ENSURING MULTIDIMENSIONAL FAIRNESS IN PUBLIC SERVICE

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Damla Akoluk
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ENSURING MULTIDIMENSIONAL FAIRNESS IN PUBLIC SERVICE

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In this study, we focus on service planning problems, in which decisions lead to distributions of multiple benefits to multiple users, hence involve fairness and efficiency concerns in a multidimensional way. We develop two mathematical modeling-based approaches that incorporate these concerns in such problems. The first formulation aggregates the multidimensional efficiency concerns and multidimensional fairness concerns in a bi-objective model. The second formulation defines an objective function for each benefit, which maximizes the total social welfare obtained from that specific benefit distribution, hence results in an n-objective model, where n is the number of benefits. We illustrate and compare these approaches on an example public service provision problem.

Keywords: Fairness, Public service provision, Public education, Knapsack problem, Equity, Epsilon constraint algorithm, Multi-criteria optimization.

ÖZET

KAMU SERVİSLERİNDE ÇOK BOYUTLU EŞİTLİKÇİLİK KAYGISI

Damla Akoluk
Endüstri Mühendisliği, Yüksek Lisans
Tez Danışmanı: Özlem Karsu
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Bu çalışmada, verilen kararların birden fazla kullanıcıya birden fazla fayda tipinde fayda sağladığı, ve her tip fayda içn hem verimlilik, hem adillik gözetildiği, yani çok boyutlu verimlilik ve eşitlikçilik kaygılarının olduğu hizmet servisi planlama problemleri ele alınmıştır. Verimlilikten kasıt, toplam faydanın en çoklanmasının istenmesi, eşitlikçilikten kasıt ise herhangi bir toplam faydanın farklı kullanıcılara eşitlikçi bir şekilde dağıtılmasının istenmesidir. Bu kaygıları bu tür problemlerde göz önüne alan, matematiksel modelleme tabanlı iki farklı yaklaşım geliştirilmiştir. İlk yaklaşım, çok boyutlu verimlilik kaygılarını ve çok boyutlu adalet kaygılarını iki amaçlı bir modelde bir araya getirir. İkinci yaklaşım, her fayda için, o faydanın dağıtımından elde edilen toplam sosyal refahı ençoklamayı amaçlar, yani n'nin fayda sayısı olduğu n-amaçlı bir model olarak tanımlanmıştır. Önerilen bu yaklaşımlar örnek bir kamu hizmet servisi planlama problemi üzerinde uygulanmış ve bu problemlerde verimlilik ve eşitlikçilik arasındaki ödünleşme incelenmiştir.

Anahtar sözcükler: Çok boyutlu eşitlikçi optimizasyon, çok amaçlı modelleme, eşitlikçi tercihler, eşitlikçi verimlilik, kamu hizmet servisleri, halk eğitim, çok amaçlı sırt çantası problemi, adillik, epsilon-kısıt yöntemi.

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Chapter 1

Introduction

In many real life applications such as public service facility location, task assignment, bandwidth allocation and scheduling, the decision makers have fairness concerns alongside efficiency concerns [1]. The need to incorporate these concerns has been acknowledged by the OR/MS community, leading to a recent increase in the number of studies on the topic. Efficiency is the concern for maximizing the total benefit whereas fairness is the concern for distributing the total benefit to users as equitable as possible. There is usually trade-off between these two concerns, making the corresponding decision making problem challenging. The solution that maximizes a system's efficiency may be a bad one in terms of fairness. Therefore, any decision support system should take such equity concerns into account when making recommendations. This usually leads to the problem being defined as a multiobjective optimization problem.

Multi-objective optimization problems (MOPs) have been studied in the Operations Research literature for a long time. These problems involve multiple competing aims, implying that there is a trade-off between them. There is no specific solution that concurrently optimizes all objectives, hence the optimality concept in single objective optimization is replaced with Pareto optimality in multiobjective optimization.

To give an example of the trade-off between efficiency and equity, let us consider a hospital location problem where users' travel distance to possible hospital locations is the output of concern. In this setting, the decision maker tries to locate the hospital such that the resulting total distance is as minimum as possible (efficiency) and there is not much imbalance in the distances traveled by the users (fairness). A solution that maximizes the system's efficiency (minimizing the users' total distance to hospital location) may require some users to travel longer distances compared to others. On the other hand, a solution that maximizes equity (trying to equalize each user's distance to the hospital location) may lead to a huge decrease in efficiency. To show this trade-off, suppose that the hospital will be located in one of the two possible locations and there are three users (the users could be different demand points, nodes or population groups). Assume that for the two candidate locations, the distances that the three users will travel are (5,5,5) and (8,2,3), respectively. While the total distance is less in the second alternative (more efficient), the first alternative is equidistant to all neighbors (more equitable). Which alternative will be chosen will depend on the priority and preferences of the decision maker. As seen in this example, the alternative that minimizes total distance can lead to an unbalanced distribution of resources or utilities. On the other hand, an alternative that maximizes equity, where all users benefit from equally or equal amount of resources, may not be the alternative that maximizes total utility.

If the decision-maker is focused on the distribution of a single benefit to multiple users, each distribution alternative corresponds to a vector as seen in the above example. This allocation vector presents the distribution of that single benefit to users/entities enjoys. In this case, there are single efficiency and single equity concerns.

On the other hand, if the decisions will lead to allocation of multiple types of benefits to multiple users, there will be a concern for efficiency (maximizing the total utility) and equity (distributing that benefit equitably) for each benefit type. To illustrate, let us expand the scope of the above example. Assume that the healthcare facility provides service in two different areas, such as dental and standard services. Moreover, this time, assume that the output of interest is not the distance; rather the benefit that the users (entities) will receive from each service type. The outputs for the beneficiaries (users) change according to the location of the hospital. In this kind of setting, alternatives become matrices where entities and benefits correspond to rows and columns, respectively. To illustrate, let us consider two alternatives, $\left(\begin{smallmatrix} 6 & 9 \\ 5 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 5 & 5 \\ 5 & 5 \end{smallmatrix}\right)$, in which two different healthcare services are allocated to two different beneficiaries. While total output in both benefits are better in first alternative, the second alternative is more equitable. Which alternative will be chosen depends on the priorities of the decision maker, in a similar manner to the single benefit example above. In this thesis, we consider this type of problems, which we call multidimensional equitable optimization problems.

Multidimensional equitable optimization problems are multiobjective optimization problems by nature due to the existence multiple efficiency and fairness concerns. However they are different from classical multi-objective optimization problems due to additional properties assumed for the underlying dominance relation. Therefore handling such problems requires customized approaches. To be able to incorporate these (multidimensional) efficiency and fairness concerns in the decision making process, we propose multi-objective mathematical modeling based approaches in this thesis. These approaches are generic in the sense that they can be adapted to various settings, in which the decisions result in allocations of multiple goods to multiple users. To keep the cognitive burden to the DM (Decision Maker) at a reasonable level, we first formulate bi-objective programming problems, solving which would provide the DM with a set of Pareto solutions. Hence we allow the DM to analyze the trade-off between efficiency and fairness and choose the solution that she will implement. Our second approach maximizes welfare functions, each of which is associated with one of the benefits distributed. These functions are concave (Schur-concave), hence they encourage efficiency and fairness in the allocations. We illustrate their use as solution approaches for service planning problems that explicitly consider multidimensional fairness alongside multidimensional efficiency.

Most of the current work in the literature focuses on settings where a single benefit is allocated; hence we believe that this study extends these and contributes to the literature by suggesting ways of addressing these concerns in a multidimensional domain.

The rest of the thesis is organized as follows:

In Chapter 2, we define the problem we address, and provide dominance rules in agreement with the assumptions in equitable decision problems.

In Chapter 3, we provide a literature review on different types of problems involving fairness concerns. We provide a categorization of such problems and discuss how the problem we consider fits into this categorization. We discuss relevant studies from the OR literature and mention the key contributions of our work.

In Chapter 4, we elaborate on the two frameworks to incorporate multidimensional equity concerns into optimization settings, where the decisions lead to allocations of multiple benefits to multiple entities. Both approaches utilize multi-objective optimization models with different structures: The first approach trades fairness off against efficiency, while the second approach trades the welfare from one benefit off against those of others.

In Chapter 5, we demonstrate the usage of our approaches on a case study. The specifics of the case study are described, followed by a detailed analysis of the results.

We conclude our work in Chapter 6 and and give recommendations for future research that could be pursued.

Chapter 2

Preliminaries and Problem Definition

Equitable optimization is a method that is used when a decision maker is faced with the problem of allocating scarce resources to multiple users in an equitable manner. Various applications of equitable optimization are observed in the operations research literature, most of which consider settings where a single resource or benefit is allocated to multiple users.

Problems where a single benefit is allocated to multiple entities can be conceptualized as multicriteria decision making problems, where each criterion corresponds to the amount that each entity receives. Dominance rules help to compare the performance score vectors of alternatives for each criterion in multi-criteria decision making literature. These rules are for identifying "bad" alternatives that are not preferred by rational decision-makers. The dominance relation used in the literature is rational (vector) dominance. If performance score vector of alternative a is at least as good as that of alternative b in terms of all criteria and is strictly better in at least one criterion, alternative a dominates alternative b. A similar dominance rule can be defined for equitable problems where a single benefit is distributed [2].

In the definitions below, $Z \subseteq \mathbb{R}^m$ represents a set of alternatives and $z \in Z, z^t = (z_1^t, z_2^t, ..., z_m^t)$ represents a typical alternative. Each alternative is a m dimensional vector, showing the allocated amounts to m users and t shows the index of the alternative. The aim is to maximize the output/benefit distributed to each user (i.e. the problem is considered as a multi-criteria problem) [1]. Throughout the text we will use the terms user and entity interchangeably.

The decision maker's choice model can be characterized by a weak preference relation denoted by $\preceq$ [3]. For two alternatives z^t and $z^h \in Z$, we can have z^t is preferred to z^h (in the weak sense), is strictly preferred to z^h or is indifferent to z^h. It is assumed in the literature that the preference relationship of a rational decision-maker satisfies the following three axioms;

1. Reflexivity: $z^t \preceq z^t \quad \forall z^t \in Z$

The reflexivity axiom suggests that each alternative is at least as good as itself.

2. Transitivity: $z^h \preceq z^t$ and $z^g \preceq z^h \Rightarrow z^g \preceq z^t \quad \forall z^t,\ z^h,\ z^g\ \in Z$

The transitivity axiom implies that if alternative z^t is preferred to alternative z^h and alternative z^h is preferred to alternative z^g, then alternative z^t is preferred to alternative z^g.

3. Strict Monotonicity: $z^t \prec z^t + e_i\varepsilon \quad \forall z^t \in Z$, e_i : m dimensional unit vector whose the i^{th} element is 1 and the other elements are 0, ε is a small number.

Strict monotonicity means that increasing the benefit of one entity while keeping other amounts the same is better. For example, between two alternative allocation vectors, $(4, 8, 15, 16, 23)$ and $(4, 8, 15, 16, 24)$, the second alternative is preferred since the last entity gets more benefit and other entities are not worse-off.

In classical (i.e. asymmetric problems), if the preference relation satisfies these three axioms, it is a rational preference relation. If an alternative vector z^t is rationally preferred to an alternative vector z^h with respect to all rational preference relations, z^t rationally dominates z^h, $z^h \preceq_r z^t$. We need two more axioms to incorporate fairness concerns in equitable problems [3].

4. Symmetry (anonymity): $z^t \approx \Pi^i(z^t)$ $\forall i = 1, 2, ..., m!$ and $\forall z^t \in Z$. $\Pi^i(z^t)$ shows the permutation vector of z^t.

Symmetry (anonymity) axiom ensures that there is no difference between users. As an example, alternative allocation vectors, $(13, 11, 23)$ and $(23, 11, 13)$ should be the same for the decision maker since the alternatives are equivalent.

5. Pigou-Dalton transfers principle: $z^t_j \geq z^t_i \Rightarrow z^t \preceq z^t - \varepsilon e_j + \varepsilon e_i \;\; \forall z^t \in \mathbb{R}^m$, where $0 \leq \varepsilon \leq z^t_j - z^t_i$, where e_i and e_j are the i^{th} and j^{th} unit vectors in $\mathbb{R}^m$.

Pigou-Dalton transfer principle is widely used in the economics literature focusing on income inequalities. This principle states that, for an allocation, any new allocation created by taking some benefit from a relatively better off entity and transferring it to the other(s), should be a more preferred alternative. As an example, consider the allocation vector $(60, 100, 70, 80)$. Transferring 20 units of benefit from the second entity to the first one while keeping the other entities the same, hence obtaining $(80, 80, 70, 80)$, is more preferable since the resulting allocation is more equitable than the first allocation.

If a preference relation satisfies the above five axioms, it is called "equitable rational preference relation" [3]. If an alternative z^t is preferred to alternative z^h for all equitable rational preference relations, then z^t dominates z^h in equitable rational manner, and this is shown as $z^h \preceq_e z^t$. Hence equitable dominance is the intersection relation of all equitable preference relations.

The above relations consider allocations of a single benefit to multiple users and hence are defined for vectors. In many real life applications, the decisions may result in the distribution of benefits (outcomes) to the users. In such cases the concerns for efficiency and fairness are multidimensional.

In this multi-benefit framework the rules and definitions should be extended from vector dominance to matrix dominance, which is relatively harder since the alternatives (z vectors) become matrices. In this work, we address such problems, which we briefly introduce as follows:

Consider an optimization setting where any decision made results in allocations of a set of benefits over a set of entities and the decision maker has efficiency and fairness concerns for all benefits.

