ESSAYS ON NETWORK THEORY AND APPLICATIONS

A Ph.D. Dissertation

by
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To my precious nuclear family...

ESSAYS ON NETWORK THEORY AND APPLICATIONS

The Graduate School of Economics and Social Sciences
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by

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ABSTRACT

ESSAYS ON NETWORK THEORY AND APPLICATIONS

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This thesis consists of three essays centering on network theory. In the first essay, we use a network model to show how homophily, conjoined with conformity, may shape political divisions along ethnic lines in multi-ethnic societies. We find that the decisive factor is not simply the degree of homophily but the presence of *monotone agents*, who are only connected with their own types. When there is no monotone agent, even if the level of homophily is unbounded, ethnic divisions can be avoided. The presence of a few monotone agents necessarily divides a sparsely integrated society along ethnic lines.

The second essay examines both theoretically and empirically (strong) Nash equilibrium of the free labor mobility network formation game. First, we design a network formation game in which each country's action is a choice of a mobility network between a subset of countries. The utility of each country is determined by a country specific threshold level of *absorption ratio* and *net labor flows*. We

theoretically characterize all stable and optimal mobility networks under specific assumptions. In our empirical analysis, we focus on EU and EFTA member countries. We observe that some specific countries incur the maximum loss in the grand mobility network according to our model. These countries turnout to be the ones in which reintroduction of quotas on migration is approved via referendum.

In the third essay, we examine a normal form game of network formation due to Myerson (1991). All players simultaneously announce the links they wish to form. A link is created if and only if there is mutual consent for its formation. The empty network is always a Nash equilibrium of this game. We define a refinement of Nash equilibria that we call trial perfect. We show that the set of networks which can be supported by a pure strategy trial perfect equilibrium coincides with the set of pairwise-Nash equilibrium networks, for games with link-responsive payoff functions.

Keywords: Equilibrium Refinement, Homophily, Labor Mobility, Network Formation, Pairwise-stability.

ÖZET

AĞ TEORİSİ VE UYGULAMALARI ÜZERİNE DENEMELER

İkizler, Hüseyin

Doktora, Ekonomi Bölümü

Tez Yöneticisi: Dr. Öğr. Üyesi Kemal Yıldız

Ocak 2019

Bu tez çalışması, ağ teorisini merkez alan üç makaleden oluşmaktadır. İlk makalede, çok etnikli toplumlarda, aynı türlülüğün uyumluluk ile siyasi bölünmeleri etnik kesimler boyunca nasıl şekillendirebileceğini göstermek için ağ modeli kullanılmıştır. Belirleyici faktörün, sadece bireylerin kendilerine benzer kişilerle olan bağlantı sayısından kaynaklanmadığını, karşı türden bireylerle bağlantısı olmayan monoton bireylerin varlığının da olduğu görülmüştür. Monoton bireyin olmadığı durumda, homojenlik seviyesi sınırsız olsa dahi, etnik bölünmeler önlenebilmektedir. Birkaç monoton bireyin varlığı zorunlu olarak nadiren bütünleşmiş bir toplumu etnik hatlar boyunca bölmektedir.

İkinci makale, serbest iş ağı oluşturma oyununun hem teorik hem de ampirik olarak (güçlü) Nash dengelerini incelemektedir. Makalede tasarlanan ağ oluşturma oyununda ülkeler herhangi bir ülke grubu arasında kurulan ağ yapısını seçmektedir. Her ülkenin faydası, ülkeye özgü bir absorbe oranı ve net işgücü akışı eşik seviyesi

ile belirlenmektedir. Belirli varsayımlar altında tüm kararlı ve optimal mobilite ağları teorik olarak karakterize edilmiştir. Ampirik analizde ise, AB ve EFTA üye ülkelerine odaklanılmıştır. Bazı ülkelerin, modele göre büyük hareketlilik ağında maksimum zarara uğradıkları gözlenmiştir. Bu ülkelerin, göç kotalarının yeniden oluşturulmasının referandum yoluyla onaylandığı ülkeler olduğu görülmüştür.

Üçüncü makalede, Myerson (1991)'dan kaynaklanan normal bir ağ oluşumu oyununu incelemekteyiz. Tüm oyuncular aynı anda oluşturmak istedikleri bağlantıları duyururlar. Bir bağlantı ancak ve ancak onun oluşumu için karşılıklı rıza varsa oluşturulur. Boş ağ her zaman bu oyunun Nash dengesidir. Deneme mükemmel olarak adlandırdığımız Nash dengelerinin iyileştirilmesini tanımlarız. Saf bir strateji denemesiyle desteklenebilen ağlar kümesinin mükemmel dengeyi desteklediğini, bağlantıya duyarlı ödeme fonksiyonuna sahip oyunlar için ikili-Nash denge ağları kümesine denk geldiğini gösterdik.

Anahtar Kelimeler: Ağ Oluşumu, Benzer Tür, Denge Düzeltmesi, İkili-kararlılık, İşgücü Hareketliliği.

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CHAPTER 1

INTRODUCTION

Social networks are important in many aspects of our lives. In many circumstances agents act by imitating their neighbors who are similar to them, in that similar agents contact more frequently. For example, the decision of an agent to whether adopt or not a new product, find a job is often influenced by the choices of his linkages. There is a growing literature that motivates the theoretical explanation of network effects (Jackson and Zenou, 2015).

The tendency of agents to contact with agents who are similar to them is referred to as "homophily" (Currarini *et al.* (2009), Jackson and López-Pintado (2013)). When agents contact with others who have the same characteristics like race, ethnicity, religion, age or education this is called "*Status Homophily*". When agents contact with others who think similarly (e.g. political views, consumption behavior, etc.) this is called "*Value Homophily*". As it can be evident from their definitions, status homophily does not have any direct outcome relevant implication, on the other hand value homophily does. It is observed that status homophily leads to value homophily (DiPrete *et al.* (2011)). For example, Marsden (1988) shows that Americans tend to discuss important issues mostly with friends of the same race, age and education. Although the status homophily seems to decrease in the last two decades (Mcpherson *et al.* (2006)), it still persists. In Chapter 2, we use a network model to show how status homophily derives value homophily.

The importance of network occurs also in economic settings, such as trading alliances, research partnership, etc. As Jackson and van den Nouweland (2005) suggest it is important to understand which networks are likely to form. In Chapter 3, we consider a network in which nodes are interpreted as countries and a link connecting two countries represents the free labor mobility agreement between the two countries. This will give rather a simple network structure to work with. As an application, we analyze enlargement of EU using a game theoretic network formation model for the free labor mobility area.

The mutual consent requirement of the free labor mobility agreement game in Chapter 3 creates coordination problems. In general, the game has a multiplicity of Nash equilibrium. In Chapter 3, we use strong Nash equilibrium, because both pairwise stability and Nash equilibrium concepts result multiple stable networks. A way of addressing this issue, pairwise-Nash equilibrium is commonly used in the literature.[1] It requires that, on top of the standard Nash equilibrium conditions, any mutually beneficial link be formed at equilibrium[2], without spec-

[1]Pairwise-Nash equilibrium was used, among others, in Bloch and Jackson (2007), Calvó-Armengol (2004), Goyal and Joshi (2006), Buechel and Hellmann (2012) and Joshi and Mahmud (2016).

[2]But, this is not demanding robustness to bilateral moves, as pairwise-Nash equilibrium does not allow pairs of players to coordinate fully in their strategies.

ifying any process through which players might coordinate such a deviation. The aim of Chapter 4 is to redefine pairwise-Nash equilibrium as a non-cooperative refinement. If the concept can be rephrased without referring to any implicit cooperation, then its use in non-cooperative games would be justified.

CHAPTER 2

HOMOPHILY, CONFORMITY AND THE DYNAMICS OF SEGREGATION

In many circumstances agents act by imitating their neighbors who are similar to them, in that similar agents contact more frequently. The tendency of agents to contact with agents who are similar to them is referred to as "homophily" (Currarini *et al.* (2009), Jackson and López-Pintado (2013)). When agents contact with others who have the same characteristics like race, ethnicity, religion, age or education this is called "*Status Homophily*". When agents contact with others who think similarly (e.g. political views, consumption behavior, parental attitudes, etc.) this is called "*Value Homophily*". As it can be evident from their definitions, status homophily does not have any direct outcome relevant implication, on the other hand value homophily does. It is observed that status homophily leads to value homophily (DiPrete *et al.* (2011)). For example, Marsden (1988) shows that Americans tend to discuss important issues mostly with friends of the same race, age and education. Although the status homophily seems to decrease in the last two decades (Mcpherson *et al.* (2006)), it still persists. We use a network model to show how status homophily derives value homophily.

In our model, we have two types of agents. These types might be different races or ethnicities which stand for status homophily. Agents can choose between two opinions, **A** or **B**. In our benchmark model, we assume that agents only care about conformity and would like to have the same opinion with a majority of their neighbors. This is inline with Schelling (1969)'s Segregation Model in which an agent is said to be *happy* if half or more of his 10 nearest neighbors are of the same type. In other words, agents go along with the majority of the group regardless of what they themselves think. Also we extend this model in which agents can possibly get an intrinsic utility from adopting opinion **A** or **B** alone.[1]

In our context, the more connections of an agent who adopts opinion **A** (**B**), the more likely that agent will adopt opinion **A** (**B**). Precisely, let *internal links* of an agent be the links that he forms with the same type of agents and *external links* of an agent be the links that he forms with the opposite type of agents. We refer the degree of *status homophily* as the internal/external link ratio of an agent, so status homophily takes value between $[1, \infty)$. We say there is no status homophily when ratio equals to 1 and there is extreme status homophily when ratio converges to ∞.

In our model, if a group of agents adopt the same opinion at an equilibrium then this corresponds to *value homophily* for that group. In addition, if the same type

[1] Jahoda (1959) points out that there is an ample evidence for the existence of independence in decision process. Also Terry and Hogg (1996) show that personal factors utilize a larger influence on decision process when an agent has a lower number of connections adopting the opinion. So we formulate our extended model as such: if the number of an agent's connections who adopt an opinion **B** is limited then personal factors make the agent adopt opinion **A**.

of agents adopt the same opinion at an equilibrium then status homophily *derives* value homophily. We analyze if there is a threshold degree of status homophily that makes status homophily derive value homophily at any equilibrium. We prove that for any degree of status homophily there is a network with almost the same degree of status homophily such that status homophily does not derive value homophily at some equilibria.

Our main finding shows that the decisive factor is not simply the degree of status homophily, but the presence of *monotone agents*, who are only connected with their own types. When there is no monotone agent, even if the level of status homophily is unbounded, ethnic divisions can be avoided. The presence of a few monotone agents necessarily divides a sparsely integrated society along ethnic lines.

The rest of the essay is organized as follows. We introduce our theoretical framework in Section 2.1. In Section 2.2 we present results. We extend our base model in Section 2.3. In Section 2.4 we conclude.

2.1 The Base Model

Consider a population consisting of two equally sized groups of agents. We assume that each group has n agents and each agent of the same type are linked to each other, i.e. there is a complete network within types. Also, each agent has the same number of external links with the opposite type denoted by p. So, the degree of status homophily of each agent equals to $\dfrac{n-1}{p}$.

Agents can choose between two opinions, **A** or **B**. They only care about conformity and would like to have the same opinion with a majority of their neighbors. A state of the system is a vector $s = (s^1, \ldots, s^n) \in S \equiv \{0,1\}^n$ where $s^i = 0$ if agent i adopts opinion **A**, whereas $s^i = 1$ if agent i adopts opinion **B**. A state s^* is an *equilibrium* if each agent adopts the same opinion with the majority of his connections.

The main question is how does homophily in the initial network (status homophily) affect the segregation of agents between the two opinions at an equilibrium (value homophily).

