)='Case3.2';
```

```
    elseif KLowerBound36<=K && K<=KUpperBound36
        i=i+1;
        OBJcode(i,1:7)='Case3.6';

    elseif KLowerBound33<=K && K<=KUpperBound33
        i=i+1;
        OBJcode(i,1:7)='Case3.3';
    end

        %Feasibility check K 3.2 < 3.4
    if KLowerBound32<=K && K<=KUpperBound32
        i=i+1;
        OBJcode(i,1:7)='Case3.2';

    elseif KLowerBound34XY<=K && K<=KUpperBound34XY
        i=i+1;
        OBJcode(i,1:7)='Case3.4';
    end

%Feasibility check h 1.4 < 2.4
    if hLowerBound14XY<=h && h<=hUpperBound14XY && T*lo<C && C<T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.4';

    elseif hLowerBound24<=h
        i=i+1;
```

```
    elseif hLowerBound24<=h
        i=i+1;
        OBJcode(i,1:7)='Case2.4';
    end

%Feasilibity check K of 1.5
    if KLowerBound15<=K && K<=KUpperBound15 && C>T*lo
        i=i+1;
        OBJcode(i,1:7)='Case1.5';
    end

%Feasilibity check K of 1.6
    if KLowerBound16<=K && K<=KUpperBound16 && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.6';
    end

%Feasiblity of 1.7
    i=i+1;
    OBJcode(i,1:7)='Case1.7';

%Feasibility check K of 2.5
    if KLowerBound25<=K && K<=KUpperBound25
        i=i+1;
        OBJcode(i,1:7)='Case2.5';
    end
```

```
%Feasiblity of 2.7
    i=i+1;
    OBJcode(i,1:7)='Case2.7';

%Feasilibity check K of 3.1
    if 0<K && K<=KUpperBound31
        i=i+1;
        OBJcode(i,1:7)='Case3.1';
    end

    %Feasilibity check K of 3.5
    if KLowerBound35YZ<=K && K<=KUpperBound35YZ
        i=i+1;
        OBJcode(i,1:7)='Case3.5';
    end

%CASE BZ
elseif ri*li+ro*l1-ro*lo>=0 && ri*li-ro*lo<=0 && lo-(l1+li)>=0
        disp('CASE: BZ')

%Feasilibity Check K 4.2 < 2.2 < 2.6
    if 0<K && K<=KUpperBound42
        i=i+1;
        OBJcode(i,1:7)='Case4.2';

    elseif KLowerBound22<=K && K<=KUpperBound22
        i=i+1;



        i=i+1;
        OBJcode(i,1:7)='Case2.2';

    elseif KLowerBound26<=K && K<=KUpperBound26
        i=i+1;
        OBJcode(i,1:7)='Case2.6';
    end

%Feasilibity Check K 4.2 < 3.6 < 3.3
    if 0<K && K<=KUpperBound42
        i=i+1;
        OBJcode(i,1:7)='Case4.2';

    elseif KLowerBound36<=K && K<=KUpperBound36
        i=i+1;
        OBJcode(i,1:7)='Case3.6';

    elseif KLowerBound33<=K && K<=KUpperBound33
        i=i+1;
        OBJcode(i,1:7)='Case3.3';
    end

%Feasibility check h 1.4
    if hLowerBound14Z<=h && h<=hUpperBound14Z && T*lo>C && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.4';
    end
```

```
%Feasilibity check K of 1.5
    if KLowerBound15<=K && K<=KUpperBound15 && C>T*lo
        i=i+1;
        OBJcode(i,1:7)='Case1.5';
    end

%Feasilibity check K of 1.6
    if LowerBound16<=K && K<=KUpperBound16 && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.6';
    end

%Feasiblity of 1.7
    i=i+1;
    OBJcode(i,1:7)='Case1.7';

%Feasibility check h 2.4
    if KLowerBound24<=K && K<=KUpperBound24
        i=i+1;
        OBJcode(i,1:7)='Case2.4';
    end

%Feasibility check K of 2.5
    if KLowerBound25<=K && K<=KUpperBound25
        i=i+1;
        OBJcode(i,1:7)='Case2.5';
    end
```

```
%Feasiblity of 2.7
    i=i+1;
    OBJcode(i,1:7)='Case2.7';

%CASE CX
    elseif ro*lo-(ro*l1+ri*li)>=0 && li-lo>=0
        disp('CASE: CX')

%Feasibility check K of 2.1 < 2.5
    if 0<K && K<=KUpperBound21
        i=i+1;
        OBJcode(i,1:7)='Case2.1';

    elseif KLowerBound25<=K && K<=KUpperBound25
        i=i+1;
        OBJcode(i,1:7)='Case2.5';
    end