Let x denote the (generic) decision variable vector. Any decision x that belongs to a feasible region X results in allocation of multiple benefits to multiple users. Let sets I and K denote the sets of benefits and users, respectively. Then the benefit allocations are defined through constraints of the following form;

$$z_{ik} = g_{ik}(x) \qquad\qquad \forall i \in I,\ \forall k \in K$$

where z_{ik} shows the amount of benefit i enjoyed by entity k.

We deliberately keep this formulation in a generic form and note that functions $g_{ik}(x)$ may have different forms depending on the dynamics of the specific problem considered. Denoting the set of feasible decisions of the problem as X, our problem becomes detecting good solutions in set X such that the resulting benefit allocation (represented by the z matrices) is efficient and fair. Note that due to existence of multiple beneficiaries and multiple benefits, the problem is a multicriteria decision making problem by its nature. Moreover, the symmetry in the vectorial domain is extended to row symmetry in our alternative matrices, which is illustrated in Example 1 below. Similar to the single objective setting, we can compare some pairs of matrices and conclude that one should always be preferred over the other. However, for some pairs, trade offs will be observed, hence the choice would depend on the decision maker, as shown in Examples 1 and 2 below.

Example 1. *Consider an example of public healthcare projects selection problem, where the DM tries to maximize the benefits for different patient groups and ensure equity between these groups.* $I = \{1, 2\}$ *denotes the different kind of benefits provided from health packages, while* $K = \{1, 2\}$ *denotes patient types (entity). In this case, the alternatives will be matrices where rows indicate the entities and columns denote the benefits. This problem incorporates the multidimensional efficiency and multidimensional fairness concerns.*

Let us consider two alternatives, $\left(\begin{smallmatrix}6 & 6\\ 6 & 6\end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix}7 & 7\\ 7 & 7\end{smallmatrix}\right)$. *Clearly, the second alternative is preferred to the first one by the DM since it distributes more benefit while all beneficiaries in both alternatives take equal amount of output.*

Let us consider other alternatives: $\left(\begin{smallmatrix}6 & 6\\ 8 & 8\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}7 & 7\\ 7 & 7\end{smallmatrix}\right)$. *The DM will prefer the second alternative since it distributes same total amount of outputs more equitably. Let us now consider the following alternatives:* $\left(\begin{smallmatrix}3 & 4\\ 5 & 6\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}5 & 6\\ 3 & 4\end{smallmatrix}\right)$. *The DM will be indifferent since we assume impartiality over users, rendering any row permutation of a matrix equally desirable. The user anonymity in single benefit setting (alternatives are vectors) is equivalent to row symmetry in multi-benefit setting where the alternatives are matrices.*

We will try to explain difficulties of considering matrices as alternatives with the following example.

Example 2. *Let us say there are two different alternatives such as,* $\left(\begin{smallmatrix}2 & 8\\ 6 & 2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}4 & 8\\ 4 & 2\end{smallmatrix}\right)$. *The second alternative is more equitable according to Pigou-Dalton principle. However, in order to reach this, two units of benefit 1 are transferred from the second entity to the first entity. And as a result of this transfer, the second entity becomes worse-off than the first in terms of both benefits. In this case, the second alternative is not necessarily preferred to the first one.*

In the cases where alternatives have different efficiencies for both benefits, using dominance rules becomes even more difficult. Consider two alternatives, $\left(\begin{smallmatrix}3 & 8\\ 4 & 2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}6 & 4\\ 2 & 4\end{smallmatrix}\right)$. *The first alternative is more equitable and less efficient in terms of first benefit, and more efficient and less equitable in terms of second benefit.*

We develop two different approaches to contribute the current know-how and help improving the quality of the decisions made by the decision makers in such settings. In both approaches, we work on incorporating fairness and efficiency concerns in a multidimensional manner.

Chapter 3

Literature Review

In this chapter, we provide a review of the relevant literature. The problem considered in the thesis can be classified as multidimensional equitable optimization problem. We first mention equitable optimization problems, in which single benefit distributions are considered. Then, we discuss the solution methods used in solving multiobjective resource allocation problems with efficiency concern.

The important role of fairness in real-life decisions has been acknowledged in many operational research (OR) problems studied in recent years [1]. Especially in applications related to social welfare, including fairness in the proposed solution methods is a must. In line with this, there is notable increase in the reported studies in the OR literature incorporating fairness concerns in various areas such as supply chain [4, 5], logistics [6], allocation problems [7, 8, 9, 10, 11, 12], equitable choice [13], network [14, 15, 16] and portfolio selection [17].

A significant challenge occurring in such applications is the fact that fairness arises as an additional criterion to other, mostly system efficiency-related, criteria such as total cost or total benefit, and that there is trade-off between these concerns. This calls for the use of multiobjective programming approaches, which enables the decision makers analyze such trade-offs. There are studies in the literature that acknowledge and address the trade-off between efficiency and fairness concerns (see [18], [19], [20] and [21]).

Figure 3.1 demonstrates a categorization of the equitable decision making problems. We refer to problems aiming equitable allocations of benefits (or resources) to a set of entities as equitable decision making problems, which can be categorized into two main sets based on whether a choice or an optimization setting is considered. In the choice settings, the options (alternatives) are explicitly given, hence the problem is choosing the alternative to implement (see e.g, [22] for a choice problem over alternative allocations of a single benefit and [13] for an extension to settings with multiple benefits). When alternatives are implicitly denoted by constraints, the problem becomes an equitable optimization problem.

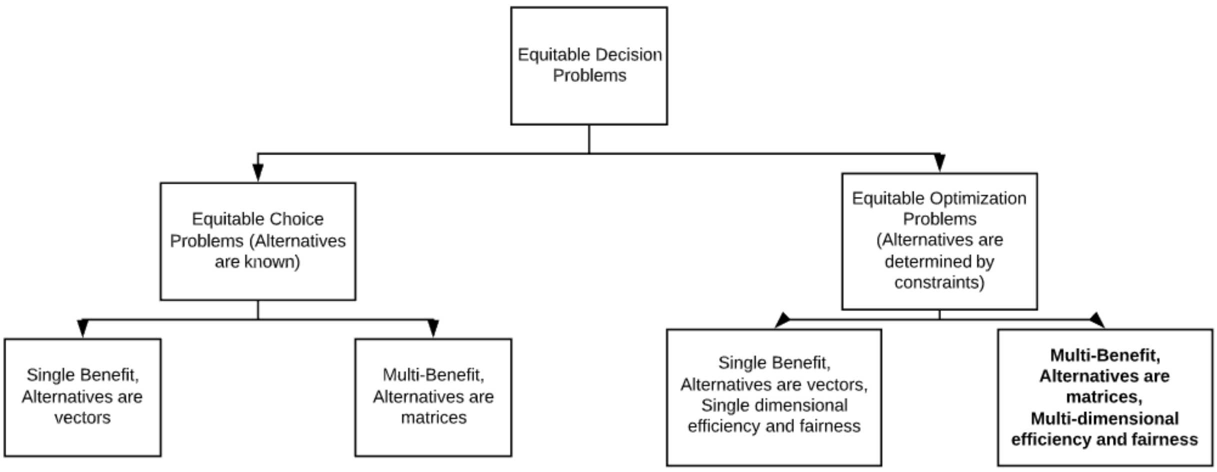

Figure 3.1: Categorization of equitable decision making problems in the literature

We can categorize our study under equitable optimization problems in multi benefit domain (see Figure 3.1). Most of the current work focuses on equitable optimization problems with single benefit concern and there are not many works on the multiple benefit domain. For that reason, we look at the equitable optimization problems in the single benefit domain in the next section.

3.1 Equitable Optimization Problems (Single Benefit)

In these problems if the decisions are associated with distributions of a single benefit across multiple entities, then efficiency and fairness concerns are single dimensional and the allocation alternatives are vectors, showing how the benefit is distributed across entities. Most of the current literature is concerned with such settings and uses mainly three methods: using a fairness-related function in addition to an efficiency-related one; combining these two in a welfare function; or treating the problem as a multi-objective optimization problem and finding the equitably nondominated solutions (see [3]).

3.1.1 Inequality Index Based Approaches

In this section, we discuss inequality index based approaches in equitable optimization problems. These studies attempt to quantify the fairness degree of an allocation using specific index, some of which are borrowed from the income inequality measurement literature. Our first approach follows this line of thought and is an index based approach. In this method, various measures of inequality (fairness) are used to assess the fairness level of a given distribution. In the mathematical models, this index is either optimized as an objective function or restricted through constraints in an efficiency maximizing setting. In the latter approach, the inequality measure is ensured to be less than a certain threshold. Below, we provide some examples from different application domains.

[23] considers optimal traffic assignment problems (vehicle routing problem) with two concerns: minimizing total travel time on a set of paths and minimizing the maximum travel time of the arc in the network of vehicle routing problem. They use an index based approach so as to ensure a balance between minimizing the average arc congestion travel time and the road network's worst arc congestion travel time.

[24] presents sequential appointment scheduling with service criteria. Different fairness steps are recommended, such as decreasing the disparity between the

amount of patients in each slot at the beginning of the system and minimizing the deviation between arrival times of the patients at each slot.

[25] focuses on different kinds of information flow (bottom-up and top-down) in a healthcare case study. The former is an analyst-driven mechanism that explores the trade-off between the overall utility of a population and a particular utility enhancement for the least well-off people.

[26] introduces a logistic problem with fairness concern. A case analysis of the recyclable waste management scheme is the foundation of their work. Their goal is to ensure equal distribution of the depots' workloads while minimizing the variable costs. They include an objective function to minimize the workload variates between depots.

[21] shows scarce resource scheduling in a hospital case study, where there is a unfair scheduling among physicians in long term. They introduce a satisfaction indicator for preference fulfillment to eliminate imbalance issues in the hospital.

Examples of index-based approaches can be extended. It also can be seen in the location problems, in which the total distance is minimized while limiting the longest distance to any demand point by a certain threshold. In that setting, the longest distance determines how unfair a distribution is [27]. Further examples of this Rawlsian type of approach can be given from various applications including but not limited to location [28, 29, 30], scheduling [31], logistics , resource allocation [32, 33] and project selection [34].

3.1.2 Social Welfare Function Based Approaches

Rather than using separate inequality and efficiency functions in a model, some studies use special types of functions that incorporate both efficiency and fairness concerns [35, 36, 37]. Such functions are called social welfare (equitable aggregation) functions. Maximizing a function which incorporates both efficiency and equity is analogous to the multicriteria decision making methods that assume that the decision-maker has a known utility function and maximize this utility function. The welfare function must be an increasing function (to encourage efficiency) and must be symmetric and satisfy the Pigou-Dalton transfers principle (see eg. [38]), to promote egalitarian allocations. Such functions are selected from the set of Schur-concave functions, which are symmetric by definition [39]. Ordered weighted averaging (OWA) functions, in which weights are ordered such that relatively worse-off entities receive relatively higher weights, are typical examples of this type of social-welfare functions [40].

[41] studies ordered weighted averaging (OWA) aggregation for multicriteria problems. They introduce two linear programming formulations to linearize OWA type objective functions. [37] use ordered median functions which are symmetric concave to address fairness concerns. [42] tries to allocate indivisible tasks while using Minmax share approach. [43] proposes a social welfare function which combines equity and efficiency concerns for two-person problem and many-person problem. [36] works on a nurse rostering model with a welfare function that determines shifts of nurses according to their skills.

Our second method is based on the same idea of using welfare functions, but since there are multiple benefit allocations, the resulting optimization problems are multiobjective.

The third group of methods formulate the a single benefit distribution problem as a multi-objective optimization problem and suggest finding solutions that are equitably nondominated. We will not discuss these in detail. The interested reader is referred to [44] for more information on such approaches.

In this research, we focus on problems, where decisions result in distributions of multiple benefits to multiple entities. Examples of this setting could be seen in many public and private sector decision making problems, such as location, task assignment, scheduling, bandwidth allocation, health investment, health-care systems and course design. In these problems, the decision makers (DMs) have efficiency-related concerns and try to maximize the total benefits. Moreover, in most of these settings they also have fairness concerns, and hence would like to distribute the benefits to users as fair as possible. Since multiple benefits exist, the concerns of efficiency and fairness are multidimensional. A typical example occurs in public education course allocation settings, where a DM decides which courses to be offered in different neighborhoods. The DM wants to open courses so that the population can benefit as much as possible subject to a given budget, while ensuring that fair service is offered to different population groups.

In such settings the DM is faced with the problem of evaluating alternative distributions of multiple benefits to multiple users (see Figure 3.1). This problem can be considered as extension of two different problem types in the literature: The first one is the equitable optimization problems, whose applications in the literature have focused on distribution of a single good or bad as fair and efficient as possible. We mentioned this type above. The second one is multiobjective resource allocation problems with only efficiency concern.

3.2 Multiobjective Resource Allocation Problems with Only Efficiency Concern

So far, most of the optimization settings in the operations research literature formulated multiobjective resource allocation problems so as to maximize the total amount for each type of good (benefit), therefore, considering only efficiency.

In the optimization field, problems with more than one outcome, have been studied for multiobjective knapsack problems, though without considering fairness. [45] works on a branch and cut algorithm, and implements two-phase algorithm to generate efficient solutions for bi-objective knapsack problems. [46] provides a new dynamic programming algorithm and experimentally compares the method with other exact solution methods proposed, and shows that their algorithm works faster than the best algorithm ([47]) known to date.

These problems generalize the single benefit knapsack problems to multi-benefit knapsack problems. However, in these problems, only the total output is maximized, while distribution is not addressed. In many real-life decision making processes, however, maximizing efficiency may not be convenient in terms of equity.

Our problem extends these problems as well, as it incorporates fairness concerns into these settings.

The problem we define extends multiobjective resource allocation problems as it concerns the distributions of multiple goods (bundles) across entities, hence alternatives are matrices.