To see the relationships between status homophily and value homophily let us explore the following three networks (Figure 2.1):

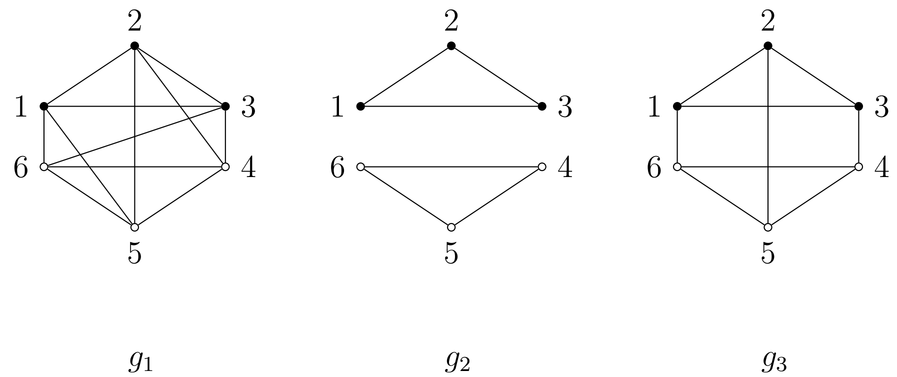

Figure 2.1: Networks g_1-g_3

In network g_1, agents have half of their links with the opposite type, i.e. there is no homophily. In network g_1, there are eight equilibria at which there are different opinions; only at two of them status homophily derives value homophily. Suppose, as in network g_2, agents are initially connected only within types. There are four Nash equilibria of the opinion game. In two of the equilibria every agent adopts the same opinion. In the other two equilibria agents have different opinions. At g_2, obviously, at every equilibria status homophily derives value homophily. This continues to be true at g_3 as well. Although at g_3 one third of each agent's links are with the opposite type, still at any equilibrium with two different opinions, status homophily derives value homophily. In the following section we analyze whether there is a threshold value for the degree of status homophily that makes status homophily derive value homophily at every equilibria.

2.2 Results

There is always an equilibrium at which status homophily leads to value homophily. But, the following proposition asserts that there is no threshold value for the degree of status homophily such that value homophily arises at each equilibrium.

Proposition 1. *For any degree of status homophily, $h \in \mathbb{R}_+$, there exists a network g with almost the same degree of status homophily, $y \in [h, h+1)$, such that status homophily does not derive value homophily at some equilibria of the opinion game.*

Proof. See Appendices. □

Proposition 1 shows that for status homophily to derive value homophily the decisive factor is not the degree of status homophily. Our next proposition, Proposition 2, shows that the presence of *monotone agents*, agents who are only connected with their own types, plays a key role for the status homophily to derive value homophily.

Proposition 2. *Let p be the number of external links of an agent and m_i be the number of monotone agents of Type i. If $m_i > p$ then for any network* g *status homophily derives value homophily at every equilibria.*

Proof. See Appendices. □

Proposition 2 shows that a sparsely integrated society, in which status homophily degree is rather high, can be polarized along ethnic lines by a few monotone agents. On the contrary, highly integrated societies, in which status homophily degree is rather low, are resilient to ethnic divisions if for each type the number of monotone agents is rather limited.

2.3 The Extended Model

Different from the Base Model in Section 2.1, each agent i has a separable utility function:

$$u_i^A(A_i) = u_A + f(A_i) \quad \text{or} \quad u_i^B(B_i) = u_B + f(B_i)$$

where $u_i^A(A_i)[u_i^B(B_i)]$ is the total utility from adopting opinion $\mathbf{A}(\mathbf{B})$ when $A_i(B_i)$ number of agent i's connections adopt opinion $\mathbf{A}(\mathbf{B})$, $u_A(u_B)$ is the utility from adopting opinion $\mathbf{A}(\mathbf{B})$ alone. We assume $f(.)$ is a strictly increasing with the number of connections who adopt the same opinion. For simplicity, we assume $f(0) = 0$ and $f(n-1+p) = 1$ where $n-1+p$ is the number of total links in the network g defined in the base model. With out loss of generality, we assume $u_A > u_B$, otherwise the model will be similar with the base model. Moreover, we assume $u_B + 1 > u_A$ to insure that agents can possibly adopt opinion $\mathbf{B}$ if there is enough connections who adopt opinion $\mathbf{B}$.

For a given network g,

- if $u_i^A(A_i) > u_i^B(B_i)$, agent i will adopt opinion $\mathbf{A}$,
- if $u_i^B(B_i) > u_i^A(A_i)$, agent i will adopt opinion $\mathbf{B}$,
- if $u_i^A(A_i) = u_i^B(B_i)$, agent i will adopt opinion $\mathbf{A}$ with probability $\frac{1}{2}$ and will adopt opinion $\mathbf{B}$ with probability $\frac{1}{2}$.

A state s^* is an *equilibrium* if each agent maximizes his utility. In the following subsection we generalize our earlier results to this extended model. A key observation in this generalization is the following:

Remark 1. Note that $f(.)$ is a strictly increasing function and $u_A > u_B$, so for some $n^* \in \mathbb{Z}_+$ such that $\frac{n^*}{n-1+p} \leq \frac{1}{2}$, $u_i^A(n^*) > u_i^B(n^*)$, i.e. agent i adopts opinion $\mathbf{A}$ if he has more than or equal to n^* connections who adopt opinion $\mathbf{A}$. Thus, a state s^* is an **equilibrium** if each agent has at least $\frac{n^*}{n-1+p}$ fraction of links who adopt the same opinion.

2.3.1 The Generalization of the Results

The class of networks that satisfy Proposition 1 is a superset of the class of networks that satisfy Proposition 1*. Unlike in Proposition 1, in Proposition 1* the structure of links between types in the network g matters.

Proposition 1*. *See Proposition 1.*

Proof. See Appendices. □

Proposition 2*. *See Proposition 2.*

Proof. See Appendices. □

2.4 Conclusion

We introduce a simple network model to analyze if there is a threshold degree of status homophily that makes status homophily derive value homophily at every equilibria. The main takeaway of the essay is that the decisive factor is not simply the degree of status homophily (Proposition 1), but the presence of monotone agents, who are only connected with their own types (Proposition 2). It follows that if there is no monotone agent, even if the level of status homophily is unbounded, ethnic divisions can be avoided. On the other hand, the presence of a few monotone agents necessarily divides a sparsely integrated society along ethnic lines. Our results do generalize to an extended model in which agents can possibly get an intrinsic utility from adopting an opinion alone.

CHAPTER 3

FREE LABOR MOBILITY NETWORK BETWEEN EU AND EFTA COUNTRIES

There is a growing body of literature questioning how does the enlargement of a free labor mobility network (or simply mobility network) affect the incumbent countries (e.g. Kahanec and Zimmermann (2010), Barrell *et al.* (2010) and Galgoczi *et al.* (2013)). However, little is known about the process of mobility network formation. This essay contributes to the debate by analyzing enlargement of EU using a game theoretic network formation model for the free labor mobility area.

We consider a network in which nodes are interpreted as countries and a link connecting two countries represents the free labor mobility agreement between the two countries. For our analysis, we need a measure of a country's ability to accommodate the labor inflows in a mobility network. For this purpose, we propose the "***absorption ratio***", which is the ratio of the total amount of net labor flows to the total amount of post-migration labor force. There is ample evidence in labor literature indicating that labor inflows increase the unemployment rate and reduce the real wages (Harris and Todaro (1970), Stalder (2010) and Oh *et al.* (2011)). We assume that country benefits from any amount of labor outflows but tolerates labor inflows up to some country specific threshold level. In our model, each country joins the mobility network if absorption ratio in the mobility network is less than his threshold.

On the theoretical side, we characterize all stable and optimal mobility networks under specific assumptions. Typically in each stable networks, the mobility clique always occur. The size of these cliques changes according to the assumptions. In taking these theoretical results to data, we focus on the mobility network within EU and EFTA countries.

A major difficulty in taking our theoretical model to data is the identification of the countries' threshold levels. EU-EFTA data is particularly appropriate for this. Following the enlargement, the majority of incumbent countries maintained tough restrictions on labor immigration from the EU-10 countries. These restrictions have been applied up to seven years after the enlargement (Sedelmeier (2014)). Therefore, we choose **2004-2011** as a controlled migration period (i.e. countries revealed their threshold levels) and **2012-2015** as a free mobility period. Our main empirical question is whether the free mobility network in **2012-2015** is stable compared to controlled mobility network in **2004-2011**. We observe that some specific countries incur the maximum loss in the grand mobility network according to our model. These countries turnout to be the ones in which reintroduction of quotas on migration is approved via referendum.

3.1 Basic Model

Let $N = \{1, \ldots, n\}$ be the set of countries. The labor mobility relations among these countries are formally represented via a network in which nodes are interpreted as countries. Having a link connecting two countries means there is a labor mobility agreement between these countries.

A network g is an undirected graph connecting countries.[1] For each pair of countries i and j, let ij denote the link between i and j, so $ij \in g$ indicates that countries i and j are linked in the network g. The complete network in which any pair of countries are linked to each other is denoted by g^N. For each network g, $N(g)$ denotes the set of non-isolated nodes in g, i.e. $N(g) = \{i \mid \exists j \in N \text{ with } ij \in g\}$. The set of country i's direct links in the network g is $N_i(g) = \{j \in N \setminus \{i\} \mid ij \in g\}$.

Given any nonempty $C \subset N$, g^C is the complete network among the countries in C. We refer to g^C as the **free mobility clique** among the countries in C. Let $G = \{g^C \mid \emptyset \neq C \subset N\}$ be the set of all possible complete networks on nonempty subsets of N.

Initially countries constitute a complete network on N. Each country's action is choice of a free mobility clique. That is, the action space of each country i is $S_i = \{g^C \in G \mid i \in C\}$. An action profile is thus a vector $s = (s_1, \ldots, s_n)$ indicating desired free mobility clique for each country. A free mobility clique g^C will be formed if and only if for all $i \in C$, $s_i = g^C$. An action profile $s = (s_1, \ldots, s_n)$

[1] Notations are from Bloch and Jackson (2006).

induces a set of free mobility cliques $g^{C_1}, \ldots, g^{C_k}$ such that $N = \bigcup_{l=1}^{k} C_l$.

We identify induced network structure of the countries with action profile s such that the induced network structure $g(s)$ satisfies

$$g(s) = \bigcup_{l=1}^{k} g^{C_l}$$

We assume that each country i has a unit of citizens. Based on the random utility model, it is possible to express the probability that citizens of country i will migrate to country j conditional on the set of available countries. Under the IIA assumption for a given network g, we have

$$P_{ij}(N_i(g)) = \frac{P_{ij}(N)}{\sum_{l \in N_i(g) \cup \{i\}} P_{il}(N)}$$

Let $PRLF_i$ be the pre-migration labor force in country i's labor market. Given a network g the total amount of labors that prefer to migrate from the country is $[1 - P_{ii}(N_i(g))] \times PRLF_i$. The post-migration labor force $POLF_i(g)$ is then given by $\sum_{j \in N_i(g) \cup \{i\}} P_{ji}(N_j(g)) \times PRLF_j$. The "***absorption ratio***" is the ratio of the total amount of net labor flows to the total amount of post-migration labor force. We consider the absorption ratio $AR_i(g)$ as a measure of a country's ability to accommodate the labor inflows in a mobility network g. If $AR_i(g)$ is positive (negative) this means that there is a net labor inflows from (outflows to) the country.

$$AR_i(g) = \frac{POLF_i - PRLF_i(g)}{POLF_i(g)} \times 100$$

We design a normal form game $\Gamma =< N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} >$ where N is the set of countries, S_i is the action space of each country i and $u_i : S_1 \times \ldots \times S_n \to \mathbb{R}$ is country i's payoff function. For any action profile $s = (s_1, \ldots, s_n)$, the utility function is:

$$u_i(s) = \begin{cases} \overline{AR}_i - AR_i(g(s)) & \text{, if } N_i(g(s)) \neq \emptyset \\ 0 & \text{, otherwise.} \end{cases}$$

In our model, each country decides to join the mobility network if absorption ratio in the mobility network is less than his threshold which can take the value at least 0. For simplicity, we assume that a country will choose to be in a coalition even if he gets zero utility.

For given free mobility cliques $g^{C_1}, \ldots, g^{C_k}$ induced by the action profile s, a nonempty coalition $C \subset N$ is an **admissible coalition** if there exists $l \in \{1, \ldots, k\}$ such that $C \subset C_l$. We define a strong Nash equilibrium as an action profile that is stable against deviations by admissible coalitions. An action profile s is a **strong Nash equilibrium** if there is no free mobility cliques $g^{C_1}, \ldots, g^{C_k}$ induced by the action profile s and no admissible coalition C with action profile $s' = (s'_C, s_{-C})$ such that $u_i(s') \geq u_i(s)$ for all $i \in C$ and for some $i \in C$ we have $u_i(s') > u_i(s)$. We say that a network g is **stable** if the state which induces the network g is a strong Nash equilibrium. Additionally, a network g is **optimal** if it is a stable network containing the largest number of links in the set of stable networks.