%Feasilibity check K of 1.6
    if LowerBound16<=K && K<=KUpperBound16 && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.6';
    end

%Feasibility check K of 2.6
    if KLowerBound26<=K && K<=KUpperBound26
        i=i+1;
```

```
%Feasibility check K of 2.6
    if KLowerBound26<=K && K<=KUpperBound26
        i=i+1;
        OBJcode(i,1:7)='Case2.6';
    end

    %Feasilibity check K of 3.6
    if C*ro+((C^2*h*li)/((li+l1)^2))<=K && K<=C*ro+min((C^2)*h/li,h*((T*l1-C)^2)/li)
        i=i+1;
        OBJcode(i,1:7)='Case3.6';
    end

    %CASE: CY
    elseif ro*lo-(ro*l1+ri*li)>=0 && li+l1-lo>=0 && li-lo<=0
        disp('CASE: CY')

%Feasibility check K of 2.1 < 2.5
    if 0<K && K<=KUpperBound21
        i=i+1;
        OBJcode(i,1:7)='Case2.1';

    elseif KLowerBound25<=K && K<=KUpperBound25
        i=i+1;
        OBJcode(i,1:7)='Case2.5';
    end
```

```
%Feasibility check K of 3.6 < 3.3
    if KLowerBound36<=K && K<=KUpperBound36
        i=i+1;
        OBJcode(i,1:7)='Case3.6';

    elseif KLowerBound33<=K && K<=KUpperBound33
        i=i+1;
        OBJcode(i,1:7)='Case3.3';
    end

%Feasilibity check K of 1.5
    if KLowerBound15<=K && K<=KUpperBound15 && C>T*lo
        i=i+1;
        OBJcode(i,1:7)='Case1.5';
    end

%Feasilibity check K of 1.6
    if KLowerBound16<=K && K<=KUpperBound16 && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.6';
    end

%Feasibility check K of 2.6
    if KLowerBound26<=K && K<=KUpperBound26
        i=i+1;
        OBJcode(i,1:7)='Case2.6';
    end
```

```
%CASE CZ
elseif ro*lo-(ro*l1+ri*li)>=0 && lo-(l1+li)>0
    disp('CASE: CZ')

%Feasibility check K of 2.1 < 2.5

    if 0<K && K<=KUpperBound21
        i=i+1;
        OBJcode(i,1:7)='Case2.1';

    elseif KLowerBound25<=K && K<=KUpperBound25
        i=i+1;
        OBJcode(i,1:7)='Case2.5';
    end

 %Feasibility Check K of 3.2 < 3.6 < 3.3
    if KLowerBound32<=K && K<=KUpperBound32
        i=i+1;
        OBJcode(i,1:7)='Case3.2';

    elseif KLowerBound36<=K && K<=KUpperBound36
        i=i+1;
        OBJcode(i,1:7)='Case3.6';

    elseif KLowerBound33<=K && K<=KUpperBound33
        i=i+1;
        OBJcode(i,1:7)='Case3.3';



    if 0<K && K<=KUpperBound31
        i=i+1;
        OBJcode(i,1:7)='Case3.1';

    elseif KLowerBound34Z<=K && K<=KUpperBound34Z
        i=i+1;
        OBJcode(i,1:7)='Case3.4';
    end

%Feasibility check h of 1.4
    if  hLowerBound14Z<=h && h<=hUpperBound14Z && T*lo>C && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.4';
    end

%Feasilibity check K of 1.5
    if KLowerBound15<=K && K<=KUpperBound15 && C>T*lo
        i=i+1;
        OBJcode(i,1:7)='Case1.5';
    end

%Feasilibity check K of 1.6
    if KLowerBound16<=K && K<=KUpperBound16 && C>T*(li+l1)
        i=i+1;
        OBJcode(i,1:7)='Case1.6';
    end
```

```
%Feasibility check K of 2.5
    if KLowerBound25<=K && K<=KUpperBound25
        i=i+1;
        OBJcode(i,1:7)='Case2.5';
    end