Chapter 4

Mathematical Programming Formulations

In this Chapter, we first introduce *aggregate efficiency-fairness* framework. It helps to observe the trade-offs between efficiency and fairness. Then, we propose *concave welfare framework* which aims to increase total social welfare in each benefit type separately.

In the first approach, we aggregate the multidimensional efficiency concerns, that is the concern for maximizing the sums of all types of benefits, using an efficiency-related aggregation function. Similarly, the multidimensional fairness concerns, that is the concern for distributing each type of benefit as equitable as possible, are aggregated using a fairness-related aggregation function, resulting in a bi-objective programming problem as follows:

$$\max \quad \text{“}Efficiency, \quad Fairness\text{”}$$

$$Subject\ to:$$

$$z_{ik} = g_{ik}(x) \qquad \forall i \in I,\ \forall k \in K \tag{4.1}$$

$$x \in X \tag{4.2}$$

In this generic formulation x is the decision variable vector and X is the set of feasible decisions. Each decision x results in a distribution of multiple benefits across a set of entities. z_{ik} is the amount of benefit type i enjoyed by entity k. $g_{ik}(x)$ is a function that determines the value of the enjoyed benefit of z_{ik}. "Efficiency" and "Fairness" refer to the efficiency and fairness aggregation functions, the explicit forms of which will be provided in the upcoming sections. This approach explicitly focuses on the trade-off between these concerns.

In the second approach, we investigate the case where the aggregation is performed over the efficiency and fairness concerns of each benefit allocation, resulting in the following n-objective programming problem:

$$\begin{aligned}
&\max \quad \text{``}Welfare_1, \quad Welfare_2, ..., \quad Welfare_n\text{''} \\
&Subject\ to: \\
&z_{ik} = \ g_{ik}(x) && \forall i \in I,\ \forall k \in K && (4.3) \\
&x \in X && && (4.4)
\end{aligned}$$

$(Welfare_1, \quad Welfare_2, ..., \quad Welfare_n)$ is the vector of n objective functions where $Welfare_i$ is the aggregation function used for benefit i, the exact form of which will be given later.

We now provide the detailed descriptions of these two approaches.

4.1 Aggregate efficiency-fairness framework (AEF)

This approach aggregates the multidimensional efficiency concerns in one objective and multidimensional fairness concerns in the other objective. The overall fairness and efficiency levels of a decision are calculated as the sum of fairness and efficiency scores assigned to each benefit allocation.

The efficiency score function is a function of *total benefits* distributed from each benefit type. Since the benefits would typically take values on different ranges and are measured in different units, a scalarization would be needed to aggregate the total amounts of different benefits. For scalarization purposes, we define lower and upper bounds on the total amount of benefit i that could be enjoyed by the entities (the upper bound is determined by solving the problem as if the only concern were maximizing the total amount of that specific benefit) and denote these as L_i and H_i, respectively. Moreover, we assume that the DM would like to avoid cases with very high level of total efficiency score but this score coming only from a small subset of the benefit types, indicating very low totals in other benefit types. We propose using an increasing concave function as aggregation function to ensure balance in efficiency score values across multiple benefits. We define this aggregation efficiency function as $k_i(x)$ which determines the value of the efficiency score of the decision vector x. As mentioned above, we define L_i and H_i to calculate the scalarized value of each benefit type i (see (4.5)). We use these scalarized values for each benefit type i, t_i, to measure efficiency score contribution. In other words, we show the aggregate efficiency function's score contribution for each benefit type i with e_i which is the value of t_i in the function $k_i(x)$ (see (4.6)). (Note that $k_i(x) = k_i(t_i)$, t_i is defined as in (4.5).)

$$t_i = \left(\frac{\sum_{k \in K} z_{ik}}{H_i - L_i} \right) \qquad \forall i \in I \tag{4.5}$$

$$e_i = k_i(t_i) \qquad \forall i \in I \tag{4.6}$$

We use piecewise linearization method to calculate score values of t_i (e_i) in the aggregate efficiency function. To do so we use the following constraints, parameters and decision variables;

Parameters for linearization method

K : the set of entities, $K = \{1, 2, ..., l\}$, index k

I : the set of benefits, $I = \{1, 2, ..., n\}$, index i

M : number of thresholds used for piecewise linearization of the concave functions

L_i : a lower bound on the total amount of benefit i enjoyed by the entities

H_i : an upper bound on the total amount of benefit i enjoyed by the entities

ΔT^e_m : difference between two consecutive normalized benefit thresholds defining

interval m in efficiency aggregation function, $m = 1, ..., (M-1)$
ΔU_m^e : difference between efficiency scores of benefit thresholds defining interval m in efficiency aggregation function, $m = 1, ..., (M-1)$
W_m^e : $\Delta U_m^e / \Delta T_m^f$, $m = 1, ..., (M-1)$.

Decision variables for linearization method

t_i : scalarized value of total amount of benefit i
e_i : efficiency score contribution of t_i
$\overline{X}_{im}^e$: amount of normalized total benefit obtained within interval m from benefit type i in efficiency aggregation function

Linearization method for efficiency function

$$t_i = \sum_{m=1}^{M-1} \overline{X}_{im}^e \qquad \forall i \in I \tag{4.7}$$

$$e_i = \sum_{m=1}^{M-1} W_m^e \overline{X}_{im}^e \qquad \forall i \in I \tag{4.8}$$

$$0 \leq \overline{X}_{im}^e \leq \Delta T_m^e \qquad \forall i \in I \ , \ m = 1, ..., (M-1) \tag{4.9}$$

Constraint set (4.7) shows the equivalence of the total scalarized benefit in type i's in terms of $\overline{X}_{im}^e$ values. Constraint set (4.8) is used to calculate the aggregate efficiency scores. Constraint set (4.9) makes sure that $\overline{X}_{im}^e$ is in the given range, $[0, \Delta T_m^e]$. ΔT_m^e shows the difference between threshold values in $x - axis$ for efficiency score function.

The above three sets of constraints should be added to the mathematical model so that the aggregate efficiency score contribution of the t_i, e_i can be found.

The fairness score function is a function of the *benefit proportions*, showing the proportion of the total benefit distributed to each entity. Similar to the efficiency case, we assume that the DM would like to avoid cases in which some benefits are equitably distributed while there is extreme inequity in other benefit distributions. Hence, we use an increasing concave function as an aggregation function to ensure equity in fairness score values across multiple benefits. We illustrate the aggregation function for fairness score as $h_{ik}(x)$, where the benefit type is i and entity is k. In this generic formulation x denotes the decision variable vector.

We find the proportion of the total amount of benefit i enjoyed by a specific entity k, z_{ik}, to the total amount of benefit i enjoyed by all of the entities, which is shown as z_{ik}^N (see (4.10)). We calculate the z_{ik}^N's contribution to fairness score,f_{ik}, using aggregate fairness function (see (4.11)).

$$z_{ik}^N = \left(\frac{z_{ik}}{\sum_{k \in K} z_{ik}} \right) \qquad \forall i \in I \ , \ \forall k \in K \tag{4.10}$$

$$f_{ik} = h_{ik}(z_{ik}^N) \tag{4.11}$$

We linearize aggregate fairness score function by using piecewise linearization. The linearization method for the fairness score function contribution is as follows;

Additional parameters for linearization method

ΔT_m^f : difference between two consecutive normalized benefit thresholds defining interval m in fairness aggregation function, $m = 1, ..., (M-1)$

ΔU_m^f : difference between fairness scores of benefit thresholds defining interval m, in fairness aggregation function, $m = 1, ..., (M-1)$

W_m^f : $\Delta U_m^f / \Delta T_m^f$, $m = 1, ..., (M-1)$.

Additional decision variables for linearization method

z_{ik} : amount of benefit i enjoyed by entity k

z_{ik}^N : normalized z_{ik} $(\frac{z_{ik}}{\sum_{k \in K} z_{ik}})$

f_{ik} : fairness score contribution of z_{ik}^N

$\overline{X}_{ikm}^f$: amount of normalized benefit obtained within interval m for entity k from benefit type i in fairness aggregation function

Linearization method for fairness function

$$z_{ik}^N = \sum_{m=1}^{M-1} \overline{X}_{ikm}^f \qquad \forall i \in I \ , \ \forall k \in K \tag{4.12}$$

$$f_{ik} = \sum_{m=1}^{M-1} W_m^f \overline{X}_{ikm}^f \qquad \forall i \in I \ , \ \forall k \in K \tag{4.13}$$

$$0 \leq \overline{X}_{ikm}^f \leq \Delta T_m^f \qquad \forall i \in I \ , \ \forall k \in K, \quad m = 1, ..., (M-1) \tag{4.14}$$

Constraints (4.12) are used to identify the benefit values within the intervals. Constraints (4.13) demonstrate the calculation of fairness score value of z_{ik}^N. In other words, constraint sets (4.12) and (4.13) are used to calculate the fairness score value of the entity k's share from benefit type i (z_{ik}^N). Constraint set (4.14) makes sure that $\overline{X}_{ikm}^f$ is less than or equal to the corresponding interval, [0, ΔT_m^f]. ΔT_m^f shows the difference between threshold values for fairness score function in $x - axis$. For that purpose, the range of normalized benefit values, [0,1], is divided into $M - 1$ intervals as seen in Figure 4.1, which shows an example fairness score function with 10 intervals.

The above three sets of constraints should be added to the mathematical model so that the aggregate fairness score contribution of the z_{ik}^N, f_{ik} can be found.

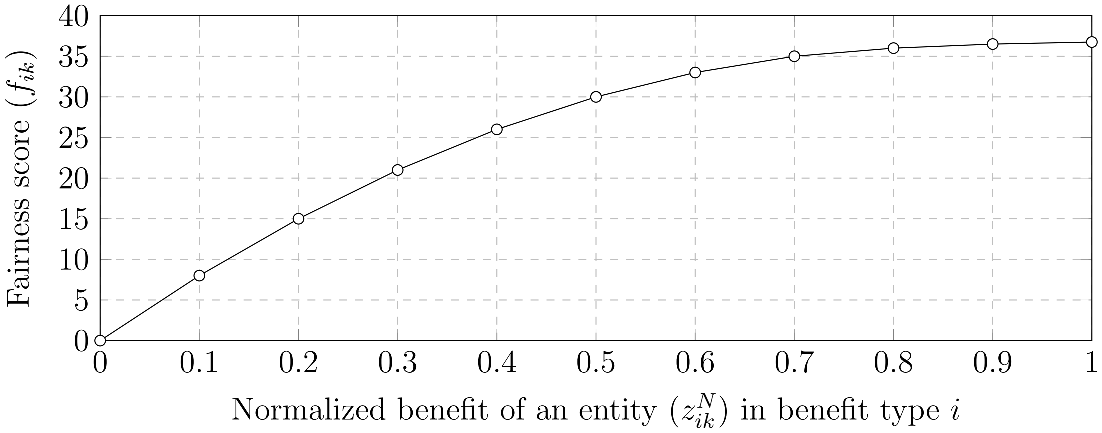

Figure 4.1: An example fairness score function

Note that the aggregated fairness function considers proportions and hence does not incorporate any efficiency concerns, i.e. there is no difference between alternative allocations with different totals as long as the users enjoy the total with the same proportions. To illustrate, the following two benefit distributions over three entities would have the same fairness score : (10, 10, 20) and (25, 25, 50), although their efficiency levels are different. We incorporate the (multidimensional) efficiency preferences only in the first objective.

The formulation is as follows:

Aggregate efficiency-fairness model (AEF-C)

$$Max \ \sum_{i \in I} e_i, \quad Max \ \sum_{k \in K} \sum_{i \in I} f_{ik} \tag{4.15}$$

$$Subject \ to:$$

$$x \in X \tag{4.16}$$

$$z_{ik} = \ g_{ik}(x) \qquad \forall i \in I \ , \ \forall k \in K \tag{4.17}$$

$$z_{ik}^N = \left(\frac{z_{ik}}{\sum_{k \in K} z_{ik}} \right) \qquad \forall i \in I \ , \ \forall k \in K \tag{4.18}$$

$$f_{ik} = h_{ik}(z_{ik}^N) \qquad \forall i \in I \ , \ \forall k \in K \tag{4.19}$$

$$t_i = \left(\frac{\sum_{k \in K} z_{ik}}{H_i - L_i} \right) \qquad \forall i \in I \tag{4.20}$$

$$e_i = k_i(t_i) \qquad \forall i \in I \tag{4.21}$$

Constraint set (4.16) ensures that x, the decision vector, is an element of X, which is the feasible set in the decision space. The decisions and the feasible decision space are problem specific, hence we give them in generic form. Constraint set (4.17) is used to calculate total amount of benefit i enjoyed by entity k as a result of decision x. Constraint set (4.18) scales the total amount of benefit i enjoyed by entity k to a [0,1] interval by converting z_{ik} to the proportion of the total amount, z_{ik}^N. Note that a concave fairness function is used to ensure that these fairness scores are allocated equitably across entities. This function is linearized using piecewise linearization as explained above.

We perform a similar linearization for the concave efficiency function. Constraint set (4.20) is used to calculate scalarized total value for each benefit. To linearize, we replace constraints (4.19) and (4.21) with constraint groups [(4.12), (4.13), (4.14)] and [(4.7), (4.8), (4.9)] respectively. For the sake of completeness, we give the full formulation in Appendix A.

We call the above model AEF-C (Aggregate efficiency-fairness model with concave efficiency score function). For comparative purposes we also consider another variant (AEF-L), where the efficiency aggregation function is a linear function instead of a concave one. This implies that the DM is only concerned with the total amounts, hence decisions with very high totals in one benefit and low totals in others are also deemed acceptable. Parameters and decision variables of AEF-L are the same as in AEF-C. We delete constraints (4.7), (4.8), (4.9), i.e. (4.21), and replace the first objective function with $\sum_{i \in I} t_i$.