3.2 Theoretical Results

In this section, we will characterize strong Nash equilibrium of the mobility network formation game under the following assumptions.

Assumption 1. Each country has equal size of pre-migration labor force, i.e. for all $i, j \in N$, we have $PRLF_i = PRLF_j$.

Proposition 3. *Under **A1**, every action profile is a strong Nash equilibrium.*

Proof. See Appendices. □

Proposition 3 asserts that **A1** guarantees that every action profile is a strong Nash equilibrium, even though absorption threshold levels and migration probability distributions of countries are different.

Corollary. *Under **A1**, the grand coalition network is the unique optimal network.*

Assumption 2. Each country has the same migration probability distribution defined on host countries. That is, with equal probability, citizens of countries i and j stay in the home country, i.e. $P_{ii}(N) = P_{jj}(N)$ and citizens of country i

migrate to any host country with the same probability, i.e. $P_{ij}(N) = P_{ik}(N)$, for $j, k \in N$.

Under **A2**, the migration probability parameters of a country reduce to $P_{ii}(N)$ where $P_{ij}(N) = \frac{1-P_{ii}(N)}{n}$.

Assumption 3. No two countries has equal size of pre-migration labor force, i.e. there exists no $i, j \in N$ such that $PRLF_i = PRLF_j$.

Proposition 4. *Under **A2-A3**, an action profile s is a strong Nash equilibrium if and only if s induces a network g with free mobility cliques $g^{C_1}, \ldots, g^{C_k}$ such that for every $l \in \{1, \ldots, k\}$, $|C_l| \leq 2$ and for any $i \in C_l$, $u_i(s) \geq 0$.*

Proof. See Appendices. □

Under **A2-A3**, if an action profile s is a strong Nash equilibrium, then s induces a network g which has free mobility cliques containing at most two countries.

Remark 2. Under **A2**, there exists a strong Nash equilibrium which induces a network that has free mobility cliques containing more than two countries. In such free mobility cliques, all countries must have equal size of pre-migration labor force.

Remark 3. Under **A2-A3**, the grand coalition network is not an optimal network. There may exist more than two optimal networks. The common characteristic of the optimal networks is that all of them contains at most $\frac{n}{2}$ free mobility cliques.

Proposition 5 gives the necessary and sufficient condition for the existence of stable network with free mobility cliques containing at least two members. Note that Proposition 5 allows any type of heterogeneity in the countries.

Proposition 5. *A stable network with free mobility cliques containing at least two members exists if and only if for some action profile s there exists $C \subset N$ such that $|C| \geq 2$ and for all $i \in C$, $u_i(s) \geq 0$.*

Proof. See Appendices. $\square$

3.3 Free Labor Mobility Networks between EU and EFTA Countries

3.3.1 Data

We use OECD labor mobility data for periods 2004-2015. In this era, there are 32 countries who have joint labor mobility agreement. 28 of them are EU members and 4 of them are EFTA members. Among these, there is no available labor mobility data for 6 countries (Estonia, Croatia, Cyprus, Lithuania, Malta, Liechtenstein).[2] Since the total population of these 6 countries is nearly 2 percent of the total population of 32 countries, the non-existence of data for these countries is assumed to be insignificant.

For the 2004-2015 period, data consist of: **(1)** total nationals in countries, **(2)**

[2]Croatia became EU member in 2013, the effect of the absence of Croatia's data can be negligible.

total immigrants from, to EU/EFTA countries and **(3)** labor force participation rate of citizens of countries. We use 2012-2015 period averages of the stocks of EU/EFTA immigrants of working age by citizenship and by country. We calculate the percentage of citizens who leave country i but prefer to participate in another country j's labor market in the mobility network g^N, i.e. $P_{ij}(N)$. These percentages are shown in Table 3.1.

3.3.2 Stability and Optimality

Following the 2004 enlargement, the majority of incumbent countries maintained tough restrictions on labor immigration from the EU-10 countries. These restrictions have been applied up to seven years after the enlargement. Therefore, we choose **2004-2011** as a controlled migration period (i.e. countries revealed their threshold levels) and **2012-2015** as a free mobility period. Our main question is if the free mobility network in **2012-2015** is stable compared to controlled mobility network in **2004-2011**. We use **2004-2011** period data to calculate country specific threshold level of absorption ratio ($\overline{AR}_i$). We normalize the threshold levels that are greater than zero as zero.

Table 3.1: Percentage of participating in host country of each country's citizen

		Home Country																									
		AT	**BE**	**DK**	**FI**	**FR**	**DE**	**EL**	**IE**	**IT**	**LU**	**NL**	**PT**	**ES**	**SE**	**UK**	**CZ**	**LV**	**HU**	**PL**	**SK**	**SI**	**BG**	**RO**	**IS**	**NO**	**CH**
Host Country	**Austria**	95.9	0.0	0.0	0.0	0.0	0.2	0.0	0.0	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.4	0.1	0.4	0.5	0.2	0.2	0.1	0.0	0.1
	Belgium	0.0	97.4	0.1	0.1	0.3	0.1	0.2	0.1	0.3	1.5	0.9	0.3	0.1	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.0	0.3	0.2	0.1	0.0	0.0
	Denmark	0.0	0.0	97.2	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.0	0.0	0.2	0.0	0.1	0.0	0.0	0.1	0.1	2.8	0.4	0.0
	Finland	0.0	0.0	0.0	97.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0
	France	0.1	1.1	0.1	0.1	98.6	0.1	0.1	0.3	0.3	1.7	0.3	4.3	0.3	0.1	0.3	0.0	0.1	0.1	0.1	0.1	0.0	0.2	0.3	0.1	0.1	0.8
	Germany	2.6	0.3	0.4	0.3	0.2	98.4	3.1	0.3	1.0	4.9	1.0	1.1	0.3	0.2	0.2	0.4	1.3	1.2	1.5	0.7	1.3	1.7	1.1	0.6	0.2	0.7
	Greece	0.0	0.0	0.0	0.0	0.0	0.0	95.8	0.0	0.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.7	0.2	0.0	0.0	0.0
	Ireland	0.0	0.0	0.0	0.0	0.0	0.0	0.0	88.9	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.1	1.1	0.1	0.3	0.2	0.0	0.0	0.1	0.0	0.0	0.0
	Italy	0.1	0.1	0.0	0.0	0.0	0.1	0.1	0.1	97.0	0.1	0.1	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.2	0.1	0.1	0.6	4.2	0.0	0.0	0.1
	Luxembourg	0.0	0.2	0.0	0.0	0.1	0.0	0.0	0.0	0.0	90.4	0.0	0.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0
	Netherlands	0.1	0.3	0.1	0.0	0.0	0.1	0.1	0.1	0.0	0.1	96.7	0.2	0.1	0.0	0.1	0.0	0.2	0.1	0.2	0.1	0.0	0.2	0.0	0.2	0.1	0.0
	Portugal	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	89.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2	0.0	0.0	0.0
	Spain	0.1	0.3	0.2	0.2	0.2	0.2	0.0	0.4	0.3	0.2	0.3	1.1	98.6	0.2	0.6	0.1	0.2	0.1	0.2	0.2	0.1	1.9	3.4	0.4	0.4	0.3
	Sweden	0.0	0.0	0.7	1.3	0.0	0.0	0.1	0.1	0.0	0.0	0.1	0.0	0.0	97.9	0.0	0.0	0.2	0.1	0.1	0.0	0.0	0.1	0.0	1.5	0.9	0.0
	UK	0.2	0.2	0.4	0.4	0.2	0.2	0.4	9.5	0.3	0.1	0.4	1.1	0.2	0.4	98.2	0.3	5.1	0.8	2.1	1.4	0.0	0.6	0.6	0.0	0.5	0.2
	Czech	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	98.6	0.0	0.0	0.1	1.6	0.0	0.1	0.0	0.0	0.0	0.0
	Latvia	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	90.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	Hungary	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	96.8	0.0	0.1	0.0	0.0	0.2	0.0	0.0	0.0
	Poland	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	94.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	Slovakia	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.1	0.0	94.6	0.0	0.0	0.0	0.0	0.0	0.0
	Slovenia	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	97.6	0.0	0.0	0.0	0.0	0.0
	Bulgaria	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	93.0	0.0	0.0	0.0	0.0
	Romania	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	89.1	0.0	0.0	0.0
	Iceland	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	91.2	0.0	0.0
	Norway	0.0	0.0	0.4	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.4	0.0	0.2	0.1	0.0	0.0	0.0	2.6	97.4	0.0
	Switzerland	0.5	0.1	0.1	0.1	0.2	0.4	0.1	0.1	0.5	0.5	0.1	2.0	0.2	0.1	0.1	0.1	0.1	0.1	0.0	0.2	0.2	0.1	0.0	0.1	0.0	97.5
	Total	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100

Table 3.2: Absorption ratio period average levels

Country	Absorption Ratio 2004-2011	Absorption Ratio 2012-2015	Utility	Country	Absorption Ratio 2004-2011	Absorption Ratio 2012-2015	Utility
NO (EFTA)	0.4	2.6	-2.2	**NL** (EU15)	0.0	-1.1	1.1
CH (EFTA)	**10.0**	**12.1**	**-2.1**	**FI** (EU15)	0.0	-2.0	2.0
UK (EU15)	**1.2**	**2.4**	**-1.2**	**SI** (EU10)	0.0	-2.1	2.1
LU (EU15)	35.9	36.9	-1.0	**HU** (EU10)	0.0	-2.5	2.5
BE (EU15)	4.8	5.7	-0.9	**EL** (EU15)	0.0	-2.8	2.8
ES (EU15)	2.9	3.5	-0.6	**IE** (EU15)	0.0	-2.8	2.8
DE (EU15)	2.0	2.3	-0.3	**SK** (EU10)	0.0	-4.9	4.9
CZ (EU10)	0.1	0.3	-0.2	**IS** (EFTA)	0.0	-5.0	5.0
SE (EU15)	1.3	1.2	0.1	**PL** (EU10)	0.0	-6.0	6.0
FR (EU15)	1.4	1.3	0.1	**BG** (EU10)	0.0	-7.6	7.6
AT (EU15)	0.0	1.4	-1.4	**LV** (EU10)	0.0	-10.8	10.8
DK (EU15)	0.0	0.6	-0.6	**PT** (EU15)	0.0	-11.2	11.2
IT (EU15)	0.0	-0.3	0.3	**RO** (EU10)	0.0	-12.5	12.5

To check the stability of the grand mobility network in the post-enlargement period with no migration restrictions, we calculate the utility levels of each country.[3] From Table 3.2 one can observe that 10 of the 26 countries get negative utility. That is, the grand free mobility network is not stable compared to controlled mobility network in 2004-2011. If we rank utilities of the countries with strictly positive absorption threshold levels, then we have Norway, Switzerland, and UK in the top three. First in 2014 Switzerland, then in 2016 UK put whether to stay in the free mobility network or not to a vote. These findings seem to be consistent with the implications of our model.

[3]AT:Austria, BE:Belgium, DK:Denmark, FI:Finland, FR:France, DE:Germany, EL:Greece, IE:Ireland, IT:Italy, LU:Luxembourg, NL:Netherlands, PT:Portugal, ES:Spain, SE:Sweden, UK:United Kingdom, CZ:Czech Republic, LV:Latvia, HU:Hungary, PL:Poland, SK:Slovakia, SI:Slovenia, BG:Bulgaria, RO:Romania, IS:Iceland, NO:Norway, CH:Switzerland.

As our Proposition 5 suggests, we observe that countries form free mobility cliques in stable networks. By our Proposition 4, under **A2-A3**, the maximum size of the cliques should be 2. We observe that **A3** is satisfied, however since country specific migration probability distributions are different **A2** is violated. This leads to the maximum size of the cliques changes to be 5 instead of 2.