%Feasibility check K of 2.6
    if KLowerBound26<=K && K<=KUpperBound26
        i=i+1;
        OBJcode(i,1:7)='Case2.6';
    end
end


char(OBJcode);

if i==1
    char(OBJcode)
    caseResults(char(OBJcode(1,1:7)),ri,ro,li,l1,lo,K,h,C,T)
elseif i>=2
    if strcmp(char(OBJcode(1,1:7)),char(OBJcode(2,1:7)))==1
        char(OBJcode(2:i,1:7))
        for k=2:i
            char(OBJcode(k,1:7))
            caseResults(char(OBJcode(k,1:7)),ri,ro,li,l1,lo,K,h,C,T)
        end



    elseif strcmp(char(OBJcode(1,1:7)),char(OBJcode(2,1:7)))==0
        for k=1:i
            char(OBJcode(k,1:7))
            caseResults(char(OBJcode(k,1:7)),ri,ro,li,l1,lo,K,h,C,T)
        end
    end
end
```

Appendix G

Verification of Little's Law for the Stochastic Demand Problem - Regular Service Only

$$\mathrm{E}\left[\mathrm{IS}\right] = \frac{1}{\lambda_0} + \int_{n=0}^{C-2} \frac{e^{-\lambda_0 \tau}\ (\lambda_0 \tau)^n}{n!} * \tau + \int_0^{\tau} t\ \frac{e^{-\lambda_0 t} {\lambda_0}^{C-1} t^{C-2}}{\Gamma(C-1)}\ dt \tag{1}$$

E[# units per shipment] =

$$1 + \int_{n=0}^{C-2} \frac{e^{-\lambda_0 \tau}\ (\lambda_0 \tau)^n}{n!} * n + (C-1) * \int_0^{\tau} \frac{e^{-\lambda_0 t} {\lambda_0}^{C-1} t^{C-2}}{\Gamma(C-1)}\ dt \tag{2}$$

$$(2) = \lambda_0 * (1) \tag{3}$$

$$\begin{aligned}
\lambda_0 * (1) &= \left(\frac{1}{\lambda_0} + \int_{n=0}^{C-2} \frac{e^{-\lambda_0 \tau}\ (\lambda_0 \tau)^n}{n!} * \tau + \int_0^{\tau} t\ \frac{e^{-\lambda_0 t} {\lambda_0}^{C-1} t^{C-2}}{\Gamma(C-1)}\ dt \right) * \lambda_0 \\
&= 1 + \int_{n=0}^{C-2} \frac{e^{-\lambda_0 \tau}\ (\lambda_0 \tau)^{n+1}}{n!} + \int_0^{\tau} \frac{e^{-\lambda_0 t} {\lambda_0}^{C} t^{C-1}}{\Gamma(C-1)}\ dt \\
&= 1 + \left((\lambda_0 \tau) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0 \tau}\ (\lambda_0 \tau)^n}{n!} \right) + \frac{\Gamma(C) - \Gamma(C,\ \lambda_0 \tau)}{\Gamma(C-1)} \\
&= 1 + \left((\lambda_0 \tau) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0 \tau}\ (\lambda_0 \tau)^n}{n!} \right) + \frac{(C-1)! - \Gamma(C,\ \lambda_0 \tau)}{(C-2)!}
\end{aligned}$$

$$= 1 + \left((\lambda_0\tau) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) + (C-1) - \frac{\Gamma(C,\ \lambda_0\tau)}{(C-2)!}$$

$$= 1 + \left((\lambda_0\tau) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) + (C-1) - \frac{(C-1)! * \int_{n=0}^{C-1} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!}}{(C-2)!}$$

$$= 1 + \left((\lambda_0\tau) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) + (C-1) - \left((C-1) * \int_{n=0}^{C-1} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right)$$

$$= C + \left((\lambda_0\tau) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) - \left((C-1) * \int_{n=0}^{C-1} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right)$$

$$= C + \left((\lambda_0\tau) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) - \left((C-1) * \int_{n=0}^{C-1} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) \tag{4}$$

$$(2) = 1 + \left(e^{-\lambda_0\tau} \left(\frac{(\lambda_0\tau)^0}{0!} 0 + \frac{(\lambda_0\tau)^1}{1!} 1 + .. + \frac{(\lambda_0\tau)^{C-3}}{(C-3)!} (C-3) + \frac{(\lambda_0\tau)^{C-2}}{(C-2)!} (C-2) + \right) \right) + (C-1) \left(\frac{\Gamma(C-1) - \Gamma(C-1,\ \lambda_0\tau)}{\Gamma(C-1)} \right)$$

$$= 1 + \left(e^{-\lambda_0\tau} * \left(0 + \frac{(\lambda_0\tau)^1}{0!} + .. + \frac{(\lambda_0\tau)^{C-3}}{(C-4)!} + \frac{(\lambda_0\tau)^{C-2}}{(C-3)!} \right) \right) + \left((C-1) * \left(\frac{(C-2)! - (C-2)! \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!}}{(C-2)!} \right) \right)$$

$$= 1 + \left((\lambda_0\tau) * \int_{n=0}^{C-3} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) + \left((C-1) * \left(1 - \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) \right)$$