4.2 Concave welfare framework

Recall that, when allocation alternatives are vectors and fairness concerns exist, the DM's preference relation is assumed to be an equitable preference relation, which satisfies axioms of symmetry and Pigou-Dalton principle of transfers. If the DM's equitable preference relation can be represented by a function, the function should be an equitable aggregation function, which satisfies Pigou-Dalton transfer, anonymity and strict monotonicity properties. It is well-known that equitable aggregation functions should be Schur-concave [3], which are defined as follows [1]:

Definition 1. *A bistochastic (doubly stochastic) matrix is a square matrix* (Q) *of nonnegative numbers in which each of rows and columns sums up to 1.*

A function $f(.) : \mathbb{R}^m \to \mathbb{R}$ *is Schur-concave if and only if for all doubly stochastic matrices* Q*,* $f(Qz) \geq f(z)$*.*

Note that Schur-concave functions are symmetric and they satisfy the Pigou-Dalton principle of transfers, by definition. In this approach, we use Schur-concave welfare functions for single benefit allocations to incorporate efficiency and fairness concerns. Also, we define the objective functions based on the benefit types and separately maximize social welfare obtained from each benefit distribution.

To encourage fairness, each such function is defined as a Schur-concave function of the form;

$$SW_i = \sum_k u_i(z_{ik}) \tag{4.22}$$

Where $u_i(.)$ shows the welfare gained by providing z_{ik} units of benefit i to entity k. Note that any entity k receives utility from benefit i with respect to the same function $u_i(.)$, hence the function is a symmetric function of the allocation vector of benefit i. Using concave welfare ($u_i(.)$) functions encourages fair distribution of the benefits.

We use piecewise linearization to calculate the $u_i(.)$ values for the benefits from concave welfare function. In the linearization method, additional decision variables u_{ik} and $\overline{Y}_{ikm}$ are defined so that the contribution to social welfare score can be calculated. In other words, Let M be the number of thresholds for the linearization where the possible intervals are $m = 1, ..., (M-1)$.

Additional parameters for linearization method

ΔT_m : difference between two consecutive benefit thresholds defining interval m, $m = 1, ..., (M-1)$

ΔU_m : difference between welfare scores of benefit thresholds defining interval m, $m = 1, ..., (M-1)$

W_m : $\Delta U_m / \Delta T_m$, $m = 1, ..., (M-1)$.

Additional decision variables for linearization method

u_{ik}: contribution to social welfare from the share of user k in benefit i, i.e. $u_i(z_{ik})$

$\overline{Y}_{ikm}$: amount of benefit type i obtained by entity k within interval m

Linearization method for concave welfare function

$$z_{ik} = \sum_{m=1}^{M-1} \overline{Y}_{ikm} \qquad \forall i \in I,\ \forall k \in K \tag{4.23}$$

$$u_{ik} = \sum_{m=1}^{M-1} W_m \overline{Y}_{ikm} \qquad \forall i \in I,\ \forall k \in K \tag{4.24}$$

$$0 \leq \overline{Y}_{ikm} \leq \Delta T_m \qquad \forall i \in I,\ \forall k \in K,\ \ m = 1, ..., (M-1) \tag{4.25}$$

As in the aggregate efficiency-fairness model, $\overline{Y}_{ikm}$ denotes the amount of benefit gained within the interval m for entity k in benefit type i. Constraint set (4.23) determines the $\bar{Y}$ values for each interval with respect to gained total benefit, z_{ik}. Constraint set (4.24) is used to calculate the score contribution. Constraint set (4.25) ensures that $\bar{Y}$ values are all in the corresponding interval.

The above three sets of constraints should be included to the mathematical model so that the z_{ik}'s contribution to social welfare score, u_{ik} can be found.

The formulation of the mathematical model, in general, is as follows;

Concave welfare model (CW)

$$Max\ \text{“}\sum_{k \in K} u_{1k}, \dots, \sum_{k \in K} u_{nk}\text{”} \tag{4.26}$$

$$Subject\ to:$$

$$x \in X \tag{4.27}$$

$$z_{ik} = g_{ik}(x) \qquad \forall i \in I,\ \forall k \in K \tag{4.28}$$

$$u_{ik} = u_i(z_{ik}) \qquad \forall i \in I,\ \forall k \in K \tag{4.29}$$

As in the aggregate efficiency-fairness model, x and X denote the decision vector and feasible decision space, respectively. Constraint set (4.28) is used to calculate the amount of benefit i enjoyed by the entity k as a result of decision x. To linearize, we replace (4.29) with constraints (4.23), (4.24) and (4.25). For the sake of completeness, we give the full formulation in Appendix A.

The following section will discuss a real-life based problem that can be tackled using the structure discussed above.

Chapter 5

Case Study

In this Chapter, we demonstrate the two approaches on a real-life based public service provision problem, in which the decision makers seek to allocate various public education services as fair and efficient as possible.

We now introduce a real-life case study on a problem that public service planners face in Turkey. Public Education Centers (PEC) organize courses in almost all provinces of Turkey. These courses are offered all over the year and are run for seven days of the week, and are free of charge. They can be offered as full day sessions as well as morning, lunch, evening and weekend sessions (groups).

Ankara, as the capital city of Turkey, hosts many of these courses. With a population of nearly 5.5 million [48], the city consists of many districts, each with residents with various demographic characteristics. Hence, planning public education to ensure that the residents are offered courses based on their needs, in a fair and efficient manner is a substantial concern. We consider the problem of planning PECs in the districts of Ankara such that the benefits of the courses are distributed among different district groups in an equitable and efficient manner. For demonstration purposes we define the entities as district groups categorized based on poverty rates (see Figure 5.1). It is, however, possible to use the proposed methodology on the same planning problem with different entity definitions (e.g.

population groups can be constructed based on neighborhood or based on age.)

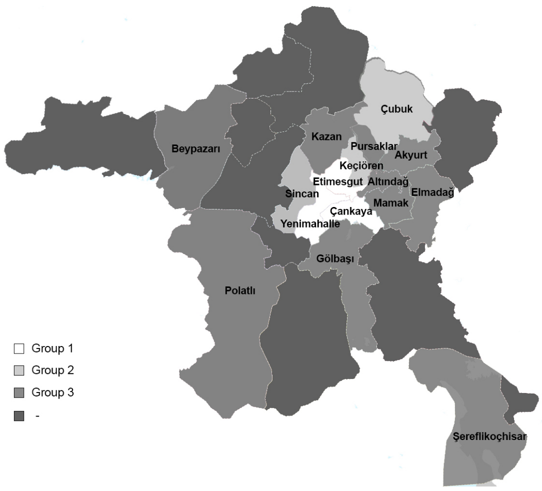

Figure 5.1: Poverty rate level map of districts of Ankara

The problem is also challenging as there are multidimensional trade-offs, both between efficiency and fairness levels of alternative plans for the same type of service, as well as across different types of services. Since the resources are limited, it is important for public planners to evaluate alternative solutions and make decisions based on a transparent mechanism that reveals *gains* and *losses*.

5.1 Data for the case study

There are different types of possible courses that can be offered (see webpages [49] and [50] for the list of courses). These courses can be gathered into two main groups: hobby courses (H) and vocational assistance courses (VA).

The data for the model are based on factors deemed important for the illustrative case study and are estimated using publicly available information [51]. We consider 16 districts with residents that are in the first and second most socio-economically developed categories[1] [53]. These are: Etimesgut, Çankaya, Yenimahalle, Sincan, Çubuk, Keçiören, Polatlı, Pursaklar, Mamak, Beypazarı, Gölbaşı, Altındağ, Şereflikoçhisar, Kahramankazan, Elmadağ and Akyurt (see Appendix B). We divide these districts into three groups of similar population size (see Figure 5.1) based on relative poverty rate[2] [53] and consider these groups as entities of the allocation problem, as shown in Table 5.1. We use abbreviations to increase the legibility. Please see Appendix B.1 for full names. The sizes of district-based population groups are estimated using publicly available data [54].

Table 5.1: Categorization of districts into income level based groups[1]

District	Poverty Rate	Group	District	Poverty Rate	Group
ET	0.4		MA	10	
ÇA	2	1	BE	17	
YE	4		GÖ	17.1	
Sİ	5.9		AL	17.9	
ÇU	6.5	2	ŞE	25.9	3
KE	6.7		KA	30	
PO	9.6		EL	32.9	
PU	9.7	3	AK	36.5	

Each public educational unit serves only the region it is located in. We assume that the demand for the courses will be 1% of the district's population and the courses have capacity to satisfy this demand. Cost of opening a new course mainly consists of venue and instructor costs, and varies from district to district. We estimate the venue cost by calculating an approximate rental cost of the area that would be required for the course, which is directly proportional to the district's population. We also use district-based rental rates for this estimation. The teaching cost is calculated by multiplying permanent instructors' salaries by the number of instructors that would be needed in the district [3]. In districts with

[1]Since the population of third and fourth groups are below 20000, their poverty rates cannot be calculated and they are referred to as upper-rural and lower-rural, respectively [52].

[2]The individual or household that has income and spending below a certain limit(a specific rate of the average welfare level of the society) is defined as relatively poor. The relative poverty rate, in our data, is the share of individuals or households living with less than half of the median disposable personal income in Ankara.

[3]We assume that the instructors are already recruited. Hence costs associated with recruiting and training teachers are not taken into account.

higher population, the course classrooms are assumed to accommodate 25 people, otherwise they are assumed to serve 20 people. We assume that the available budget is approximately 50% of the total cost of the courses.

We expect that whether people would prefer hobby or vocational courses would depend on the average poverty level of the district residents. Therefore, we estimate attendance rates to the hobby and vocational assistance courses accordingly, as presented in Table 5.2. To be more specific, we assume that around 1% of the district's population (see Table 5.3) would be attending the courses and the participants' preference for hobby and vocational courses would be determined by the rates given in Table 5.2. For example, in the second group, a total of 15,177 people would be participating, 8,965 of whom are expected to chase hobby courses (see Table 5.3). The cost of opening a new course is district-specific since it is evaluated by examining both the rental prices and the salaries of the course instructors in the districts. We measure the benefit of a course in terms of the number of participants.

Table 5.2: Rates used to reflect course participation preferences of districts

District Group 1 (G1)			**District Group 2 (G2)**			**District Group 3 (G3)**								
District	**VA**	**H**	**District**	**VA**	**H**	**District**	**VA**	**H**	**District**	**VA**	**H**	**District**	**VA**	**H**
ET	0.35	0.65	Sİ	0.4	0.6	PO	0.45	0.55	GÖ	0.45	0.55	EL	0.45	0.55
ÇA	0.35	0.65	ÇU	0.4	0.6	PU	0.45	0.55	AL	0.45	0.55	AK	0.45	0.55
YE	0.35	0.65	KE	0.4	0.6	MA	0.45	0.55	ŞE	0.45	0.55			
						BE	0.45	0.55	KA	0.45	0.55			

Table 5.3: Cost and benefit (outcome) values of courses

Hobby Courses					**Vocational Assistance Courses**				
		Outcome					**Outcome**		
District	**Cost (1000 TL)**	**G1**	**G2**	**G3**	**District**	**Cost (1000 TL)**	**G1**	**G2**	**G3**
ET	1947	3339	0	0	ET	1947	1998	0	0
ÇA	3740	5387	0	0	ÇA	3740	3223	0	0
YE	2429	4313	0	0	YE	2429	2323	0	0
Sİ	1480	0	3061	0	Sİ	1480	0	2127	0
ÇU	275	0	525	0	ÇU	275	0	365	0
KE	2803	0	5368	0	KE	2803	0	3730	0
PO	379	0	0	673	PO	379	0	0	550
PU	514	0	0	787	PU	514	0	0	644
MA	1877	0	0	3560	MA	1877	0	0	2913
BE	153	0	0	266	BE	153	0	0	217
GÖ	584	0	0	739	GÖ	584	0	0	605
AL	1128	0	0	2035	AL	1128	0	0	1665
ŞE	106	0	0	188	ŞE	106	0	0	154
KA	173	0	0	294	KA	173	0	0	241
EL	140	0	0	249	EL	140	0	0	204
AK	100	0	0	190	AK	100	0	0	156

We organize the courses so that the benefits are distributed efficiently and fairly to district groups generated based on the poverty rate. This choice is without loss of generality, one can easily use the same structure for another problem in which the entities are defined in a different manner. We do not allow a course to be partially initiated, hence formulate the problem as a binary knapsack problem.

5.2 Mathematical models solved in the case study

We now give the mathematical models that are solved in the two frameworks we consider.