Table 3.3: Number of distinct cliques containing the country in a stable network

Country	Size 5 cliques	Size 4 cliques	Size 3 cliques	Size 2 cliques	Country	Size 5 cliques	Size 4 cliques	Size 3 cliques	Size 2 cliques
UK	1	5	12	22	**EL**	0	0	1	9
FR	1	4	7	24	**IT**	0	0	1	9
DE	1	3	11	24	**CH**	0	0	0	25
SE	1	2	4	25	**CZ**	0	0	0	16
IS	1	2	3	8	**PT**	0	0	0	10
NO	0	2	4	24	**LV**	0	0	0	10
AT	0	2	2	8	**HU**	0	0	0	10
LU	0	1	6	25	**BG**	0	0	0	10
SI	0	1	5	9	**RO**	0	0	0	10
ES	0	1	4	25	**FI**	0	0	0	9
DK	0	1	1	8	**IE**	0	0	0	9
BE	0	0	1	25	**SK**	0	0	0	9
NL	0	0	1	10	**PL**	0	0	0	7

The common characteristic of the optimal networks is that all of them contains 6 free mobility cliques (Table 3.4). In each optimal network, Austria-France-Spain-Norway, Sweden-United Kingdom-Slovenia-Iceland and Germany-Denmark-Luxembourg form free mobility cliques. Belgium, Czech Republic, and Switzerland can only form 2-member cliques in each optimal network.

Table 3.4: Structure of the optimal networks

1^{st} **Clique**	2^{nd} **Clique**	3^{rd} **Clique**	4^{th} **Clique**	5^{th} **Clique**	6^{th} **Clique**
AT (EU15)	**SE** (EU15)	**DK** (EU15)	**BE** (EU15)	**CZ** (EU10)	**CH** (EFTA)
FR (EU15)	**UK** (EU15)	**DE**(EU15)	?	?	?
ES (EU15)	**SI** (EU10)	**LU** (EU15)	⇓	⇓	⇓
NO (EFTA)	**IS** (EFTA)				
			Candidates		
			FI (EU15)		**FI** (EU15)
			EL (EU15)		**EL** (EU15)
			IE (EU15)		**IE** (EU15)
			IT (EU15)		**IT** (EU15)
			NL (EU15)	**NL** (EU15)	**NL** (EU15)
			PT (EU15)	**PT** (EU15)	**PT** (EU15)
			LV (EU10)	**LV** (EU10)	**LV** (EU10)
			HU (EU10)	**HU** (EU10)	**HU** (EU10)
			PL (EU10)		**PL** (EU10)
			SK (EU10)		**SK** (EU10)
			BG (EU10)	**BG** (EU10)	**BG** (EU10)
			RO (EU10)	**RO** (EU10)	**RO** (EU10)

3.4 Concluding Remarks

We observe that implications of our theoretical model for labor mobility network formation within EU and EFTA countries are consistent with some of the observed facts. As a main takeaway from these observations, we can suggest that country specific conditions should be applied that restrict the free labor mobility flows between member countries. The network structure between countries might lead the migration issue to spread to all EU and EFTA countries. We consider extending our analysis with different skills of labors as a promising future work.

CHAPTER 4

EQUILIBRIUM REFINEMENTS FOR THE NETWORK FORMATION GAME

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To understand which networks can emerge when players strategically decide with whom to establish links, a model of network formation needs to specify the process through which players set up links, together with a notion for network equilibrium compatible with this process. We will analyze a normal form game of network formation due to Myerson (1991). All players simultaneously announce the links they wish to form, and a link is formed if and only if there is mutual consent for its formation.

The mutual consent requirement of the Myerson game creates coordination problems. Nash equilibrium does not lead to sharp predictions. The empty network can always be supported by a Nash equilibrium, when nobody announces any link, and in general the game has a multiplicity of Nash equilibria. To address this multiplicity, pairwise-Nash equilibrium is commonly used in the literature.[1] It requires that, on top of the standard Nash equilibrium conditions, any mutually beneficial link be formed at equilibrium[2], without specifying any process through which players might coordinate such a deviation.

The aim of this essay is to redefine pairwise-Nash equilibrium as a non-cooperative refinement. If the concept can be rephrased without referring to any implicit cooperation, then its use in non-cooperative games would be justified.

One thing needs to be cleared before one begins to talk about non-cooperative "equilibrium networks". In this game, there usually exists many pure strategy equilibria that support the same network.[3] So, when we refer to the set, for example, of "Nash equilibrium networks", we mean the set of networks for which there exists a pure strategy Nash equilibrium that leads to that network structure. Hence, the existence of one Nash equilibrium for the network qualifies it as a Nash equilibrium network.

We define a new non-cooperative equilibrium, trial perfect equilibrium. In a trial perfect equilibrium players best respond to trembles of their opponents, where all best responses are given a strictly positive probability and trembles are ordered so that more costly mistakes are made with less or zero probability. Hence it is a

[1] Pairwise-Nash equilibrium was used, among others, in Bloch and Jackson (2007), Calvó-Armengol (2004), Goyal and Joshi (2006), Buechel and Hellmann (2012) and Joshi and Mahmud (2016).

[2] But, this is not demanding robustness to bilateral moves, as pairwise-Nash equilibrium does not allow pairs of players to coordinate fully in their strategies.

[3] Any network, except the complete network and networks where all absent links are beneficial to both parties involved, can be supported by multiple pure strategy Nash equilibria.

non-cooperative equilibrium in the spirit (and an extension) of Myerson's (1978) proper equilibrium and does not presume any coordination between players.

We show that trial perfect equilibria coincide with pairwise-Nash equilibria for network formation games with link-responsive payoffs. This shows that it is unnecessary to refer to any bilateral coordination to eliminate networks where players fail to form mutually beneficial links.

Link responsiveness requires that a change in the network changes the payoffs of the players whose links change. It is generically satisfied by network payoffs with some exogenous parameters (such as a constant marginal link cost).

Section 4.1 introduces the model and describes the network formation game and the equilibrium concepts. The main result is provided in Section 4.2. Section 4.3 concludes with a discussion of our contribution. The proofs are in Appendices.

4.1 The Model

Networks $N = \{1, \ldots, n\}$ is the set of players who may be involved in a network. A network[4] g is a list of pairs of players who are linked to each other. We denote the link between two players i and j by ij, so $ij \in g$ indicates that i and j are linked in the network. Let g^N be the set of all subsets of N of size 2. The network g^N is referred to as the complete network. The set $\mathcal{G} = \left\{g \subseteq g^N\right\}$ denotes the set of all possible networks on N. The set of i's direct links in g is

[4]We adopt the network and link notation from Bloch and Jackson (2006).

$L_i(g) = \{jk \in g : j = i \text{ or } k = i\}$ and $L_i(g^N \backslash g) = \{ij : j \neq i \text{ and } ij \notin g\}$ is the set of i's direct links not in g. That is, $ij \notin g$ is equivalent to $ij \in L_i(g^N \backslash g)$.

We let $g + ij$ denote the network obtained by adding the link ij to the network g and $g - ij$ denote the network obtained by deleting the link ij from the network g. More generally, given $i \in N$, for every collection of links $\ell \subseteq L_i(g)$, $g - \ell$ is the network obtained from g by eliminating all the links in ℓ, while for every collection of links $\ell \subseteq L_i(g^N \backslash g)$, $g + \ell$ is the network obtained from g by adding all the links in ℓ.

Network payoffs A network payoff function is a mapping $u : \mathcal{G} \to \mathbb{R}^N$ that assigns to each network g a payoff $u_i(g)$ for each player $i \in N$.

Link marginal payoffs Let $g \in \mathcal{G}$. For all $i, j \in N$ such that $ij \in g$:

$$m_{ij} u_i(g) = u_i(g) - u_i(g - ij)$$

is the marginal payoff to i from the link ij in g. More generally, consider a set of links $\ell \subseteq L_i(g)$. The joint value to i of ℓ is:

$$m_\ell u_i(g) = u_i(g) - u_i(g - \ell).$$

Consider now some link $ij \notin g$. Then, $m_{ij} u_i(g + ij)$ is the marginal payoff accruing to i from the new link ij being added to g. More generally, consider a collection of i's links absent from g, $\ell \subseteq L_i(g^N \backslash g)$. The joint value to i of these new links added to g is $m_\ell u_i(g+\ell) = u_i(g+\ell) - u_i(g)$.

Definition 1 (link-responsiveness)**.** *The network payoff function u is link-responsive on g if and only if we have $u_i(g+\ell'-\ell) - u_i(g) \neq 0$, for all $i \in N$, and for all $\ell \subseteq L_i(g)$ and $\ell' \subseteq L_i\left(g^N \backslash g\right)$ such that $g+\ell'-\ell \neq g$.*

Link-responsiveness requires that no player is indifferent to a change in his set of direct links, whether due to formation, link removal, or a combination of both.

A positive theory of network formation needs to specify the process through which players set up links, together with a notion for network equilibrium compatible with this process. We formulate a simultaneous move game of network formation due to Myerson (1991), defined originally in the context of cooperative games with communication structures.[5] This game is simple and intuitive, but generally displays a multiplicity of Nash equilibria.

A simultaneous move game of network formation The set of players is N. All players $i \in N$ individually and simultaneously announce the direct links they wish to form. Formally, $S_i = \{0,1\}^n$ is the set of pure strategies available to i and let $s_i = (s_{i1}, \ldots, s_{in}) \in S_i$ with the restriction that $s_{ii} = 0$. Then, $s_{ij} = 1$ if and only if i wants to set up a direct link with $j \neq i$ (and thus $s_{ij} = 0$, otherwise). The game due to Myerson (1991) assumes that mutual consent is needed to create

[5]To quote Myerson: "Now consider a link-formation process in which each player independently writes down a list of players with whom he wants to form a link (...) and the payoff allocation is (...) for the graph that contains a link for every pair of players who have named each other."(p. 448)

a direct link, that is, the link ij is created if and only if $s_{ij}.s_{ji} = 1$.[6]

A pure strategy profile $s = (s_1, \ldots, s_n)$ induces an undirected network $g(s)$ where $ij \in g(s)$ if and only if $s_{ij}.s_{ji} = 1$. The set of pure strategy profiles are denoted by $S = S_1 \times ... \times S_n$ and by $\Sigma = \Sigma_1 \times ... \times \Sigma_n$ the set of mixed strategy profiles, where Σ_i is the set of the mixed strategies available to player i. For $n = 2$, a mixed strategy for a player is simply a binomial distribution, the probability of announcing the single possible link, and the probability of not announcing it. For more players, a mixed strategy profile becomes a multivariate binomial probability distribution. A mixed strategy profile generates a probability distribution over $\mathcal{G}$. Thus, like the result of a pure strategy profile is a single network, the outcome of a mixed strategy profile is a random graph.[7]

For a network $g \in \mathcal{G}$, let $D(g) = \{s \in S | g(s) = g\}$ be the set of pure strategy profiles that induce g. Given $\sigma \in \Sigma$, let $p_\sigma(s)$ be the probability that s is played under the mixed strategy profile σ. Then the probability, $p_\sigma(g)$, that σ induces a network $g \in \mathcal{G}$ is

$$p_\sigma(g) = \sum_{s \in D(g)} p_\sigma(s)$$

[6]Although this is a very simple game, the number of pure strategies of a player, 2^{n-1}, increases exponentially with the number of players. Baron *et al.* (2008) shows that it is NP-hard to check whether there exists a Nash equilibrium that guarantees a minimum payoff to all players.

[7]Jackson and Rogers (2004) deals with random graphs in strategic network formation, though in a different context.

and the expected utility of player i is:

$$Eu_i(\sigma) = \sum_{g \in \mathcal{G}} u_i(g).p_\sigma(g)$$

Pairwise-Nash equilibrium A pure strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a Nash equilibrium of the simultaneous move game of network formation if and only if $u_i(g(s^*)) \geq u_i\left(g\left(s_i, s_{-i}^*\right)\right)$, for all $s_i \in S_i$, $i \in N$. The Nash equilibrium, though, is too weak an equilibrium concept to single out equilibrium networks. For instance, the empty network is always a Nash equilibrium.[8] To remedy this, following Goyal and Joshi (2006), we define *pairwise-Nash equilibrium*[9], which has a coalitional flavor as players are allowed to deviate by pairs.[10] Beyond the standard Nash equilibrium conditions it further requires that any mutually beneficial link be formed at equilibrium. Pairwise-Nash equilibrium networks are robust to bilateral and commonly agreed one-link creation, and to unilateral multi-link severance.

Formally,

Definition 2. *A network $g \in \mathcal{G}$ is a pairwise-Nash equilibrium network with respect to the network payoff function u if and only if there exists a Nash equilibrium strategy profile s^* that supports g, that is, $g = g(s^*)$, and, for all $ij \notin g$, if $m_{ij}u_i(g+ij) > 0$, then $m_{ij}u_j(g+ij) < 0$, for all $i \in N$.*

For a given network payoff function u, we denote by $PN(u)$ the set of pairwise-

[8] When nobody announces any link.