$$= 1 + \left((\lambda_0\tau) * \int_{n=0}^{C-3} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) + (C-1) - \left((C-1) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right)$$

$$= C + \left((\lambda_0\tau) * \int_{n=0}^{C-3} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) - (C-1) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \tag{5}$$

No consider (4)-(5)

$$= C + \left((\lambda_0\tau) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) - \left((C-1) * \int_{n=0}^{C-1} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) - \left(C + \left((\lambda_0\tau) * \int_{n=0}^{C-3} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right) - (C-1) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} \right)$$

$$= \left((\lambda_0\tau) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!}\right) - \left((C-1) * \int_{n=0}^{C-1} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!}\right) - \left(\left((\lambda_0\tau) * \int_{n=0}^{C-3} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!}\right) - (C-1) * \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!}\right)$$

$$\left((\lambda_0\tau) * \left(\int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} - \int_{n=0}^{C-3} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!}\right)\right) - \left((C-1) * \left(\int_{n=0}^{C-1} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} - \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!}\right)\right)$$

$$= \left(\frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^{C-1}}{(C-2)!}\right) - \left((C-1) * \left(\int_{n=0}^{C-1} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!} - \int_{n=0}^{C-2} \frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^n}{n!}\right)\right)$$

$$= \left(\frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^{C-1}}{(C-2)!}\right) - \left((C-1) * \left(\frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^{C-1}}{(C-1)!}\right)\right)$$

$$= \left(\frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^{C-1}}{(C-2)!}\right) - \left(\frac{e^{-\lambda_0\tau}\ (\lambda_0\tau)^{C-1}}{(C-2)!}\right) = 0$$

Appendix H

Derivation of The First Order Condition

Let ${C_P}^*$ be the integer value that maximizes the objective function above.

$$O(C_P) = r_0\lambda_0 + \frac{r_P C_P - h * \frac{C_P(C_P+1)}{2\lambda_P} - K}{\frac{C_0}{\lambda_0} + e^{-\lambda_0 \tau}\left(\sum_{n=0}^{C_0-3} \frac{\lambda_0^n \tau^{n+1}}{n!} + \frac{(1-C_0)}{\lambda_0}\sum_{n=0}^{C_0-2}\frac{\lambda_0^n \tau^n}{n!}\right)}$$

If $C_P = {C_P}^*$, then

- $O(C_P) - O(C_P+1) \geq 0$

$$O(C_P) - O(C_P+1) = r_0\lambda_0 + \frac{r_P C_P - h\frac{C_P(C_P+1)}{2\lambda_P} - K}{E[IS]} - \left(r_0\lambda_0 + \frac{r_P C_P - h\frac{(C_P+1)(C_P+2)}{2\lambda_P} - K}{E[IS]}\right)$$

$$= \frac{r_P C_P - h\frac{C_P(C_P+1)}{2\lambda_P}}{E[IS]} - \left(\frac{r_P C_P - h\frac{(C_P+1)(C_P+2)}{2\lambda_P}}{E[IS]}\right)$$

$$= \frac{-h/(2\lambda_P)\Big(C_P(C_P+1) - (C_P+1)(C_P+2)\Big) - r_P}{E[IS]}$$

$$= \frac{-h/(2\lambda_P)(C_P+1)(-2) - r_P}{E[IS]} = \frac{h/\lambda_P(C_P+1) - r_P}{E[IS]} \geq 0 \quad \textit{where } E[IS] > 0$$

$$C_P \geq \frac{r_P \lambda_P}{h} - 1$$

- $O(C_P) - O(C_P - 1) \geq 0$

$$O(C_P) - O(C_P - 1) = r_0\lambda_0 + \frac{r_P C_P - h\frac{C_P(C_P+1)}{2\lambda_P} - K}{E[IS]} - \left(r_0\lambda_0 + \frac{r_P(C_P - 1) - h\frac{C_P(C_P-1)}{2\lambda_P} - K}{E[IS]} \right)$$

$$= \frac{-h/(2\lambda_P)C_P\Big(C_P + 1 - (C_P - 1)\Big) + r_P}{E[IS]} = \frac{-h/\lambda_P C_P + r_P}{E[IS]} \geq 0 \quad \textit{where } E[IS] > 0$$

$$C_P \leq \frac{r_P \lambda_P}{h}$$

As a result ${C_P}^*$ be the integer value between $\left[\frac{r_P\lambda_P}{h} - 1, \frac{r_P\lambda_P}{h}\right]$.