Problem Parameters

K : the set of district groups, index k
I : the set of course types, index i
J : the set of districts, index j
M : number of thresholds used for piecewise linearization
L_i : a lower bound on the total amount of enrollment in course i
H_i : an upper bound on the total amount of enrollment in course i
ΔT^f_m : difference between two consecutive normalized benefit thresholds defining interval m in fairness aggregation function, $m = 1, ..., (M-1)$
ΔT^e_m : difference between two consecutive normalized benefit thresholds defining interval m in efficiency aggregation function, $m = 1, ..., (M-1)$
ΔU^f_m : difference between fairness scores of benefit thresholds defining interval m, in fairness aggregation function, $m = 1, ..., (M-1)$
ΔU^e_m : difference between efficiency scores of benefit thresholds defining interval m in efficiency aggregation function, $m = 1, ..., (M-1)$
W^f_m : $\Delta U^f_m / \Delta T^f_m$, $m = 1, ..., (M-1)$.
W^e_m : $\Delta U^e_m / \Delta T^f_m$, $m = 1, ..., (M-1)$.
p_{ijk} : number of participants from district group k to course type i in district j
c_{ij} : cost of opening course type i in district j
C : total available budget

Decision Variables

z_{ik} : total number of people in district group k that are enrolled in course type i
z_{ik}^N : normalized z_{ik} ($\frac{z_{ik}}{\sum_{k\in K} z_{ik}}$)
f_{ik} : fairness score contribution of z_{ik}^N
t_i : scalarized value of total amount of benefit in course type i
e_i : efficiency score contribution of t_i
$\overline{X}_{ikm}^f$: amount of normalized benefit obtained within interval $m = 1, ..., (M-1)$ for district group k from course type i in fairness aggregation function
$\overline{X}_{im}^e$: amount of normalized total benefit obtained within interval $m = 1, ..., (M-1)$ from course type i in efficiency aggregation function

$$y_{ij} = \begin{cases} 1, & \text{if course type } i \text{ is offered at district } j \\ 0, & \text{otherwise.} \end{cases}$$

a_{ikj} : auxiliary variable ($y_{ij} \times z_{ik}^N$)

Aggregate efficiency-fairness model (AEF-C) ;

$$\max \sum_{i\in I} e_i, \quad \max \sum_{k\in K}\sum_{i\in I} f_{ik}$$

$$Subject\ to:$$

$$(4.5), (4.7), (4.8), (4.9), (4.12), (4.13), (4.14), (4.18), (4.20)$$

$$\sum_{j\in J}\sum_{i\in I} c_{ij}y_{ij} \leq C \tag{5.1}$$

$$z_{ik} = \sum_{j\in J} p_{ijk}y_{ij} \qquad \forall i \in I\ ,\ \forall k \in K \tag{5.2}$$

$$\sum_{j\in J}\sum_{k'\in K} p_{ijk'}a_{ikj} = z_{ik} \qquad \forall i \in I\ ,\ \forall k \in K \tag{5.3}$$

$$a_{ikj} \leq y_{ij} \qquad \forall i \in I\ ,\ \forall k \in K,\ \ \forall j \in J \tag{5.4}$$

$$a_{ikj} \leq z_{ik}^N \qquad \forall i \in I\ ,\ \forall k \in K,\ \ \forall j \in J \tag{5.5}$$

$$a_{ikj} \geq z_{ik}^N - (1 - y_{ij}) \qquad \forall i \in I\ ,\ \forall k \in K,\ \ \forall j \in J \tag{5.6}$$

$$y_{ij} \in \{0, 1\} \qquad \forall i \in I\ ,\ \forall j \in J \tag{5.7}$$

Constraint (5.1) ensures that the budget is not exceeded. The set of constraints (5.2) is used to calculate the total number of district group members benefiting from a specific course type, for each course type-district group pair. Constraint sets (5.3) - (5.6) are for linearization. We set $\Delta T^f_m = \Delta T^e_m = 0.1 \quad \forall m = 1, ..., (M-1)$ in constraint sets (4.14) and (4.9). Finally, constraints (5.7) define the binary variables.

Parameters and decision variables are the same as in the concave efficiency model for variant (AEF-L). We delete constraints (4.20), (4.7), (4.8), (4.9) and replace the first objective function with $\sum_{i\in I} t_i$.

Additional parameters for concave welfare model

ΔT_m : difference between two consecutive benefit thresholds defining interval m, $m = 1, ..., (M-1)$

ΔU_m : difference between welfare scores of benefit thresholds defining interval m, $m = 1, ..., (M-1)$

W_m : $\Delta U_m/\Delta T_m$, $m = 1, ..., (M-1)$.

Additional decision variables for concave welfare model

u_{ik}: contribution to social welfare from the share of district group k in service related to course type i, i.e. $u_i(z_{ik})$

$\overline{Y}_{ikm}$: amount of benefit from course type i obtained by district group k within interval m

Concave welfare model (CW);

$$\begin{aligned}
&\max \quad \text{“}\sum_{k\in K} u_{1k}, \quad \sum_{k\in K} u_{2k}\text{”} && \\
&\textit{Subject to}: && \\
&(4.23),\ \ (4.24),\ \ (4.25),\ \ (5.2) && \\
&\sum_{j\in J}\sum_{i\in I} c_{ij}y_{ij} \le C && (5.8)\\
&y_{ij} \in \{0,1\} \qquad\qquad \forall i\in I\ ,\ \forall j \in J && (5.9)
\end{aligned}$$

Constraint (5.8) ensures that the budget is not exceeded. Constraint set (5.9) defines the binary variables.

5.3 Solution method implemented in the case study

Note that the aggregation based framework is a bi-objective framework, irrespective of the number of benefit types. In the second framework, though, the number of objective functions is equal to the number of benefit types. Since we consider two benefit types (hobby and vocational-related benefit), all formulations result in bi-objective programming problems, the details of which are provided in the above section.

We solve these models using the epsilon-constraint approach [55], as described in Algorithm 1 below. The models are coded in Eclipse JAVA Oxygen and solved by CPLEX 12.6 on a dual core (Intel Core i7 2.81 GHz) computer with 16 GB RAM. All solution times are reported in central processing unit (CPU) seconds.

Given a biobjective programming problem:

P: max $z_1(x)$, max $z_2(x)$
subject to;
$x \in X$

The pseudo-code of the ϵ-constraint method is provided in Algorithm 1. The algorithm returns the set of nondominated points ($\mathcal{N}$) of problem P.

Algorithm 1: ϵ-constraint method

Set $\epsilon = 0$ and determine a desired *stepsize*. Set $\mathcal{N} = \emptyset$.
Let $x^i \in \arg\max_{x \in X} z_i(x),\ z_i^I = z_i(x^i),\ z_i^N = \arg\min_{x \in X} z_i(x)$ for i=1,2;
Let $\delta = 10^{-5}[\frac{z_1^I - z_1^N}{z_2^I - z_2^N}]$
Solve P_1**:** max $z_1(x)+\ \delta z_2(x)$
    subject to;
    $x \in X$
if P_1 *is feasible* **then**
    Set feasible = true
    Let x^* be an optimal solution to P_1.
    $\mathcal{N} = \mathcal{N} \cup \{(z_1(x^*), z_2(x^*)).$
else
    Set feasible = false
end
while *feasible* **do**
    Set $\epsilon = z_2(x^*)$ + *stepsize*
    Solve P_2**:** max $z_1(x)+\ \delta z_2(x)$
    subject to;
        $x \in X$
        $z_2(x) \geq \epsilon$
    if P_2 *is feasible* **then**
        Let x^* be an optimal solution to P_2.
        $\mathcal{N} = \mathcal{N} \cup \{(z_1(x^*), z_2(x^*)).$
    else
        Set feasible= false
    end
end

We use this algorithm to solve both of our models. At the initialization step, we determine stepsize and set epsilon to zero (see line 1). At the beginning of the algorithm we find initial solution with the help of P_1 (see line 4-6), then we implement the main loop to find new Pareto-optimal solutions (lines 14-26) in which we optimize $z_1(x)$ (with augmentation term to avoid dominated solutions) while restricting the $z_2(x)$ with a lower bound (lines 16-19). In each iteration we systematically change ϵ values (see line 15) to find different nondominated solutions. Algorithm terminates when the solution of P_1 is infeasible (line 24).

5.4 Results

In this section, we discuss the results of our case study. Before providing the details, we first explain a potential issue with the aggregated efficiency-fairness approach and how we address this issue.

A common drawback to all approaches relying on inequality indices is the fact that these indices only focus on how the total benefit is distributed regardless of the level of total benefit: even an allocation not distributing any benefit to any of the entities is a perfectly fair one.

The aggregated efficiency-fairness approaches suffer from the same drawback as they quantify fairness based on proportions of benefit distributed to the entities. To give an example, consider the following two solutions that allocate two benefits and three entities (the rows correspond the allocation vectors of the benefits): $\left[\begin{smallmatrix} 100 & 300 & 200 \\ 100 & 300 & 200 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 150 & 100 & 100 \\ 150 & 100 & 100 \end{smallmatrix}\right]$ when AEF-C approach is used, the aggregated fairness and efficiency scores of these matrices are (73.50, 130.64) and (65.00, 134.86), respectively; hence none dominates the other in the bi-criteria sense. However, when the allocation matrices are investigated in detail, it is observed that the first allocation is better. This is because anonymity is assumed across entities, implying that an allocation matrix is equally good as its column permutations (Anonymity assumption implies column permutation in a multi-benefit setting where benefits correspond to rows and entities correspond to columns).

When we permute the second matrix swapping the first and the second columns, it is seen that in the first alternative, each element is at least as much as its counterpart in the second one. Since the second matrix distributes the benefits in a less efficient but relatively fairer way, the fairness score of this option is higher than that of the first, making it a nondominated alternative. This can also be observed looking at the fractional distributions as follows: $\left[\begin{smallmatrix} 0.17 & 0.5 & 0.33 \\ 0.17 & 0.5 & 0.33 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0.42 & 0.29 & 0.29 \\ 0.42 & 0.29 & 0.29 \end{smallmatrix}\right]$, leading to the second matrix having a higher aggregated fairness score. This is the reason why any approach incorporating fairness concerns using inequality indices should also account for efficiency.

The relation between the above allocation matrices is called equitable matrix dominance [13] and is defined below. Note that in the definitions below, rows and columns correspond to benefits and entities, respectively.

Definition 2. *Given two alternatives f^j, $f^{j'} \in \mathbb{R}^{(n \times l)}$ where n is the number of benefits and l is the number of entities, $I = \{1, 2, ..., n\}$ and $K = \{1, 2, ..., l\}$:*

$$f^{j'} \preceq_d f^j (f^j \text{ weakly matrix dominates } f^{j'}) \iff f^{j'}_{ik} \leq f^j_{ik} \text{ for all } i \in I,\ k \in K.$$

Let $\pi_r(f^{j'})$ be a column permutation of $f^{j'}$ and $R = \{1, 2, ..., l!\}$:
$f^{j'} \preceq_{em} f^j (f^j$ equitably matrix weak dominates (em-dominates) $f^{j'}) \iff \pi_r(f^{j'}) \preceq_d f^j$ for at least one $r \in R$.

To remedy the issue of obtaining em-dominated alternatives, we perform post-processing and eliminate such solutions. All the results reported in the following analysis correspond to the solutions obtained after post-processing; hence to non-dominated solutions in the em-dominance sense. For each method, we perform dominance check among the solution set returned by that specific method, i.e. we do not cross-check with the solutions returned by the other methods. That is, if a solution is em-dominated by solutions returned by an alternative approach we do not eliminate it since it would not be traceable in a real-life application.

We first investigate the Pareto solutions returned by the aggregate efficiency-fairness approach. Recall that we have two variants of this approach based on the form of aggregate efficiency function: concave and linear. Figures 5.2 and

5.3 show the Pareto solutions of concave (AEF-C) and linear (AEF-L) efficiency function variants, respectively and Figure 5.4 shows the solutions of the concave welfare approach (CW). In each figure, in addition to the Pareto solutions of the corresponding approach, we provide the images of the solutions returned by the other two approaches so as to enable a detailed comparative analysis.

AEF-C, AEF-L and CW return 29, 17 and 17 solutions, with total solution times of 12.03, 8.21 and 0.88 seconds, respectively. For clarity purposes we excluded two of the fairest solutions of AEF-C and AEF-L from all of the graphs. These two solutions were also problematic as they failed to provide any vocational benefit to the groups, hence are removed from the analysis.[4] One of these problematic solutions was also obtained in CW (as the hobby welfare maximizing extreme), hence it is also removed from the CW set. We provide the results of the three approaches in detail in Tables D.1 and D.2 (Appendix D).

When aggregate efficiency-fairness approach variants are compared, it is seen that the variant using a concave efficiency aggregation function outperforms the other variant in terms of the number and diversity of the solutions obtained. Specifically, AEF-C helps us to find more solutions on both edges of the Pareto frontier. Moreover, while CW provides solutions covering the range between the vocational benefit maximizing solution and the hobby benefit maximizing one, hence illustrating the trade-off between the two benefit types; AEF-C and AEF-L reveal the trade-off between (aggregated) fairness and efficiency.

It is observed that all approaches return a core set of solutions, which are common, as well as additional solutions which perform worse with respect to the objective functions of the other methods (as seen in Figures 5.2, 5.3 and 5.4.)

One sees in Figure 5.4 that AEF-C avoids extreme solutions that put too much emphasis on one benefit while sacrificing from the other one and provides solutions which have similar welfare scores for the two benefit types, hence the range of AEF-C solutions are narrower than that of CW. The proposed solutions are around the

[4]Recall the discussion we had on the drawback of using a fairness indicator that only focuses on the proportions. Even an allocation in which no one receives anything is considered perfectly fair.

center of the CW Pareto frontier. Similarly, since CW does not put emphasis on balance, the resulting aggregated efficiency and fairness scores of CW solutions are lower compared to AEF variants, making most of these solutions dominated in Figures 5.2 and 5.3.