[9] See, also, Calvó-Armengol (2004) for an application of this equilibrium notion.

[10] See Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2005) for alternatives to pairwise-Nash equilibrium that allow for coalitional moves.

Nash equilibrium networks with respect to u.

Trial perfect equilibrium We now define trial perfect equilibrium which requires that players best respond to their opponents trials of, other than equilibrium, best responses. Moreover their costly mistakes, like in proper equilibrium (Myerson 1978), are ordered so that more costly mistakes are made with less probability. The set of trial perfect equilibria, by definition, includes the set of proper equilibria.[11]

Definition 3. *A strategy profile $\sigma \in \Sigma$ is a trial perfect equilibrium if there exists a sequence of strategy profiles $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ with limit σ and a sequence of strictly positive reals $\{\varepsilon_t\}_{t\in\mathbb{N}}$ with limit 0 such that, for all $i \in N$, $s_i', s_i'' \in S_i$, and $t \in \mathbb{N}$:*

(i) $s_i' \in \arg\max_{s_i\in S_i} u_i(s_i, \sigma_{-i}^{\varepsilon_t})$ *implies that* $\sigma_i^{\varepsilon_t}(s_i') \neq 0$, *and*

(ii) $Eu_i(s_i', \sigma_{-i}^{\varepsilon_t}) > Eu_i(s_i'', \sigma_{-i}^{\varepsilon_t})$ *implies that* $\sigma_i^{\varepsilon_t}(s_i'') \leq \varepsilon_t \cdot \sigma_i^{\varepsilon_t}(s_i')$.

A trial perfect equilibrium is the limit of mixed strategies where a positive probability is assigned to all the best responses, but unlike a proper equilibrium, those strategies which are not best responses need not be assigned a positive probability. We call a network $g' \in \mathcal{G}$ a trial perfect equilibrium network, if there exists a pure strategy trial perfect equilibrium $s \in S$ such that $g(s) = g'$. For a given network payoff function u, we denote by $TPE(u)$ the set of trial perfect equilibrium networks with respect to u.

[11] See Calvó-Armengol and İlkılıç (2009) for a characterization of proper equilibria of the Myerson network formation game.

4.2 Result

Theorem 1. *If the network payoff u is link-responsive, then $PN(u) = TPE(u)$.*

Proof. See Appendices. □

The equivalence between pairwise-Nash equilibrium and trial perfect equilibrium qualifies the first as a non-cooperative equilibrium concept. It is attainable without assuming any implicit cooperation between players.

Link-responsiveness is enough to show that a network g is a pairwise-Nash equilibrium network if and only if it is also a trial perfect equilibrium network. We separate the equivalence into two inclusion relations, which are given as Propositions 6 and 7, in Appendices, where the proof of Theorem 1 is. Proposition 6 declares the set of trial perfect equilibrium networks as a subset of pairwise-Nash equilibrium networks.[12] Proposition 7, vice versa.

To prove Proposition 6, first consider a network g which is not a pairwise-Nash equilibrium network, then either, g is not a Nash equilibrium network, or there exists $ij \notin g$, which would have benefited both parties had it been formed. If the first of these conditions hold, then g is not a trial perfect equilibrium network. So assume the first holds and it is the latter that fails to hold. Then, it must be the case that neither i nor j has announced this link. We show that this cannot be a trial perfect equilibrium. In a Nash equilibrium profile, if neither i nor j

[12] Though the technique used in the proof is similar to that of Proposition 3 of Calvó-Armengol and İlkılıç (2009), in fact, the result in this essay is stronger and implies that proposition.

announces the link ij, then for both i and j, there exists a best response, where they announce this link. Hence, there cannot be a sequence of equilibria that converges to this strategy profile, where each player uses all his best responses with positive probability.

To prove Proposition 7 we first define the minimal strategy profile that supports g. This is the profile where players announce only their existing links in g. Then we provide a sequence of profiles. In those profiles all players always announce all their existing links in g. Plus, if a player gains from the formation of a non-existing link, with probabilities that converge to zero, he announces these links.

Next, we index the players from 1 to n. For those links which are not in g due to the fact that the link marginal returns are negative for both parties, we let the lower indexed player involved in such a link announce the link with probabilities that converge to zero. This announcement is not to be reciprocated in a best response by the other party, as the formation of the link would have harmed. Hence, none of the extra announcements incorporated into the converging sequence of equilibria are reciprocated, making the network g the only possible outcome of any realization of the strategy profiles that constitute the sequence.

We show that this sequence satisfies the conditions of the definition of trial perfect equilibrium. Hence the strategy profile it converges is a trial perfect equilibrium. So, any pairwise-Nash equilibrium network can be supported by a trial perfect equilibrium.

4.3 Discussion

Pairwise-Nash equilibria, although a strict subset of Nash equilibria, is not a non-cooperative equilibrium refinement. It is a conceptual drawback to use this notion for a non-cooperative game. We remedy this by defining a non-cooperative equilibrium refinement, trial perfect equilibrium. We show that this new equilibrium notion coincides with pairwise-Nash equilibrium for games of network formation with link responsive payoffs. Adding pairwise-Nash equilibrium (trial perfect equilibrium) to the list of non-cooperative equilibrium concepts justifies its use in non-cooperative analysis of network formation.

Calvó-Armengol and İlkılıç (2009) and this essay introduce mixed strategies to the analysis of the network formation game. Although the results are for pure strategy equilibria, the analysis can not do without mixed strategies. As each mixed strategy profile gives a probability distribution over the set of possible networks, the use of mixed strategies brings into focus the formation of random graphs, which arise naturally via players whose best responses are mixed strategies.

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APPENDICES

A Proofs of Chapter 2

Lemma 1. *Suppose there are n agents on both sides in network g. Agents form complete networks within their types and each agent is connected to two and only two agents of the opposite type. At some equilibria status homophily does not derive value homophily when n is odd.*

Proof. Since n is odd, $\exists k \in \mathbb{Z}_+$ such that $n := 2k + 1$. Note that each agent is connected to two and only two agents of the opposite type, then there exists an onto correspondence from a set N_1 of agents with cardinality k to a set N_2 of agents with cardinality *2k*.

Each agent has $2k + 2$ links in the network, so at an equilibrium necessarily each agent has at least $k + 1$ links. Since agents form a complete network within types (i.e. each has $2k$ links in N_i) there is always two equilibria at which status homophily derives value homophily.

For some $j_0 \in N_2$ there exist $N_1^1, N_1^2 \subset N_1$ such that $|N_1^1| = k$ and $|N_1^2| = k + 1$

where[13]

$$g(N_1^1) = N_1 \bigcup N_2 \setminus \{j_0\}$$

In particular, $\forall k \in N_2 \setminus \{j_0\}\ \exists\ i^1 \in N_1^1$ such that $(i^1, k) \in g$ and i^1 is unique. Similarly, $\forall k \in N_2 \setminus \{j_0\}\ \exists\ i^2 \in N_1^2$ such that $(i^2, k) \in g$ and i^2 is unique.

Since agent j_0 is connected to two agents of the opposite type, these agents must be in N_1^2.

$\exists\ i_0^2, i_1^2 \in N_1^2$ such that $(i_0^2, j_0), (i_1^2, j_0) \in g$ and (i_0^2, i_1^2) is a unique pair. By construction, each agent in $N_2 \setminus \{j_0\}$ is connected to only one agent in N_1^1 and only one agent in N_1^2. So, there exist $i_2^1 \in N_1^1$ and $j_1 \in N_2 \setminus \{j_0\}$ and such that (i_2^1, j_1), $(i_1^2, j_1) \in g$ and $(i_2^1,\ j_1)$ is a unique pair. Similarly, there exist $i_3^2 \in N_1^2$ and $j_2 \in N_2 \setminus \{j_0, j_1\}$ and such that (i_3^2, j_2), $(i_2^1, j_2) \in g$ and $(i_3^2,\ j_2)$ is a unique pair (Figure 4.1).

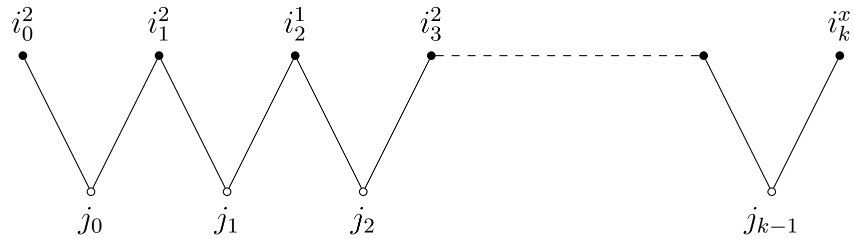

Figure 4.1: Sketch of the proof

Continuing this, we get $\{j_0, j_1, \ldots, j_{k-1}\} \subset N_2$ such that each agent has two links with $\{i_0^2, i_1^2, i_2^1, i_3^2, \ldots, i_k^x\} \subset N_1$ and each agent in $\{i_0^2, i_1^2, i_2^1, i_3^2, \ldots, i_k^x\}$ has at least

[13] g(A) is the set of agents linked with the agents in A.

one link with $\{j_0, j_1, \ldots, j_{k-1}\}$.[14]

Let $D_1 := \{i_0^2, i_1^2, i_2^1, i_3^2, \ldots, i_k^x\}$ and $D_2 := \{j_0, j_1, \ldots, j_{k-1}\}$. Except i_0^2 and i_k^x each agent in D_1 has k+2 links in the subnetwork g_1 of g in which the vertices are the elements of D_1 and D_2. In this subnetwork i_0^2, i_k^x and each agent in D_1 has $k+1$ links.

Also each agent in $N_1 \setminus D_1$ has two links with $N_2 \setminus D_2$ and each agent in $N_2 \setminus D_2$ has at least one link with $N_1 \setminus D_1$. That is, each agent in subnetwork g_2 of g in which the vertices are the element of $(N_1 \bigcup N_2) \setminus (D_1 \bigcup D_2)$ has at least $k+1$ links.

Thus, g is partitioned into two subnetworks g_1 and g_2. A state s where each agent in g_1 adopts opinion $\mathbf{A}(\mathbf{B})$ and each agent in g_2 adopts opinion $\mathbf{B}(\mathbf{A})$ is an equilibrium such that status homophily does not derive value homophily.

$\square$

Proof of Proposition 1.

Lemma 1 tells that for any degree of status homophily, $h \in \mathbb{Z}_+ \setminus \{1\}$, there exists a network g such that status homophily does not derive value homophily at some equilibria. Also when $h = 1$, trivially any state at which for each type half of the same type of agents adopt opinion $\mathbf{A}$ and the rest of the same type of agents adopt opinion $\mathbf{B}$ is an equilibrium such that status homophily does not derive value homophily.

[14] x equals 2 if k is odd, 1 otherwise.

Let h be a degree of status homophily. There exists $y \in \mathbb{Z}_+$ such that $y \in [h, h+1)$, then by Lemma 1 there exists a network g such that we may not end up with value homophily at some equilibria. Thus h is not a threshold degree of status homophily, that is, for any network with a degree of status homophily larger than h, status homophily derives value homophily at every equilibria. $\square$

Lemma 2. *Each monotone agent of the same type adopts the same opinion at every equilibria.*

Proof. Suppose there is an equilibrium at which there exists a monotone agent i who adopts the opposite opinion with the other monotone agents of his type. So, he has at least same number of links who adopt different opinions. But then, each of other monotone agents will have more links with the agents of the opposite opinion than they have with the agents of the same opinion, i.e. this state is not an equilibrium. Thus, each monotone agents must adopt the same opinion at every equilibrium. $\square$

Proof of Proposition 2.

Let D^i_j be a subset of non-monotone agents of Type i where $j = 1, 2$ such that $D^i_1 \cup D^i_2$ is the set of agents of Type i. Let n^i_j be the number of agents in D^i_j where $i = 1, 2$ and $j = 1, 2$.

Without loss of generality (WLOG) let $m_1 > 0$ and $m_2 = 0$.

Case 1: $m_1 \leq p$

Let $n^i_j = p$ for any $i, j \in \{1, 2\}$. Let g be a network such that each type of agents forms a complete network. Also let agents in $D_1^1 \cup D_1^2$ form a complete network and agents in $D_2^1 \cup D_2^2$ form a complete network. Let s be a state such that agents in $D_1^1 \cup D_1^2$ and agents in $D_2^1 \cup D_2^2$ adopt the opposite opinion. So, s is an equilibrium at which status homophily does not derive value homophily.

Case 2: $m_1 > p$

Let s be a state such that agents in $D_1^1 \cup D_1^2$ and agents in $D_2^1 \cup D_2^2$ adopt the opposite opinion.

Case 2-a: (WLOG) $n_1^1 > n_2^1$

Then by Lemma 2 and conformity assumption monotone agents of Type 1 will adopt the same opinion with the agents in D_1^1. Note that agents in D_2^1 have at most p external links. So, agents in D_2^1 will have more links with the agents of the opposite opinion than they have with the agents of the same opinion, i.e. s is not an equilibrium.

Case 2-b: $n_1^1 = n_2^1$

Again by Lemma 2 and conformity assumption monotone agents of Type 1 will adopt either D_1^1's opinion or D_2^1's opinion. In both cases, the agents who have the different opinion with the monotone agents will have more links with the agents of the opposite opinion, i.e. s is not an equilibrium.

Therefore, when $m_1 > p$ at every equilibria status homophily derives value homophily. $\square$

Lemma 1*. *See Lemma 1.*

Proof. Note that for some $n^* \in \mathbb{Z}_+$ such that $\frac{n^*}{n-1+p} \leq \frac{1}{2}$, $u_i^A(n^*) > u_i^B(n^*)$, i.e. agent i adopts opinion **A**. Let g be a network in which $n^* - 1$ agents of Type 1 are only connected to some $n^* - 1$ agents of Type 2. So, each agent in this subnetwork has n^* links, i.e. they all can adopt opinion **A**. Also, let the rest of Type 1 or Type 2 agents have no links with these agents who adopt opinion **A** and they all adopt opinion **B**. Such a state s is an equilibrium, since each agent who adopts opinion **B** has $n^* - 1$ links who adopts opinion **A** and each agent who adopts opinion **A** has n^* links who adopts opinion **A**. Therefore, this will result a network such that status homophily does not derive value homophily at some equilibrium s. □

Proof of Proposition 1*.

One can prove using the same logic of the proof of Proposition 1 and Lemma 1*. □

Lemma 2*. *See Lemma 2.*

Proof. Let s be an equilibrium such that there exist two monotone agents who adopt opposite opinions. So, by construction of the model, the monotone agent who adopts opinion **A** has at least n^* links who adopts opinion **A**. Then, the rest of the monotone agents will also adopt opinion **A** as a result of the decision criteria. So, s cannot be an equilibrium (Contradiction). Therefore, every monotone agents adopt the same opinion in every equilibria. □

Proof of Proposition 2*.

WLOG let $m_1 > 0$ and $m_2 = 0$. Suppose $m_1 > p$. By Lemma 2* every monotone agents adopt the same opinion. Suppose there is a group of Type 1 agents who adopt opinion **A**. By the decision criteria, the number of this group can be at least $n^* - p + 1$ since each Type 1 agent in this group can have p links with Type 2 agents who adopt opinion **A**. So, with p external links each agent in that group has n^* links. For each monotone agent to adopt opinion **B**, it must be the case that:

$$(n^* - p + 1) + (m_1 - 1) < n^*$$

$$\Downarrow$$

$$n^* - p + m_1 < n^*$$

But since $m_1 > p$, this cannot be the case, i.e. each monotone agent adopts opinion **A**. Then each agent who adopt opinion **B** has $(n^* - p + 1 + m_1) > n^*$ links who adopt opinion **A**. Hence they will also adopt opinion **A**. Therefore, status homophily derives value homophily at every equilibria. $\square$

B Proofs of Chapter 3

Proof of Proposition 3.

Under **A1**, for a given action profile s, let g be the induced network structure with the free labor mobility cliques $g^{C_1}, \ldots, g^{C_k}$. For any $i \in N$,

$$\begin{aligned}
POLF_i(g) &= Pii(g) \times PRLF_i + \Sigma_{j \in N_i(g)}(Pij(g) \times PRLF_j) \\
&\text{A1} \Downarrow \\
&= PRLF_i \times [Pii(g) + \Sigma_{j \in N_i(g)} Pij(g)] \\
&= PRLF_i
\end{aligned}$$

That is, $AR_i(g) = 0$. So, for any $i \in N(g)$, $u_i(s) = \overline{AR}_i$. Similarly, for any admissible coalition $C \subset N_i(g)$ with any action profile s' and for any $i \in C$, $u_i(s') = \overline{AR}_i$ where $s' = (s'_C, s_{-C})$. So, there is no admissible coalition $C \subset N_i(g)$ with an action profile s' such that $u_i(s') \geq u_i(s)$ for all $i \in C$ and for some $i \in C$, $u_i(s') > u_i(s)$, i.e. action profile s is a strong Nash equilibrium. □

Proof of Proposition 4.

$(\Leftarrow)$

Suppose an action profile s induces a network g with the free labor mobility cliques $g^{C_1}, \ldots, g^{C_k}$ such that for any $l \in \{1, \ldots, k\}$, $|C_l| \leq 2$ and for any $i \in C_l$, $u_i(s) \geq 0$. Since there is no non-singleton admissible coalition for any C_l the action profile s is a strong Nash equilibrium.

($\Rightarrow$)

Under **A2-A3**, let's assume an action profile s is a strong Nash equilibrium. Assume by contradiction, action profile s induces a network g with free labor mobility cliques $g^{C_1}, \ldots, g^{C_k}$, such that there exists $l \in \{1, \ldots, k\}$, $|C_l| > 2$. Note that by definition for every $l \in \{1, \ldots, k\}$ and for any $i \in C_l$ we have $u_i(s) \geq 0$.

Let $\tilde{j}$ denotes the country in C that has the largest labor force among the countries in C and let $\tilde{C} = C \setminus \{\tilde{j}\}$. By **A3**, country $\tilde{j}$ is unique.

Claim: For a given action profile s and for any $C \subset N$, we have for every $i \in \tilde{C}$, $u_i(s') > u_i(s)$ where $s' = (s'_{\tilde{C}}, s_{-\tilde{C}})$.

Proof of Claim.

For a given action profile s and for any $C \subset N$, let $|C \setminus \{i\}| = m$ and let for every $i, j, k \in C$,

$$P_{ij}(N_i(g^C)) = P_{ik}(N_i(g^C)) = P(m) \tag{4.1}$$

$$P_{ij}(N_i(g^{\tilde{C}})) = P_{ik}(N_i(g^{\tilde{C}})) = P(m-1) \tag{4.2}$$

$$P_{ii}(N_i(g^C)) = P_{jj}(N_i(g^C)) = 1 - m \times P(m) \tag{4.3}$$

$$P_{ii}(N_i(g^{\tilde{C}})) = P_{jj}(N_j(g^{\tilde{C}})) = 1 - (m-1) \times P(m-1) \tag{4.4}$$

Want to show: For every $i \in \tilde{C}$ $u_i(s') > u_i(s)$.

Note that,

$$
\begin{aligned}
u_i(s') &= \overline{AR}_i - AR_i(g^{\tilde{C}}) && (4.5)\\
u_i(s) &= \overline{AR}_i - AR_i(g^{C}) && (4.6)
\end{aligned}
$$

Using Equations (4.5) and (4.6),

$$
\begin{aligned}
u_i(s') - u_i(s) &= AR_i(g^C) - AR_i(g^{\tilde{C}})\\
&= \frac{POLF_i(g^C) - PRLF_i}{POLF_i(g^C)} - \frac{POLF_i(g^{\tilde{C}}) - PRLF_i}{POLF_i(g^{\tilde{C}})}\\
&= \frac{PRLF_i \times [POLF_i(g^C) - POLF_i(g^{\tilde{C}})]}{POLF_i(g^C) \times POLF_i(g^{\tilde{C}})}
\end{aligned}
$$

Thus,

$$u_i(s') - u_i(s) > 0 \Leftrightarrow POLF_i(g^C) - POLF_i(g^{\tilde{C}}) > 0$$

From Equations (4.1)-(4.4), we have

$$
\begin{aligned}
POLF_i(g^C) - POLF_i(g^{\tilde{C}}) &= \frac{P(m) - P(m-1)}{P(m-1)} \times [\Sigma_{j \in C} PRLF_j - (m+1) \times PRLF_i] + PRLF_{\tilde{j}} - PRLF_i\\
&= -P(m) \times [\Sigma_{j \in C} PRLF_j - (m+1) \times PRLF_i] + PRLF_{\tilde{j}} - PRLF_i\\
&= -P(m) \times [\Sigma_{j \in C} PRLF_j - (m+1) \times PRLF_i] + PRLF_{\tilde{j}} - PRLF_i\\
&= -P(m) \times (m+1) \times [\overline{PRLF}_C - PRLF_i] + PRLF_{\tilde{j}} - PRLF_i
\end{aligned}
$$

where $\overline{PRLF}_C$ is the mean of the pre-migration labor force of the countries in C.

Note that if $P_{ii}(N) = 0$, then for any $j \in C \setminus \{i\}$, $P_{ij}(N) = \frac{1}{m}$.

So, for any $m \in \mathbb{Z}^+$ $P(m) \leq \frac{1}{m}$. This implies;

$$\begin{aligned}
POLF_i(g^C) - POLF_i(g^{\tilde{C}}) &> -\frac{m+1}{m} \times [\overline{PRLF}_C - PRLF_i] + PRLF_{\tilde{j}} - PRLF_i \\
&= -\frac{\overline{PRLF}_C - PRLF_i}{m} + PRLF_{\tilde{j}} - \overline{PRLF}_C \\
&= -\frac{\overline{PRLF}_C - PRLF_i + m \times PRLF_{\tilde{j}} - m \times \overline{PRLF}_C}{m} \\
&= \frac{PRLF_i + m \times PRLF_{\tilde{j}} - \Sigma_{j \in C} PRLF_j}{m} \\
&= \frac{m \times PRLF_{\tilde{j}} - \Sigma_{j \in C \setminus \{i\}} PRLF_j}{m} \\
&> 0
\end{aligned}$$

Hence, for a given action profile s and for any $C \subset N$, we have for every $i \in \tilde{C}$, $u_i(s') > u_i(s)$ where $s' = (s'_{\tilde{C}}, s_{-\tilde{C}})$. □

Therefore, for an action profile s which induces a network g with free labor mobility cliques $g^{C_1}, \ldots, g^{C_k}$, if there exists $l \in \{1, \ldots, k\}$ such that $|C_l| > 2$, then by the claim, action profile s cannot be a strong Nash equilibrium (Contradiction).

□

Proof of Proposition 5.

($\Rightarrow$) This part trivially holds.

($\Leftarrow$) Let for a given action profile s with an induced network structure g, there exists a coalition $C \subset N$ such that $|C| \geq 2$ and for all $i \in C$, $u_i(s) \geq 0$. With out loss of generality, let for all $j \in N \setminus C$, $N_j(g) = \emptyset$, that is, every $j \in N \setminus C$ is isolated country.

Case 1: There is no admissible coalition $C' \subset C$ with an action profile s' such that for every $j \in C'$, $u_i(s') > u_i(s)$, i.e. s is a strong Nash equilibrium leading to stable network with free mobility cliques containing at least two members.

Case 2: There exists an admissible coalition $C' \subset C$ with an action profile s' such that for every $j \in C'$, $u_i(s') > u_i(s)$.

Case 2a: s' is a strong Nash equilibrium leading to stable network with free mobility cliques containing at least two members as in Case 1.

Case 2b: s' is not a strong Nash equilibrium.

This can continue until there exists $C'' \subset C$ with an action profile s'' such that $|C''| = 2$ and for every $j \in C''$, $u_i(s'') > 0$, i.e. s'' is a strong Nash equilibrium leading to stable network with free mobility cliques containing at least two members. $\square$

C Proofs of Chapter 4

Proposition 6. *If the network payoff u is link-responsive, then $TPE(u) \subseteq PN(u)$.*

Proof. Let u be link-responsive. We show that $g \notin PN(u)$ implies that $g \notin TPE(u)$.

If g^* is not a Nash equilibrium network, then $g \notin PN(u)$ and $g \notin TPE(u)$. Let g^* be a Nash equilibrium outcome of the simultaneous move game of network formation such that $m_{ij}u_i(g^* + ij) > 0$ and $m_{ij}u_j(g^* + ij) > 0$, for some $ij \notin g^*$. Then, $g^* \notin PN(u)$. Suppose that $g^* \in TPE(u)$, and let s^* be a pure strategy trial perfect equilibrium that supports g^*. Then, $g^* = g(s^*)$. Let $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ be a sequence of $\varepsilon-$trial equilibria such that $\lim_{t\to+\infty} \sigma^{\varepsilon_t}(s^*) = 1$.