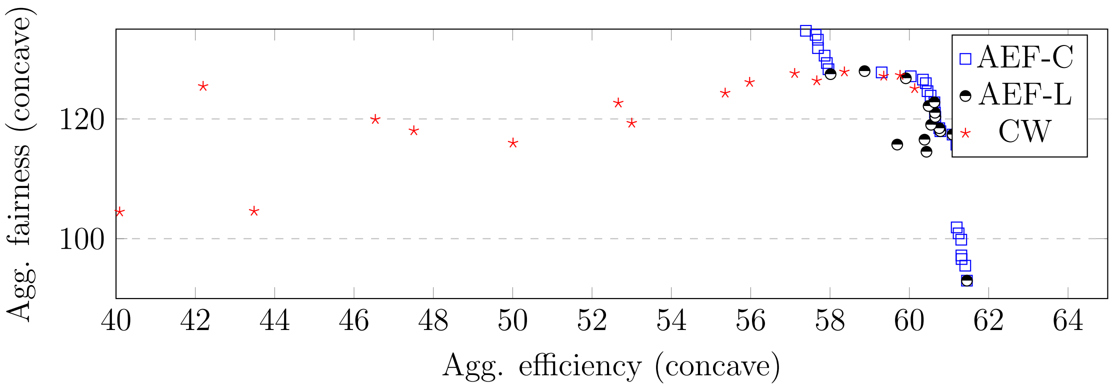

Figure 5.2: Pareto optimal solutions of the aggregate efficiency-fairness model: AEF-C

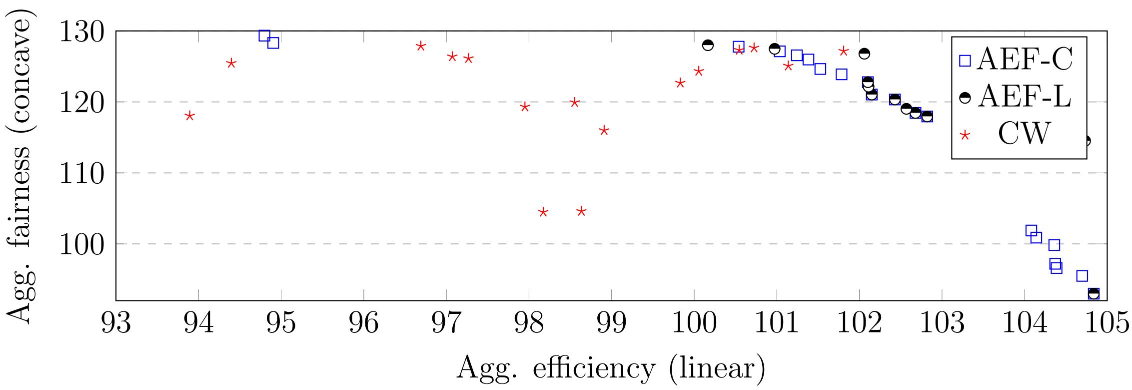

Figure 5.3: Pareto optimal solutions of the aggregate efficiency-fairness Model: AEF-L

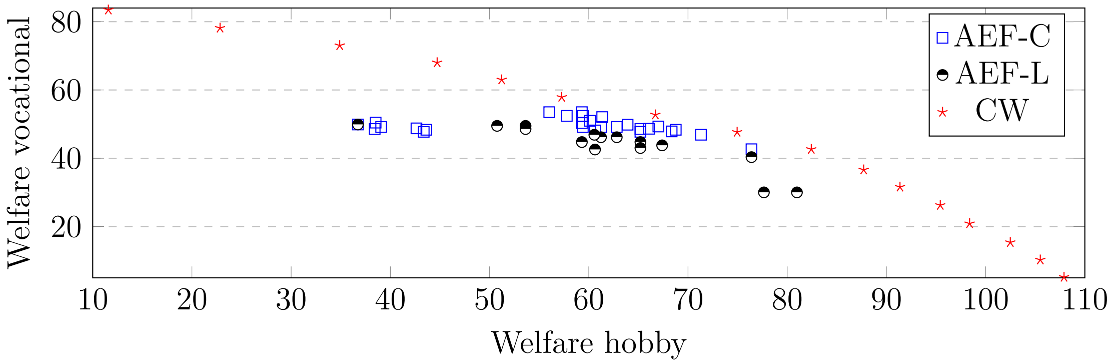

Figure 5.4: Pareto optimal solutions of the concave welfare model: CW

To investigate the trade-off between efficiency and fairness, we present the two extreme solutions (1 and 27) for AEF-C, which correspond to the solutions with the best levels of efficiency and fairness in Figure 5.2, respectively, as well as a moderate solution (15) in Figure 5.5.

$$\begin{array}{cc} & \textit{Solution } 1 \\ & \begin{array}{ccc} G1 & G2 & G3 \end{array} \\ \begin{array}{c} H \\ VA \end{array} & \begin{bmatrix} 0 & 8429 & 0 \\ 0 & 2127 & 4782 \end{bmatrix} \end{array} \quad \begin{array}{cc} & \textit{Solution } 15 \\ & \begin{array}{ccc} G1 & G2 & G3 \end{array} \\ \begin{array}{c} H \\ VA \end{array} & \begin{bmatrix} 4313 & 3061 & 1317 \\ 0 & 2492 & 3836 \end{bmatrix} \end{array} \quad \begin{array}{cc} & \textit{Solution } 27 \\ & \begin{array}{ccc} G1 & G2 & G3 \end{array} \\ \begin{array}{c} H \\ VA \end{array} & \begin{bmatrix} 3339 & 3061 & 1611 \\ 1998 & 2127 & 1665 \end{bmatrix} \end{array}$$

Figure 5.5: Extreme and moderate solutions of AEF-C

There are striking differences between the solutions. Solution 27 uses the budget to ensure a more equal distribution of the benefits across groups. As one approaches to the maximum efficiency extreme, solution 1, the total number of people served increases (15,338 vs. 13,801), but this occurs at the expense of fairness as we see that no benefit is provided to group 1 and no hobby courses are offered to group 3. Solution 15 is in between the two extremes: it offers a more balanced distribution of hobby course service to the population groups compared to solution 1, while still suffering from imbalance in vocational assistance courses. This solution uses a higher portion of the budget on hobby courses compared to solution 27, resulting in an increase in the number of people offered hobby courses, while sacrificing from the benefit of vocational assistance courses.

The extreme solutions of AEF-L are shown in Figure 5.6. Similar observations can be made regarding this variant. However, unlike AEF-C solutions, even the fairest solution (Solution 15) cannot ensure that all groups have at least benefited from both benefit types.

$$\begin{array}{cc} & \textit{Solution } 1 \\ & \begin{array}{ccc} G1 & G2 & G3 \end{array} \\ \begin{array}{c} H \\ VA \end{array} & \begin{bmatrix} 0 & 3586 & 3560 \\ 0 & 2127 & 5729 \end{bmatrix} \end{array} \quad \begin{array}{cc} & \textit{Solution } 9 \\ & \begin{array}{ccc} G1 & G2 & G3 \end{array} \\ \begin{array}{c} H \\ VA \end{array} & \begin{bmatrix} 4313 & 3061 & 922 \\ 0 & 2127 & 4578 \end{bmatrix} \end{array} \quad \begin{array}{cc} & \textit{Solution } 15 \\ & \begin{array}{ccc} G1 & G2 & G3 \end{array} \\ \begin{array}{c} H \\ VA \end{array} & \begin{bmatrix} 3339 & 3061 & 3826 \\ 0 & 2492 & 2483 \end{bmatrix} \end{array}$$

Figure 5.6: Extreme and moderate solutions of AEF-L

Solutions 2 and 17, which correspond to the solutions with the best hobby welfare and vocational welfare in Figure 5.4 are given below in Figure 5.7. Solution 8 is an example solution lying in the middle of the Pareto frontier. This approach moves from a hobby welfare maximizing solution towards a vocational welfare maximizing one, hence the interpretation of the extreme solutions is different. The Pareto frontier shows the trade-off between a hobby course service prioritizing approach and a vocational course service prioritizing one, rather than the trade-off between overall fairness and efficiency. As expected, in solution 2 almost all budget is devoted to hobby courses while in solution 17, almost all of it is used for vocational courses. Solution 8 provides a compromise between these two extremes, offering a more balanced distribution of benefits.

$$\begin{array}{cc} & \textit{Solution } 2 \\ & \begin{array}{ccc} G1 & G2 & G3 \end{array} \\ \begin{array}{c} H \\ VA \end{array} & \begin{bmatrix} 4313 & 5893 & 5595 \\ 0 & 0 & 551 \end{bmatrix} \end{array} \quad \begin{array}{cc} & \textit{Solution } 8 \\ & \begin{array}{ccc} G1 & G2 & G3 \end{array} \\ \begin{array}{c} H \\ VA \end{array} & \begin{bmatrix} 4313 & 3061 & 3999 \\ 0 & 2127 & 2060 \end{bmatrix} \end{array} \quad \begin{array}{cc} & \textit{Solution } 17 \\ & \begin{array}{ccc} G1 & G2 & G3 \end{array} \\ \begin{array}{c} H \\ VA \end{array} & \begin{bmatrix} 0 & 525 & 705 \\ 1998 & 4095 & 4888 \end{bmatrix} \end{array}$$

Figure 5.7: Extreme and moderate solutions of CW

To observe how the two benefits (H and VA) are allocated across the three groups, we summarize all allocations in Figures 5.8 and 5.9, which show allocations of the hobby course and vocational course benefits in AEF-C solutions, respectively. As seen in these figures, the shares of the groups get closer as one moves from most of efficient solution (1) to the fairest one (27) in both benefit types. We show how the benefits are distributed for each group in the solutions of the AEF-C model (see Appendix C). Since AEF-L model has a similar pattern to AEF-C with a narrower range, we do not demonstrate the figures of AEF-L.

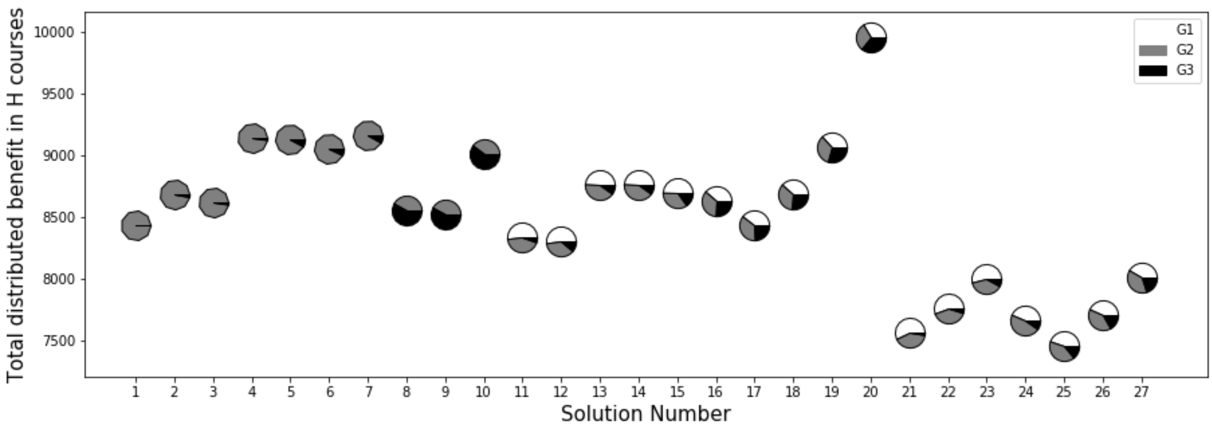

Figure 5.8: Hobby course benefit distribution across groups in AEF-C solutions

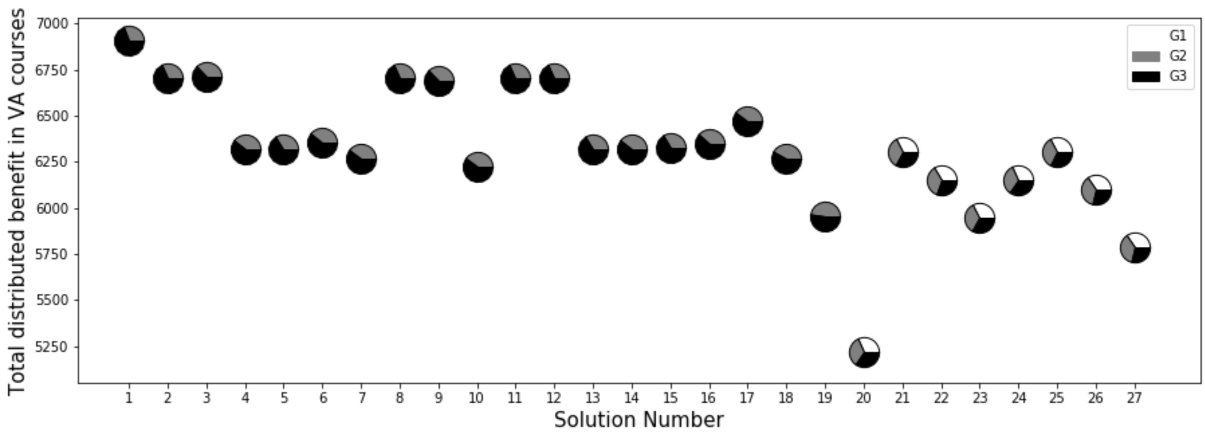

Figure 5.9: Vocational assistance course benefit distribution across groups in AEF-C solutions

Figures 5.10 and 5.11 show the allocations of two benefits in CW solutions. As expected, starting from the first solution, which has the highest hobby course welfare (resp. the lowest level of vocational course welfare) the level of total hobby benefit decreases (resp. total vocational course benefit increases) as one moves toward the other edge of the frontier. There is one exception to this observation: solution 12. In solution 12, for example, vocational welfare increases as a result of obtaining a fairer allocation of a benefit rather than a higher total. This is desired and is a result of using a concave welfare function: fairer allocations with little sacrifice from the total amount result in higher welfare.

The figures also provide information on the range of the benefit level enjoyed by each group. For example, we see that the range of hobby benefits for district group 3 is relatively wider across Pareto solutions of AEF-C compared to solutions of CW. We show how the benefits are distributed for each group in the solutions of the CWC model (see Appendix C).