Given that s^* is also a Nash equilibrium strategy and that $ij \notin g^*$, necessarily, $s^*_{ij} = s^*_{ji} = 0$.

As $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ is a sequence of $\varepsilon-$trial equilibria, for all $t \in \mathbb{N}$, either, there exists $s_i \in S_i$ such that $s_{ij} = 1$ and $\sigma_i^{\varepsilon_t}(s_i) > 0$, or there exists $s_j \in S_j$ such that $s_{ji} = 1$ and $\sigma_j^{\varepsilon_t}(s_j) > 0$. Given a $t \in \mathbb{N}$, w.l.o.g., assume the latter holds.

For all $j \neq i$, define $e(ij) = (0, ..., s_{ij} = 1, 0, ..., 0)$. With the pure strategy $e(ij)$, player i only announces the link with j. Let $s'_i = s^*_i \vee e(ij)$. With s'_i, player i announces exactly the same links announced in the pure equilibrium strategy s^*_i plus an extra link with player j. This extra link is not reciprocated by player j

in s^*.

For all $t \in \mathbb{N}$, define:

$$\Delta_i(s_i', s_i^*; \sigma_{-i}^{\varepsilon_t}) = Eu_i(g(s_i', \sigma_{-i}^{\varepsilon_t})) - Eu_i(g(s_i^*, \sigma_{-i}^{\varepsilon_t})) = \sum_{\widetilde{s}_{-i} \in S_{-i}} \sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i}).\Delta_i(s_i', s_i^*; \widetilde{s}_{-i}), \tag{4.7}$$

where

$$\Delta_i(s_i', s_i^*; \widetilde{s}_{-i}) = u_i(g(s_i', \widetilde{s}_{-i})) - u_i(g(s_i^*, \widetilde{s}_{-i})).$$

For all $\widetilde{s}_{-i}$ such that $\widetilde{s}_{ji} = 0$, we have $g(s_i', \widetilde{s}_{-i}) = g(s_i^*, \widetilde{s}_{-i})$, and $\Delta_i(s_i', s_i^*; \widetilde{s}_{-i}) = 0$. Therefore,

$$\Delta_i(s_i', s_i^*; \sigma_{-i}^{\varepsilon_t}) = \sum_{\widetilde{s}_{-i} \in S_{-i} \,:\, \widetilde{s}_{ji}=1} \sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i}).\Delta_i(s_i', s_i^*; \widetilde{s}_{-i}).$$

Let $\widetilde{s}_{-i} \in S_{-i}$ such that $\widetilde{s}_{ji} = 1$. Define $\widetilde{g} = g(s_i^*, \widetilde{s}_{-i})$. Note that $ij \notin \widetilde{g}$, and that $g(s_i', \widetilde{s}_{-i}) = \widetilde{g} + ij$. Also, $s_{ik} = 0$ implies that $ik \notin \widetilde{g}$. Define

$$\mathcal{G}(s_i^*) = \{g \in \mathcal{G} : s_{ik}^* = 0 \Rightarrow g_{ik} = 0\}.$$

It is readily checked that

$$\mathcal{G}(s_i^*) = \{g(s_i^*, \widetilde{s}_{-i}) : \widetilde{s}_{-i} \in S_{-i}, \widetilde{s}_{ji} = 1\}.$$

Therefore, we can write:

$$\Delta_i(s_i', s_i^*; \sigma_{-i}^{\varepsilon_t}) = \sum_{\widetilde{g} \in \mathcal{G}(s_i^*)} \mu_{\varepsilon_t}(\widetilde{g}).m_{ij}u_i(\widetilde{g} + ij),$$

where

$$\mu_{\varepsilon_t}(\widetilde{g}) = \sum_{\substack{\widetilde{s}_{-i} \in S_{-i} \,:\, \widetilde{s}_{ji}=1 \\ g(s_i^*, \widetilde{s}_{-i}) = \widetilde{g}}} \sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i}).$$

Given that $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ be a sequence of $\varepsilon-$trial equilibria that converges to s^*, there exists $T \in \mathbb{N}$ such that, for all $t \geq T$, $\mu_{\varepsilon_t}(g^*) > 0$. Therefore, $\Delta_i(s_i', s_i^*; \sigma_{-i}^{\varepsilon_t}) > 0$ is equivalent to

$$m_{ij}u_i(g^* + ij) + \sum_{\widetilde{g} \in \mathcal{G}(s_i^*)\ ,\ \widetilde{g} \neq g^*} \frac{\mu_{\varepsilon_t}(\widetilde{g})}{\mu_{\varepsilon_t}(g^*)} . m_{ij}u_i(\widetilde{g} + ij) > 0.$$

Since $\Delta_i(s_i', s_i^*; \sigma_{-i}^{\varepsilon_t})$ is continuous in $\sigma_{-i}^{\varepsilon_t}$, and given that $m_{ij}u_i(g^* + ij) > 0$, it suffices to show that $\lim_{t\to+\infty} \mu_{\varepsilon_t}(\widetilde{g})/\mu_{\varepsilon_t}(g^*) = 0$, for all $\widetilde{g} \in \mathcal{G}(s_i^*)$, for $\widetilde{g} \neq g^*$. Note that $\lim_{t\to+\infty} \sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i}) = 0$, for all $\widetilde{s}_{-i} \in S_{-i}$ such that $\widetilde{s}_{ji} = 1$. Therefore, $\lim_{t\to+\infty} \mu_{\varepsilon_t}(\widetilde{g}) = 0$, for all $\widetilde{g} \in \mathcal{G}(s_i^*)$, including $\widetilde{g} = g^*$.

Establishing that

$$\lim_{t\to+\infty} \frac{\mu_{\varepsilon_t}(\widetilde{g})}{\mu_{\varepsilon_t}(g^*)} = 0, \text{ for all } \widetilde{g} \in \mathcal{G}(s_i^*),\ \widetilde{g} \neq g^*,$$

is thus equivalent to showing that the rate of convergence of $\mu_{\varepsilon_t}(\widetilde{g})$, $\widetilde{g} \neq g^*$ is at least one order of magnitude higher than that of $\mu_{\varepsilon_t}(g^*)$. This will be implied by the definition of an $\varepsilon-$trial equilibrium, as detailed below.

For each player $k \in N$, we partition the strategy set S_k into two disjoint sets S_k^+

and S_k^- defined as follows:

$$\begin{cases} S_k^+ = \{s_k \in S_k : u_k(g(s_k, s_{-k}^*)) \geq u_k(g^*)\} \\ S_k^- = \{s_k \in S_k : u_k(g(s_k, s_{-k}^*)) < u_k(g^*)\} \end{cases}.$$

It is plain that $S_k = S_k^+ \cup S_k^-$ and that $S_k^+ \cap S_k^- = \emptyset$. Given that u is link-responsive together with the fact that s^* is a Nash equilibrium strategy supporting g^* implies that $g(s_k', s_{-k}^*) = g^*$, for all $s_k' \in S_k^+$. Moreover, as $\lim_{t \to +\infty} \sigma^{\varepsilon_t} = s^*$, and given that each player's expected payoff is continuous in the vector of other players' mixed strategies, there exists some t_k such that, for all $t \geq t_k$, we have $u_k(g(s_k^+, \sigma_{-k}^{\varepsilon_t})) > u_k(g(s_k^-, \sigma_{-k}^{\varepsilon_t}))$, for all $s_k^+ \in S_k^+$ and $s_k^- \in S_k^-$. Given that $\{\sigma^{\varepsilon_t}\}_{t \in \mathbb{N}}$ is a sequence of ε_t-trial equilibria, this implies that, for all $t \geq t_k$, $s_k^+ \in S_k^+$ and $s_k^- \in S_k^-$ we have:

$$\sigma_k^{\varepsilon_t}(s_k^-) \leq \varepsilon_t . \sigma_k^{\varepsilon_t}(s_k^+).$$

Note, also, that $s_j' \in S_j^+$.

We assumed w.l.o.g that there exists $s_j \in S_j$ such that $s_{ji} = 1$ and $\sigma_j^{\varepsilon_t}(s_j) > 0$. Now, let's show that there exists some $T \in \mathbb{N}$ such that, for some $t \geq T$, there exists $s_j^+ \in S_j^+$ such that $s_{ji}^+ = 1$ and $\sigma_j^{\varepsilon_t}(s_j^+) > 0$. Assume not, then there exists $s_j^- \in S_j^-$such that $s_{ji}^- = 1$ and $\sigma_j^{\varepsilon_t}(s_j^-) \neq 0$, and for all $s_j^+ \in S_j^+$ such that $s_{ji}^+ = 1$ and $\sigma_j^{\varepsilon_t}(s_j^+) = 0$. But this contradicts with the result above that there exists some t_j such that, for all $t \geq t_j$, $s_k^+ \in S_k^+$ and $s_k^- \in S_k^-$ we have $\sigma_j^{\varepsilon_t}(s_j^-) \leq \varepsilon_t . \sigma_j^{\varepsilon_t}(s_j^+)$.

Hence, there exists $s_j^+ \in S_j^+$ such that $s_{ji}^+ = 1$ and $\sigma_j^{\varepsilon_t}(s_j^+) > 0$. Fix $\overline{s}_j$, as the strategy such that $s^+_{ji} = 1$ implies $\sigma^{\varepsilon_t}_j(\overline{s}_j) \geqslant \sigma^{\varepsilon_t}_j(s^+_j)$. The strategy $\overline{s}_j$ is well defined as S^+_j is finite.

Define,

$$\mathcal{G}^{-1}(g) = \{(s^*_i, \widetilde{s}_{-i}) = s \in S : g(s) \in \mathcal{G}(s^*_i)\},$$

as the set of strategy profiles that support the networks in $\mathcal{G}(s^*_i)$.

We now define

$$\begin{aligned}
&\mathcal{G}^{-1}_1(g) = \{(s^*_i, \widetilde{s}_{-i}) = s \in S : g(s) = g^*\} \\
&\mathcal{G}^{-1}_2(g) = \{(s^*_i, \widetilde{s}_{-i}) = s \in S : s = (\widetilde{s}_j, s^*_{-j}),\ s^*_{-j} \in S_j, \widetilde{s}_{ji} = 1,\ \text{and } g(s) \neq g^*\} \\
&\mathcal{G}^{-1}_3(g) = \{(s^*_i, \widetilde{s}_{-i}) = s \in S : s = (s^*_i, \widetilde{s}_{-i}),\ \widetilde{s}_{-i} \in S_{-i}, \widetilde{s}_{ji} = 1,\ \widetilde{s}_k \neq s^*_k \text{ for some } k \neq j \\
&\quad \text{and } g(s) \neq g^*\}
\end{aligned}$$

In words, the profiles in $\mathcal{G}^{-1}_1(g)$ always lead to g^*, where only player j makes a mistake (including always the announcement of the link ij, in particular $(\overline{s}_j, s^*_{-j}) \in \mathcal{G}^{-1}_1(g)$), whereas the profiles in $\mathcal{G}^{-1}_2(g)$ are the ones where only player j makes a mistake, but this mistake changes the network structure, and $\mathcal{G}^{-1}_3(g)$ corresponds to the set of profiles where additional mistakes by at least one other player is committed. Clearly, $\mathcal{G}^{-1}(g) = \mathcal{G}^{-1}_1(g) \cup \mathcal{G}^{-1}_2(g) \cup \mathcal{G}^{-1}_3(g)$.

But, for all $\widetilde{s}_j \in S_j$ such that $\widetilde{s} = (\widetilde{s}_j, s^*_{-j}) \in \mathcal{G}^{-1}_2(g)$, necessarily, $\widetilde{s}_j \in S^-_j$ (since s^* is a Nash equilibrium strategy), implying in turn that $\sigma^{\varepsilon_t}_j(\widetilde{s}_j) \leq \varepsilon_t.\sigma^{\varepsilon_t}_j(\overline{s}_j)$, for all $t \geq t_j$. Therefore, for all $t \geq t_j$, we have:

$$\sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i}) \leq \varepsilon_t.\sigma^{\varepsilon_t}_{-i}(\overline{s}_j, s^*_{-i-j}).$$

Hence, for all $\widetilde{s} \in \mathcal{G}_2^{-1}(g)$, $\lim_{t\to+\infty} \frac{\sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i})}{\sigma_j^{\varepsilon_t}(\overline{s}_j, s^*_{-i-j})} = 0$.

Let now $\widetilde{s} \in \mathcal{G}_3^{-1}(g)$. Define $L = \{k \neq j : \widetilde{s}_k \neq s^*_k\}$. By definition, $L \neq \emptyset$. Now,

$$\sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i}) = \sigma_j^{\varepsilon_t}(\widetilde{s}_j).\sigma_L^{\varepsilon_t}(\widetilde{s}_L).\sigma_{-i-j-L}^{\varepsilon_t}(s^*_{-i-j-L}),$$

and, thus,

$$\begin{aligned}\lim_{t\to+\infty} \frac{\sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i})}{\sigma_{-i}^{\varepsilon_t}(\overline{s}_j, s^*_{-i-j})} &= \lim_{t\to+\infty} \frac{\sigma_j^{\varepsilon_t}(\widetilde{s}_j).\sigma_L^{\varepsilon_t}(\widetilde{s}_L).\sigma_{-i-j-L}^{\varepsilon_t}(s^*_{-i-j-L})}{\sigma_j^{\varepsilon_t}(\overline{s}_j).\sigma_L^{\varepsilon_t}(s^*_L).\sigma_{-i-j-L}^{\varepsilon_t}(s^*_{-i-j-L})} \\ &= \lim_{t\to+\infty} \frac{\sigma_j^{\varepsilon_t}(\widetilde{s}_j)}{\sigma_j^{\varepsilon_t}(\overline{s}_j)}.\lim_{t\to+\infty} \frac{\sigma_L^{\varepsilon_t}(\widetilde{s}_L)}{\sigma_L^{\varepsilon_t}(s^*_L)}\end{aligned}$$

Now, since for all $t \geq t_j$, $\sigma_j^{\varepsilon_t}(\overline{s}_j) \geqslant \sigma_j^{\varepsilon_t}(\widetilde{s}_j)$ if $\widetilde{s}_j \in S_j^+$ and $\sigma_j^{\varepsilon_t}(\widetilde{s}_j) \leq \varepsilon_t.\sigma_j^{\varepsilon_t}(\overline{s}_j)$ if $\widetilde{s}_j \in S_j^-$)

$$\lim_{t\to+\infty} \frac{\sigma_j^{\varepsilon_t}(\widetilde{s}_j)}{\sigma_j^{\varepsilon_t}(\overline{s}_j)} \leqslant 1$$

and since $\lim_{t\to+\infty} \sigma_L^{\varepsilon_t}(\widetilde{s}_L) = 0$ and $\lim_{t\to+\infty} \sigma_L^{\varepsilon_t}(s^*_L) = 1$

$$\lim_{t\to+\infty} \frac{\sigma_L^{\varepsilon_t}(\widetilde{s}_L)}{\sigma_L^{\varepsilon_t}(s^*_L)} = 0,$$

then

$$\lim_{t\to+\infty} \frac{\sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i})}{\sigma_{-i}^{\varepsilon_t}(\overline{s}_j, s^*_{-i-j})} = 0.$$

Then, since there exists only a finite set of strategy profiles $s \in S$ that supports a $g \in \mathcal{G}$, and for $\widetilde{g} \in \mathcal{G}(s_i^*)$, $\mu_{\varepsilon_t}(\widetilde{g}) = \sum_{\substack{\widetilde{s}_{-i} \in S_{-i} \,:\, \widetilde{s}_{ji}=1 \\ g(s_i^*, \widetilde{s}_{-i}) = \widetilde{g}}} \sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i})$, $\lim_{t \to +\infty} \frac{\mu_{\varepsilon_t}(\widetilde{g})}{\mu_{\varepsilon_t}(g^*)} = 0$, for all $\widetilde{g} \in \mathcal{G}(s_i^*), \widetilde{g} \neq g^*$.

But then, given that σ^{ε_t} is an ε_t-trial equilibrium, there exists some $T \in \mathbb{N}$, such that $\sigma_i^{\varepsilon_t}(s_i^*) \leq \varepsilon_t.\sigma_i^{\varepsilon_t}(s_i')$, for all $t \geq T$, implying that $\lim_{t \to +\infty} \sigma_i^{\varepsilon_t}(s_i^*) \neq 1$, which is a contradiction. □

Proposition 7. *If the network payoff u is link-responsive, then $PN(u) \subseteq TPE(u)$.*

Proof. Let u be link-responsive. Let $g^* \in PN(u)$, let $s^0 \in S$ be a strategy that supports g^*, that is $g^* = g(s^0)$, such that $ij \notin g^*$ implies $s_{ij}^0 = s_{ji}^0 = 0$. As g^* is a pairwise-Nash equilibrium network, s^0 is a Nash equilibrium.

Fix a labeling of players with positive integers, from 1 to n.

For each $i \in N$, define,

$$\begin{aligned} S_i(s^0) &= \{s_i \in S_i : \text{ for } j \in N,\, j \neq i, [s_{ij}^0 = 1 \Rightarrow s_{ij} = 1] \\ \text{and } [[m_{ij}u_i(g^* + ij) &< 0 \text{ and } m_{ij}u_j(g^* + ij) > 0] \text{ implies } s_{ij} = 0] \\ \text{and } [[m_{ij}u_i(g^* + ij) &< 0 \text{ and } m_{ij}u_j(g^* + ij) < 0 \text{ and } j < i] \text{ implies } s_{ij} = 0]\} \end{aligned}$$

Define, $\{\sigma^{\varepsilon_t}\}_{t \in \mathbb{N}}$, so that, for all $i \in N$:

(i) $\sigma_i^{\varepsilon_t}(s_i^0) = 1 - (\#S_i(s^0) - 1).\varepsilon_t$, and

(ii) for $s_i \in S_i(s^0)$, $s_i \neq s_i^0$, $\sigma_i^{\varepsilon_t}(s_i) = \varepsilon_t$.

As there exists only a finite number of strategies in $S_i(s^0)$, the above sequence of strategies is well-defined.

Now, let's show that $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ has a subsequence of $\varepsilon-$trial equilibria that converges to s^0.

By definition, as $\varepsilon_t \to 0$, $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ converges to s^0.

For $g \in \mathcal{G}$, given a mixed strategy profile σ, define,

$$\mu(g,\sigma) = \sum_{\substack{s\in S \\ g(s)=g}} \sigma(s),$$

as the probability of g being formed when σ is played.

Then, by definition, for all $t \in \mathbb{N}$, $\mu(g^*, \sigma^{\varepsilon_t}) = 1$.

To show that $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ has a subsequence of $\varepsilon-$trial equilibria that converges to s^0, we will establish that there exists $T \in \mathbb{N}$ such that for all $t \geqslant T$, for all $i \in N$, $s_i \notin S_i(s^0)$, implies $Eu_i(g(s_i, \sigma_{-i}^{\varepsilon_t})) - Eu_i(g(s^0)) < 0$.

Take $i \in N$, take $s_i \notin S_i(s^0)$, then:

- (i) there exists $j \in N$ such that $s_{ij}^0 = 1$ and $s_{ij} = 0$, or
- (ii) there exists $j \in N$ such that $m_{ij}u_i(g^* + ij) < 0$ and $m_{ij}u_j(g^* + ij) > 0$ and $s_{ij} = 1$, or
- (iii) there exists $j \in N$ such that $j < i$ and $m_{ij}u_i(g^* + ij) < 0$ and $m_{ij}u_j(g^* + ij) < 0$ and $s_{ij} = 1$.

If (i) holds, then $s_i \in S_i^-$, as $Eu_i(g(s_i, \sigma_{-i}^{\varepsilon_t}))$ is continuous in $\sigma_{-i}^{\varepsilon_t}$, there exists $T \in \mathbb{N}$ such that for all $t \geqslant T$, $Eu_i(g(s_i, \sigma_{-i}^{\varepsilon_t})) - Eu_i(g(s^0)) < 0$, and we are done.

Suppose (i) does not hold, then there exist $\{j_1, ..., j_l\} \subseteq N$ such that, for all $j_p \in \{j_1, ..., j_l\}$ there exists $s_{j_p} \in S_{j_p}$, $s_{j_p i} = 1$, $\sigma_{j_p}^{\varepsilon_t}(s_{j_p}) = \varepsilon_t$ and $m_{ij} u_i(g^* + ij) < 0$.

For this $\{j_1, ..., j_l\} \subseteq N$, let:

$$\begin{aligned}
G_0 &= \{g^*\}, \\
G_1 &= \{g \in \mathcal{G} : g = g^* + ij_p, \text{ for some } j_p \in \{j_1, ..., j_l\}\}, \\
G_2 &= \{g \in \mathcal{G} : g = g^* + ij_p + ij_q, \text{ for some } j_p, j_q \in \{j_1, ..., j_l\}, j_p \neq j_q\}, \\
&\quad ... \\
G_l &= \{g \in \mathcal{G} : g = g^* + ij_1 + ... + ij_l\}.
\end{aligned}$$

Then, for $p \in \{1, ..., l\}$, for $g \in G_p$, $\mu(g, (s_i, \sigma_{-i}^{\varepsilon_t})) = \varepsilon_t^p.(1 - \varepsilon_t)^{l-p}$. Hence,

$$\begin{aligned}
Eu_i(g(s_i, \sigma_{-i}^{\varepsilon_t})) - Eu_i(g(s^0)) &= \sum_{g \in G_0 \cup ... \cup G_l} \mu(g, (s_i, \sigma_{-i}^{\varepsilon_t})).(u_i(g) - u_i(g^*)) \\
&= \sum_{g \in G_1 \cup ... \cup G_l} \mu(g, (s_i, \sigma_{-i}^{\varepsilon_t})).(u_i(g) - u_i(g^*))
\end{aligned}$$

For $g \in G_1$, $\mu(g, (s_i, \sigma_{-i}^{\varepsilon_t})) = \varepsilon_t.(1 - \varepsilon_t)^{l-1}$.

Then, for $l \geq p > 1$, $g_p \in G_p$ implies $\lim_{t \to +\infty} \frac{\mu(g, (s_i, \sigma_{-i}^{\varepsilon_t}))}{\varepsilon_t.(1-\varepsilon_t)^{l-1}} = 0$.

Hence, there exists $T \in \mathbb{N}$ such that for all $t \geqslant T$, for all $i \in N$, $s_i \notin S_i(s^0)$,

$$\sum_{g \in G_1 \cup \ldots \cup G_l} \mu(g, (s_i, \sigma_{-i}^{\varepsilon_t})).(u_i(g) - u_i(g^*))$$

is equivalent to

$$\sum_{g \in G_1} (u_i(g) - u_i(g^*)).$$

But $g \in G_1$ implies $u_i(g) - u_i(g^*) < 0$. So,

$$\sum_{g \in G_1} (u_i(g) - u_i(g^*)) < 0.$$

Hence, there exists $T \in \mathbb{N}$ such that for all $t \geqslant T$, for all $i \in N$, $s_i \notin S_i(s^0)$, $Eu_i(g(s_i, \sigma_{-i}^{\varepsilon_t})) - Eu_i(g(s^0)) < 0$.

Then, there exists $T \in \mathbb{N}$ such that for all $t \geqslant T$, for all $i \in N$, $s_i \in S_i(s^0)$ implies $Eu_i(g(s_i, \sigma_{-i}^{\varepsilon_t}) = Eu_i(g^*) \geqslant Eu_i(g(s_i', \sigma_{-i}^{\varepsilon_t}))$, for all $s_i' \in S_i$, and $s_i \notin S_i(s^0)$ implies $Eu_i(g(s_i, \sigma_{-i}^{\varepsilon_t})) < Eu_i(g^*)$.

Accordingly, in $\{\sigma^{\varepsilon_t}\}_{t \in \mathbb{N}}$, $s_i \in S_i(s^0)$ implies $\sigma_i^{\varepsilon_t}(s_i) \geqslant \varepsilon_t$, and $s_i \notin S_i(s^0)$ implies $\sigma_i^{\varepsilon_t}(s_i) = 0$.

Hence, $\{\sigma^{\varepsilon_t}\}_{t \in \mathbb{N}}$ has a subsequence of $\varepsilon-$trial equilibria that converges to s^0, meaning s^0 is a trial perfect equilibrium. $\square$