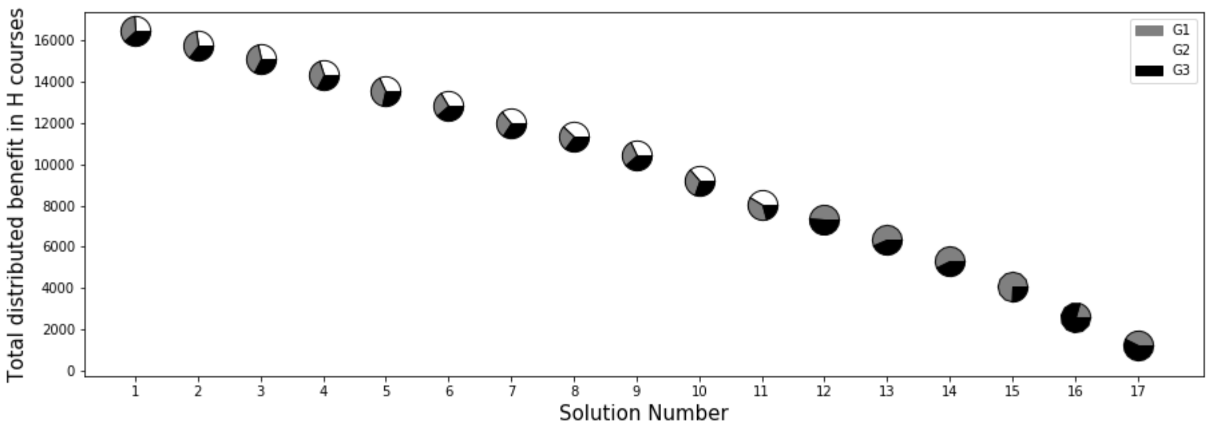

Figure 5.10: Hobby course benefit distribution across groups in CW solutions

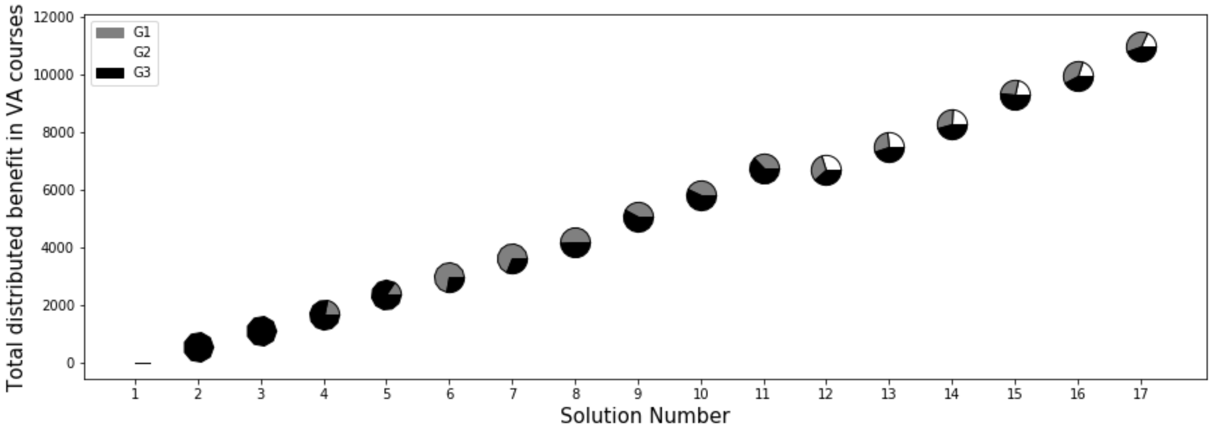

Figure 5.11: Vocational assistance course benefit distribution across groups in CW solutions

We also investigate the solutions in more detail to observe the district-specific recommendations of the alternative approaches. For each district-course pair, we calculated in how many of the Pareto optimal solutions the corresponding course is offered in that district. We then find the frequency of that decision by dividing it by the total number of Pareto solutions found. For example, in AEF-C, 29 Pareto solutions are found and in 19 of these, a vocational course is opened in Etimesgut (ET), resulting in a percentage value of 65.52. We report these percentage values in Table 5.4. We highlight the cases for which a district-course pair's percentage is considerably high (higher than 40%) and considerably low (lower than 20%) across all solutions in boldface. Such an analysis may help the decision makers to determine district-course pairs to prioritize when making decisions. For example, there is a tendency to open hobby and vocational courses to Elmadağ, Beypazarı, Polatlı in all models; while no courses are offered in Sincan, Mamak and Altındağ in any of the solutions.

Table 5.4: District-specific course recommendations in each model

District	H(%)			VA(%)		
	AEF-C	**AEF-L**	**CW**	**AEF-C**	**AEF-L**	**CW**
YE	34.48	**41.18**	29.41	**44.83**	35.29	**41.18**
ÇA	31.03	**11.76**	17.65	24.14	**0.00**	35.29
ET	20.69	**58.82**	**58.82**	**65.52**	**76.47**	**47.06**
ÇU	20.69	**11.76**	35.29	**17.24**	35.29	**47.06**
KE	**17.24**	**5.88**	35.29	**44.83**	**52.94**	29.41
Sİ	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
EL	**96.55**	**94.12**	**58.82**	**93.10**	**88.24**	**58.82**
BE	**75.86**	**41.18**	**70.59**	**72.41**	**64.71**	**70.59**
PO	**55.17**	**52.94**	**41.18**	**51.72**	**64.71**	**47.06**
KA	34.48	**52.94**	**47.06**	**0.00**	**0.00**	**0.00**
ŞE	31.03	**52.94**	**52.94**	**41.38**	35.29	**47.06**
AK	31.03	**17.65**	29.41	**0.00**	**0.00**	**11.76**
PU	31.03	**5.88**	**5.88**	**44.83**	29.41	**52.94**
GÖ	20.69	29.41	**52.94**	**17.24**	35.29	**17.65**
MA	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
AL	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**

Overall, we observe that one can obtain solutions with various degrees of efficiency and fairness using the proposed methodology. Among the aggregate efficiency-fairness variants, AEF-C (which relies on concave functions for aggregating the normalized efficiency and fairness scores) and AEF-L (which relies on linear and concave functions for aggregating the normalized efficiency and fairness scores,respectively) find similar solutions and both effectively distribute the total benefit performances across different benefit types. However, AEF-C finds more solutions on both efficiency and fairness edges of the Pareto-frontier. When aggregation based methods and the concave welfare method, which relies on defining a welfare function for each benefit, are compared, we observe that the solution sets may differ, in line with how the method is structured. Indeed, they all suggest a set of core solutions, which are (almost) the same.

The results exhibit the computational feasibility of the suggested methods. The case study also demonstrates that considering both fairness and efficiency in the resulting benefit distributions can have a significant impact on how the resources are allocated.

Chapter 6

Conclusion

We consider optimization problems in which the decisions made result in allocations of multiple benefits to multiple entities in various degrees. Such problems are highly relevant in many public sector decision making problems and they are generalizations of equitable optimization problems, in which only a single good is allocated.

Ensuring equity is important for obtaining implementable solutions that will be accepted for all stakeholders (these may be population groups) in many real life resource allocation settings. We solve these problems under the assumption that entities are indistinguishable, hence no group is prioritized over another.

We suggest two modeling approaches to be used in any such optimization problem. The first of these approaches tackles the efficiency and fairness concerns separately, hence provides a tool to observe the trade-off between these. The second approach aggregates efficiency and fairness concerns of each benefit using a concave (hence Schur-concave) function, therefore it defines a welfare function that demonstrates how good a decision is with respect to the allocation of the corresponding benefit. The trade-off is observed between welfares of different benefit types.

We demonstrate the usability of the approaches on a real-life based application example, in which the decision makers seek to allocate various public education services as fair and efficient as possible. The results exhibit the computational feasibility of the suggested methods. The case study also demonstrates that considering both fairness and efficiency in the resulting benefit distributions can have a significant impact on how the resources are allocated. Ignoring fairness can lead to some district groups suffering from lack of educational services. Since the problem setting is relevant for most real-life public sector decision making settings, the suggested models would provide useful insights and hence contribute to the relevant literature.

There can be several future research directions to be considered. One research challenge is developing exact and or heuristic solution algorithms to be able to solve the resulting multiobjective optimization models of larger-scale problems. Moreover, how to ensure balance in the asymmetric case in which the groups are not anonymous is an interesting, yet challenging question for both research and application aspects.

Alternative ways of addressing fairness concerns can be explored. One possibility is using the deprivation cost function in humanitarian logistics models. Deprivation cost is defined as the economic estimation of human suffering related to the absence of access to a service or asset [56]. By definition, this cost is a function of deprivation time (and in some cases also dependent on the socio-economic characteristics of people). Also, it should be convex, monotonic and non-linear to the deprivation time [56]. In some settings, it can be used in the objective, to penalize for the time that supply did not arrive at the demand node.

Let us assume, an extension of the planning problem that we consider in Chapter 5, in which there will be periodical course (service) openings to possible locations. In this problem the decision makers try to determine the service frequencies in different locations. A deprivation intensity or similarly, a reward intensity function can be used in this setting. Note that the form of such a reward intensity function should be selected so as to encourage frequent course openings and fair service concurrently. Future research can be performed on the determining the properties and forms of such functions and using them in mathematical models, in which both efficiency and equity concerns are addressed.

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Appendix A

AEF-C and CW models in full formulation

Concave welfare model in open form (CW)

$$Max \ \text{``}\sum_{k \in K} u_{1k}, \dots, \sum_{k \in K} u_{nk}\text{''} \tag{A.1}$$

$$Subject\ to:$$

$$x \in X \tag{A.2}$$

$$z_{ik} = g_{ik}(x) \qquad \forall i \in I,\ \forall k \in K \tag{A.3}$$

$$z_{ik} = \sum_{m=1}^{M-1} \overline{Y}_{ikm} \qquad \forall i \in I,\ \forall k \in K \tag{A.4}$$

$$u_{ik} = \sum_{m=1}^{M-1} W_m \overline{Y}_{ikm} \qquad \forall i \in I,\ \forall k \in K \tag{A.5}$$

$$0 \leq \overline{Y}_{ikm} \leq \Delta T_m \qquad \forall i \in I,\ \forall k \in K, \quad m = 1, ..., (M-1) \tag{A.6}$$

Aggregate efficiency-fairness model in open form (AEF-C)

$$Max \ \sum_{i\in I} e_i, \quad Max \ \sum_{k\in K}\sum_{i\in I} f_{ik} \tag{A.7}$$

$$Subject\ to:$$

$$x \in X \tag{A.8}$$

$$z_{ik} = \ g_{ik}(x) \qquad \forall i \in I\ ,\ \forall k \in K \tag{A.9}$$

$$z_{ik}^N = \left(\frac{z_{ik}}{\sum_{k\in K} z_{ik}}\right) \qquad \forall i \in I\ ,\ \forall k \in K \tag{A.10}$$

$$z_{ik}^N = \sum_{m=1}^{M-1} \overline{X}_{ikm}^f \qquad \forall i \in I\ ,\ \forall k \in K \tag{A.11}$$

$$f_{ik} = \sum_{m=1}^{M-1} W_m^f \overline{X}_{ikm}^f \qquad \forall i \in I\ ,\ \forall k \in K \tag{A.12}$$

$$t_i = \left(\frac{\sum_{k\in K} z_{ik}}{H_i - L_i}\right) \qquad \forall i \in I \tag{A.13}$$

$$t_i = \sum_{m=1}^{M-1} \overline{X}_{im}^e \qquad \forall i \in I \tag{A.14}$$

$$e_i = \sum_{m=1}^{M-1} W_m^e \overline{X}_{im}^e \qquad \forall i \in I \tag{A.15}$$

$$0 \le \overline{X}_{ikm}^f \le \Delta T_m^f \qquad \forall i \in I\ ,\ \forall k \in K, \ \ m = 1, ..., (M-1) \tag{A.16}$$

$$0 \le \overline{X}_{im}^e \le \Delta T_m^e \qquad \forall i \in I\ ,\ m = 1, ..., (M-1) \tag{A.17}$$

Appendix B

Data for the case study

The districts of Ankara can be categorized into four groups based on socio-economic level as follows [57]:

1. Group 1: Sincan, Etimesgut, Gölbaşı, Yenimahalle, Çankaya, Mamak, Altındağ, Keçiören and Pursaklar.
2. Group 2: Polatlı, Kazan, Beypazarı, Çubuk, Akyurt, Şereflikoçhisar and Elmadağ.
3. Group 3: Nallıhan, Ayaş, Güdül, Çamlıdere, Kızılcahamam, Kalecik and Evren.
4. Group 4: Haymana and Bala.

Ankara has 25 districts. These districts are divided into 5 groups according to socio-economic levels. For this case study, districts in the first and second groups (consist of 16 districts) are considered as possible course locations which are in the first and second most socio-economically developed groups of Ankara. Table B.1 presents abbreviation of the district names.

Table B.1: Abbreviation of the district names

District	Abbreviation	District	Abbreviation
Etimesgut	ET	Mamak	MA
Çankaya	ÇA	Beypazarı	BE
Yenimahalle	YE	Gölbaşı	GÖ
Sincan	Sİ	Altındağ	AL
Çubuk	ÇU	Şereflikoçhisar	ŞE
Keçiören	KE	Kahramankazan	KA
Polatlı	PO	Elmadağ	EL
Pursaklar	PU	Akyurt	AK

Appendix C

Detailed outcome graphics for AEF-C and CW models

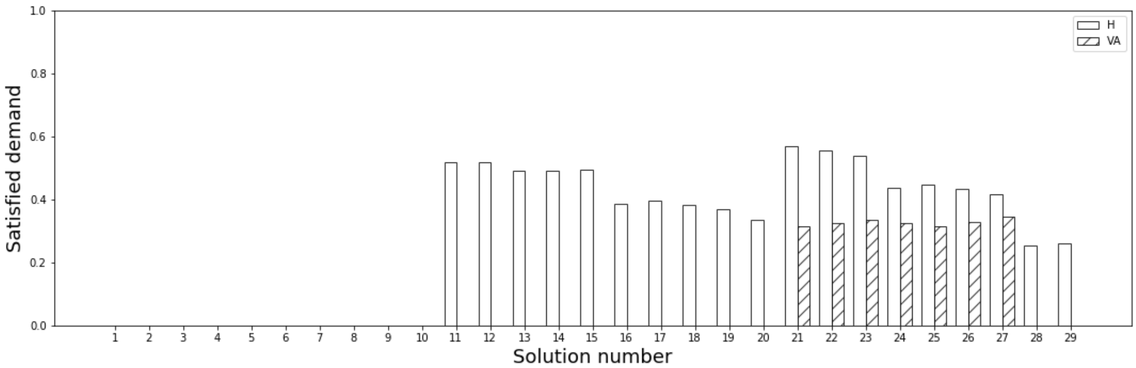

Figure C.1: Satisfied demand for G1 in both benefit types in AEF-C model

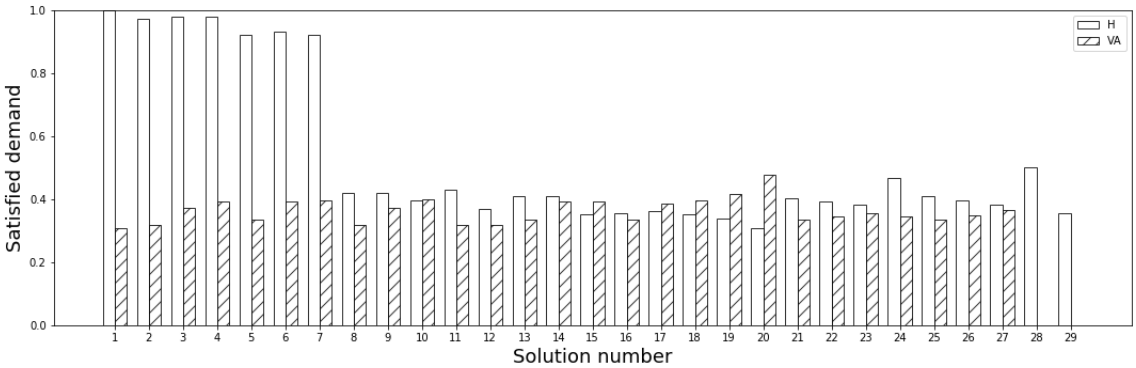

Figure C.2: Satisfied demand for G2 in both benefit types in AEF-C model

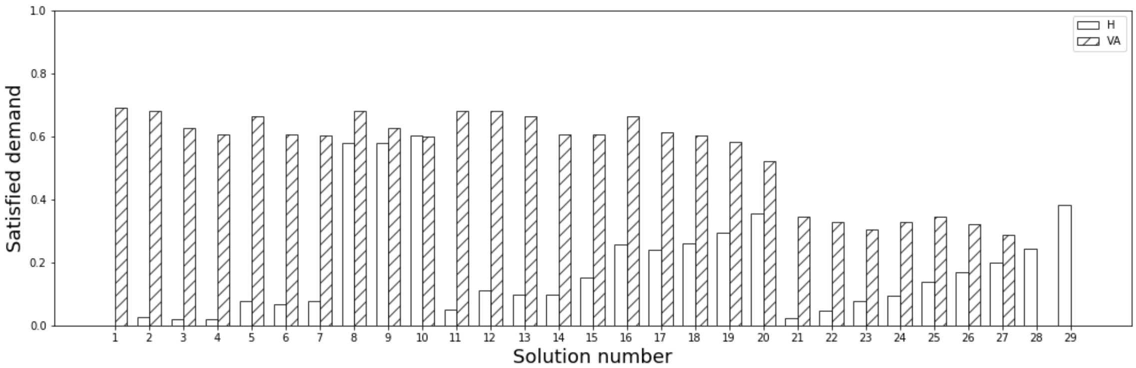

Figure C.3: Satisfied demand for G3 in both benefit types in AEF-C model

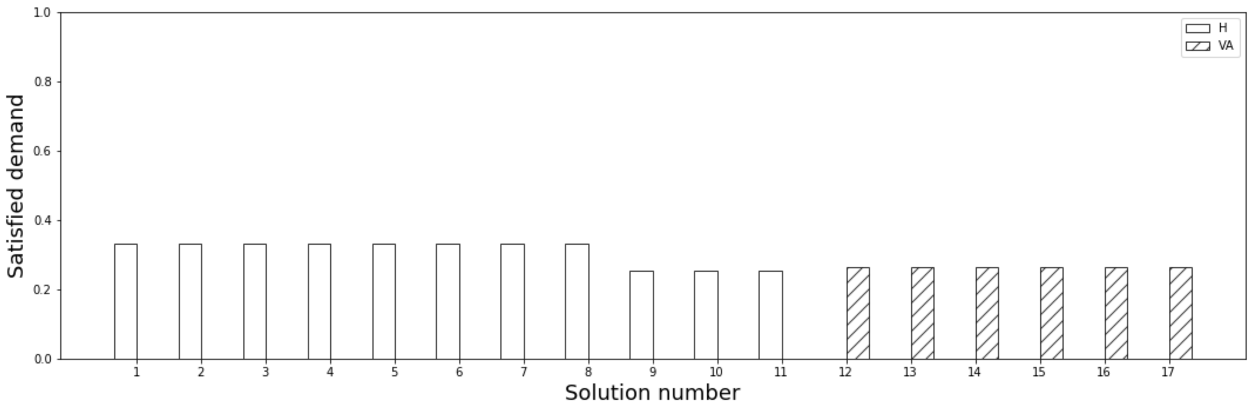

Figure C.4: Satisfied demand for G1 in both benefit types in CW model

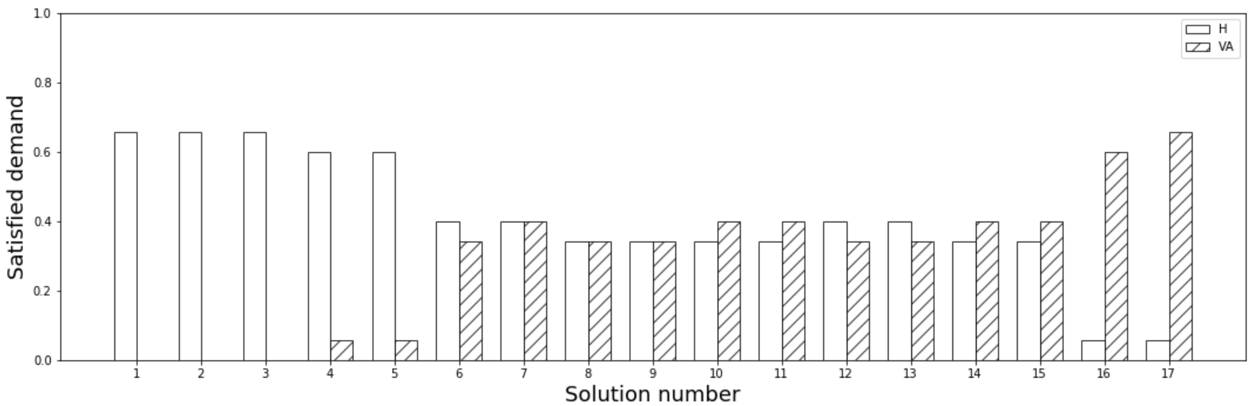

Figure C.5: Satisfied demand for G2 in both benefit types in CW model

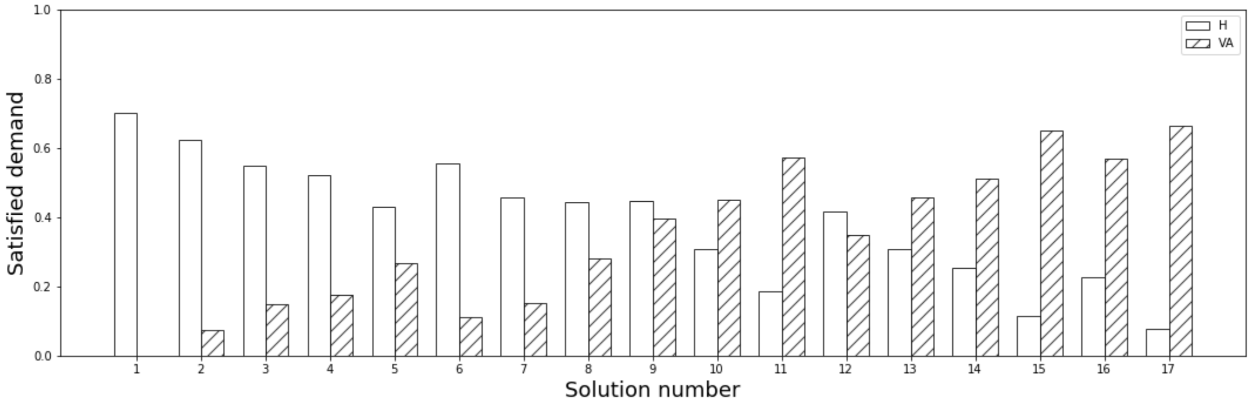

Figure C.6: Satisfied demand for G3 in both benefit types in CW model

Appendix D

Results of the case study

Table D.1: Output values of solution sets of AEF-C method in terms of all objective functions considered[1]

Sol of AEF-C	AE-C	AF-C		AE-L	AF-L		CW_1	CW_2
1	61.451	92.986		104.837	92.986		36.729	49.892
2	61.409	95.491		104.697	95.491		39.073	49.172
3	61.316	96.600		104.387	96.600		38.517	50.440
4	61.311	97.199		104.370	97.199		38.422	48.581
5	61.308	99.827		104.359	99.827		43.364	47.751
6	61.242	100.885		104.141	100.885		42.630	48.755
7	61.193	101.883		104.081	101.883		43.628	48.346
8	61.185	115.710		103.949	115.710		59.402	49.172
9	61.090	117.389		103.632	117.389		59.304	50.327
10	60.782	117.950		102.821	117.950		60.628	48.148
11	60.754	118.450		102.679	118.450		61.229	49.172
12	60.654	120.342		102.430	120.342		62.821	49.172
13	60.645	121.035		102.150	121.035		65.205	47.751
14	60.631	122.778		102.103	122.778		65.205	48.581
15	60.535	123.882		101.784	123.882		66.074	48.642
16	60.458	124.636		101.527	124.636		68.353	47.864
17	60.415	125.969		101.383	125.969		67.012	49.306
18	60.342	126.563		101.243	126.563		68.769	48.346
19	60.035	127.122		101.036	127.122		71.306	46.892
20	59.302	127.773		100.538	127.773		76.400	42.644
21	57.961	128.303		94.904	128.303		56.016	53.499
22	57.919	129.335		94.798	129.335		57.786	52.412
23	57.863	130.570		94.658	130.570		60.129	50.972
24	57.696	131.829		94.240	131.829		59.415	52.412
25	57.693	133.232		94.233	133.232		59.302	53.499
26	57.637	134.040		94.093	134.040		61.353	52.059
27	57.392	134.719		93.480	134.719		63.914	49.829
28	36.750	176.238		100.000	176.238		96.260	0.000
29	36.700	177.863		97.988	177.863		100.500	0.000

[1] CW = Concave Welfare, AE-L = Linear Aggregate Efficiency, AE-C = Concave Aggregate Efficiency, AF-C = Concave Aggregate Fairness

Table D.2: Output values of solution sets of AEF-L and CW methods in terms of all objective functions considered [2]

Sol of AEF-L	AE-C	AF-C		AE-L	AF-L		CW_1	CW_2
1	61.451	92.986		104.837	92.986		36.729	49.892
2	60.431	114.519		104.732	114.519		53.628	49.484
3	59.694	115.717		104.510	115.717		50.759	49.487
4	60.382	116.536		104.486	116.536		53.628	48.555
5	61.090	117.389		103.632	117.389		59.304	44.743
6	60.782	117.950		102.821	117.950		60.628	42.564
7	60.754	118.450		102.679	118.450		61.229	46.161
8	60.548	119.025		102.569	119.025		60.562	46.881
9	60.654	120.342		102.430	120.342		62.821	46.161
10	60.645	121.035		102.150	121.035		65.205	44.740
11	60.485	122.155		102.111	122.155		67.395	43.780
12	60.631	122.778		102.103	122.778		65.205	42.997
13	59.911	126.788		102.060	126.788		76.400	40.333
14	58.014	127.477		100.974	127.477		80.981	30.010
15	58.871	127.984		100.166	127.984		77.652	30.013
16	36.750	176.238		100.000	176.238		96.260	0.000
17	36.700	177.863		97.988	177.863		100.500	0.000
Sol of CW								
1	36.700	177.863		97.988	177.863		110.036	0.000
2	40.091	104.480		98.173	104.480		107.901	5.186
3	43.481	104.598		98.635	104.598		105.509	10.244
4	46.537	119.928		98.553	119.928		102.485	15.359
5	50.009	115.986		98.910	115.986		98.384	20.876
6	52.662	122.672		99.832	122.672		95.428	26.242
7	55.357	124.326		100.055	124.326		91.367	31.578
8	57.108	127.611		100.725	127.611		87.708	36.614
9	59.354	127.159		101.809	127.159		82.428	42.635
10	59.765	127.295		100.545	127.295		74.953	47.649
11	60.134	125.076		101.139	125.076		66.711	52.734
12	58.362	127.870		96.691	127.870		57.271	57.880
13	57.666	126.381		97.072	126.381		51.221	62.986
14	55.973	126.125		97.268	126.125		44.716	68.001
15	53.001	119.307		97.951	119.307		34.896	73.019
16	47.508	118.041		93.893	118.041		22.835	78.129
17	42.196	125.456		94.397	125.456		11.576	83.433

[2] CW = Concave Welfare, AE-L = Linear Aggregate Efficiency, AE-C = Concave Aggregate Efficiency, AF-C = Concave Aggregate Fairness